Scales in hybrid mice over $\mathbb{R}$

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Abstract

We analyze scales in $\mathrm{Lp}^{{}^{\mathrm{G}}\Omega}(\mathbb{R},\Omega\restriction\mathrm{HC})$, the stack of sound, projecting, Θ-g-organized Ω-mice over $\Omega\restriction\mathrm{HC}$, where Ω is either an iteration strategy or an operator, Ω has appropriate condensation properties, and $\Omega\restriction\mathrm{HC}$ is self-scaled. This builds on Steel's analysis of scales in $L(\mathbb{R})$ and $\mathrm{Lp}(\mathbb{R})$ (also denoted $K(\mathbb{R})$). As in Steel's analysis, we work from optimal determinacy hypotheses. One of the main applications of the work is in the core model induction.

1 Introduction

There has been significant progress made in the core model induction in recent years. Pioneered by W. H. Woodin and further developed by J. R. Steel, R. D. Schindler and others, it is a powerful method for obtaining lower-bound consistency strength for a large class of theories. One of the key ingredients is the scales analysis in $L(\mathbb{R})$, and further, in $\mathrm{Lp}(\mathbb{R})$ (also denoted $K(\mathbb{R})$); see Steel's [16], [18] and [19]. Applications include Woodin's proof of $\mathsf{AD}^{L(\mathbb{R})}$ from an ω_1-dense ideal on ω_1 and Steel's proof that PFA implies $\mathsf{AD}^{L(\mathbb{R})}$, amongst many others.

To use the core model induction for stronger results (for example, to construct models of "$\mathsf{AD}^+ + \Theta > \Theta_0$") one would like to have a scales analysis

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for *hybrid* mice over $\mathbb{R}$ – structures beyond $\mathrm{Lp}(\mathbb{R})$. In this paper we present such an analysis. There have been recent works that make use of methods and results from this paper, for example [21], [4], and [7].

This paper owes a strong debt to Steel's scale constructions in [16], [18] and [19], and to Sargsyan's notion of *reorganized hod premouse*, [5, §3.7]. Indeed, these are the two main components, and the main work here is in putting them together.

For the purposes mentioned above, one would particularly like to have a scales analysis for something like $\mathrm{Lp}^{\Sigma}(\mathbb{R})$, the stack of "projecting Σ-mice over $\mathbb{R}$", where Σ is an iteration strategy with hull condensation. Unfortunately, the usual definition[1] of "Σ-premouse over $\mathbb{R}$" doesn't make sense, because $\mathbb{R}$ is not wellordered. One might try to get around this particular issue by arranging Σ-premice by simultaneously feeding in multiple branches instead of feeding them in one by one. But it seems difficult to define an amenable predicate achieving this, as discussed in 3.52. Even if one could arrange this amenably, the scale constructions in [18] and [19] do not appear to generalize well with such an approach, because of their dependence on the close relationship between a mouse over $\mathbb{R}$ and its local HOD.

We deal with these problems here by using the hierarchy of Θ-*g-organized* Σ-*premice*, a kind of strategy premouse. The definition is a simple variant of *g-organization*, which is essentially due to Sargsyan; its main content is just that of the *reorganization* of hod premice. We similarly define (Θ-)g-*organized* $\mathcal{F}$-*premice* for operators $\mathcal{F}$, where *operators* are defined in [11].[2] Given either $\Omega = \Sigma$ or $\Omega = \mathcal{F}$ as above, we only define (Θ-)g-organization assuming that (Ω, X) is *nice* for some $X \in \mathrm{HC}$; this demands both a degree of *condensation* and of *generic determination* of Ω; see 3.8.

Given a nice (Ω, X) and *self-scaled* $\Upsilon \subseteq \mathrm{HC}$ (see 3.45; this holds for $\Upsilon = \emptyset$) we define $\mathrm{Lp}^{\mathrm{G}\Omega}(\mathbb{R}, \Upsilon)$ as the stack of all sound, countably iterable Θ-g-organized Ω-premice built over (HC, Υ), projecting to $\mathbb{R}$. We will ana-

[1] Roughly, that is: Given Σ-premice $\mathcal{N} \trianglelefteq \mathcal{M}$, with $\mathcal{N}$ reasonably closed, and letting $\mathcal{T}$ be the $<_{\mathcal{N}}$-least iteration tree for which $\mathcal{N}$ lacks instruction regarding the branch $b = \Sigma(\mathcal{T})$, then b is the next piece of information fed in to $\mathcal{M}$ after $\mathcal{N}$.

[2] Many readers will probably be comfortable reading the present paper without knowledge of [11], as the particulars of [11] are not strongly related to our purposes here. In fact, one could completely ignore the role of operators and focus entirely on strategy mice, without losing any of the main ideas. There is significant overlap between [11] and §2 of the present paper. For better readability, the common themes are generally presented in both papers. A few things are omitted in one, but can be seen in the other.

lyze scales in this structure. If $\Upsilon = \Omega{\restriction}\mathrm{HC}$, the analysis can be done from optimal determinacy assumptions. We remark that when $\mathrm{Lp}^{\Omega}(\mathbb{R}, \Upsilon)$ is well-defined (such as when Ω is a mouse operator), we usually have $\mathrm{Lp}^{\Omega}(\mathbb{R}, \Upsilon) \neq \mathrm{Lp}^{\mathrm{G}\Omega}(\mathbb{R}, \Upsilon)$, but if Ω *relativizes well* (or something similar to this; see [15, Definition 1.3.21(?)]), the two hierarchies agree on their $\mathfrak{P}(\mathbb{R})$, and actually have identical extender sequences (see 4.11).

The scale constructions themselves are mostly a fairly straightforward generalization of Steel's work in [16], [18], [19]; we assume that the reader is familiar with these.[3] Let (Ω, X) be nice and Υ self-scaled, and let $\mathcal{M}$ end a weak gap of $\mathrm{Lp}^{\mathrm{G}\Omega}(\mathbb{R}, \Upsilon)$. The construction of new scales over such $\mathcal{M}$ breaks into three cases, covered in Theorems 5.17, 5.22 and 5.26; these are analogous to [18, Theorems 4.16, 4.17] and [19, Theorem 0.1] respectively. Thus, for the first we must assume that $\mathcal{J}(\mathcal{M}) \models \mathsf{AD}$. In the context of our primary application (core model induction), this assumption *will* hold *if* $\Omega{\restriction}\mathrm{HC} \notin \mathcal{M}|\alpha$ and there are *no divergent* AD *pointclasses*; see 5.55. For the latter two the determinacy assumption is just that $\mathcal{M} \models \mathsf{AD}$, but there are also other assumptions necessary. If $\Upsilon = \Omega{\restriction}\mathrm{HC}$ then the latter two theorems cover all weak gaps, and so one never needs to assume that $\mathcal{J}(\mathcal{M}) \models \mathsf{AD}$.

We won't reproduce all the details of the proofs in [18] and [19], but will focus on the new features, and fill in some omissions. The most significant of the new features are as follows. First, we must generalize the local HOD analysis of a level $\mathcal{M}$ of $\mathrm{Lp}(\mathbb{R})$ to that of a level $\mathcal{M}$ of $\mathrm{Lp}^{\mathrm{G}\Omega}(\mathbb{R}, \Upsilon)$. As in [18], we establish a level-by-level fine-structural correspondence between $\mathcal{H}$, the local HOD of $\mathcal{M}$, and $\mathcal{M}$ itself, above $\Theta^{\mathcal{M}}$. The fact that we are using Θ-g-organization is very important in establishing this correspondence (and as for $\mathrm{Lp}(\mathbb{R})$, the correspondence itself is very important in the scales analysis). Second, an issue not dealt with in [19], but with which we deal here, is that a short tree $\mathcal{T}$ on a k-suitable premouse $\mathcal{N}$ may introduce Q-structures with extenders overlapping $\delta(\mathcal{T})$. Third, a new case arises in the scale constructions – at the end of a gap $[\alpha, \beta]$ of $\mathcal{M}$ where $\mathcal{M}|\beta$ is *P-active*; that is, strategy information is encoded in the predicate of $\mathcal{M}|\beta$. (It seems this case could have been avoided, however, if we had arranged our strategy premice slightly differently.)

The paper is organized as follows. In §2 we discuss *strategy premice* (in

[3]One needs familiarity with said papers for §§4,5 of this paper. If the reader has familiarity with just [16], one might read the present paper, referring to [18] and [19] as needed to fill in details we omit here.

the sense of *iteration strategy*) in detail, give a new presentation of these, and prove some condensation properties thereof, assuming that the iteration strategy involved has hull condensation and has a simply definable domain. In §3 we discuss g-organized and Θ-g-organized Ω-premice, and prove related condensation facts. In §4 we analyse the local HOD of $\mathcal{M} \triangleleft \mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R}, \Upsilon)$. In §5 we analyse the pattern of scales in $\mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R}, \Upsilon)$.

1.1 Conventions and Notation

We use **boldface** to indicate a term being defined (though when we define symbols, these are in their normal font). Citations such as [10, Theorem 3.1(?)] are used to indicate a referent that may change in time – that is, at the time of writing, [10] is a preprint and its Theorem 3.1 is the intended referent.

We work under ZF throughout the paper, indicating choice assumptions where we use them ($\mathsf{DC}_{\mathbb{R}}$ in particular will be assumed for various key facts). We write DC_A for the restriction of DC to relations on A. Ord denotes the class of ordinals. Given a transitive set M, $\mathrm{o}(\mathcal{M})$ denotes $\mathrm{Ord} \cap M$. We write $\mathrm{card}(X)$ for the cardinality of X, $\mathfrak{P}(X)$ for the power set of X, and for $\theta \in \mathrm{Ord}$, $\mathfrak{P}(<\theta)$ is the set of bounded subsets of θ and $\mathscr{H}_\theta$ the set of sets hereditarily of size $<\theta$. We write $f : X \dashrightarrow Y$ to denote a partial function.

We identify $\in [\mathrm{Ord}]^{<\omega}$ with the strictly decreasing sequences of ordinals, so given $p, q \in [\mathrm{Ord}]^{<\omega}$, $p{\restriction}i$ denotes the upper i elements of p, and $p \trianglelefteq q$ means that $p = q{\restriction}i$ for some i, and $p \triangleleft q$ iff $p \trianglelefteq q$ but $p \neq q$. The default ordering of $[\mathrm{Ord}]^{<\omega}$ is lexicographic (largest element first), with $p < q$ if $p \triangleleft q$.

Let $\mathcal{M} = (X, A_1, \ldots)$ be a first-order structure with universe X and predicates, constants, etc, $A_1, \ldots$. We write $\lfloor \mathcal{M} \rfloor$ for X. If $\mathcal{L}$ is the first-order language of $\mathcal{M}$, then definability over $\mathcal{M}$ uses $\mathcal{L}$, unless otherwise specified. If $\mathcal{L}' \subseteq \mathcal{L}$, then, for example, $\Sigma_1(\mathcal{L}')$ denotes the Σ_1 formulas of $\mathcal{L}'$, and if $X \subseteq \mathcal{M}$, then $\Sigma_1^{\mathcal{M}}(\mathcal{L}', X)$ denotes the relations which are $\Sigma_1(\mathcal{L}')$-definable over $\mathcal{M}$ from parameters in X. A **transitive structure** is a first-order structure with with transitive universe. We sometimes blur the distinction between the terms *transitive* and *transitive structure*. For example, when we refer to a transitive structure as being **rud closed**, it means that its universe is rud closed. For $\mathcal{M}$ a transitive structure, $\mathrm{o}(\mathcal{M}) = \mathrm{o}(\lfloor \mathcal{M} \rfloor)$. An arbitrary transitive set X is also considered as the transitive structure (X). We write $\mathrm{trancl}(X)$ for the transitive closure of X.

Given a transitive structure $\mathcal{M}$, we write $\mathcal{J}_\alpha(\mathcal{M})$ for the α^{th} step in Jensen's $\mathcal{J}$-hierarchy over $\mathcal{M}$ (for example, $\mathcal{J}_1(\mathcal{M})$ is the rud closure of $\text{trancl}(\{\mathcal{M}\})$. We similarly use $\mathcal{S}$ to denote the function giving Jensen's more refined $\mathcal{S}$-hierarchy. And $\mathcal{J}(\mathcal{M}) = \mathcal{J}_1(\mathcal{M})$.

We take (standard) **premice** as in [20], and our definition and theory of *strategy premice* is modelled on [20],[3]. Throughout, we define most of the notation we use, but hopefully any unexplained terminology is either standard or as in those papers. The article also uses a small part of the theory (and notation) of hod mice, as covered in the first parts of [5]. (However, the main scale calculations are not related particularly to hod mice, and can be understood without knowing any theory thereof.) For discussion of generalized solidity witnesses, see [24].

Our notation pertaining to iteration trees is fairly standard, but here are some points. For $\mathcal{T}$ a putative iteration tree, we write $\leq_\mathcal{T}$ for the tree order of $\mathcal{T}$ and $\text{pred}^\mathcal{T}$ for the $\mathcal{T}$-predecessor function. Let $\alpha + 1 < \text{lh}(\mathcal{T})$ and $\beta = \text{pred}^\mathcal{T}(\alpha + 1)$. Then $M^{*\mathcal{T}}_{\alpha+1}$ denotes the $\mathcal{N} \trianglelefteq M^\mathcal{T}_\beta$ such that $M^\mathcal{T}_{\alpha+1} = \text{Ult}_n(\mathcal{N}, E)$, where $n = \deg^\mathcal{T}(\alpha + 1)$ and $E = E^\mathcal{T}_\alpha$, and $i^{*\mathcal{T}}_{\alpha+1}$ denotes $i^\mathcal{N}_E$, for this $\mathcal{N}, E$. And for $\alpha + 1 \leq_\mathcal{T} \gamma$, $i^{*\mathcal{T}}_{\alpha+1,\gamma} = i^\mathcal{T}_{\alpha+1,\gamma} \circ i^{*\mathcal{T}}_{\alpha+1}$. Also let $M^{*\mathcal{T}}_0 = M^\mathcal{T}_0$ and $i^{*\mathcal{T}}_0 = \text{id}$. If $\text{lh}(\mathcal{T}) = \gamma + 1$ then $M^\mathcal{T}_\infty = M^\mathcal{T}_\gamma$, etc, and $b^\mathcal{T}$ denotes $[0, \gamma]_\mathcal{T}$.

A premouse $\mathcal{P}$ is **η-sound** iff for every $n < \omega$, if $\eta < \rho^\mathcal{P}_n$ then $\mathcal{P}$ is n-sound, and if $\rho^\mathcal{P}_{n+1} \leq \eta$ then letting $p = p^\mathcal{P}_{n+1}$, $p \backslash \eta$ is $(n + 1)$-solid for $\mathcal{P}$, and $\mathcal{P} = \text{Hull}^\mathcal{P}_{n+1}(\eta \cup p)$. The **$\eta$-core** of $\mathcal{P}$ is $\text{cHull}^\mathcal{P}_{n+1}(\eta \cup p^\mathcal{P}_{n+1})$. Here Hull and cHull are as defined in 2.21.

2 Strategy premice

Definition 2.1. Let Y be transitive. Then $\varrho_Y : Y \to \text{rank}(Y)$ denotes the rank function. And $\hat{Y}$ denotes $\text{trancl}(\{(Y, \omega, \varrho_Y)\})$. For M transitive, we say that M is **rank closed** iff for every $Y \in M$, we have $\hat{Y} \in M$ and $\hat{Y}^{<\omega} \in M$. Note that if M is rud closed and rank closed then $\text{rank}(M) = \text{Ord} \cap M$. $\dashv$

Definition 2.2 ($\mathcal{J}$-structure)**.** Let $\alpha \in \text{Ord}\backslash\{0\}$, let y be transitive, $Y = \hat{y}$,

$$D = \text{Lim} \cap [\text{o}(Y) + \omega, \text{o}(Y) + \omega\alpha)$$

and let $\vec{P} = \langle P_\beta \rangle_{\beta \in D}$ be given.

We define $\mathcal{J}^{\vec{P}}_\beta(Y)$ for $\beta \in [1, \alpha]$, if possible, by recursion on β, as follows. We set $\mathcal{J}^{\vec{P}}_1(Y) = \mathcal{J}(Y)$ and take unions at limit β. For $\beta + 1 \in [2, \alpha]$, let

$R = \mathcal{J}_\beta^{\vec{P}}(Y)$ and suppose that $\vec{P}_{\mathrm{o}(R)} = (P_0, \ldots, P_{n-1})$ for some $n < \omega$, and that for each $i < n$, $P_i \subseteq R$ and is amenable to R. In this case we define

$$\mathcal{J}_{\beta+1}^{\vec{P}}(Y) = \mathcal{J}(R, \vec{P}{\restriction}R, P_0, \ldots, P_{n-1}).$$

Note then that by induction, $\vec{P}{\restriction}R \subseteq R$ and $\vec{P}{\restriction}R$ is amenable to R.

For $m < \omega$ let $\mathcal{L}_{\mathcal{J},m}$ be the language with binary relation symbol $\dot{\in}$, predicate symbols $\dot{\vec{P}}$ and $\dot{P}_i$ for $i < m$, and constant symbol cb.

Let $m < \omega$. An m**-$\mathcal{J}$-structure over** Y is an amenable $\mathcal{L}_{\mathcal{J},m}$-structure

$$\mathcal{M} = (\mathcal{J}_\alpha^{\vec{P}}(Y), \in^{\mathcal{M}}, \vec{P}, Y; P_0, \ldots, P_{m-1}),$$

where $\alpha \in \mathrm{Ord}\backslash\{0\}$ and $\vec{P} = \left\langle \vec{P}_\gamma \right\rangle_{\gamma \in D}$ with domain D defined as above, the universe $\lfloor \mathcal{M} \rfloor = \mathcal{J}_\alpha^{\vec{P}}(Y)$ is defined, $\dot{\in}^{\mathcal{M}} = \in \cap \lfloor \mathcal{M} \rfloor$, $\mathrm{lh}(\vec{P}_\gamma) = n$ for each $\gamma \in D$, $\dot{\vec{P}}^{\mathcal{M}} = \vec{P}$, $\dot{P}_i^{\mathcal{M}} = P_i$, and $cb^{\mathcal{M}} = Y$.

Let $\mathcal{M}$ be a m-$\mathcal{J}$-structure over Y, and adopt the notation above. Let $l(\mathcal{M})$ denote α. For $\beta \in [1, \alpha]$ and $R = \mathcal{J}_\beta^{\vec{P}}(Y)$ and $\gamma = \mathrm{o}(R)$, let

$$\mathcal{M}|\beta = (R, \in \cap R, \vec{P}{\restriction}R, Y; \vec{P}_{\gamma,0}, \ldots, \vec{P}_{\gamma,m-1})$$

where $\vec{P}_{\mathrm{o}(\mathcal{M}),i} = P_i$. We write $\mathcal{N} \trianglelefteq \mathcal{M}$, and say that $\mathcal{N}$ is an **initial segment** of $\mathcal{M}$, iff $\mathcal{N} = \mathcal{M}|\beta$ for some β. Clearly if $\mathcal{N} \trianglelefteq \mathcal{M}$ then $\mathcal{N}$ is an m-$\mathcal{J}$-structure over Y. We write $\mathcal{N} \triangleleft \mathcal{M}$, and say that $\mathcal{N}$ is a **proper segment** of $\mathcal{M}$, iff $\mathcal{N} \trianglelefteq \mathcal{M}$ but $\mathcal{N} \neq \mathcal{M}$.

A **$\mathcal{J}$-structure** is an m-$\mathcal{J}$-structure, for some m. ⊣

Definition 2.3. A $\mathcal{J}$-structure $\mathcal{M}$ over A is **acceptable** iff for all $\mathcal{N} \triangleleft \mathcal{M}$ and all $\alpha < \mathrm{o}(\mathcal{N})$, if there is $X \subseteq A^{<\omega} \times \alpha^{<\omega}$ such that $X \in \mathcal{J}(\mathcal{N})\backslash\mathcal{N}$, then in $\mathcal{J}(\mathcal{N})$ there is a map $A^{<\omega} \times \alpha^{<\omega} \stackrel{\text{onto}}{\to} \mathcal{N}$. ⊣

The following lemma is proven just like the corresponding fact for L.

Lemma 2.4. *Let $\mathcal{M}$ be a $\mathcal{J}$-structure over A. Then there is a map, which we denote $h^{\mathcal{M}}$, such that*

$$h^{\mathcal{M}} : A^{<\omega} \times l(\mathcal{M})^{<\omega} \stackrel{\text{onto}}{\to} \mathcal{M}$$

whose graph is $\Sigma_1^{\mathcal{M}\restriction\mathcal{L}_{\mathcal{J},0}}$, uniformly in $\mathcal{M}$. Moreover, for $\mathcal{N} \trianglelefteq \mathcal{M}$, we have $h^{\mathcal{N}} \subseteq h^{\mathcal{M}}$.

Definition 2.5. Let $\mathcal{M}$ be an acceptable $\mathcal{J}$-structure over A and $\rho < \mathrm{o}(\mathcal{M})$. Then ρ is an **A-cardinal** of $\mathcal{M}$ iff $\mathcal{M}$ has no map $A^{<\omega} \times \gamma^{<\omega} \overset{\text{onto}}{\to} \rho$ where $\gamma < \rho$. Let $\Theta^{\mathcal{M}}$ denote the least A-cardinal of $\mathcal{M}$, if such exists. We say that ρ is **A-regular** in $\mathcal{M}$ iff $\mathcal{M}$ has no map $A^{<\omega} \times \gamma^{<\omega} \overset{\text{cof}}{\to} \rho$ where $\gamma < \rho$. We say that ρ is an **ordinal-cardinal** of $\mathcal{M}$ iff $\mathcal{M}$ has no map $\gamma^{<\omega} \overset{\text{onto}}{\to} \rho$ where $\gamma < \rho$. $\dashv$

Lemma 2.6. *Let $\mathcal{M}$ be an acceptable $\mathcal{J}$-structure over A and $0 < \xi < l(\mathcal{M})$. Let κ be an A-cardinal of $\mathcal{M}$ such that $\kappa \leq \mathrm{o}(\mathcal{M}|\xi)$. Then $\mathrm{rank}(A) < \kappa \leq \xi$ and $\kappa = \mathrm{o}(\mathcal{M}|\kappa)$.*

Lemma 2.7. *There is a Σ_1 formula $\varphi \in \mathcal{L}_{\mathcal{J},0}$ such that, for any acceptable $\mathcal{J}$-structure $\mathcal{M}$ over A, we have the following.*

Suppose $\Theta = \Theta^{\mathcal{M}}$ exists. Then:

1. *Θ is the least α such that $\mathfrak{P}(A^{<\omega})^{\mathcal{M}} \subseteq \mathcal{M}|\alpha$.*
2. *$\lfloor \mathcal{M}|\Theta \rfloor$ is the set of all $x \in \mathcal{M}$ such that $\mathrm{trancl}(x)$ is the surjective image of $A^{<\omega}$ in $\mathcal{M}$.*
3. *Over $\mathcal{M}|\Theta$, $\varphi(0,\cdot,\cdot)$ defines a function $G : \Theta \to \mathcal{M}|\Theta$ such that for all $\alpha < \Theta$, we have $G(\alpha) : A^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\alpha$.*
4. *Θ is A-regular in $\mathcal{M}$.*

Let $\kappa_0 < \kappa_1$ be consecutive A-cardinals of $\mathcal{M}$. Then:

5. *κ_1 is the least α such that $\mathfrak{P}(A^{<\omega} \times \kappa_0^{<\omega})^{\mathcal{M}} \subseteq \mathcal{M}|\alpha$.*
6. *$\lfloor \mathcal{M}|\kappa_1 \rfloor$ is the set of all $x \in \mathcal{M}$ such that $\mathrm{trancl}(x)$ is the surjective image of $A^{<\omega} \times \kappa_0^{<\omega}$ in $\mathcal{M}$.*
7. *Over $\mathcal{M}|\kappa_1$, $\varphi(\kappa_0,\cdot,\cdot)$ defines a map $G : \kappa_1 \to \mathcal{M}|\kappa_1$ such that for all $\alpha < \kappa_1$, we have $G(\alpha) : A^{<\omega} \times \kappa_0^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\alpha$.*
8. *κ_1 is A-regular in $\mathcal{M}$.*

Proof. We just prove parts 1–4; the others are similar.

Let γ be least such that $\mathfrak{P}(A^{<\omega}) \cap \mathcal{M} \subseteq \mathcal{M}|\gamma$. Then γ is a limit ordinal. By acceptability, for every $\alpha < \gamma$, $\mathcal{M}|\gamma$ has a map $A^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\alpha$.

Now suppose that $\gamma < \Theta$, and let $g : A^{<\omega} \stackrel{\text{onto}}{\to} \gamma^{<\omega}$ be in $\mathcal{M}$. Let $h = h^{\mathcal{M}|\gamma}$. Then because $g, h \in \mathcal{M}$, clearly $\mathcal{M}$ has a map $f : A^{<\omega} \stackrel{\text{onto}}{\to} \mathcal{M}|\gamma$, so $\mathcal{M}$ has a map $A^{<\omega} \stackrel{\text{onto}}{\to} \mathfrak{P}(A^{<\omega})^{\mathcal{M}}$, a contradiction.

So $\gamma = \Theta$, which gives parts 1,2.

Now consider part 3. Let $\alpha < \Theta$. We will define $g : A^{<\omega} \times A^{<\omega} \stackrel{\text{onto}}{\to} \mathcal{M}|\alpha$, and the uniformity in the definition will yield the result. Let $\beta \in [\alpha, \Theta)$ be least such that

$$\mathfrak{P}(A^{<\omega}) \cap \mathcal{M}|\beta \not\subseteq \mathcal{M}|\alpha.$$

Let $h = h^{\mathcal{M}|\beta}$. Let $x \in A^{<\omega}$ be such that for some y, $f = h(x, y)$ is such that $f : A^{<\omega} \to \mathcal{M}|\alpha$ is a surjection (such x exists by acceptability). Let y be least such, and $f = h(x, y)$. Then for $z \in A^{<\omega}$, define $g(x, z) = f(z)$. For all other (x, z), $g(x, z) = \emptyset$. This completes the definition of g, and the uniformity is clear.

Part 4 now follows. $\square$

Corollary 2.8. *Let $\mathcal{M}$ be an acceptable $\mathcal{J}$-structure over A and let γ be an A-cardinal of $\mathcal{M}$. If γ is a limit of A-cardinals of $\mathcal{M}$ then $\mathcal{M}|\gamma$ satisfies Separation and Power Set. If γ is not a limit of A-cardinals of $\mathcal{M}$ then $\mathcal{M}|\gamma \models \mathsf{ZF}^-$. In particular, $\mathcal{M}|\Theta^{\mathcal{M}} \models \mathsf{ZF}^-$.*

Lemma 2.9. *Let $\mathcal{M}$ be an acceptable $\mathcal{J}$-structure over A such that $\Theta^{\mathcal{M}}$ exists. Let $\kappa \in [\Theta^{\mathcal{M}}, \mathrm{o}(\mathcal{M}))$. Then κ is an A-cardinal of $\mathcal{M}$ iff κ is an ordinal-cardinal of $\mathcal{M}$.*

Proof. Suppose $\kappa > \Theta = \Theta^{\mathcal{M}}$ and κ is an ordinal-cardinal, but $\mathcal{M}$ has a map

$$f : A^{<\omega} \times \gamma^{<\omega} \stackrel{\text{onto}}{\to} \kappa$$

where $\gamma < \kappa$. For each $y \in \gamma^{<\omega}$, let $f_y : A^{<\omega} \to \kappa$ be $f_y(x) = f(x, y)$, and let g_y be the norm associated to f_y (that is, $f_y(x) < f_y(x')$ iff $g_y(x) < g_y(x')$, and $\mathrm{rg}(g_y)$ is an ordinal). Then $g_y \in \mathcal{M}$ and $\mathrm{rg}(g_y) < \Theta$, because the prewellorder on $A^{<\omega}$ determined by f_y is in $\mathcal{M}|\Theta$ and $\mathcal{M}|\Theta \models \mathsf{ZF}^-$. Similarly, the function $y \mapsto (f_y, g_y)$ is in $\mathcal{M}$. Let

$$h : \Theta \times \gamma^{<\omega} \stackrel{\text{onto}}{\to} \kappa$$

be as follows. Let $(\alpha, y) \in \Theta \times \gamma^{<\omega}$. If $\alpha \notin \mathrm{rg}(g_y)$ then $h(\alpha, y) = 0$; otherwise $h(\alpha, y) = f(x, y)$ where $g_y(x) = \alpha$. Then $h \in \mathcal{M}$, a contradiction. $\square$

Definition 2.10. Let $\mathcal{M}$ be an acceptable $\mathcal{J}$-structure over A and let $\kappa < \mathrm{o}(\mathcal{M})$. Then $(\kappa^+)^{\mathcal{M}}$ denotes either the least ordinal-cardinal γ of $\mathcal{M}$ such that $\gamma > \kappa$, if there is such, and denotes $\mathrm{o}(\mathcal{M})$ otherwise. By 2.9, if $\Theta^{\mathcal{M}} \leq \kappa$, then $(\kappa^+)^{\mathcal{M}}$ is the least A-cardinal γ of $\mathcal{M}$ such that $\gamma > \kappa$, if there is such, or is $\mathrm{o}(\mathcal{M})$ otherwise. This applies when $E \neq \emptyset$ in 2.11 below. $\dashv$

Definition 2.11. Let $\mathcal{L} = \mathcal{L}_{\mathcal{J},2} \cup \{\dot{cp}, \dot{\Psi}\}$, where $\dot{cp}, \dot{\Psi}$ are constant symbols. Let $\dot{E} = \dot{P}_0$ and $\dot{P} = \dot{P}_1$.

Let a be transitive and $A = \hat{a}$. A **potential hybrid premouse (hpm) over** A is an amenable $\mathcal{L}$-structure

$$\mathcal{M} = (\mathcal{J}_\alpha^{\vec{P}}(A), \in^{\mathcal{M}}, \vec{P}, A; E, P; cp, \Psi)$$

where $\dot{E}^{\mathcal{M}} = E$, etc, with the following properties:

1. $\bar{\mathcal{M}} = \mathcal{M}{\restriction}\mathcal{L}_{\mathcal{J},2}$ is a 2-$\mathcal{J}$-structure.
2. Either $P = \emptyset$ or $E = \emptyset$.
3. If $E \neq \emptyset$ then α is a limit and there is an extender F over $\mathcal{M}$ such that:
   - $\mathrm{rank}(A) < \mu = \mathrm{crit}(F)$,
   - F is $A^{<\omega} \times \gamma^{<\omega}$-complete for all $\gamma < \mu$,
   - E is the amenable code for F, as in [20], and the premouse axioms [22, Definition 2.2.1] hold for $(\lfloor \mathcal{M} \rfloor, \vec{P}, E)$.

   (It follows that $\mathcal{M}$ has a largest cardinal δ, and $\delta \leq i_F(\mu)$, and $\mathrm{o}(\mathcal{M}) = (\delta^+)^U$ where $U = \mathrm{Ult}(\mathcal{M}, F)$, and $i_F(\vec{P}{\restriction}(\mu^+)^{\mathcal{M}}){\restriction}\mathrm{o}(\mathcal{M}) = \vec{P}$.)
4. For every $\bar{\mathcal{N}} \trianglelefteq \bar{\mathcal{M}}$, $\mathcal{N} = (\bar{\mathcal{N}}; cp, \Psi)$ is a potential hybrid premouse over A (so $cp, \Psi \in \mathcal{J}(A)$).

Let $\mathcal{M}$ be a potential hpm. We write $\mathcal{N} \trianglelefteq \mathcal{M}$ iff $\mathcal{N}$ is as above. Likewise $\mathcal{N} \triangleleft \mathcal{M}$. For $\alpha \leq l(\mathcal{M})$, $\mathcal{M}|\alpha$ denotes the $\mathcal{N} \trianglelefteq \mathcal{M}$ such that $l(\mathcal{N}) = \alpha$, and $\mathcal{M}||\alpha$ denotes the potential hpm $\mathcal{N}$ which is the same as $\mathcal{M}|\alpha$, except that $E^{\mathcal{N}} = \emptyset$. (So $P^{\mathcal{M}||\alpha} = P^{\mathcal{M}|\alpha}$ always, which will help ensure that $P^{\mathcal{M}||\alpha}$ is the kind of structure we want to consider.) $\dashv$

Remark 2.12. Let $\mathcal{N}$ be a potential hpm over A. Suppose $E^{\mathcal{N}}$ codes an extender F. Clearly $\kappa = \mathrm{crit}(F) > \Theta^{\mathcal{M}} > \mathrm{rank}(A)$. By [22, Definition 2.2.1], we have $(\kappa^+)^{\mathcal{M}} < \mathrm{o}(\mathcal{M})$; cf. 2.7. Note that *we allow F to be of superstrong type* (see 2.14) in accordance with [22], not [20, Definition 2.4].[4]

Remark 2.13. From now on we will omit "$\in^{\mathcal{M}}$" from the list of predicates for $\mathcal{J}$-structures $\mathcal{M}$.

Definition 2.14. Let $\mathcal{M}$ be a potential hpm over A. We say that $\mathcal{M}$ is ***E*-active** iff $E^{\mathcal{M}} \neq \emptyset$, and ***P*-active** iff $P^{\mathcal{M}} \neq \emptyset$. **Active** means either E-active or P-active. ***E*-passive** means not E-active. ***P*-passive** means not P-active. **Passive** means not active. **Type 0** means passive. **Type 4** means P-active. **Type 1, 2** or **3** mean E-active, with the usual distinctions.

We write $F^{\mathcal{M}}$ for the extender F coded by $E^{\mathcal{M}}$ (where $F = \emptyset$ if $E^{\mathcal{M}} = \emptyset$). We write $\mathbb{E}^{\mathcal{M}}$ for the function with domain $l(\mathcal{M})$, sending $\alpha \mapsto F^{\mathcal{M}|\alpha}$. Likewise for $\mathbb{E}_+^{\mathcal{M}}$, but with domain $l(\mathcal{M}) + 1$.

If $F = F^{\mathcal{M}} \neq \emptyset$, we say $\mathcal{M}$, or F, is **superstrong** iff $i_F(\mathrm{crit}(F)) = \nu(F)$. We say that $\mathcal{M}$ is **super-small** iff $\mathcal{M}$ has no superstrong whole segment. We define $\mathcal{M}^{\mathrm{sq}}$ as in [3]. (Unless $\mathcal{M}$ is type 3, we have $\mathcal{M}^{\mathrm{sq}} = \mathcal{M}$.) ⊣

Definition 2.15. Let $\mathcal{L}^- = \mathcal{L}\backslash\{\dot{E}, \dot{P}\}$. Let $\mathcal{L}^+ = \mathcal{L} \cup \{\dot{\mu}, \dot{e}\}$, where $\dot{\mu}, \dot{e}$ are constant symbols.

Let $\mathcal{N}$ be a potential hpm over A.

If $\mathcal{N}$ is E-active then $\mu^{\mathcal{N}} =_{\mathrm{def}} \mathrm{crit}(F^{\mathcal{N}})$, and otherwise $\mu^{\mathcal{N}} =_{\mathrm{def}} \emptyset$.

If $\mathcal{N}$ is E-active type 2 then $e^{\mathcal{N}}$ denotes the trivial completion of the largest non-type Z proper segment of F; otherwise $e^{\mathcal{N}} =_{\mathrm{def}} \emptyset$.[5]

If $\mathcal{N}$ is not type 3 then $\mathfrak{C}_0(\mathcal{N}) = \mathcal{N}^{\mathrm{sq}}$ denotes the $\mathcal{L}^+$-structure $(\mathcal{N}, \mu^{\mathcal{N}}, e^{\mathcal{N}})$ (with $\dot{\mu}^{\mathcal{N}} = \mu^{\mathcal{N}}$ etc).

If $\mathcal{N}$ is type 3 then define the $\mathcal{L}^+$-structure $\mathfrak{C}_0(\mathcal{N}) = \mathcal{N}^{\mathrm{sq}}$ essentially as in [3]; so letting $\vec{P} = \vec{P}^{\mathcal{N}}$ and $\nu = \nu(F^{\mathcal{N}})$,

$$\mathcal{N}^{\mathrm{sq}} = (\mathcal{J}_\nu^{\vec{P}\restriction\nu}(A), \vec{P}\restriction\nu, A; E', \emptyset; cp^{\mathcal{N}}, \Psi^{\mathcal{N}}, \mu^{\mathcal{N}}, \emptyset)$$

where E' is defined as usual. We also let $(\mathcal{N}^{\mathrm{sq}})^{\mathrm{unsq}} = \mathcal{N}$. ⊣

[4] The main point of permitting superstrong extenders is that it simplifies certain things. But it complicates others. If the reader prefers, one could instead require that F *not* be superstrong, but various statements throughout the paper regarding condensation would need to be modified, along the lines of [3, Lemma 3.3].

[5] In [3], the (analogue of) e is referred to by its code $\gamma^{\mathcal{M}}$. We use e instead because this does not depend on having (and selecting) a wellorder of $\mathcal{M}$.

Definition 2.16. **$\mathcal{L}^+$-Q-formulas** and **$\mathcal{L}^+$-P-formulas** are defined analogously to in [3, §§2,3], using the language $\mathcal{L}^+$, but with the rΣ_1 of [3] replaced by Σ_1. $\dashv$

Lemma 2.17. *There are $\mathcal{L}^+$-Q-formulas $\varphi_0, \varphi_1, \varphi_2, \varphi_4$ and an $\mathcal{L}^+$-P-formula φ_3, such that for all wellfounded $\mathcal{L}^+$-structures $\mathcal{N}$ with $\mu^{\mathcal{N}} \in \mathrm{Ord}(\mathcal{N})$:*

- *For $i \in \{0, 1, 2, 4\}$, $\mathcal{N} \models \varphi_i$ iff $\mathcal{N} = \mathfrak{C}_0(\mathcal{M})$ for some type i potential hpm $\mathcal{M}$.*
- *If $\mathcal{N} = \mathfrak{C}_0(\mathcal{M})$ for a type 3 potential hpm $\mathcal{M}$ then $\mathcal{N} \models \varphi_3$, and if $\mathcal{N} \models \varphi_3$ then $E^{\mathcal{N}}$ codes an extender F over $\mathcal{N}$ and if $\mathrm{Ult}(\mathcal{N}, F)$ is wellfounded then $\mathcal{N} = \mathfrak{C}_0(\mathcal{M})$ for a type 3 potential hpm.*

Proof. This is a routine adaptation of the analogues [3, Lemma 2.5], [3, Lemma 3.3] respectively, with the added point that we can drop the clause "or $\mathcal{N}$ is of superstrong type" of [3, Lemma 3.3], because we allow extenders of superstrong type. $\square$

Definition 2.18. Let $\mathcal{N}$ be a potential hpm. Let $\mathcal{R}$ be an $\mathcal{L}^+$-structure (possibly illfounded). Let $\pi : \mathcal{R} \to \mathfrak{C}_0(\mathcal{N})$.

We say that π is a **weak 0-embedding** iff π is Σ_0-elementary (therefore $\mathcal{R}$ is extensional and wellfounded, so assume $\mathcal{R}$ is transitive) and there is $X \subseteq \mathcal{R}$ such that X is $\in$-cofinal in $\mathcal{R}$ and π is Σ_1-elementary on elements of X, and if $\mathcal{N}$ is type 1 or 2, then letting $\mu = \mu^{\mathcal{R}}$, there is $Y \subseteq \mathcal{R}|(\mu^+)^{\mathcal{R}} \times \mathcal{R}$ such that Y is $\in \times \in$-cofinal in $\mathcal{R}|(\mu^+)^{\mathcal{R}} \times \mathcal{R}$ and π is Σ_1-elementary on elements of Y.

Let $\mathcal{M}, \mathcal{N}$ be type i potential hpms. A **weak 0-embedding** π from $\mathcal{M}$ to $\mathcal{N}$, denoted $\pi : \mathcal{M} \to \mathcal{N}$, is a weak 0-embedding $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{N})$. (So for example, if $i = 3$ then $\mathrm{dom}(\pi) \neq \lfloor \mathcal{M} \rfloor$).) $\dashv$

Lemma 2.19. *Let $\mathcal{M}$ be a potential hpm, let $\mathcal{R}$ be an $\mathcal{L}^+$-structure and let $\pi : \mathcal{R} \to \mathfrak{C}_0(\mathcal{M})$ be a weak 0-embedding.*

Suppose $\mathcal{M}$ is type $i \neq 3$. Then $\mathcal{R} = \mathfrak{C}_0(\mathcal{N})$ for some type i potential hpm $\mathcal{N}$. In fact, for any $\mathcal{L}^+$-Q-formula φ, if $\mathfrak{C}_0(\mathcal{M}) \models \varphi$ then $\mathcal{R} \models \varphi$.

Suppose $\mathcal{M}$ is type 3. For any $\mathcal{L}^+$-P-formula φ, if $\mathfrak{C}_0(\mathcal{M}) \models \varphi$ then $\mathcal{R} \models \varphi$. If $\mathrm{Ult}(\mathcal{M}, F^{\mathcal{M}})$ is wellfounded then $\mathcal{R} = \mathfrak{C}_0(\mathcal{N})$ for some type 3 potential hpm $\mathcal{N}$.

The proof is routine, so we omit it.

Definition 2.20. Let $\mathcal{R}$ be an $\mathcal{L}^+$-structure. Let Γ be a collection of $\mathcal{L}^+$-formulas with "$x = \dot{c}$" in Γ for each constant $\dot{c} \in \mathcal{L}^+$. Let $X \subseteq \lfloor \mathcal{R} \rfloor$. Then

$$\mathrm{Hull}_\Gamma^{\mathcal{R}}(X) =_{\mathrm{def}} (H, \in', \vec{P}', cb^{\mathcal{R}}; E', P'; \dot{cp}^{\mathcal{R}}, \dot{\Psi}^{\mathcal{R}}, \dot{\mu}^{\mathcal{R}}, \dot{e}^{\mathcal{R}}),$$

where H is the set of all $y \in \lfloor \mathcal{R} \rfloor$ such that for some $\varphi \in \Gamma$ and $\vec{x} \in X^{<\omega}$, y is the unique $y' \in \mathcal{R}$ such that $\mathcal{R} \models \varphi(\vec{x}, y')$; and $\in' = \in^{\mathcal{R}} \cap H^2$ and $\vec{P}' = \vec{P}^{\mathcal{R}} \cap H$, etc. If $\mathcal{R}$ is transitive, then $\mathcal{C} = \mathrm{cHull}_\Gamma^{\mathcal{R}}(X)$ denotes the $\mathcal{L}^+$-structure which is the transitive collapse of $\mathrm{Hull}_\Gamma^{\mathcal{R}}(X)$. (That is, $\lfloor \mathcal{C} \rfloor$ is the transitive collapse of H, and letting $\pi : \lfloor \mathcal{C} \rfloor \to H$ be the uncollapse, $E^{\mathcal{C}} = \pi^{-1}(E^{\mathcal{R}})$, etc.) $\dashv$

Definition 2.21. Let $\mathcal{M}$ be a potential hpm and $\mathcal{R} = \mathfrak{C}_0(\mathcal{M})$. The **fine structural notions** for $\mathcal{M}$ are just those of $\mathcal{R}$. We sketch the definition of the **fine structural notions** for $\mathcal{R}$. For extra details refer to [3],[20]; we also adopt some simplifications explained in [9].[6] Let $A = cb^{\mathcal{R}}$.

We say that $\mathcal{R}$ is 0-**sound** and let $\rho_0^{\mathcal{R}} = \mathrm{o}(\mathcal{R})$ and $p_0^{\mathcal{R}} = \emptyset$ and $\mathfrak{C}_0(\mathcal{R}) = \mathcal{R}$ and $\mathrm{r}\Sigma_1^{\mathcal{R}} = \Sigma_1^{\mathcal{R}}$. (Here and in what follows, definability uses $\mathcal{L}^+$.)

Now let $n < \omega$ and suppose that $\mathcal{R}$ is n-sound (which will imply that $\mathcal{R} = \mathfrak{C}_n(\mathcal{R})$) and that $\omega < \rho_n^{\mathcal{R}}$. We write $\vec{p}_n^{\mathcal{R}} = (p_1^{\mathcal{R}}, \ldots, p_n^{\mathcal{R}})$. Then $\rho_{n+1}^{\mathcal{R}}$ is the least ordinal $\rho \geq \omega$ such that for some $X \subseteq A^{<\omega} \times \rho^{<\omega}$, X is $\mathrm{r}\Sigma_{n+1}^{\mathcal{R}}$ but $X \notin \lfloor \mathcal{R} \rfloor$. And $p_{n+1}^{\mathcal{R}}$ is the least tuple $p \in \mathrm{Ord}^{<\omega}$ such that some such X is

$$\mathrm{r}\Sigma_{n+1}^{\mathcal{R}}(A \cup \rho_{n+1}^{\mathcal{R}} \cup \{p, \vec{p}_n^{\mathcal{R}}\}).$$

For any $X \subseteq \lfloor \mathcal{R} \rfloor$, let

$$\mathrm{Hull}_{n+1}^{\mathcal{R}}(X) = \mathrm{Hull}_{\mathrm{r}\Sigma_{n+1}}^{\mathcal{R}}(X),$$

and $\mathrm{cHull}_{n+1}^{\mathcal{R}}(X)$ be its transitive collapse. Then we let

$$\mathcal{C} = \mathfrak{C}_{n+1}(\mathcal{R}) = \mathrm{cHull}_{n+1}^{\mathcal{R}}(A \cup \rho_{n+1}^{\mathcal{R}} \cup \vec{p}_{n+1}^{\mathcal{R}}),$$

and the uncollapse map $\pi : \mathcal{C} \to \mathcal{R}$ is the associated **core embedding**. Define $(n+1)$-**solidity** and $(n+1)$-**universality** for $\mathcal{R}$ as usual (putting all elements of A into every relevant hull). We say that $\mathcal{R}$ is $(n+1)$-**sound** iff $\mathcal{R}$ is $(n+1)$-solid and $\mathcal{C} = \mathcal{R}$ and $\pi = \mathrm{id}$.

[6] The simplifications involve dropping the parameters u_n, and replacing the use of generalized theories with pure theories. These changes are not important, and if the reader prefers, one could redefine things more analogously to [3],[20].

Now suppose that $\mathcal{R}$ is $(n+1)$-sound and $\rho_{n+1}^{\mathcal{R}} > \omega$ (so $\rho_{n+1}^{\mathcal{R}} > \mathrm{rank}(A)$). Define $T = T_{n+1}^{\mathcal{R}} \subseteq \mathcal{R}$ by letting $t \in T$ iff for some $q \in \mathcal{R}$ and $\alpha < \rho_{n+1}^{\mathcal{R}}$,

$$t = \mathrm{Th}_{\mathrm{r}\Sigma_{n+1}}^{\mathcal{R}}(A \cup \alpha \cup \{q\}).$$

(This denotes the *pure* $\mathrm{r}\Sigma_{n+1}$ theory, as opposed to the *generalized* $\mathrm{r}\Sigma_{n+1}$ theory of [3].[7]) Define $\mathrm{r}\Sigma_{n+2}^{\mathcal{R}}$ from T as usual. ⊣

Definition 2.22. Let $k \leq \omega$ and let $\mathcal{M}, \mathcal{N}$ be a k-sound potential hpms.

A **(near) k-embedding** $\pi : \mathcal{M} \to \mathcal{N}$, literally a **(near) k-embedding** $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{N})$, is analogous to the corresponding notion in [20] (but the elementarity is with respect to the language $\mathcal{L}^+$ the fine structure is that of $\mathfrak{C}_0(\mathcal{M})$ and $\mathfrak{C}_0(\mathcal{N})$). If $k \geq 1$, a **weak k-embedding** $\pi : \mathcal{M} \to \mathcal{N}$ is likewise, but analogous to the corresponding notion in [12, Definition 2.1(?)].[8] Recall that when $k = \omega$, each of these notions are equivalent with full elementarity.

A **(weakly, nearly) k-good** embedding $\pi : \mathcal{M} \to \mathcal{N}$ is a (weak, near) k-embedding $\pi : \mathcal{M} \to \mathcal{N}$ such that $cb^{\mathcal{M}} = cb^{\mathcal{N}}$ and $\pi \restriction cb^{\mathcal{M}} = \mathrm{id}$. ⊣

Definition 2.23. Let $\mathcal{N}$ be an ω-sound potential hpm. We say that $\mathcal{N}$ is **$< \omega$-condensing** (or satisfies **$< \omega$-condensation**) iff for every $k < \omega$, every $(k+1)$-sound potential hpm $\mathcal{M}$, every weak k-embedding $\pi : \mathcal{M} \to \mathcal{N}$ such that $\rho = \rho_{k+1}^{\mathcal{M}} \leq \mathrm{crit}(\pi)$ and $\rho < \rho_{k+1}^{\mathcal{N}}$, either $\mathcal{M} \triangleleft \mathcal{N}$ or $\mathcal{M} \triangleleft \mathrm{Ult}(\mathcal{N}|\rho, F^{\mathcal{N}|\rho})$. ⊣

Definition 2.24. A **hybrid premouse (hpm)** is a potential hpm $\mathcal{M}$ such that every $\mathcal{N} \triangleleft \mathcal{M}$ is ω-sound and $< \omega$-condensing. ⊣

Lemma 2.25. *Lemmas 2.17 and 2.19 hold with every instance of* potential hpm *replaced by* hpm*.*

We now proceed to defining *strategy premice*, or, Σ*-premice*, for an iteration strategy Σ. We first define the process we use to feed in branches determined by Σ. For $\gamma \in \mathrm{Ord}$ and $b \subseteq \mathrm{Ord}$, we write $\gamma + b$ for $\{\gamma + \alpha \parallel \alpha \in b\}$. Given a structure $\mathcal{M}$, an iteration tree $\mathcal{T} \in \mathcal{M}$ of length $\omega\lambda$, and a $\mathcal{T}$-cofinal branch b, Woodin noticed that $\mathcal{M}$ can be extended to a structure $\mathcal{N}$ over which b is added with an amenable predicate, with $\mathcal{N} = (\mathcal{J}_\lambda(\mathcal{M}), \mathrm{o}(\mathcal{M}) + b)$. We will use a variant of this:

[7] As in [3, §2], it does not matter which we use.

[8] Note that this definition of *weak k-embedding* diverges slightly from the definitions given in [3] and [20].

Definition 2.26 ($\mathfrak{B}$, $b^{\mathcal{M}}$)**.** Let $\mathcal{Q}$ be an hpm over A with $\mathcal{N} = cp^{\mathcal{Q}}$ transitive. Let $\lambda > 0$ and let $\mathcal{T}$ be an iteration tree[9] on $\mathcal{N}$, with $\mathrm{lh}(\mathcal{T}) = \omega\lambda$ and $\mathcal{T}\restriction\beta \in \mathcal{Q}$ for all $\beta \leq \mathrm{lh}(\mathcal{T})$. Let $\zeta \in [1, \lambda]$ and $b \subseteq \omega\zeta$ be such that $b \cap \beta \in \mathcal{Q}$ for all $\beta < \omega\zeta$.

Then $\mathfrak{B}(\mathcal{Q}, \mathcal{T}, \omega\zeta, b)$ denotes the potential hpm $\mathcal{S}$ such that $\mathcal{Q} \triangleleft \mathcal{S}$, $l(\mathcal{S}) = l(\mathcal{Q}) + \zeta$, $E^{\mathcal{S}} = \emptyset$,

$$P^{\mathcal{S}} = \{\mathcal{T}\} \times (\mathrm{o}(\mathcal{Q}) + b)$$

and for each $\mathcal{R}$ such that $\mathcal{Q} \triangleleft \mathcal{R} \trianglelefteq \mathcal{S}$,

$$P^{\mathcal{R}} = \{\mathcal{T}\} \times (\mathrm{o}(\mathcal{Q}) + [0, \gamma)_{\mathcal{T}})$$

where $\mathrm{o}(\mathcal{Q}) + \gamma = \mathrm{o}(\mathcal{R})$. (Note that $\mathcal{S}$ is amenable.) We also write $b^{\mathcal{S}} = b$ and $\mathcal{T}^{\mathcal{S}} = \mathcal{T}$, and for $\mathcal{R}, \gamma$ as above, we write $b^{\mathcal{R}} = [0, \gamma)_{\mathcal{T}}$ and $\mathcal{T}^{\mathcal{R}} = \mathcal{T}$.

If $\zeta = \lambda$ then we write $\mathfrak{B}(\mathcal{M}, \mathcal{T}, b)$ for $\mathfrak{B}(\mathcal{M}, \mathcal{T}, \omega\lambda, b)$. $\dashv$

Our notion of Σ-*premouse* $\mathcal{N}$ for an iteration strategy Σ, proceeds basically as follows. For certain $\mathcal{M} \triangleleft \mathcal{N}$, we will identify an iteration tree $\mathcal{T} \in \mathcal{M}$, via Σ, such that $\Sigma(\mathcal{T})$ is not encoded into $\mathcal{M}$, but $\Sigma(\mathcal{T}\restriction\alpha)$ is encoded into $\mathcal{M}$, for all limits $\alpha < \mathrm{lh}(\mathcal{T})$. Let $\mathcal{S} = \mathfrak{B}(\mathcal{M}, \mathcal{T}, \Sigma(\mathcal{T}))$. In a common case, then either $\mathcal{S} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{S}$. (For the kind of Σ-premouse most central to this paper, we will actually need a generalization of this, in which there will be some $\mathcal{R}$ such that $\mathcal{M} \triangleleft \mathcal{R} \triangleleft \mathcal{S}$ and $\mathcal{R} \triangleleft \mathcal{N}$, but $\mathcal{N}$ disagrees with $\mathcal{S}$ above $\mathcal{R}$.) Clearly if $\mathrm{lh}(\mathcal{T}) > \omega$ then $\mathcal{S}$ codes redundant information between $\mathcal{M}$ and $\mathcal{S}$ (the branches $\Sigma(\mathcal{T}\restriction\alpha)$ for $\alpha < \mathrm{lh}(\mathcal{T})$) before coding $\Sigma(\mathcal{T})$ itself over $\mathcal{S}$. The point of this redundancy is that it smooths out the theory a little: it seems to allow one to prove slightly nicer condensation properties, given that Σ itself has nice condensation properties, while keeping the definition of Σ-*premouse* simple.[10] The key facts are given in 2.36 and 2.38 below.

We now give some terminology relating to iteration strategies we will use in this section. Typically the domain of an iteration strategy consists of some simply definable class of trees; we will assume that it is Σ_0 definable.

Definition 2.27. Let $\mathcal{P}$ be a transitive structure and $\lambda \leq \mathrm{Ord}$. A **putative λ-iteration strategy for $\mathcal{P}$** is a function Σ such that $\mathrm{dom}(\Sigma)$ is a class of iteration trees $\mathcal{T}$ on $\mathcal{P}$ of limit length $< \lambda$, and for each $\mathcal{T} \in \mathrm{dom}(\Sigma)$, $\Sigma(\mathcal{T})$

[9] We formally take an *iteration tree* to include the entire sequence $\langle M^{\mathcal{T}}_{\alpha} \rangle_{\alpha < \mathrm{lh}(\mathcal{T})}$ of models. So $\mathcal{N}$ is determined by $\mathcal{T}$, and "$\mathcal{T}$ is an iteration tree on $\mathcal{N}$" is $\Sigma_0(\mathcal{T}, \mathcal{N})$.

[10] Difficulties that arise if one codes Σ by only feeding $\Sigma(\mathcal{T})$ itself are discussed in 2.47.

is a $\mathcal{T}$-cofinal branch. Given such a Σ, we say that Σ has **recognizable domain** iff there is a Σ_0 formula ψ in the language of set theory such that for all trees $\mathcal{T}$ on $\mathcal{P}$, we have $\mathcal{T} \in \operatorname{dom}(\Sigma)$ iff $\mathcal{T}$ is via Σ and $\operatorname{lh}(\mathcal{T}) < \lambda$ and $\psi(\mathcal{T})$.[11] A **λ-iteration strategy for $\mathcal{P}$** is a putative strategy Σ such that every putative tree via Σ is in fact an iteration tree. (Note here that $\Sigma(\mathcal{T})$ might fail to be defined for some tree $\mathcal{T}$ via Σ.) A **(putative) iteration strategy for $\mathcal{P}$** is a (putative) λ-iteration strategy for $\mathcal{P}$, for some λ. $\dashv$

Definition 2.28. Let $\mathcal{M}$ be a potential hpm. Then $\mathcal{J}^{\text{hpm}}(\mathcal{M})$ denotes the unique potential hpm $\mathcal{N}$ such that $\mathcal{M} \triangleleft \mathcal{N}$ and $l(\mathcal{N}) = l(\mathcal{M})+1$ and $P^{\mathcal{N}} = \emptyset$. For ordinals α, we define $\mathcal{J}^{\text{hpm}}_\alpha(\mathcal{M})$ inductively as follows.

- $\mathcal{J}^{\text{hpm}}_0(\mathcal{M}) = \mathcal{M}$ and $\mathcal{J}^{\text{hpm}}_1(\mathcal{M}) = \mathcal{J}^{\text{hpm}}(\mathcal{M})$.
- $\mathcal{J}^{\text{hpm}}_{\beta+1}(\mathcal{M}) = \mathcal{J}^{\text{hpm}}(\mathcal{J}^{\text{hpm}}_\beta(\mathcal{M}))$.
- For λ limit, $\mathcal{J}^{\text{hpm}}_\lambda(\mathcal{M})$ is the unique passive potential hpm $\mathcal{N}$ such that $\mathcal{N} = \lim_{\beta<\lambda} \mathcal{J}^{\text{hpm}}_\beta(\mathcal{M})$.

Let a be transitive and $A = \hat{a}$ and $\mathcal{P}, \Psi \in \mathcal{J}(A)$. Then $\mathcal{J}^{\text{hpm}}(A; \mathcal{P}, \Psi)$ denotes the unique passive potential hpm $\mathcal{N}$ over A, with $cp^{\mathcal{N}} = \mathcal{P}$, $\Psi^{\mathcal{N}} = \Psi$ and $l(\mathcal{N}) = 1$. For $\alpha \geq 0$, $\mathcal{J}^{\text{hpm}}_{1+\alpha}(A; \mathcal{P}, \Psi)$ denotes $\mathcal{J}^{\text{hpm}}_\alpha(\mathcal{J}^{\text{hpm}}(A; \mathcal{P}, \Psi))$. $\dashv$

Definition 2.29. An **abstract strategy premouse (aspm)** is an hpm $\mathcal{M}$ such that $cp^{\mathcal{M}}$ is a transitive structure and $\Psi^{\mathcal{M}}$ is a putative strategy for $cp^{\mathcal{M}}$ and there is $\chi \in \text{Ord}$ and sequences $\vec{\eta} = \langle \eta_\alpha \rangle_{\alpha \leq \chi}$ and $\vec{\Sigma} = \langle \Sigma_\alpha \rangle_{\alpha \leq \chi}$ such that $\vec{\eta}$ is strictly increasing and continuous, $\eta = 1$, $\eta_\chi = l(\mathcal{M})$, $\vec{\Sigma}$ is an increasing (possibly not strictly) and continuous sequence of putative strategies for $\text{cp}^{\mathcal{M}}$, $\Sigma_0 = \Psi^{\mathcal{M}}$, and for each $\alpha < \chi$, either:

- $\mathcal{M}|\eta_{\alpha+1} = \mathcal{J}^{\text{hpm}}(\mathcal{M}|\eta_\alpha)$ and $\Sigma_{\alpha+1} = \Sigma_\alpha$; or
- There is $\mathcal{T} \in \mathcal{M}|\eta_\alpha$ such that the following holds. We have that $\mathcal{T}$ is an iteration tree via Σ_α, but no proper extension of $\mathcal{T}$ is via Σ_α. Let $\mathcal{N} = \mathcal{M}|\eta_\alpha$ and $\mathcal{N}' = \mathcal{M}|\eta_{\alpha+1}$ and $\theta = \operatorname{lh}(\mathcal{T})$. Then there is $b \subseteq \theta$ such that $\mathcal{S} =_{\text{def}} \mathfrak{B}(\mathcal{N}, \mathcal{T}, b)$ is defined[12] and either:

[11] Since $\mathcal{P} = M_0^{\mathcal{T}} \in \text{trancl}(\mathcal{T})$, ψ can reference $\mathcal{P}$, and any of the models of $\mathcal{T}$.

[12] That is, $b \cap \beta \in \mathcal{N}$ for all $\beta < \theta$. Note that possibly $b = \emptyset$ and $\mathcal{N} \triangleleft \mathcal{S}$ here. So in this case, $\mathcal{M}$ is still considered a φ-indexed spm, even if there is no $\mathcal{T}$-cofinal branch.

– $\mathcal{N}' = \mathcal{S}$, b is a $\mathcal{T}$-cofinal branch[13] and $\Sigma_{\alpha+1} = \Sigma_\alpha \cup \{(\mathcal{T}, b)\}$, or

– $\mathcal{N}' \triangleleft \mathcal{S}$ and $\Sigma_{\alpha+1} = \Sigma_\alpha$.

Given an aspm $\mathcal{M}$, we write $\chi^{\mathcal{M}} = \chi$, $\vec{\eta}^{\mathcal{M}} = \vec{\eta}$, etc, and $\Sigma^{\mathcal{M}} = \Sigma^{\mathcal{M}}_\chi$.[14] We say that $\mathcal{M}$ is a **successor** iff χ is a successor. If $\mathcal{M}$ is a successor then $\mathcal{M}^-$ denotes $\mathcal{M}|\eta_{\chi-1}$. ⊣

It is easy to see that the sequences $\vec{\eta}, \vec{\Sigma}$ above are unique,[15] so the notation $\vec{\eta}^{\mathcal{M}}$, etc, is unambiguous. We select the trees $\mathcal{T}$ for which we add $\Sigma(\mathcal{T})$ in a first-order manner:

Definition 2.30. Let $\varphi \in \mathcal{L}^+$, $\mathcal{M}$ be an hpm and $\mathcal{T} \in \mathcal{M}$. We write $\mathcal{T} = \mathcal{T}^{\mathcal{M}}_\varphi$ iff $cp^{\mathcal{M}}$ is transitive and $\mathcal{T}$ is a limit length iteration tree on $cp^{\mathcal{M}}$ and $\mathcal{T}$ is the unique $x \in \mathcal{M}$ such that $\mathcal{M} \models \varphi(x)$. ⊣

The generality of the indexing device φ in the definition below was probably influenced by Sargsyan's [5, Definitions 1.1, 1.2].

Definition 2.31. Let $\varphi \in \mathcal{L}^+$. A **$\varphi$-indexed strategy premouse (φ-spm)** is an aspm $\mathcal{M}$ such that letting $\vec{\eta} = \vec{\eta}^{\mathcal{M}}$, etc, for every $\alpha < \chi$, letting $\mathcal{N} = \mathcal{M}|\eta_\alpha$ and $\mathcal{N}' = \mathcal{M}|\eta_{\alpha+1}$, we have:

– If $\mathcal{T}^{\mathcal{N}}_\varphi$ is undefined then $P^{\mathcal{N}'} = \emptyset$ (so $\mathcal{N}' = \mathcal{J}^{\mathrm{hpm}}(\mathcal{N})$).

– Suppose $\mathcal{T} =_{\mathrm{def}} \mathcal{T}^{\mathcal{N}}_\varphi$ is defined. Then $P^{\mathcal{N}'} \neq \emptyset$ and $\mathcal{T}^{\mathcal{N}'} = \mathcal{T}$ (so $\mathcal{T}$ is the witness to the corresponding clause of 2.29) and $\mathcal{T}^{\mathcal{R}}_\varphi = \mathcal{T}$ for all $\mathcal{R}$ such that $\mathcal{N} \trianglelefteq \mathcal{R} \triangleleft \mathcal{N}'$, but if $\mathcal{N}' \triangleleft \mathcal{M}$ then $\mathcal{T}^{\mathcal{N}'}_\varphi \neq \mathcal{T}$.

Let $\mathcal{M}$ be a φ-spm, and let $\vec{\eta}$, etc, be as above. We say that $\mathcal{M}$ is **φ-whole** iff, if $\mathcal{M}$ is a successor and $\mathcal{T} =_{\mathrm{def}} \mathcal{T}^{\mathcal{M}^-}_\varphi$ is defined, then either $\mathcal{M} = \mathfrak{B}(\mathcal{M}^-, \mathcal{T}, b)$ for some b, or $\mathcal{T}^{\mathcal{M}}_\varphi \neq \mathcal{T}$.

Let Σ be a putative iteration strategy for a transitive structure $\mathcal{P}$. Let $\varphi \in \mathcal{L}^+$. A **$\varphi$-indexed Σ-premouse ((Σ, φ)-premouse)**, is a φ-spm $\mathcal{M}$ such that $cp^{\mathcal{M}} = \mathcal{P}$ and $\Sigma^{\mathcal{M}} \subseteq \Sigma$. ⊣

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[13]Note that $M^{\mathcal{T}}_b$ might be illfounded. But in this case $\mathcal{T}\,^\frown b$ is not an iteration tree, so there is no $\alpha \leq \theta$ such that $\mathcal{T}' = \mathcal{T}^{\mathcal{M}|\eta_\alpha}_\varphi$ is defined and $\mathcal{T}$ is properly extended by $\mathcal{T}'$.

[14]No particular demand is made on $\mathrm{dom}(\Sigma^{\mathcal{M}})$ (though it is closed under initial segment).

[15]Adopt the notation of 2.29 and let $\alpha < \chi$. Then $\eta_{\alpha+1}$ is the least $\eta > \eta_\alpha$ such that either $\eta = l(\mathcal{M})$ or $P^{\mathcal{M}|(\eta+1)} = \emptyset$ or $P^{\mathcal{M}|(\eta+1)} = \{\mathcal{U}\} \times B$ for some $\mathcal{U}, B$ such that $B \cap \mathrm{o}(\mathcal{M}|\eta) = \emptyset$. (This is because $0 \in b$ whenever b is a branch through an iteration tree.)

Clearly if $\mathcal{M}$ is a φ-spm then $\Sigma^{\mathcal{M}}$ is the least putative strategy Σ such that $\mathcal{M}$ is a φ-indexed Σ-pm.

It seems difficult to express φ-indexed spm-hood with Q-formulas. So we consider the more general notion of *φ-indexed possible-spm*, which we can express with Q-formulas, modulo the usual restrictions.

Definition 2.32. A **φ-indexed possible spm** is an hpm $\mathcal{M}$ such that there is a φ-indexed spm $\mathcal{N}$ such that either $\mathcal{M} = \mathcal{N}$, or $\mathcal{N}$ is a successor, $\mathcal{N}^- \triangleleft \mathcal{M}$, $\mathcal{T} =_{\text{def}} \mathcal{T}_{\varphi}^{\mathcal{N}^-}$ is defined, and letting $\mathrm{o}(\mathcal{N}) = \mathrm{o}(\mathcal{N}^-) + \zeta$, there is a $\mathcal{T}{\restriction}\zeta$-cofinal branch b such that $\mathcal{M} = \mathfrak{B}(\mathcal{N}^-, \mathcal{T}, \zeta, b)$.

We adapt terminology and notation for spms to possible-spms in the obvious manner. $\dashv$

So a φ-indexed possible spm only fails to be a φ-spm if, with notation as above, we have $\zeta < \mathrm{lh}(\mathcal{T})$ but $b \neq [0, \zeta)_{\mathcal{T}}$. The following lemma is straightforward:

Lemma 2.33. *Let $\varphi \in \mathcal{L}^+$. Then Lemma 2.17 holds with every instance of* potential hpm *replaced by* φ-indexed possible spm.

Definition 2.34. Let $\mathcal{R}, \mathcal{M}$ be E-passive possible-spms and $\pi : \mathcal{R} \dashrightarrow \mathcal{M}$. Then π is a **very weak 0-embedding** iff π is Σ_0-elementary on its domain and there is an $\in$-cofinal set $X \subseteq \mathcal{R}$ such that

$$X \cup \mathrm{o}(\mathcal{R}) \cup cp^{\mathcal{R}} \cup \{cp^{\mathcal{R}}, \Psi^{\mathcal{R}}, cb^{\mathcal{R}}\} \subseteq \mathrm{dom}(\pi),$$

$\pi{\restriction}cp^{\mathcal{R}} = \mathrm{id}$, and π is Σ_1-elementary on parameters in X.

Let C be a class of possible-spms. We say that C is **very condensing** iff for all E-passive $\mathcal{M} \in C$ and all E-passive possible-spms $\mathcal{R}$, if there is a very weak 0-embedding $\pi : \mathcal{R} \to \mathcal{M}$ then $\mathcal{R} \in C$. $\dashv$

Lemma 2.35. *The truth of $\mathcal{L}^+$-Q-formulas is preserved downward under very weak 0-embeddings.*

We next consider preservation of Σ-pms, for strategies Σ with hull condensation (see [5, Definitions 1.29–1.31]).

Lemma 2.36. *Let $\mathcal{M}$ be a φ-indexed spm, not of type 3. Let $\mathcal{R}$ be a φ-indexed possible spm.*

(1) *Let Σ be an iteration strategy with hull condensation. Suppose that $\mathcal{M}$ is a Σ-pm, $cp^{\mathcal{R}} = cp^{\mathcal{M}}$ and either (i) $\Psi^{\mathcal{R}} \subseteq \Sigma$ and there is a very weak 0-embedding $\pi : \mathcal{R} \dashrightarrow \mathcal{M}$, or (ii) there is a weak 0-embedding $\pi : \mathcal{R} \to \mathcal{M}$ above $cp^{\mathcal{R}}$. Then $\mathcal{R}$ is a Σ-pm.*

(2) *Suppose there is $\pi\colon \mathcal{M} \to \mathcal{R}$ such that either:*

(a) *π is Σ_2-elementary, or*

(b) *π is cofinal Σ_1-elementary and either $\mathcal{M}$ is a limit or $\mathcal{T}_\varphi^{\mathcal{M}^-}$ is undefined, or*

(c) *π is cofinal Σ_1-elementary, $\mathcal{M}$ is a successor and $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}^-}$ is defined and either $b^{\mathcal{M}} \in \mathcal{M}$ or π is continuous at $\mathrm{lh}(\mathcal{T})$.*[16]

Then $\mathcal{R}$ is a φ-indexed spm.

Proof. Part (1): We just consider the case (i). (So by 2.34, $\mathcal{R}, \mathcal{M}$ are E-passive and π is above $cp^{\mathcal{R}}$.) We may assume that $\mathcal{R}$ is a successor and every proper segment of $\mathcal{R}$ is a Σ-pm, since π induces very weak 0-embeddings (in fact, fully elementary on their domains) from the proper segments of $\mathcal{R}$ to proper segments of $\mathcal{M}$. It follows that $\mathcal{M}$ is a successor and $\pi(\mathcal{R}^-) = \mathcal{M}^-$. We may assume that $\bar{\mathcal{T}} = \mathcal{T}_\varphi^{\mathcal{R}^-}$ is defined, so $\pi(\bar{\mathcal{T}}) = \mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}^-}$ is defined. Let $\mathrm{o}(\mathcal{R}^-) + \bar{\gamma} = \mathrm{o}(\mathcal{R})$ and $\mathrm{o}(\mathcal{M}^-) + \gamma = \mathrm{o}(\mathcal{M})$. Then π induces a hull embedding from $(\bar{\mathcal{T}}\restriction\bar{\gamma})\frown b^{\mathcal{R}}$ to $(\mathcal{T}\restriction\gamma)\frown b^{\mathcal{M}}$. Since the latter is via Σ, as is $\bar{\mathcal{T}}$, hull condensation gives that $b^{\mathcal{R}} = \Sigma(\bar{\mathcal{T}}\restriction\bar{\gamma})$, so $\mathcal{R}$ is a Σ-pm.

We leave (2)(a) and (2)(b) to the reader. Consider (2)(c). Note that $\pi(\mathcal{M}^-) = \mathcal{R}^-$, and since $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}^-}$ is defined, so is $\pi(\mathcal{T}) = \mathcal{T}_\varphi^{\mathcal{R}^-}$. Let $\mathrm{o}(\mathcal{M}^-) + \gamma = \mathrm{o}(\mathcal{M})$, so $\mathrm{o}(\mathcal{R}^-) + \gamma' = \mathrm{o}(\mathcal{R})$, where $\gamma' = \sup \pi``\gamma$. Then $b^{\mathcal{M}}$ is $\mathcal{T}\restriction\gamma$-cofinal, and since $\pi``b^{\mathcal{M}} \subseteq b^{\mathcal{R}}$, $b^{\mathcal{R}}$ is $\pi(\mathcal{T})\restriction\gamma'$-cofinal. So we may assume that $\gamma' < \mathrm{lh}(\pi(\mathcal{T}))$, and must see that $b^{\mathcal{R}} = [0, \gamma')_{\pi(\mathcal{T})}$.

Suppose $b^{\mathcal{M}} \in \mathcal{M}$. Then because π is Σ_1-elementary, $b^{\mathcal{R}} = \pi(b^{\mathcal{M}}) \cap \gamma'$. If $\gamma' < \pi(\gamma)$ then since $\pi(b^{\mathcal{M}})$ is $\pi(\mathcal{T})\restriction\pi(\gamma)$-cofinal, we are done. If $\gamma' = \pi(\gamma)$ then since $\gamma < \mathrm{lh}(\mathcal{T})$, so $b^{\mathcal{M}} = [0, \gamma)_{\mathcal{T}}$, so $b^{\mathcal{R}} = \pi(b^{\mathcal{M}})$ and we are done.

Now suppose that $b^{\mathcal{M}} \notin \mathcal{M}$ and π is continuous at $\mathrm{lh}(\mathcal{T})$. Then $\gamma = \mathrm{lh}(\mathcal{T})$ and $\gamma' = \mathrm{lh}(\pi(\mathcal{T}))$, contradiction. □

Corollary 2.37. *For any strategy Σ with hull condensation and any $\varphi \in \mathcal{L}^+$, the class of φ-indexed Σ-pms $\mathcal{M}$ such that $\Psi^{\mathcal{M}} = \emptyset$ is very condensing.*

[16] Cf. 2.41.

A type 3 analogue of 2.36 follows easily from 2.36:

Lemma 2.38. *Let $\mathcal{M}$ be a type 3 φ-indexed spm. Let $\mathcal{R}$ be an $\mathcal{L}^+$-structure with $cp^{\mathcal{R}} = cp^{\mathcal{M}}$.*

- *Let Σ be an iteration strategy with hull condensation. Let $\kappa = \mu^{\mathcal{M}}$ and suppose $U^{\mathcal{M}} = \mathrm{Ult}(\mathcal{M}|(\kappa^+)^{\mathcal{M}}, F^{\mathcal{M}})$ is a Σ-pm. Let $\pi : \mathcal{R} \to \mathfrak{C}_0(\mathcal{M})$ be a weak 0-embedding with $\pi{\restriction}cp^{\mathcal{R}} = \mathrm{id}$. Let $\mu = \mu^{\mathcal{R}}$ and $U^{\mathcal{R}} = \mathrm{Ult}(\mathcal{R}|(\mu^+)^{\mathcal{R}}, F^{\mathcal{R}})$. Suppose there is an elementary $\pi' : U^{\mathcal{R}} \to U^{\mathcal{M}}$ with $\pi \subseteq \pi'$.*

  Then $\mathcal{R} = \mathcal{Q}^{\mathrm{sq}}$ for some type 3 φ-indexed Σ-pm $\mathcal{Q}$, and $U^{\mathcal{R}}$ is also a φ-indexed Σ-pm.

- *Suppose there is $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathcal{R}$ such that either (i) π is Σ_2-elementary, or (ii) π is cofinal and Σ_1-elementary. Let $\mu = \mu^{\mathcal{R}}$ and suppose that $U^{\mathcal{R}}$ (as above) is wellfounded.*

  Then $\mathcal{R} = \mathcal{Q}^{\mathrm{sq}}$ for some type 3, φ-indexed spm.

We now define Σ-iterability for Σ-premice $\mathcal{M}$. The main point is that the iteration strategy should produce iterates which are Σ-premice. One needs to be a little careful, however, because the iterates might contain iteration trees outside of the domain of Σ.

Definition 2.39. Let Σ be an iteration strategy, $\varphi \in \mathcal{L}^+$ and $X = (\Sigma, \varphi)$. Let $\mathcal{M}$ be a X-pm. A **putative X-iteration tree $\mathcal{T}$ on $\mathcal{M}$** is defined as usual, with the added requirement that $M^{\mathcal{T}}_\alpha$ is an X-pm for each $\alpha+1 < \mathrm{lh}(\mathcal{T})$ (and for each such α, $E^{\mathcal{T}}_\alpha \in \mathbb{E}_+(M^{\mathcal{T}}_\alpha)$). Let $\mathcal{T}$ be a putative X-tree on $\mathcal{M}$. We say that $\mathcal{T}$ is a **well-putative X-iteration tree** iff $\mathcal{T}$ is a the models of $\mathcal{T}$ are all wellfounded. We say that $\mathcal{T}$ is an **X-iteration tree** iff $M^{\mathcal{T}}_\alpha$ is an X-pm for all $\alpha + 1 \le \mathrm{lh}(\mathcal{T})$.

Let $k < \omega$ and let $\mathcal{M} \in \mathscr{B}$ be a k-sound X-pm. Let $\theta \in \mathrm{Ord}$. The iteration game $\mathcal{G}^{X,\mathcal{M}}(k, \theta)$ has the rules of $\mathcal{G}^{\mathcal{M}}(k, \theta)$, except for the following differences. Let $\mathcal{T}$ be the putative tree being produced. For $\alpha+1 \le \theta$, if both players meet their requirements at all stages $< \alpha$, then, in stage α, player II must first ensure that $\mathcal{T}{\restriction}\alpha + 1$ is a well-putative X-tree, and if $\alpha + 1 < \theta$, that $\mathcal{T}{\restriction}\alpha+1$ is an X-tree. Given this, if $\alpha+1 < \theta$, player I then selects $E^{\mathcal{T}}_\alpha$, but we replace that requirement that $\mathrm{lh}(E^{\mathcal{T}}_\beta) < \mathrm{lh}(E^{\mathcal{T}}_\alpha)$ for all $\beta < \alpha$, with the requirement that $\mathrm{lh}(E^{\mathcal{T}}_\beta) \leq \mathrm{lh}(E^{\mathcal{T}}_\alpha)$ for all $\beta < \alpha$.[17]

Let $\alpha, \theta \in \mathrm{Ord}$. The iteration game $\mathcal{G}^{X,\mathcal{M}}(k, \alpha, \theta)$ is defined just as $\mathcal{G}^{\mathcal{M}}(k, \alpha, \theta)$, with the differences that (i) the rounds are runs of $\mathcal{G}^{X,\mathcal{Q}}(q, \theta)$ for some $\mathcal{Q}, q$,[18]and (ii) if α is a limit and neither player breaks any rule, and $\vec{\mathcal{T}}$ is the sequence of trees played, then player II wins iff $M^{\vec{\mathcal{T}}}_\infty$ is defined (that is, the trees eventually do not drop on their main branches, etc) and wellfounded.

The game $\mathcal{G}^{X,\mathcal{M}}_{\max}(k, \alpha, \theta)$ is like $\mathcal{G}^{X,\mathcal{M}}(k, \alpha, \theta)$, except that player I may not drop in model or degree between rounds. (For example, in both games, after the first round has produced a successor length k-maximal tree $\mathcal{T}_0$, the second round forms a q-maximal tree $\mathcal{T}_1$ on $\mathcal{Q}$, for a certain $(\mathcal{Q}, q)$. In $\mathcal{G}^{X,\mathcal{M}}_{\max}$, $\mathcal{Q} = M^{\mathcal{T}_0}_\infty$ and $q = \deg^{\mathcal{T}_0}(\infty)$, whereas in $\mathcal{G}^{X,\mathcal{M}}$, player I chooses $\mathcal{Q} \trianglelefteq M^{\mathcal{T}_0}_\infty$ and $q \leq \omega$, with $q \leq \deg^{\mathcal{T}_0}(\infty)$ if $\mathcal{Q} = M^{\mathcal{T}_0}_\infty$. Likewise at the start of every later round.)

If α is a limit ordinal, the game $\mathcal{G}^{X,\mathcal{M}}(k, <\alpha, \theta)$ is like $\mathcal{G}^{X,\mathcal{M}}(k, \alpha, \theta)$, except that if the game runs through α rounds with no player breaking any rules within those rounds, then player II wins automatically, irrespective of whether the direct limit model is defined or wellfounded. Likewise $\mathcal{G}^{X,\mathcal{M}}_{\max}(k, <\alpha, \theta)$.

Now X-(k, θ)**-iteration strategy**, X-(k, α, θ)**-maximal iterability**, etc, are defined from these games in the obvious manner.

The game $\mathcal{G}^{X,\mathcal{M}}_{\mathrm{hod}}(k, \alpha, \theta)$ is just like $\mathcal{G}^{X,\mathcal{M}}(k, \alpha, \theta)$, except that if at the end of round β a successor length normal tree $\mathcal{T}_\beta$ has been produced, and both players have met all their obligations up to that point, and $b^{\mathcal{T}_\beta}$ drops in model or degree, then player II wins. **Hod** X-(k, α, θ)**-iteration strategy** and **-iterability** are defined using $\mathcal{G}^{X,\mathcal{M}}_{\mathrm{hod}}(k, \alpha, \theta)$. $\dashv$

Remark 2.40. The requirement, in $\mathcal{G}^{\mathcal{M}}(k, \theta)$, that $\mathrm{lh}(E^{\mathcal{T}}_\beta) \leq \mathrm{lh}(E^{\mathcal{T}}_\alpha)$ for $\beta < \alpha$, is weaker than requiring $\mathrm{lh}(E^{\mathcal{T}}_\beta) < \mathrm{lh}(E^{\mathcal{T}}_\alpha)$, because of superstrongs. See

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[17] Thus, if we reach a putative tree $\mathcal{T}$ of length θ, then II wins iff either θ is a limit or $M^{\mathcal{T}}_{\theta-1}$ is wellfounded. If $\theta = \alpha + 1$, we cannot in general expect $M^{\mathcal{T}}_\alpha$ to be an X-pm. For example, suppose that $\theta = \omega_1 + 1$ and Σ is an $(\omega_1 + 1)$-strategy for some $\mathcal{P} \in \mathrm{HC}$. Then $M^{\mathcal{T}}_{\omega_1}$ could have φ-whole successor proper segments $\mathcal{N}$ such that $\mathcal{U} = \mathcal{T}^{\mathcal{N}^-}_\varphi$ is defined, but $\mathrm{lh}(\mathcal{U}) > \omega_1 + 1$. In this case $\mathcal{U} \notin \mathrm{dom}(\Sigma)$, so $\mathcal{N}$ is not an X-pm. In applications such as comparison, in this circumstance we only need to know that $M^{\mathcal{T}}_{\omega_1}$ is wellfounded. So we still decide the game in favour of player II in this situation.

[18] Recall that (considering the rules of $G^{\mathcal{M}}(k, \alpha, \theta)$) if a round of $\mathcal{G}^{X,\mathcal{M}}(k, \alpha, \theta)$ reaches a tree of length θ, then the game finishes at that point. So $\mathcal{Q}$ here will certainly be an X-pm.

[11, Remark 2.44(?)] regarding this and changes to the comparison algorithm that are needed to accommodate superstrongs.

Remark 2.41. Lemma 2.36 left open the possibility that $\mathcal{R}$ fails to be a φ-indexed spm, when $\pi : \mathcal{M} \to \mathcal{R}$ is cofinal and Σ_1-elementary, $\mathcal{M}$ is a successor, $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}^-}$ is defined, $b^\mathcal{M} \notin \mathcal{M}$ and π is discontinuous at $\lambda = \mathrm{lh}(\mathcal{T})$, so $\mathcal{M}$ is φ-whole, $\lambda' = \sup \pi``\lambda < \mathrm{lh}(\mathcal{T}')$ where $\mathcal{T}' = \pi(\mathcal{T}) = \mathcal{T}_\varphi^{\mathcal{R}^-}$, and $b^\mathcal{R} \neq [0, \lambda')_{\mathcal{T}'}$. Now let $X = (\Sigma, \varphi)$, where Σ has hull condensation and $\varphi \in \mathcal{L}^+$, and suppose further that $\mathcal{M}$ is a X-iterable X-pm, as witnessed by some strategy Λ. We describe two standard circumstances below which will then lead to contradiction.

First, suppose that $\pi : \mathcal{M} \to \mathcal{R}$ is via Λ. Then because Λ is a X-iteration strategy, $b^\mathcal{R} = [0, \lambda)_{\mathcal{T}'}$, a contradiction.

Second, suppose that Σ has hull condensation, π is any degree 0 iteration embedding of $\mathcal{M}$ (π need not be via any iteration strategy). We will show that $b^\mathcal{M} \in \mathcal{M}$, for a contradiction.

Because π is a degree 0 iteration embedding, the discontinuity implies that $\mathcal{M} \models$"There is $E \in \mathbb{E}$ which is a total measure and $\mathrm{lh}(\mathcal{T}^\mathcal{M})$ has cofinality $\kappa = \mathrm{crit}(E)$". Let $C \in \mathcal{M}$, $C \subseteq \mathrm{lh}(\mathcal{T})$ be a club of ordertype κ. Then

$$\sigma = i_E^\mathcal{M} : \mathcal{M} \to U = \mathrm{Ult}_0(\mathcal{M}, E)$$

is continuous at all points of C. Let $\zeta = \sup \sigma``\mathrm{lh}(\mathcal{T})$. Then $\sigma``C = \sigma(C) \cap \zeta$ is club in ζ. But

$$U \models \text{``}\zeta < \mathrm{lh}(\sigma(\mathcal{T})) \text{ and } \mathrm{cof}(\zeta) = \kappa \text{ is uncountable''}.$$

So $[0, \zeta)_{\sigma(\mathcal{T})} \cap \sigma``C$ is club in ζ, and $C' \in \mathcal{M}$ where C' is the club

$$C' = C \cap \sigma^{-1}``[0, \zeta)_{\sigma(\mathcal{T})}.$$

Because $\mathcal{M}$ is X-iterable, $\sigma(\mathcal{T})$ is via Σ. But then by hull condensation, $\Sigma(\mathcal{T})$ is the downward $\leq_\mathcal{T}$-closure of C', which is in $\mathcal{M}$.

Definition 2.42. Let $\mathcal{M}$ be an hpm and $\mathcal{N} \trianglelefteq \mathcal{M}$. We say that $\mathcal{N}$ is a **cutpoint** of $\mathcal{M}$ iff for all $\mathcal{P} \trianglelefteq \mathcal{M}$, if $\mathcal{N} \triangleleft \mathcal{P}$ and $F^\mathcal{P} \neq \emptyset$ then $\mathrm{o}(\mathcal{N}) \leq \mathrm{crit}(F^\mathcal{P})$. And $\mathcal{N}$ is a **strong cutpoint** of $\mathcal{M}$ iff likewise, but with the conclusion replaced with "$\mathrm{o}(\mathcal{N}) < \mathrm{crit}(F^\mathcal{P})$". $\dashv$

Definition 2.43 ($\mathrm{Lp}^{(\Sigma,\varphi)}$)**.** Let Σ be a strategy with hull condensation for a transitive structure $\mathcal{P} \in \mathrm{HC}$, $\varphi \in \mathcal{L}^+$ and $X = (\Sigma, \varphi)$. Let a be transitive and $A = \hat{a}$, with $\mathcal{P} \in \mathcal{J}(A)$. Assume DC_A.

Let $n \leq \omega$ and let $\mathcal{M}$ be an n-sound X-pm over A (and $\eta \leq \mathrm{o}(\mathcal{M})$). We say that $\mathcal{M}$ is **countably (above-η) X-$(n, \omega_1 + 1)$-iterable** iff for every countable hpm $\bar{\mathcal{M}}$, if $\mathcal{P} = cp^{\bar{\mathcal{M}}}$ and there is an elementary $\pi : \bar{\mathcal{M}} \to \mathcal{M}$ then $\bar{\mathcal{M}}$ is (above-$\bar{\eta}$) X-$(n, \omega_1 + 1)$-iterable (where $\bar{\eta}$ is the collapse of η).

$\mathrm{Lp}^X(a)$ denotes the stack of all countably X-$(\omega, \omega_1{+}1)$-iterable X-premice $\mathcal{M}$ over A such that $\mathcal{M}$ is fully sound and projects to ω.[19] Assuming $\mathsf{DC}_{\mathbb{R}}$, and letting $B \subseteq \mathrm{HC}$, $\mathrm{Lp}^X(\mathbb{R}, B)$ denotes $\mathrm{Lp}^X((\mathrm{HC}, B))$, and $\mathrm{Lp}^X(\mathbb{R})$ denotes $\mathrm{Lp}^X(\mathrm{HC})$.[20]

Let $\mathcal{N}$ be an X-premouse. Then $\mathrm{Lp}_+^X(\mathcal{N})$ denotes the stack of all X-premice $\mathcal{M}$ such that either $\mathcal{M} = \mathcal{N}$, or $\mathcal{N} \triangleleft \mathcal{M}$, $\mathcal{N}$ is a strong cutpoint of $\mathcal{M}$, $\mathcal{M}$ is $\mathrm{o}(\mathcal{N})$-sound, and there is $n < \omega$ such that $\rho_{n+1}^{\mathcal{M}} \leq \mathrm{o}(\mathcal{N}) < \rho_n^{\mathcal{M}}$ and $\mathcal{M}$ is countably above-$\mathrm{o}(\mathcal{N})$ X-$(n, \omega_1 + 1)$-iterable. Note that $\mathrm{Lp}_+^X(\mathcal{N})$ might have a largest element, which projects strictly across $\mathrm{o}(\mathcal{N})$ and is not ω-sound. $\dashv$

Definition 2.44. Let Σ be an iteration strategy, $\varphi \in \mathcal{L}^+$, $X = (\Sigma, \varphi)$ and $\mathcal{M}$ be an X-pm. Let $k \leq \omega$. Then $\mathcal{M}$ is **X-k-fine** iff for each $j \leq k$, we have (i) $\mathfrak{C}_j(\mathcal{M})$ is a j-solid X-pm, (ii) if $j < k$ then $\mathfrak{C}_j(\mathcal{N})$ is $(j+1)$-universal, and (iii) if $k = \omega$ then $\mathfrak{C}_\omega(\mathcal{N})$ is $< \omega$-condensing. $\dashv$

Lemma 2.45. *Let $\Sigma, \mathcal{P}, \varphi, X, a, A$ be as in 2.43 (so we assume DC_A). Then:*

- *For $k < \omega$, every k-sound, countably X-$(k, \omega_1, \omega_1 + 1)$-iterable X-pm $\mathcal{M}$ over A is X-$(k+1)$-fine.*

- *Every ω-sound, countably X-$(\omega, \omega_1, \omega_1 + 1)$-iterable X-pm over A is $< \omega$-condensing.*

- *Every countably X-$(0, \omega_1, \omega_1 + 1)$-iterable X-pseudo-premouse over A is an X-pm.*

[19] DC_A is enough to prove that this is a stack. For let $\mathcal{M}, \mathcal{N}$ be such X-premice. Because $\mathcal{M}, \mathcal{N}$ are generated by ordinals and elements of A, by taking elementary substructures which do not collapse A, we may assume that there are maps $A^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}$ and $A^{<\omega} \overset{\text{onto}}{\to} \mathcal{N}$. But then by DC_A, we may assume that $A, \mathcal{M}, \mathcal{N}$ are countable, so we can compare $\mathcal{M}, \mathcal{N}$ as usual.

[20] Since $\mathbb{R}$ is not transitive, this is not an abuse of notation.

– *There is no countably* X*-*$(0,\omega_1+1)$*-iterable* X*-bicephalus over* A*.*

Proof. Consider for example the proof that $\mathcal{M}$ is X-$(k+1)$-fine. We may assume that $\mathcal{M}$ is countable, by DC_A. If AC holds (recall that our background theory is ZF) then using the condensation lemmas 2.36 and 2.38, it is straightforward to see that the proofs of the copying construction, weak Dodd-Jensen[21] and the fundamental fine structural theorems go through. But we may assume ZFC, because letting $x \in \mathbb{R}$ code $\mathcal{M}$ and Λ be an iteration strategy for $\mathcal{M}$ as hypothesized, then we can pass to $W = L^{\Lambda,\Sigma}[x]$ (where we feed Λ, Σ into W like with strategy mice; we do not care about fine structure for W), replacing Σ with $\Sigma' = \Sigma \cap W$. □

We will build Σ-mice by background construction:

Definition 2.46. Let a be transitive and $A = \hat{a}$. Let Σ be an iteration strategy for a transitive structure in $\mathcal{J}(A)$, let $\varphi \in \mathcal{L}^+$ and let $X = (\Sigma, \varphi)$. An $L^X[\mathbb{E}, A]$**-construction (of length** χ**)** is a sequence $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha<\chi}$ such that for all $\alpha < \chi$:

– $\mathcal{N}_\alpha$ is a X-pm over A and $l(\mathcal{N}_0) = 1$.

– If α is a limit then $\mathcal{N}_\alpha = \liminf_{\beta<\alpha} \mathcal{N}_\beta$.

– If $\alpha + 1 < \chi$ then letting $\mathcal{N} = \mathcal{N}_{\alpha+1}$, either:

  – $\mathcal{N}$ is E-active and $\mathcal{N}||\mathrm{o}(\mathcal{N}) = \mathcal{N}_\alpha$ and letting $\kappa = \mu^{\mathcal{N}}$, then $\mathrm{Ult}(\mathcal{N}|(\kappa^+)^{\mathcal{N}}, F^{\mathcal{N}})$ is an X-pm, or

  – $\mathcal{N}_\alpha$ is X-ω-fine and $\mathcal{M} =_{\mathrm{def}} \mathfrak{C}_\omega(\mathcal{N}_\alpha) \triangleleft \mathcal{N}$ and $l(\mathcal{N}) = l(\mathcal{M}) + 1$. ⊣

We will consider fully backgrounded $L^\Sigma[\mathbb{E}, A]$-constructions. Assume DC_A. Then given $\mathcal{N}_\alpha$ and supposing that $\mathcal{N}_\alpha$ is X-k-fine, countable X-$(k, \omega_1, \omega_1+1)$-iterability will be enough to verify that $\mathcal{N}_\alpha$ is X-$(k+1)$-fine. This iterability will be established (where we can) by the standard arguments, using the condensation lemmas.

Remark 2.47. Our definition of Σ-premice (for an iteration strategy Σ) differs a little from the standard one. The standard one is along the lines of: given $\mathcal{M}|\alpha$, letting $\mathcal{T} \in \mathcal{M}|\alpha$ be the $<_{\mathcal{M}|\alpha}$-least tree for which $\mathcal{M}|\alpha$ does not

[21] $\mathsf{DC}_{\mathbb{R}}$ seems to be used in the construction of an iteration strategy with the weak Dodd Jensen property.

know $\Sigma(\mathcal{T})$, and $\omega\lambda = \text{lh}(\mathcal{T})$, let $\mathcal{M}|(\alpha+\lambda) = (\mathcal{J}_\lambda(\mathcal{M}|\alpha), B)$, such that B codes $\Sigma(\mathcal{T})$ amenably.

Whatever one's definition of Σ-premice, one would probably like to know that an ultrapower of a Σ-premouse is also a Σ-premouse. As has been observed by others, this is not true of the hierarchy described above. For suppose $\mathcal{M}|\alpha$, $\mathcal{T}$ and λ are as above, and $\text{lh}(\mathcal{T})$ has measurable cofinality κ in $\mathcal{M}|(\alpha+\lambda)$, and E is an extender over $\mathcal{M} = \mathcal{M}|(\alpha+\lambda)$ with $\text{crit}(E) = \kappa$. Then $U = \text{Ult}_0(\mathcal{M}, E)$ is not in the hierarchy. For i_E is discontinuous at $\text{lh}(E)$, but $\text{o}(U) = \sup i_E``\text{o}(\mathcal{M})$.

There seem to have been two approaches used to correct this problem (other than the one we use) used by others. One is to feed in all initial segments of $\Sigma(\mathcal{T})$ (even though they have been fed in earlier), immediately prior to feeding in $\Sigma(\mathcal{T})$ itself. But this approach seems flawed. For $(*)$ let $\mathcal{M}'$ be a structure in this hierarchy, with $B^{\mathcal{M}'} \neq \emptyset$, but $B^{\mathcal{M}'}$ coding a non-$\mathcal{T}'$-cofinal (for the relevant tree $\mathcal{T}'$) branch $[0, \omega\gamma')_{\mathcal{T}'}$ (for some $\omega\gamma' < \text{lh}(\mathcal{T}')$). Let $\pi : \mathcal{M} \to \mathcal{M}'$ be fully elementary. Then clearly $B^{\mathcal{M}}$ codes $[0, \omega\gamma)_{\mathcal{T}}$ where $\pi(\mathcal{T}) = \mathcal{T}'$ and $\pi(\gamma) = \gamma'$, and $\omega\gamma < \text{lh}(\mathcal{T})$. But we need that $[0, \omega\gamma)_{\mathcal{T}} \subseteq \Sigma(\mathcal{T})$, and this is not clear (even if Σ has hull condensation).

The other correction, which is better, is to simply not feed in $\Sigma(\mathcal{T})$ in the case that $\text{lh}(\mathcal{T})$ has measurable cofinality in $\mathcal{M}|(\alpha+\lambda)$ (as witnessed by some measure on $\mathbb{E}^{\mathcal{M}}$). For by the argument in 2.41, $\mathcal{M}$ already has $\Sigma(\mathcal{T})$ as an element, and there is a uniform procedure which $\mathcal{M}$ can use to determine $\Sigma(\mathcal{T})$.

Thus, one must show that the relevant ultrapowers and substructures of models in the resulting hierarchy are also in the hierachy. It is easy to see that ultrapowers preserve the relevant first-order properties.

So let $\mathcal{M}'$ be a Σ-premouse and let $\pi : \mathcal{M} \to \mathcal{M}'$ be a weak 0-embedding. We want to know that $\mathcal{M}$ is a Σ-premouse, given that Σ has hull condensation. We just need to verify the first-order properties.

We need to rule out the possibility that $B^{\mathcal{M}} = \emptyset$ (and therefore $B^{\mathcal{M}'} = \emptyset$), but there is some $B \neq \emptyset$ such that $(\mathcal{M}, B)$ is a Σ-premouse. Let $\mathcal{T} \in \mathcal{M}$ be the relevant tree (with B coding $\Sigma(\mathcal{T})$). Because π is a weak 0-embedding, this implies that $\mathcal{T}' = \pi(\mathcal{T})$ is the least tree for which $\mathcal{M}'$ does not know $\Sigma(\mathcal{T}')$, and π is discontinuous at $\text{lh}(\mathcal{T})$. Suppose also that $\mathcal{M} = \mathfrak{C}_1(\mathcal{M}')$ and π is the core map, and $\mathcal{M}'$ is $(0, \omega_1, \omega_1+1)$-iterable. Then by the usual proof of solidity (with a little extra argument to deal with the possibility that $\mathcal{M}$ is not a Σ-premouse), $\mathcal{M}$ and $\mathcal{M}'$ are 1-solid and $\pi(p_1^{\mathcal{M}}) = p_1^{\mathcal{M}'}$, and then using the comparison argument in the proof of universality, and the commutativity of π with the resulting iteration embeddings, one gets that $\mathrm{lh}(\mathcal{T})$ has measurable cofinality in $\mathcal{M}$, and therefore $\mathcal{M}$ is in fact a Σ-premouse, contradiction. (For the higher degree core maps, the present situation cannot arise, just by elementarity.)

Now suppose that $B^{\mathcal{M}'} \neq \emptyset$. It is easy to see that $B^{\mathcal{M}}$ codes some branch b through $\mathcal{T}$, and that $B^{\mathcal{M}} \cap \mathcal{M}$ is cofinal in $\mathrm{o}(\mathcal{M})$ (by the Σ_1-elementarity of π on a set cofinal in $\mathrm{o}(\mathcal{M})$). But b need not be $\mathcal{T}$-cofinal. (For example, if $\mathrm{o}(\mathcal{M}')$ has uncountable cofinality, it is easy to find $\mathcal{N} \triangleleft \mathcal{M}$ such that letting $\mathcal{M} = (\mathcal{N}, B^{\mathcal{M}'} \cap \mathcal{N})$ and $\pi = \mathrm{id}$, then $\pi : \mathcal{M} \to \mathcal{M}'$ is a weak 0-embedding, and $\mathcal{T} = \mathcal{T}'$.) If we have that π is Σ_1-elementary on a set $X \subseteq \mathrm{o}(\mathcal{M})$ which is both cofinal in $\mathrm{o}(\mathcal{M})$ and cofinal in $\mathrm{lh}(\mathcal{T})$, then b will be cofinal in $\mathcal{T}$.

These arguments give that the models produced in an $L[\mathbb{E}, \Sigma]$-construction will all be Σ-mice, as long as iterates of countable elementary substructures are realizable back into models of the construction, in the usual manner. But we opted for the hierarchy for Σ-premice defined in §2 because it has stronger condensation properties, and without assuming any iterability.

3 G-organization

Let Ω be either an operator or an iteration strategy. In this section we implement some ideas of Grigor Sargsyan, defining g-*organized* Ω-*premice*. This will be useful assuming that Ω has a certain absoluteness property, which we first describe.

Definition 3.1. Let a be transitive and $A = \hat{a}$. We say that A is **self-wellordered (swo'd)** iff $a = \mathrm{trancl}(x \cup \{x, \prec\})$ for some transitive set x, and wellorder $\prec$ of x. For swo'd A and $\prec$ as above, let $\prec_A$ denote the canonical wellorder of A determined by $\prec$. ⊣

Definition 3.2. Let ψ be a Σ_0 formula in the language of set theory.[22] Then $\varphi_{\psi,\min}(x)$ denotes the formula in the free variable x asserting, over abstract spms $\mathcal{M}$ with $cb^{\mathcal{M}}$ swo'd: "Let $\prec$ be the canonical wellorder of the universe. Then x is the $\prec$-least limit length iteration tree $\mathcal{T}$ on cp according to Σ^V such that $\Sigma^V(\mathcal{T})$ is undefined and $\psi(\mathcal{T})$ holds".

Let $\varphi_{\min} = \varphi_{\text{“true”},\min}$. ⊣

[22] ψ will be used to restrict the class of iteration trees being considered; for example, $\psi(x)$ might say that "x is a normal tree"

Definition 3.3. Either:

- let Σ be an iteration strategy for a transitive structure $\mathcal{P} \in \mathrm{HC}$, let $\varphi \in \mathcal{L}^+$ and $X = (\Sigma, \varphi)$, let $a \in \mathrm{HC}$ and $A = \hat{a}$ with $\mathcal{P} \in \mathcal{J}_1(A)$, or

- let $X = \mathcal{F}$ be an operator over $\mathscr{B}$, $\kappa \leq \mathrm{o}(\mathscr{B})$ be an uncountable cardinal and $A \in \widehat{C_{\mathcal{F}}} \cap \mathrm{HC}$.

We write $\mathcal{M}_1^{X,\#}(A)$ for the (unique) sound, non-1-small X-pm $\mathcal{M}$ over A, such that $\mathcal{M}$ is X-$(0, \omega_1)$-iterable, and if $\mathrm{cof}(\omega_1) > \omega$, $\mathcal{M}$ is X-$(0, \omega_1 + 1)$-iterable (given such an $\mathcal{M}$ exists).[23] Let κ be an uncountable cardinal. We say that $\mathcal{M}_1^{X,\#}(A)$ is **X-κ-naturally iterable** iff $\mathfrak{M} = \mathcal{M}_1^{X,\#}(A)$ exists and either:

(a) $\mathrm{cof}(\kappa) > \omega$ and $\mathfrak{M}$ is X-$(0, \kappa + 1)$-iterable, or

(b) $\mathrm{cof}(\kappa) = \omega$ and $\mathfrak{M}$ is X-$(0, \kappa)$-iterable.

When this holds, let $\Lambda_{\mathfrak{M}}^{X,\kappa}$ denote the unique[24] X-$(0, \kappa)$-strategy for $\mathfrak{M}$ which, if $\mathrm{cof}(\kappa) > \omega$, extends to an X-$(0, \kappa{+}1)$-strategy; also if $\mathrm{cof}(\kappa) > \omega$ let $\Lambda_{\mathfrak{M}}^{X,\kappa+1}$ denote the unique[25] X-$(0, \kappa + 1)$-strategy for $\mathfrak{M}$.

Let Σ be an iteration strategy for a transitive structure $\mathcal{P} \in \mathrm{HC}$, with recognizable domain, as witnessed by a Σ_0 formula ψ (in the language of set theory), with ψ least such. Then $\varphi_{\min}^{\Sigma}$ denotes $\varphi_{\psi,\min}$. Let $\varphi = \varphi_{\min}^{\Sigma}$. Then we abbreviate the pair (Σ, φ) with Σ. So a Σ-pm is a (Σ, φ)-pm, etc. $\dashv$

Definition 3.4. We say that $(\Omega, \varphi, X, A, \kappa)$ is **suitable** iff κ is an uncountable cardinal, $A = \hat{a}$ for some transitive $a \in \mathrm{HC}$, $\mathcal{M}_1^{X,\#}(A)$ exists and is X-κ-naturally iterable, and either:

- $\Omega = \Sigma$ is a κ-strategy with hull condensation and recognizable domain, for a transitive structure $\mathcal{P} \in \mathrm{HC} \cap \mathcal{J}(A)$, $\varphi \in \mathcal{L}^+$, and $X = (\Sigma, \varphi)$, or

- $\Omega = X = \mathcal{F}$ is a total operator over $\mathscr{B}$, where $\mathscr{B}$ is an operator background with $\kappa = \mathrm{o}(\mathscr{B})$, $C_{\mathcal{F}}$ is the (possibly swo'd) cone of $\mathscr{B}$ above a, and $\mathcal{F}$ condenses finely above a.

[23] ZF proves uniqueness. For let $\mathcal{M} \neq \mathcal{N}$ be such X-pms. Let $(\mathcal{T}, \mathcal{U})$ be their length ω_1 comparison if $\mathrm{cof}(\omega_1) = \omega$, or length $\omega_1 + 1$ comparison otherwise. Let $z \in \mathbb{R}$ code $(\mathcal{M}, \mathcal{N})$ and let $W = L[z, \mathcal{T}, \mathcal{U}]$. Then $\mathcal{M}, \mathcal{N} \in \mathrm{HC}^W$ and $W \models \mathsf{AC}$, and therefore if $\mathrm{cof}(\omega_1) = \omega$ then $W \models$"γ is a limit cardinal", where $\gamma = \omega_1$. So working in W we can reach a contradiction as usual.

[24] Much as before, ZF proves uniqueness.

[25] Likewise.

For suitable $t = (\Omega, \varphi, X, A, \kappa)$, let $\Omega_t = \Omega$, etc.

Let (Ω, A) be given. We say that (Ω, A) is **suitable** iff either (i) Ω is a κ-strategy Σ, with recognizable domain, for some transitive structure $\mathcal{P} \in \mathrm{HC}$ and uncountable cardinal κ, $A = \hat{a}$ for some $a \in \mathrm{HC}$, A is swo'd, and

$$t_{\Sigma,A} =_{\text{def}} (\Sigma, \varphi^{\Sigma}_{\min}, (\Sigma, \varphi^{\Sigma}_{\min}), A, \kappa)$$

is suitable, or (ii) Ω is an operator $\mathcal{F}$ over $\mathscr{B}$, and letting $\kappa = \mathrm{o}(\mathscr{B})$,

$$t_{\mathcal{F},A} =_{\text{def}} (\mathcal{F}, 0, \mathcal{F}, A, \kappa)$$

is suitable. ⊣

Lemma 3.5. *Let $t = (\Omega, \varphi, X, A, \kappa)$ be suitable, $\mathfrak{M} = \mathcal{M}_1^{X,\#}(A)$ and $\eta = \mathrm{cof}(\kappa)$. Then:*

1. *$\Lambda^{X,\kappa}_{\mathfrak{M}}$ has branch condensation and hull condensation.*
2. *If $\eta = \omega$ then $\mathfrak{M}$ is X-$(0, < \omega, \kappa)$-maximally iterable.*
3. *If $\eta > \omega$ then $\mathfrak{M}$ is X-$(0, \eta, \kappa + 1)$-maximally iterable.*[26]

Proof. These facts come from the uniqueness of $\Lambda^{X,\kappa}_{\mathfrak{M}}$, together with the the condensation properties proved in this section for strategy mice, and the condensation properties and copying arguments of [11] in the case that X is an operator. Part 1 follows routinely from these items. Parts 2 and 3 are essentially by [10, Theorem 3.1(?)]. The latter results are literally stated and proved only for standard premice, but the arguments there go through using the properties and arguments just mentioned. □

Remark 3.6. What is behind the foregoing proof (in terms of the details contained in [10]), is as follows. If $\eta = \omega$ let $\Lambda = \Lambda^{X,\kappa}_{\mathfrak{M}}$ and $\theta = \kappa$. If $\eta > \omega$ let $\Lambda = \Lambda^{X,\kappa+1}_{\mathfrak{M}}$ and $\theta = \kappa + 1$. An X-$(0, < \eta, \theta)$-maximal strategy Ψ for $\mathcal{M}$ is computed, extending Λ (and therefore $\Lambda^{X,\kappa}_{\mathfrak{M}}$), and such that the restriction of Ψ to an X-$(0, < \eta, \kappa)$-maximal strategy Ψ', lifts to $\Lambda^{X,\kappa}_{\mathfrak{M}} \subseteq \Lambda$. (Stacks via Ψ which have a last tree $\mathcal{T}$ of length $\kappa + 1$ can lift to a normal tree $\mathcal{U}$ of length $> \kappa + 1$, in which case $\mathcal{U}$ cannot literally be via Λ, but for instance, $\mathcal{U} \restriction \kappa + 1$ is via Λ.)

[26] We also get X-$(0, \eta, \kappa + 1)$-iterability (without *maximal*) for the strategy case, but for reasons covered in [11], we cannot expect the same if X is an operator.

If $\mathcal{T}$, via Ψ', has successor length, then $M^{\mathcal{T}}_\infty$ is X-$(\deg^{\mathcal{T}}(\infty), <\eta, \theta)$-maximally iterable, via the tail Ψ^* of Ψ. Moreover, given a normal tree $\mathcal{U}$ on $M^{\mathcal{T}}_\infty$ of limit length $< \kappa$, via Ψ^*, and $c = \Psi^*(\mathcal{U})$, either there is a Q-structure for $M(\mathcal{T})$ in $L^X_\kappa(M(\mathcal{T}))$, which determines c as usual, or else neither $b^{\mathcal{T}}$ nor c drop in model or degree and $i^{\mathcal{U}}_c \circ i^{\mathcal{T}}(\delta^{\mathfrak{M}}) = \delta(\mathcal{U})$.

If $\eta > \omega$ then clearly any X-$(n, <\eta, \kappa+1)$-maximal strategy extends to an X-$(n, \eta, \kappa+1)$-maximal strategy. So part 3 follows readily from the above. Note also that any strategy witnessing part 2 (part 3) must extend $\Lambda^{X,\kappa}_{\mathfrak{M}}$ (must extend $\Lambda^{X,\kappa+1}_{\mathfrak{M}}$).

Definition 3.7. In the preceding context, let $\Lambda^{X,(<\eta,\kappa)}_{\mathfrak{M}}$ denote Ψ'. ⊣

The following absoluteness property ensures that g-organization is useful:

Definition 3.8. Let $t = (\Omega, \varphi, X, A, \kappa)$ be suitable and $\mathfrak{M} = \mathcal{M}^{X,\#}_1(A)$. We say that t **determines itself on generic extensions** iff there are formulas Φ, Ψ in $\mathcal{L}^+$ and some $\gamma > \delta^{\mathfrak{M}}$ such that $\mathfrak{M}|\gamma \vDash \Phi$ and for any non-dropping $\Lambda^{X,\kappa}_{\mathfrak{M}}$-iterate $\mathcal{N}$ of $\mathfrak{M}$ via a countable tree $\mathcal{T}$ based on $\mathfrak{M}|\delta^{\mathfrak{M}}$, any $\mathcal{N}$-cardinal δ, any $\gamma \in \mathrm{Ord}$ such that $\mathcal{N}|\gamma \vDash \Phi$ & "δ is Woodin", and any g which is set-generic over $\mathcal{N}|\gamma$ (with $g \in V$), we have that $\mathcal{R} =_{\mathrm{def}} (\mathcal{N}|\gamma)[g]$ is closed under Ω, and $\Omega{\restriction}\mathcal{R}$ is defined over $\mathcal{R}$ by Ψ. We say such a pair (Φ, Ψ) **generically determines** t.

Let $A \in \mathrm{HC}$ and let Ω be either an operator or an iteration strategy. We say that (Ω, A) is **nice** iff (Ω, A) is suitable and $t_{\Omega,A}$ determines itself on generic extensions. We say that (Φ, Ψ) **generically determines** (Ω, A) iff (Φ, Ψ) generically determines $t_{\Omega,A}$. ⊣

Lemma 3.9. *Let $\mathcal{N}, \delta$, etc, be as in 3.8, except that we allow $\mathcal{T}$ to have any length $< \kappa$, and allow g to be in a set-generic extension of V. Then $\mathcal{R}$ is closed under Ω and $\Omega'{\restriction}\mathrm{dom}(\Omega) = \Omega{\restriction}\mathcal{R}$ where Ω' is the interpretation of Ψ over $\mathcal{R}$.*

Proof. We first give the proof assuming that $\Omega = \Sigma$ is a strategy, and then point out the differences for the other case. Suppose the lemma fails. Let $x \in \mathcal{R}$ be a counterexample to the claimed agreement between Σ, Σ'. So $\mathcal{U} =_{\mathrm{def}} x \in \mathrm{dom}(\Sigma) \subseteq V$. Let $\mathbb{P}$ be some forcing, and $H \subseteq \mathbb{P}$ be V-generic, such that $g \in V[H]$. Because $a \in \mathrm{HC}$, $\mathcal{N}$ is wellorderable, and so by Σ^1_1-absoluteness, we may assume $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{o}(\mathcal{N}))$. Moreover, letting $z \in \mathbb{R}$ code $a, M^{\mathcal{U}}_0, \mathfrak{M}$, we may assume that $g \in W =_{\mathrm{def}} L[z, \mathcal{T}, \mathcal{U}, \Sigma(\mathcal{U})]$.

Work in W, where AC holds. Let $\dot{g}$ be a $\mathbb{P}$-name for g. Let $\dot{\mathcal{U}} \in \mathcal{N}|\gamma$ be such that $\dot{\mathcal{U}}^g = \mathcal{U}$. Fix $p \in H$ forcing "$\dot{g}$ is $\check{\mathcal{N}}|\check{\gamma}$-generic and $\check{\dot{\mathcal{U}}}^{\dot{g}} = \check{\mathcal{U}}$"; for simplicity assume that $p = \emptyset$. Let α be large and let

$$\pi : M \to L_\alpha[z, \mathcal{T}, \mathcal{U}, \Sigma(\mathcal{U})]$$

be elementary, with M countable and all relevant objects in $\mathrm{rg}(\pi)$. Write $\pi(\bar{\mathcal{T}}) = \mathcal{T}$, etc.

Now work in V. Note that $\bar{\mathcal{U}}$ is via Σ and $\bar{\mathcal{U}} \in \mathrm{dom}(\Sigma)$ because Σ has hull condensation and recognizable domain. By 3.5, $\bar{\mathcal{T}}$ is via $\Lambda^X_{\mathfrak{M}}$. For any H^* which is $\bar{\mathbb{P}}$-generic over M, letting $g^* = \bar{\dot{g}}^{H^*}$, we then have

$$\bar{\dot{\mathcal{U}}}^{g^*} = \bar{\mathcal{U}} \in \overline{\mathcal{N}|\gamma}[g^*],$$

and letting Σ^* be the interpretation of Ψ over $\overline{\mathcal{N}|\gamma}[g^*]$, by 3.8 we have

$$\Sigma(\bar{\mathcal{U}}) = \Sigma^*(\bar{\mathcal{U}}) \in \overline{\mathcal{N}|\gamma}[g^*]. \tag{3.1}$$

So $\mathcal{U} \in \mathrm{dom}(\Sigma')$ (by the above, this is forced by $\mathbb{P}$), and so $\Sigma'(\mathcal{U}) \neq \Sigma(\mathcal{U})$, by choice of $\mathcal{U}$. By hull condensation, $\overline{\Sigma(\mathcal{U})} = \Sigma(\bar{\mathcal{U}})$, and so by line (3.1), $\overline{\Sigma(\mathcal{U})} = \Sigma^*(\bar{\mathcal{U}})$ for any H^*. So in M, $\bar{\mathbb{P}}$ forces that $\overline{\Sigma(\mathcal{U})} = \Sigma^*(\bar{\mathcal{U}})$. Therefore $\mathbb{P}$ forces that $\Sigma(\mathcal{U}) = \Sigma'(\mathcal{U})$. Contradiction.

Now consider the case that $\Omega = \mathcal{F}$ is an operator. The argument is almost the same. The coarse condensation (a component of fine condensation) of $\mathcal{F}$ above a, and the fact that $a \in \mathrm{HC}$, replaces the use of hull condensation and the recognizability of $\mathrm{dom}(\Sigma)$. Much as before, we can assume that $\mathbb{P} = \mathrm{Col}(\omega, Z)$ for some transitive $Z \in \mathscr{B}$. Because $\mathscr{B} \vDash \mathsf{DC}$ we can form an appropriate countable elementary substructure M of some large enough set in $\mathscr{B}$. We omit further detail. □

We next consider some issues pertaining to hod mice; see [5] for background.[27]

Definition 3.10. A pointclass is **smooth** iff it contains all open sets and is closed under continuous preimage, intersections, unions and real quantifiers. ⊣

[27] We assume only a basic knowledge of hod mice; more than enough is covered in the first sections of [5]. As mentioned earlier, the actual analysis of scales does not depend particularly on the theory of hod mice, and is developed in parallel for standard mice.

Remark 3.11. Assume that ω_1 is regular and let Γ be smooth pointclass. Let $a \in \mathrm{HC}$ be swo'd. Let $\vec{\Sigma}$ be the join of a sequence of strategies for a sequence $\vec{\mathcal{P}}$ of transitive structures in $\mathcal{J}(a)$ (possibly the sequence has length 0 or 1). As in [5, Definition 2.26], $\mathrm{Lp}^{\Gamma,\vec{\Sigma}}(a)$ denotes the stack of all sound $\vec{\Sigma}$-premice $\mathcal{M}$ over a which project to a, such that in Γ there is a $\vec{\Sigma}$-$(\omega, \omega_1, \omega_1)$-iteration strategy for $\mathcal{M}$ which extends to a $\vec{\Sigma}$-$(\omega, \omega_1, \omega_1+1)$-strategy.[28] Here we are demanding a full $\vec{\Sigma}$-$(\omega, \omega_1, \omega_1 + 1)$-strategy, not just a hod strategy. This is somewhat at odds with our usual practice in this paper, of dealing only with hod strategies for hod premice; it is done for consistency with [5]. Fortunately, if each strategy in $\vec{\Sigma}$ has hull condensation, we could have actually defined $\mathrm{Lp}^{\Gamma,\vec{\Sigma}}$ using hod strategies, or in fact using normal strategies, and gotten the same result:

Lemma 3.12. *Suppose ω_1 is regular and let $\Gamma, a, \vec{\mathcal{P}}, \vec{\Sigma}$ be as in 3.11. Suppose that every strategy in $\vec{\Sigma}$ has hull condensation. Then $\mathrm{Lp}^{\Gamma,\vec{\Sigma}}(a)$ is the stack of all sound $\vec{\Sigma}$-premice over a which project to a and such that there is a $\vec{\Sigma}$-(ω, ω_1)-strategy for $\mathcal{N}$ in Γ which extends to a $\vec{\Sigma}$-$(\omega, \omega_1 + 1)$-strategy.*

Proof. This is by the proof of 3.5, together with [10, §3(?)] and the closure of Γ under real quantifiers. □

Definition 3.13. Let $\mathcal{P}$ be a hod premouse and $\mathcal{R} \triangleleft \mathcal{S} \triangleleft \mathcal{P}$ be such that $\mathcal{R}$ is a cutpoint of $\mathcal{S}$ and $\mathcal{S} \triangleleft \mathcal{P}(\alpha)$ where α is least such that $\mathcal{R} \triangleleft \mathcal{P}(\alpha)$. Suppose either $\mathcal{S}$ projects $\leq \mathrm{o}(\mathcal{R})$, or $\mathrm{o}(\mathcal{R})$ is the largest cardinal of $\mathcal{S}$. Then $\mathcal{S}^*(\mathcal{R})$ denotes the $*$-*translation* of $\mathcal{S}$ above $\mathcal{R}$ (much as in [17, §7]; so $\mathcal{S}^*(\mathcal{R})$ is approximately a strategy premouse over $\mathcal{R}$, and in particular, $\mathrm{o}(\mathcal{R})$ is a *strong* cutpoint of $\mathcal{S}^*(\mathcal{R})$). If $\mathrm{o}(\mathcal{R})$ is the largest cardinal of $\mathcal{S}$ then $\mathcal{S}^*$ denotes $\mathcal{S}^*(\mathcal{R})$. ⊣

We now define Γ-*fullness* preserving* much as Γ-*fullness preserving* is defined in [5, Definition 2.27], but with a few modifications, the most significant of which is that we make requirements regarding dropping iterates, and related to this, the fact that we consider all cutpoints, not just strong cutpoints. (thus, because $\mathcal{R}$ is, by definition, a strong cutpoint of $\mathrm{Lp}^{\Gamma,\Sigma}(\mathcal{R})$, we must use $\mathcal{S}^*$ where $\mathcal{S}$ is used in [5]).

[28] In [5], the definition is stated in the context of AD^+, so the extension to $\omega_1 + 1$ exists. Here as elsewhere, a $\vec{\Sigma}$-$(\omega, \omega_1, \omega_1 + 1)$-strategy is only required to ensure wellfoundedness of the last model of successor length trees of size ω_1, not $\vec{\Sigma}$-correctness.

Definition 3.14. Suppose ω_1 is regular and $(\mathcal{P},\Sigma)$ is a hod pair with $\mathcal{P}\in\mathrm{HC}$ and Γ is a smooth pointclass. Then Σ is **Γ-fullness* preserving** iff the following two conditions hold:

1. Let $(\vec{\mathcal{T}},\mathcal{Q})\in I(\mathcal{P},\Sigma)\cap\mathrm{HC}$ and $\gamma\leq\lambda^{\mathcal{Q}}$. Then

   – for all cutpoints (not just strong) η of $\mathcal{Q}(0)$,
$$(\mathcal{Q}|(\eta^+)^{\mathcal{Q}(0)})^*=\mathrm{Lp}^{\Gamma}(\mathcal{Q}|\eta),$$

   – if $\gamma=\alpha+1$ then for all cutpoints η of $\mathcal{Q}(\alpha+1)$ with $\eta\geq\mathrm{o}(\mathcal{Q}(\alpha))$,
$$(\mathcal{Q}|(\eta^+)^{\mathcal{Q}(\alpha+1)})^*=\mathrm{Lp}^{\Gamma,\Sigma_{\mathcal{Q}(\alpha),\vec{\mathcal{T}}}}(\mathcal{Q}|\eta),$$

   – and if γ is a limit then for all cutpoints η of $\mathcal{Q}(\gamma)$ with $\eta\geq\delta^{\mathcal{Q}}_{\gamma}$,
$$(\mathcal{Q}|(\eta^+)^{\mathcal{Q}(\gamma)})^*=\mathrm{Lp}^{\Gamma,\oplus_{\beta<\gamma}\Sigma_{\mathcal{Q}(\beta),\vec{\mathcal{T}}}}(\mathcal{Q}|\eta).$$

2. Let $(\vec{\mathcal{T}},\mathcal{T})$ be a countable tree via Σ, consisting of a stack $\vec{\mathcal{T}}$ followed by a normal tree $\mathcal{T}$, such that $\mathcal{T}$ has successor length and $b^{\mathcal{T}}$ drops. Let $\mathcal{Q}=M^{\mathcal{T}}_{\infty}$ and $\lambda=\lambda^{\mathcal{Q}}$. Let γ be least such that $\mathrm{o}(\mathcal{Q}(\lambda))<\mathrm{lh}(E^{\mathcal{T}}_{\gamma})$ and let $\mathcal{U}=\vec{\mathcal{T}}\,{}^\frown(\mathcal{T}\restriction(\gamma+1))$. (Note $b^{\mathcal{U}}$ does not drop.) Let $\mathcal{R},\mathcal{S}$ be such that $\mathcal{Q}(\lambda)\trianglelefteq\mathcal{R}\triangleleft\mathcal{S}\trianglelefteq\mathcal{Q}$ and $\mathcal{R}$ is a cutpoint of $\mathcal{S}$ and $\mathcal{S}$ projects $\leq\mathrm{o}(\mathcal{R})$ and is $\mathrm{o}(\mathcal{R})$-sound (so either $\mathcal{S}\triangleleft\mathcal{Q}$ or all generators of $\mathcal{T}$ are $<\mathrm{o}(\mathcal{R})$). Then
$$\mathcal{S}^*(\mathcal{R})\triangleleft\mathrm{Lp}^{\Gamma,\Sigma_{\mathcal{Q}(\lambda),\mathcal{U}}}(\mathcal{R}).$$ ⊣

Definition 3.15. Let $(\mathcal{P},\Sigma)$ be a hod pair with $\mathcal{P}\in\mathrm{HC}$. We say that Σ has **weak hull condensation** iff for all transitive W,X satisfying ZF^-, with $W\in\mathrm{HC}$, and fully elementary $\pi:W\to X$, if $\mathcal{P}\in\mathrm{HC}^W$ and $\vec{\mathcal{T}}\in W$ and $\pi(\vec{\mathcal{T}})$ is a stack on $\mathcal{P}$ via Σ, then $\vec{\mathcal{T}}$ is via Σ. ⊣

Definition 3.16. A premouse or hod premouse $\mathcal{P}$ is **reasonable** iff $\mathcal{P}$ is super-small, all $\mathcal{N}\trianglelefteq\mathcal{P}$ (including $\mathcal{N}=\mathcal{P}$) satisfy the conclusions of [13, 4.11, 4.12, 4.15], and if $\mathcal{P}$ is a premouse then all $\mathcal{N}\triangleleft\mathcal{P}$ are $<\omega$-condensing, and if $\mathcal{P}$ is a hod premouse then for all $\mathcal{N}\triangleleft\mathcal{P}$, $\mathcal{N}$ is $<\omega$-condensing with respect to embeddings $\pi:\mathcal{H}\to\mathcal{N}$ such that $\mathrm{crit}(\pi)\geq\delta^{\mathcal{P}}_{\alpha}$ for all α such that $\delta^{\mathcal{P}}_{\alpha}\leq\mathrm{o}(\mathcal{N})$. Reasonableness is preserved by fine structural iteration, as super-smallness is $\mathrm{r}\Sigma_2$ and the other conditions are $\mathrm{r}\Pi_1$.

A hod pair $(\Sigma, \mathcal{P})$ is **within scope** iff $\mathsf{DC}_{\mathbb{R}}$ holds, $\mathcal{P} \in \mathrm{HC}$ is reasonable, is below $\mathsf{AD}_{\mathbb{R}}$+"Θ is regular", Σ is a *hod* $(\omega, \kappa, \kappa+1)$-strategy for $\mathcal{P}$, where κ is some regular uncountable cardinal, Σ is Γ-fullness* preserving for some smooth pointclass Γ, Σ has branch condensation, and if $\kappa > \omega_1$ then Σ has weak hull condensation. ⊣

Definition 3.17. Let $(\mathcal{P}, \Sigma)$ be a hod pair. We say that Σ has **factor hull condensation** iff whenever:

- $\vec{\mathcal{T}}, \vec{\mathcal{U}}$ are stacks via Σ and $i^{\vec{\mathcal{T}}}, i^{\vec{\mathcal{U}}}$ exist; let $\mathcal{M} = M_\infty^{\vec{\mathcal{T}}}$ and $\mathcal{N} = M_\infty^{\vec{\mathcal{U}}}$,
- $\pi : \mathcal{M} \to \mathcal{N}$ is elementary and $\pi \circ i^{\vec{\mathcal{T}}} = i^{\vec{\mathcal{U}}}$,
- $\vec{\mathcal{W}}$ is a stack on $\mathcal{N}$ via $\Sigma_{\mathcal{N}, \vec{\mathcal{U}}}$, and
- $\vec{\mathcal{V}}$ is a stack on $\mathcal{M}$ and $\pi\vec{\mathcal{V}}$ is a hull of $\vec{\mathcal{W}}$,

then $\vec{\mathcal{V}}$ is via $\Sigma_{\mathcal{M}, \vec{\mathcal{T}}}$. ⊣

Factor hull condensation trivially implies hull condensation. But the following lemma is more interesting; part of its proof uses ideas similar to those in Sargsyan's [5, Proposition 2.41].

Lemma 3.18. *Let $(\mathcal{P}, \Sigma)$ be a hod pair within scope. Then Σ has factor hull condensation.*

Proof. By weak hull condensation, we may assume that all trees we deal with are countable. (If $\kappa = \omega_1$ then because ω_1 is regular, it is easy to see that we may still reduce to countable trees, without using weak hull condensation.)

Suppose the lemma fails. Let $\vec{\mathcal{T}}, \vec{\mathcal{U}}, \mathcal{M}, \mathcal{N}, \pi$ be a counterexample. A **bad system** is a countable system

$$\left(\left\langle \vec{\mathcal{V}}_i, \vec{\mathcal{V}}_i^*, \vec{\mathcal{W}}_i, \vec{\mathcal{W}}_i^*, \vec{\sigma}_i, \vec{\sigma}_i^*, \beta_i \right\rangle_{i \leq n}, \langle \alpha_i, \pi_i \rangle_{i \leq n+1}\right)$$

where

1. $(\vec{\mathcal{T}}, \vec{\mathcal{V}}_0, \ldots, \vec{\mathcal{V}}_n)$ and $(\vec{\mathcal{U}}, \vec{\mathcal{W}}_0, \ldots, \vec{\mathcal{W}}_n)$ are terminally non-dropping stacks on $\mathcal{P}$ via Σ. Let $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{N}_0 = \mathcal{N}$ and $\mathcal{M}_{i+1} = M_\infty^{\vec{\mathcal{V}}_i}$ and $\mathcal{N}_{i+1} = M_\infty^{\vec{\mathcal{W}}_i}$.
2. $\beta_0 = \lambda^{\mathcal{M}}$ and $\alpha_0 = \lambda^{\mathcal{M}} + 1$ and $\beta_i < \alpha_i$ and $\alpha_{i+1} \leq i^{\vec{\mathcal{V}}_i}(\beta_i)$.

3. $\vec{\mathcal{V}}_i$ is based on $\mathcal{M}_i(\beta_i)$.

4. $\vec{\mathcal{V}}_i = (\vec{\mathcal{V}}_i', \mathcal{V}_i)$, where $\mathcal{V}_i$ is a normal tree (so $\mathcal{V}_i$ is terminally non-dropping and $\mathcal{M}_{i+1} = M_\infty^{\mathcal{V}_i}$).

5. $\vec{\mathcal{V}}_i^* = (\vec{\mathcal{V}}_i', \mathcal{V}_i^*)$ is a stack on $\mathcal{M}_i$, based on $\mathcal{M}_i(\beta_i)$, where $\mathcal{V}_i^*$ is a normal extension of $\mathcal{V}_i$, $\mathcal{V}_i = \mathcal{V}_i^* \restriction (\gamma_i + 1)$, where γ_i is the least γ such that $\mathrm{o}(\mathcal{M}_{i+1}(\alpha)) < \mathrm{lh}(E_\gamma^{\mathcal{V}_i^*})$ for all $\alpha < \alpha_{i+1}$, $\mathcal{V}_i^* \restriction [\gamma_i, \mathrm{lh}(\mathcal{V}_i^*))$ is based on $\mathcal{M}_{i+1}(\alpha_{i+1})$, and $\mathcal{V}_i^*$ has successor length and is terminally dropping.

6. $\vec{\mathcal{W}}_i = (\vec{\mathcal{W}}_i', \mathcal{W}_i)$, where $\mathcal{W}_i$ is a normal tree.

7. $\vec{\mathcal{W}}_i^* = (\vec{\mathcal{W}}_i', \mathcal{W}_i^*)$, where $\mathcal{W}_i^*$ is a normal extension of $\mathcal{W}_i$.

8. $(\vec{\mathcal{U}}, \vec{\mathcal{W}}_0, \ldots, \vec{\mathcal{W}}_{i-1}, \vec{\mathcal{W}}_i^*)$ is via Σ.

9. $(\vec{\mathcal{T}}, \vec{\mathcal{V}}_0, \ldots, \vec{\mathcal{V}}_{i-1}, \vec{\mathcal{V}}_i^*)$ is not via Σ.

10. $(\vec{\mathcal{T}}, \vec{\mathcal{V}}_0, \ldots, \vec{\mathcal{V}}_{i-1}, \vec{\mathcal{V}}_i^* \restriction \mathrm{lh}((\vec{\mathcal{V}}_i^*) - 1))$ is via Σ.

11. $\pi_0 = \pi$ and $\pi_i : \mathcal{M}_i \to \mathcal{N}_i$, and $\pi_i \vec{\mathcal{V}}_i$ is a hull of $\vec{\mathcal{W}}_i$, as witnessed by $\vec{\sigma}_i$, with the final node of $\pi_i \vec{\mathcal{V}}_i$ corresponding to the final node of $\vec{\mathcal{W}}_i$, and π_{i+1} is the composition of the final copy and hull embedding maps.

12. $\pi_i \vec{\mathcal{V}}_i^*$ is a hull of $\vec{\mathcal{W}}_i^*$, as witnessed by $\vec{\sigma}_i^*$, and $\vec{\sigma}_i \subseteq \vec{\sigma}_i^*$, and $\vec{\mathcal{W}}_i^*$ has no proper segment $\mathcal{W}'$ such that $\pi_i \vec{\mathcal{V}}_i^*$ is a hull of $\mathcal{W}'$ as witnessed by $\vec{\sigma}_i^*$.

Because of our choice of $\vec{\mathcal{T}}, \vec{\mathcal{U}}, \pi$, it is easy to see using branch condensation and weak hull condensation (the latter to give countability, and the former to ensure a dropping branch) that there is a bad system with $n = 0$. Using $\mathsf{DC}_{\mathbb{R}}$, it follows that there is a bad system B for which no proper extension is also a bad system. Let the notation above be used to describe B.

Let $\mathcal{R} = \mathcal{M}_{n+1}(\alpha_{n+1})$ and $\varrho = \pi_{n+1} \restriction \mathcal{R}$ and $\mathcal{S} = \mathcal{N}_{n+1}(\varrho(\alpha_{n+1}))$, so $\varrho : \mathcal{R} \to \mathcal{S}$ is elementary. Let $\eta = \sup_{\alpha < \alpha_{n+1}} \mathrm{o}(\mathcal{R}(\alpha))$. Let $\eta^{\mathcal{S}} = \varrho(\eta)$. Let $\Psi^{\mathcal{S}}$ be the above-$\eta^{\mathcal{S}}$, $(\omega, \omega_1 + 1)$-strategy for $\mathcal{S}$ given by normally extending $\vec{\mathcal{W}}_n^*$, and continuing to use Σ. Let Ψ be the ϱ-pullback of $\Psi^{\mathcal{S}}$, for $\mathcal{R}$. Let $\mathcal{X} = (\vec{\mathcal{T}}, \vec{\mathcal{V}}_0, \ldots, \vec{\mathcal{V}}_n)$.

Claim 3.19. *Ψ is a $\oplus_{\alpha < \alpha_{n+1}} \Sigma_{\mathcal{R}(\alpha), \mathcal{X}}$-strategy.*

Proof. If not then, again using weak hull and branch condensation, it is easy to produce a bad system properly extending B, a contradiction. □

Now let $\mathcal{V}$ be the tree on $\mathcal{R}$ which is equivalent to $\vec{\mathcal{V}}^*_n\upharpoonright[\gamma_n, \mathrm{lh}(\vec{\mathcal{V}}^*_n)-1)$, let b, c be the $\mathcal{V}$-cofinal branches determined by $\vec{\mathcal{V}}^*_n$ and Σ respectively. So $b \neq c$ and b drops. By the claim and using Σ, we may successfully compare the phalanxes $\Phi(\mathcal{V} \frown b)$ and $\Phi(\mathcal{V} \frown c)$, producing (padded) trees $\mathcal{Y}, \mathcal{Z}$ extending $\mathcal{V}\frown b$ and $\mathcal{V}\frown c$ respectively. Moreover, all models of $\mathcal{Y}, \mathcal{Z}$ are $\oplus_{\alpha<\alpha_{n+1}}\Sigma_{\mathcal{R}(\alpha),\mathcal{X}}$-hod premice. Let $\delta = \delta(\mathcal{V})$ and $\lambda = \mathrm{lh}(\mathcal{V})$.

Claim 3.20. *c does not drop, and therefore α_{n+1} is not a limit.*

Proof. This is a standard argument, but we give it as it is not too long, and we need it elsewhere. Suppose c drops. It suffices to see that at for every $\alpha \geq \lambda$, $[0, \alpha]_{\mathcal{Y}}$ and $[0, \alpha]_{\mathcal{Z}}$ drops, since then standard fine structure yields a contradiction. Suppose this fails. Then there is $\alpha \geq \lambda$ such that either $E = E^{\mathcal{Y}}_\alpha$, or $E = E^{\mathcal{Z}}_\alpha$, has $\mathrm{crit}(E) < \delta$. Let α be least such. Then $[0, \alpha']_{\mathcal{Y}}$ and $[0, \alpha']_{\mathcal{Z}}$ drop for each $\alpha' \in [\lambda, \alpha]$. Let $\mathcal{Q}_b = \mathcal{Q}(\mathcal{V}, b)$ and $\mathcal{Q}_c = \mathcal{Q}(\mathcal{V}, c)$. Then $\mathcal{Q}_b \neq \mathcal{Q}_c$, so δ is Woodin in $M^{\mathcal{Y}}_\alpha||\mathrm{lh}(E)$. So if there is any $F \in \mathbb{E}^{M^{\mathcal{Y}}_\alpha||\mathrm{lh}(E)}$ such that $\mathrm{crit}(F) < \delta < \mathrm{lh}(F)$, we easily get that $[0, \beta]_{\mathcal{Y}}$ and $[0, \beta]_{\mathcal{Z}}$ are dropping for all $\beta > \alpha$ (as Woodins are cutpoints of hod premice). So suppose E is the least extender overlapping δ, so $\alpha = \lambda$. Let $\kappa = \mathrm{crit}(E)$. Then κ is a measurable limit of Woodins and strong cutpoints of $M(\mathcal{V})$. Let γ be least such that $\kappa < \mathrm{lh}(E^{\mathcal{V}}_\gamma)$. Then for all $\beta < \gamma$, $\mathrm{lh}(E^{\mathcal{V}}_\beta) < \kappa$. Let $\mathcal{Q} = M^{*\mathcal{Y}}_{\lambda+1}$, or $\mathcal{Q} = M^{*\mathcal{Z}}_{\lambda+1}$, according to whether E is used in $\mathcal{Y}$ or $\mathcal{Z}$. Note that κ is a cutpoint of $\mathcal{Q}$. But then $\mathcal{Y}\upharpoonright[\gamma, \mathrm{lh}(\mathcal{Y}))$ is and $\mathcal{Z}\upharpoonright[\gamma, \mathrm{lh}(\mathcal{Z}))$ are equivalent to above-κ, normal trees on $\mathcal{Q}$. So if $\mathcal{Q} \lhd M^{\mathcal{V}}_\gamma$ then we are done, and if $\mathcal{Q} = M^{\mathcal{V}}_\gamma$ then note that $[0, \gamma]_{\mathcal{V}}$ drops (as our hod premice are below $\mathsf{AD}_{\mathbb{R}}+$"$\Theta$ is measurable"), so again we are done. □

Claim 3.21. *We have:*

- *δ is Woodin in $M^{\mathcal{V}}_c$, so $\delta = \delta^{M^{\mathcal{V}}_c}_{\alpha_{n+1}}$, the largest Woodin of $M^{\mathcal{V}}_c$.*
- *δ is a strong cutpoint of $\mathcal{Q}_b$.*
- *$M^{\mathcal{V}}_c \lhd \mathcal{Q}_b$.*

Proof. Neither $\mathcal{Q}_b$, nor $\mathcal{Q}_c$ if it exists, can have overlaps of δ, since otherwise $M^{\mathcal{V}}_c$ has a measurable limit of Woodins, which implies c drops, contradiction.

But if δ is not Woodin in $M_c^{\mathcal{V}}$ then as before, $\mathcal{Q}_b \neq \mathcal{Q}_c$, so comparison gives a contradiction.

So the comparison of $\mathcal{Q}_b$ with $M_c^{\mathcal{V}}$ is above δ, and succeeds, and this easily gives that $M_c^{\mathcal{V}} \triangleleft \mathcal{Q}_b$. $\square$

Let $\tau : M_\infty^{\vec{\mathcal{V}}_n^*} \to M_\infty^{\vec{\mathcal{W}}_n^*}$ be the final map given by the hull embedding (by the minimality of $\vec{\mathcal{W}}_n^*$ with respect to $\vec{\sigma}_n^*$, the final model of $\vec{\mathcal{V}}_n^*$ does indeed correspond to the final model of $\vec{\mathcal{W}}_n^*$). Let $\mathcal{Q}'$ be the lift of $\mathcal{Q} = \mathcal{Q}_b$ under τ, and let $\tau_{\mathcal{Q}} = \tau \upharpoonright \mathcal{Q}$. Let $\delta' = \tau_{\mathcal{Q}}(\delta)$. Let $\mathcal{X}' = (\vec{\mathcal{U}}, \vec{\mathcal{W}}_0, \ldots, \vec{\mathcal{W}}_{n-1}, \vec{\mathcal{W}}_n^*)$. Let $\alpha = \alpha_{n+1} - 1$ (possibly $\alpha = -1$) and $\alpha' = \tau_{\mathcal{Q}}(\alpha)$. Using 3.14(2), let Υ' be an above-δ', $\Sigma_{\mathcal{Q}'(\alpha'),\mathcal{X}'}$-$(\omega, \omega_1, \omega_1 + 1)$-strategy for $\mathcal{Q}$, whose restriction to countable trees is in Γ. Let Υ be the τ-pullback of Υ'. Like in Claim 3.19, we then get:

Claim 3.22. *Υ is a $\Sigma_{\mathcal{Q}(\alpha),\mathcal{X}}$-strategy, and the restriction of Υ to countable trees is in Γ (where $\mathcal{Q}(-1) = \emptyset$); note $\mathcal{Q}(\alpha) = \mathcal{R}(\alpha)$).*

But then $\mathcal{Q}_b \triangleleft \mathrm{Lp}^{\Gamma, \Sigma_{\mathcal{R}(\alpha),\mathcal{X}}}$, which with Claim 3.21, contradicts Γ-fullness* preservation for Σ, completing the proof. $\square$

The following lemma, related to [8, §2], is due to Steel. However, the standard proof seems to have a gap (in the proof of Claim 3.25 below). A correct proof of what is essentially the lemma appeared in [13, §5], but that proof is somewhat buried in another context, so we give a proof here for convenience. We state and prove the lemma literally only for pure $L[\mathbb{E}]$-constructions, but it is easy to adapt it to strategy mice and other variants.

Lemma 3.23 (Stationarity of $L[\mathbb{E}]$ constructions)**.** *Let γ be an uncountable cardinal. Let $\mathcal{P}$ be a reasonable k-sound premouse, Ψ a $(k, \gamma + 1)$-strategy for $\mathcal{P}$ and $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha \leq \gamma}$ be a fully backgrounded $L[\mathbb{E}]$-construction.*

Suppose that for each active $\mathcal{N}_{\alpha+1} = (\mathcal{N}_\alpha, E)$ there is an extender E^* such that (a) $\mathrm{card}(\mathcal{P}) < \mathrm{crit}(E^*)$, (b) $F \upharpoonright \nu(E) \subseteq E^*$, (c) if $\mathcal{P}$ is non-tame then $i_{E^*}(\Psi) \upharpoonright V_\eta \subseteq \Psi$ where η is the sup of all $\delta + 1$ such that δ is Woodin in $\mathcal{N}_\alpha$.

Then there is $\xi \leq \gamma + 1$ such that:

(1) for each $\alpha < \xi$, we have $\mathcal{N}_\alpha \trianglelefteq \mathcal{P}'$ for some Ψ-iterate $\mathcal{P}'$ of $\mathcal{P}$, and

(2) if $\xi \leq \gamma$ then there is a tree $\mathcal{T}$ via Ψ, of successor length, $\mathcal{N}_\xi = M_\infty^{\mathcal{T}}$ and $b^{\mathcal{T}}$ does not drop in model.

Proof. It suffices to prove that if (1) holds at ξ, but (2) does not, then (1) holds at $\xi + 1$. This is easy in all cases except when $\xi = \alpha + 1$ and $\mathcal{N}_{\alpha+1} = (\mathcal{N}_\alpha, E)$ for some E, so suppose this is the case. Let E^* be a background extender for E and let $j = i_{E^*}$. Let $\mathcal{T}$ be the tree witnessing the lemma's conclusion for α. We assume that $\mathcal{T}$ has minimal possible length. We must show that E is used in $\mathcal{T}$. Let $\nu = \nu(E)$ and $\kappa = \mathrm{crit}(E)$. The main point is the following claim:

Claim 3.24. *There is $\beta < \mathrm{lh}(j(\mathcal{T}))$ such that $\nu \le \nu(E^{\mathcal{T}}_\beta)$ and $E{\restriction}\nu \subseteq E^{\mathcal{T}}_\beta$.*

Proof. As in the proof that comparison of premice terminates, we have $M^{j(\mathcal{T})}_\kappa = M^{\mathcal{T}}_\kappa$ and $\kappa <_{j(\mathcal{T})} j(\kappa)$ and $i^{j(\mathcal{T})}_{\kappa,j(\kappa)}$ exists and

$$i^{\mathcal{T}}_{\kappa,j(\kappa)}{\restriction}M^{\mathcal{T}}_\kappa = j{\restriction}M^{\mathcal{T}}_\kappa. \tag{3.2}$$

So let $\beta+1 <_{\mathcal{T}} j(\kappa)$ be such that $\mathrm{pred}^{\mathcal{T}}(\beta+1) = \kappa$. We claim that β works. For let

$$k : \mathrm{Ult}(\mathcal{N}_\alpha, E) \to j(\mathcal{N}_\alpha)$$

be the factor embedding. Then $\mathrm{crit}(k) \ge \nu(E)$, and if E is type 2 then $\mathrm{crit}(k) \ge \mathrm{lh}(E)$. So $\mathcal{N}_\alpha$, $M^{\mathcal{T}}_\kappa$, $M^{\mathcal{T}}_\beta$ and $M^{j(\mathcal{T})}_{j(\kappa)}$ agree below $(\kappa^+)^{\mathcal{N}_\alpha}$. So $E^{\mathcal{T}}_\beta$ measures all sets measured by E and by line (3.2) we have that $E{\restriction}\nu' \subseteq E^{\mathcal{T}}_\beta{\restriction}\nu'$, where $\nu' = \min(\nu, \nu(E^{\mathcal{T}}_\beta))$. Now if $(\kappa^+)^{\mathcal{N}_\alpha} < (\kappa^+)^{M^{\mathcal{T}}_\kappa}$ then $\mathrm{crit}(k) = (\kappa^+)^{\mathcal{N}_\alpha}$, so E is type 1 and $\nu = (\kappa^+)^{\mathcal{N}_\alpha}$, so we are done. So assume $(\kappa^+)^{\mathcal{N}_\alpha} = (\kappa^+)^{M^{\mathcal{T}}_\kappa}$, and assume $\nu' < \nu$. Since also $(\kappa^+)^{M^{\mathcal{T}}_\kappa} \le \nu'$, the ISC applies to $E{\restriction}\nu'$. So $E{\restriction}\nu' \in \mathcal{N}_\alpha$, although $E{\restriction}\nu' \notin j(\mathcal{N}_\alpha)$. So E is not type 2. So E is type 3, but then $\mathrm{lh}(E^{\mathcal{T}}_\beta) < \nu$, contradicting the fact that $\mathcal{N}_\alpha||\nu = j(\mathcal{N}_\alpha)||\nu$. ☐

Claim 3.25. *Either:*

- $E \in \mathbb{E}_+(M^{j(\mathcal{T})}_\beta)$, *or*
- $\mathcal{R} =_{\mathrm{def}} M^{j(\mathcal{T})}_\beta|\nu$ *is active with extender* F *and* $E \in \mathbb{E}_+^{\mathrm{Ult}(\mathcal{R},F)}$.

Proof. If $(\kappa^+)^{N_\alpha} = (\kappa^+)^{M^{j(\mathcal{T})}_\beta}$ this is just by the ISC. So suppose $(\kappa^+)^{N_\alpha} < (\kappa^+)^{M^{j(\mathcal{T})}_\beta}$. Then E is type 1, the normal measure derived from E is a submeasure of the normal measure derived from $E^{j(\mathcal{T})}_\beta$, and $M^{j(\mathcal{T})}_\beta||\nu = \mathcal{N}_\alpha||\nu$. Thus, we can use [13, 4.11, 4.12, 4.15] (as $\mathcal{P}$ is reasonable) given that if $\mathcal{R}$ is active with a type 3 extender F then

$$\mathrm{Ult}(\mathcal{R}, F)||\mathrm{lh}(E) = \mathcal{N}_\alpha. \tag{3.3}$$

So suppose $F = F^{\mathcal{R}} \neq \emptyset$. We have $\mathcal{T}\restriction(\kappa+1) = j(\mathcal{T})\restriction\kappa+1$, and note that $\mathcal{T}$ uses no extenders with index in the interval (κ, ν), as E is type 1, and $j(\mathcal{T})$ uses no extender with index in the interval $(\kappa, (\kappa^+)^{M_\kappa^{\mathcal{T}}})$. So $M_\kappa^{\mathcal{T}}|\nu = \mathcal{R}$, and since $N_\alpha|\nu$ is passive, therefore $E_\kappa^{\mathcal{T}} = F$. But then $\mathcal{T}$ uses no extender with index in the interval $(\nu, \mathrm{lh}(E))$, and line (3.3) is true. $\square$

Now let λ be least such that $\mathrm{lh}(E_\lambda^{j(\mathcal{T})}) \geq \mathrm{lh}(E)$, and let ξ be the largest limit ordinal such that $\xi \leq \lambda$. By the following claim, we clearly have that $j(\mathcal{T})\restriction\lambda+1$ is via Ψ, which completes the proof.

Claim 3.26. $j(\mathcal{T})\restriction\xi+1 = \mathcal{T}\restriction\xi+1$.

Proof. We have $\mathcal{N}_\alpha \trianglelefteq M_\infty^{\mathcal{T}}$ and $j(\mathcal{N}_\alpha) \trianglelefteq M_\infty^{j(\mathcal{T})}$. Let χ be the largest cardinal of $\mathcal{N}_\alpha$ and ϵ be the largest limit cardinal of $j(\mathcal{N}_\alpha)||\mathrm{lh}(E)$. Then $\epsilon \leq \chi$ and $\mathcal{N}_\alpha||(\epsilon^+)^{\mathcal{N}_\alpha} = j(\mathcal{N}_\alpha)||(\epsilon^+)^{\mathcal{N}_\alpha}$ (possibly $(\epsilon^+)^{\mathcal{N}_\alpha} < (\epsilon^+)^{j(\mathcal{N}_\alpha)}$) and $\lfloor\mathcal{N}_\alpha\rfloor \subseteq j(\mathcal{N}_\alpha)$. These things follow from $<\omega$-condensation, considering the factor embedding k. Now let $\delta = \delta(j(\mathcal{T})\restriction\xi)$; it follows that $\delta \leq \epsilon$. So $\mathcal{N}_\alpha|\delta = j(\mathcal{N}_\alpha)|\delta$, and it suffices to see that for each $\xi' \leq \xi$, we have $[0,\xi')_{j(\mathcal{T})} = [0,\xi')_{\mathcal{T}}$. We prove this by induction on ξ'. So assume $\mathcal{T}\restriction\xi' = j(\mathcal{T})\restriction\xi'$. We may assume $\xi' \geq \kappa$, so $\delta' = \delta(\mathcal{T}\restriction\xi') \geq \kappa$ also. Now if $\mathcal{N}_\alpha \models$"$\delta'$ is not Woodin" then let $\mathcal{Q} \triangleleft M_{\xi'}^{\mathcal{T}}$ be the Q-structure for δ'. Then $\mathcal{Q} \triangleleft \mathcal{N}_\alpha$, so $\mathcal{Q} \triangleleft j(\mathcal{N}_\alpha)$, so $\mathcal{Q} \triangleleft M_{\xi'}^{j(\mathcal{T})}$. Therefore $[0,\xi')_{\mathcal{T}} = [0,\xi')_{j(\mathcal{T})}$, as required. So suppose $\mathcal{N}_\alpha \models$"$\delta'$ is Woodin". Since $\kappa \leq \delta' < \mathrm{lh}(E)$, and so by Claim 3.25, $\mathcal{P}$ is non-tame. So by our hypothesis, $j(\Psi)\restriction V_{\delta'+1} \subseteq \Psi$, so $[0,\xi')_{j(\mathcal{T})} = [0,\xi')_{\mathcal{T}}$. $\square$

Definition 3.27. Let $(\mathcal{P}, \Sigma)$ be a hod pair, within scope, and κ be such that Σ is a hod $(\omega, \kappa, \kappa+1)$-strategy. Let $a \in \mathrm{HC}$ be such that $\mathcal{P} \in \mathcal{J}_1(\hat{a})$ and $\hat{a}$ is swo'd. Suppose that $\mathfrak{M} = \mathcal{M}_1^{\Sigma,\#}(\hat{a})$ exists and is Σ-κ-naturally iterable. Let $\mathfrak{N}$ be any non-dropping $\Lambda_{\mathfrak{M}}^{\Sigma,\kappa}$-iterate of $\mathfrak{M}$. Let $\delta = \delta^{\mathfrak{N}}$ and $\Sigma_{\mathcal{P}}^{\mathfrak{N}} = \Sigma\restriction\mathfrak{N}$. Let $\chi \leq \delta+1$.

A $(\mathcal{P}, \Sigma)$**-bounded hod pair construction of $\mathfrak{N}$, of length** χ, is a sequence

$$\mathbb{D} = \langle(\mathbb{C}_\beta, \mathcal{T}_\beta, \alpha_\beta, \mathcal{Q}_\beta, \mathcal{R}_\beta, \mathcal{M}_\beta, \Sigma_\beta)\rangle_{\beta<\chi}$$

with the following properties holding inside $\mathfrak{N}$ for all $\beta < \chi$:

- $\mathcal{T}_\beta$ is a terminally non-dropping, successor length, normal tree on $\mathcal{P}$ via $\Sigma_{\mathcal{P}}^{\mathfrak{N}}$, and $\mathcal{Q}_\beta = M_\infty^{\mathcal{T}_\beta}$ and $\mathcal{R}_\beta = \mathcal{Q}_\beta(\beta)$.

- $\mathcal{T}_\alpha \subsetneq \mathcal{T}_\beta$ for $\alpha < \beta$.

- $\mathcal{T}_0$ is based on $\mathcal{P}(0)|\delta_0^{\mathcal{P}}$.
- If $\beta+1<\chi$ then $\mathcal{T}_{\beta+1}$ is based on $\mathcal{Q}_\beta(\beta+1)|\delta_{\beta+1}^{\mathcal{Q}_\beta}$, and is above $\delta_\beta^{\mathcal{Q}_\beta}$.
- If β is a limit then $\mathcal{T}_\beta = \mathcal{T}_\beta^* \frown \Sigma(\mathcal{T}_\beta^*)$ where $\mathcal{T}_\beta^* = \lim_{\alpha<\beta}\mathcal{T}_\alpha$.
- Σ_β is the strategy for $\mathcal{R}_\beta$ which is the tail of $\Sigma_{\mathcal{P}}^{\mathfrak{N}}$.
- $\mathbb{C}_0$ is the maximal $L[\mathbb{E}]$-construction[29] of $\mathfrak{N}|\delta$.
- If $\beta+1<\chi$ then $\mathbb{C}_{\beta+1}$ is the maximal $L^{\Sigma_\beta}[\mathbb{E}](\mathcal{R}_\beta)$-construction of $\mathfrak{N}|\delta$.
- If β is a limit, $\mathbb{C}_\beta$ is the maximal $L^{\Sigma_\beta^*}[\mathbb{E}](\mathcal{R}_\beta^*)$-construction of $\mathfrak{N}|\delta$, where $\Sigma_\beta^* = \oplus_{\alpha<\beta}\Sigma_\alpha$ and $\mathcal{R}_\beta^* = \oplus_{\alpha<\beta}\mathcal{R}_\alpha$.
- $\alpha_\beta<\delta$ and $\mathcal{R}_\beta = \mathfrak{N}_{\alpha_\beta}^{\mathbb{C}_\beta}$.
- For all $\alpha<\alpha_0$ there is a successor length normal tree $\mathcal{T}$ on $\mathcal{P}$, via $\Sigma_{\mathcal{P}}^{\mathfrak{N}}$, based on $\mathcal{P}(0)$, such that $\mathcal{N}_\alpha^{\mathbb{C}_0} \trianglelefteq M_\infty^{\mathcal{T}}$, and either $b^{\mathcal{T}}$ drops or $\mathcal{N}_\alpha^{\mathbb{C}_0} \triangleleft M_\infty^{\mathcal{T}}(0)$.
- If $\beta+1<\chi$ then for all $\alpha<\alpha_{\beta+1}$ there is a successor length normal tree $\mathcal{T}$ on $\mathcal{Q}_\beta$, based on $\mathcal{Q}_\beta(\beta+1)$, above $\mathcal{R}_\beta = \mathcal{Q}_\beta(\beta)$, with $\mathcal{T}_\beta \frown \mathcal{T}$ via $\Sigma_{\mathcal{P}}^{\mathfrak{N}}$, and such that $\mathcal{N}_\alpha^{\mathbb{C}_{\beta+1}} \trianglelefteq M_\infty^{\mathcal{T}}$, and either $b^{\mathcal{T}}$ drops or $\mathcal{N}_\alpha^{\mathbb{C}_{\beta+1}} \triangleleft M_\infty^{\mathcal{T}}(\beta+1)$.
- If β is a limit then for all $\alpha<\alpha_\beta$, either $\mathcal{N}_\alpha^{\mathbb{C}_\beta} \triangleleft \mathcal{R}_\beta$, or there is a successor length normal tree $\mathcal{T}$ on $\mathcal{R}_\beta$, above $\delta_\beta^{\mathcal{R}_\beta}$, with $\mathcal{T}_\beta \frown \mathcal{T}$ via $\Sigma_{\mathcal{P}}^{\mathfrak{N}}$, such that $b^{\mathcal{T}}$ drops and $\mathcal{N}_\alpha^{\mathbb{C}_\beta} \trianglelefteq M_\infty^{\mathcal{T}}$.
- $\mathfrak{M}_\beta$ is the least $\mathfrak{M} \triangleleft \mathfrak{N}$ such that $\mathrm{o}(\mathfrak{M})$ is a successor cardinal and $\beta, \alpha_\gamma < \mathrm{o}(\mathfrak{M})$ for all $\gamma \leq \beta$. Let $\Lambda_{\mathfrak{M}_\beta}$ be the $(\omega, \mathrm{Ord}, \mathrm{Ord})$-maximal strategy for $\mathfrak{M}_\beta$, guided by Q-structures computed from ordinals and $\Sigma_{\mathcal{P}}^{\mathfrak{N}}$. Then Σ_β is exactly the strategy for $\mathcal{R}_\beta$ induced by lifting to $\Lambda_{\mathfrak{M}_\beta}$.

We say that such a construction is **successful** iff $\chi = \beta+1 < \delta$ and $\mathcal{R}_\beta = \mathcal{Q}_\beta$ (thus, the construction has produced a non-dropping normal Σ-iterate $\mathcal{Q}_\beta = \mathcal{N}_{\alpha_\beta}^{\mathbb{C}_\beta}$ of $\mathcal{P}$). $\dashv$

[29]Here and below, all background extenders are required to come from $\mathbb{E}^{\mathfrak{N}}$.

Lemma 3.28. *Adopt the hypotheses and notation of 3.27. Then there is a unique successful $(\mathcal{P}, \Sigma)$-bounded hod pair construction $\mathbb{D}$ of $\mathfrak{N}$.*

Moreover, let $\beta < \mathrm{lh}(\mathbb{D})$ and $\Lambda^V_{\mathfrak{M}_\beta}$ be the Q-structure guided $(\omega, \kappa, \kappa+1)$-strategy for $\mathfrak{M}_\beta$ (so $\Lambda^V_{\mathfrak{M}_\beta}$ is induced by the tail of $\Lambda^{\Sigma,\kappa,\kappa+1}_{\mathfrak{M}}$ and $\Lambda_{\mathfrak{M}_\beta} \subseteq \Lambda^V_{\mathfrak{M}_\beta}$). Let Σ^V_β be the hod $(\omega, \kappa, \kappa+1)$-strategy for $\mathcal{R}_\beta$ induced by the tail of Σ (so $\Sigma_\beta \subseteq \Sigma^V_\beta$). Let Γ^V_β be the hod $(\omega, \kappa, \kappa+1)$-strategy for $\mathcal{R}_\beta$ given by lifting to $\Lambda^V_{\mathfrak{M}_\beta}$. Then $\Sigma^V_\beta = \Gamma^V_\beta$.

Proof. This is partly proven in [5], but we cover some details not presented there; it is in these details that the distinction between hod strategies and full strategies is important.

It is easy to see that for each χ, there is at most one construction of length χ. Trivially, if $\chi = 0$, or χ is a limit and for all $\beta < \chi$, there is a construction of length β, then there is a construction of length χ. So suppose there is an unsuccessful construction of length χ; we need to see there is a construction of length $\chi + 1$.

We assume $\chi = \beta + 1$, as if $\chi = 0$ or χ is a limit it is an easy variant.

Let $\mathbb{C}$ be the maximal $L^{\Sigma_\beta}[\mathbb{E}](\mathcal{R}_\beta)$-construction of $\mathfrak{N}|\delta$. Let Ψ be the above-$\mathcal{R}_\beta$, normal strategy for $\mathcal{Q}_\beta(\beta+1)$ given by continuing $\mathcal{T}_\beta$ as a normal tree, using Σ. An easy variant of 3.23, together with universality at δ, [17, Lemma 11.1], shows that $\mathbb{C}$ reaches a non-dropping Ψ-iterate $\mathcal{R}_{\beta+1} = \mathcal{N}^{\mathbb{C}}_\alpha$ of $\mathcal{Q}_\beta(\beta+1)$, for some $\alpha < \delta$ such that for all $\xi < \alpha$, $\mathcal{N}^{\mathbb{C}}_\xi$ is either a dropping such iterate, or a proper segment of such an iterate. (With regard to universality, we don't need to iterate $\mathcal{N}^{\mathbb{C}}_\delta$ in $\mathfrak{M}$, so we don't need $\mathfrak{M}$ to know any of its own iteration strategy.) Let $\mathcal{T}_{\beta+1}$ be the corresponding tree on $\mathcal{P}$.

So a length $\chi + 1$ construction will exist given that $\Sigma_{\beta+1}$ agrees with the hod strategy Γ for $\mathcal{R}_{\beta+1}$ given by lifting to $\Lambda_{\mathcal{M}_{\beta+1}}$ (where notation is as in 3.27). This follows from the "moreover" clause of the lemma at $\beta+1$, which we now prove. Let $\mathcal{U}$ be a limit length tree via both $\Gamma^V_{\beta+1}$ and $\Sigma^V_{\beta+1}$ (notation as in the statement of the lemma). Let $b = \Gamma^V_{\beta+1}(\mathcal{U})$ and $c = \Sigma^V_{\beta+1}(\mathcal{U})$. Because Σ and $\Lambda^{\Sigma,\kappa}_{\mathfrak{M}}$ have hull condensation, by taking a hull here we may assume everything is countable.

Sargsyan's argument showing that if b does not drop then $b = c$ (using branch condensation for Σ) goes through here (cf. [5, Lemma 2.15]). So assume that b drops. Then because we are dealing with hod strategies, $\mathcal{U}$ has the form $\vec{\mathcal{V}} \frown \mathcal{V}$, where $\vec{\mathcal{V}}$ does not drop, $\mathcal{V}$ is normal and $\mathcal{V} \frown b$ drops.

Let $\gamma < \mathrm{lh}(\mathcal{V})$ and α be such that $[0, \gamma]_{\mathcal{V}}$ does not drop, and letting

$\mathcal{N} = M^{\mathcal{V}}_\gamma$, such that $\alpha \leq \lambda^{\mathcal{N}}$ and for all $\tau < \gamma$, we have

$$\mathrm{lh}(E^{\mathcal{V}}_\tau) < \epsilon =_{\mathrm{def}} \bigcup_{\xi<\alpha} \mathrm{o}(\mathcal{N}(\xi)) < \mathrm{lh}(E^{\mathcal{V}}_\gamma),$$

and $\mathcal{V}{\restriction}[\gamma, \mathrm{lh}(\mathcal{V}))$ is based on $\mathcal{N}(\alpha)$ (and is above ϵ). Let Ω be the above-ϵ, hod $(\omega, \kappa, \kappa+1)$-strategy for $\mathcal{N}(\alpha)$, given by normally extending $\mathcal{V}{\restriction}\gamma + 1$, continuing to use $\Gamma^V_{\beta+1}$. Let

$$\mathcal{X} = \mathcal{T}_{\beta+1} \frown \vec{\mathcal{V}} \frown (\mathcal{V}{\restriction}\gamma + 1)$$

and $\Sigma' = \oplus_{\beta<\alpha}\Sigma_{\mathcal{N}(\beta),\mathcal{X}}$. Let Υ be the above-ϵ, hod $(\omega, \kappa, \kappa+1)$-strategy for $\mathcal{N}(\alpha)$, given by normally extending $\mathcal{V}{\restriction}\gamma + 1$, continuing to use Σ. So Υ is a Σ'-strategy as $(\mathcal{P}, \Sigma)$ is a hod pair. But Ω is also a Σ'-strategy, because Σ has factor-hull condensation by 3.18.

Let $\widetilde{\mathcal{V}}$ be the tree on $\mathcal{N}(\alpha)$ which is equivalent to $\mathcal{V}{\restriction}[\gamma, \mathrm{lh}(\mathcal{V}))$ (the latter tree is on $\mathcal{N}$). Let $\tilde{b}, \tilde{c}$ be the branches determined by b, c. By the previous paragraph we can use Ω and Υ to compare the phalanxes $\Phi(\widetilde{\mathcal{V}} \frown \tilde{b})$ and $\Phi(\widetilde{\mathcal{V}} \frown \tilde{c})$. This leads to contradiction almost as in the proof of 3.18. The only slight difference is in showing that $\mathcal{Q}'$ has an iteration strategy in Γ when $\tilde{c}$ does not drop, where $\mathcal{Q}'$ is as in the proof of 3.18, so consider this. As before, δ' is a cutpoint of $\mathcal{Q}'$. We have a normal tree $\mathcal{Y}$ via Σ of successor length, such that $\mathcal{Q}' \trianglelefteq M^{\mathcal{Y}}_\infty$. If $b^{\mathcal{Y}}$ drops then we can argue as in 3.18, so suppose $b^{\mathcal{Y}}$ does not drop. Then $\mathcal{Q}' \triangleleft M^{\mathcal{Y}}_\infty$. If δ' is a cutpoint of $M^{\mathcal{Y}}_\infty$ then we can use 3.14(1), so suppose otherwise. Let $E \in \mathbb{E}^{M^{\mathcal{Y}}_\infty}$ be the extender of least index overlapping δ'. So $\mathrm{o}(\mathcal{Q}') < \mathrm{lh}(E)$. Consider the tree $\mathcal{Z}$ on $M^{\mathcal{Y}}_\infty$, using only E. So $\mathcal{Q}' \triangleleft M^{\mathcal{Z}}_1$, and note that 3.14 applies to the stack $(\mathcal{Y}, \mathcal{Z})$ and $\mathcal{Q}', \delta'$. □

The next lemma, and much of its proof, are similar to Sargsyan's [5, Lemma 3.35].

Lemma 3.29. *Let κ be an uncountable cardinal. Let $(\mathcal{P}, \Sigma)$ be such that $\mathcal{P}$ is countable and reasonable and either*

(i) $\mathcal{P}$ is an n-sound premouse and Σ is the unique (n, κ)-strategy Σ' for $\mathcal{P}$ such that if $\mathrm{cof}(\kappa) > \omega$ then Σ' extends to an $(n, \kappa+1)$-strategy, or

(ii) $(\mathcal{P}, \Sigma)$ is a hod pair, within scope, and Σ is a hod $(\omega, \kappa, \kappa+1)$-strategy.

(So in case (ii) κ is regular.) Let $a \in \mathrm{HC}$ be such that $\mathcal{P} \in \mathcal{J}_1(\hat{a})$ and $\hat{a}$ is swo'd. Suppose that $\mathcal{M}_1^{\Sigma,\#}(\hat{a})$ exists and is Σ-κ-naturally iterable. Then (Σ, a) is nice.

Proof. Σ has hull condensation, by the uniqueness of Σ in case (i), and because $(\mathcal{P}, \Sigma)$ is within scope in case (ii). [30] It remains to see that $t_{(\Sigma,a)}$ determines itself on generic extensions.

We describe a process by which $\mathfrak{N}[g]$ can compute $\Sigma{\restriction}\mathfrak{N}[g]$ whenever $\mathfrak{N}$ is a non-dropping $\Lambda_{\mathfrak{M}}^{X,\kappa}$-iterate of $\mathfrak{M} = \mathcal{M}_1^{\Sigma,\#}(\hat{a})$ and g is set-generic over $\mathfrak{N}$. The result will then be a straightforward corollary. So fix $\mathfrak{N}$ and let $\delta = \delta^{\mathfrak{N}}$. Let $\mathcal{X}$ be the tree on $\mathfrak{M}$ whose last model is $\mathfrak{N}$.

Consider case (i). If $\mathrm{cof}(\kappa) = \omega$ let $\tau = \kappa$; otherwise let $\tau = \kappa + 1$. Let $\Lambda_{\mathfrak{M}}$ be the Σ-$(0, <\omega, \tau)$-maximal strategy for $\mathfrak{M}$ given by 3.5. So $\mathfrak{N}$ is a $\Lambda_{\mathfrak{M}}$-iterate. Let $\Lambda_{\mathfrak{N}}$ be the Σ-$(0, <\omega, \tau)$-maximal strategy for $\mathfrak{N}$ which is the tail of $\Lambda_{\mathfrak{M}}$. Let $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha \leq \delta}$ be the maximal $L[\mathbb{E}]$-construction of $\mathfrak{N}|\delta$, where background extenders are required to be in $\mathbb{E}^{\mathfrak{N}}$. Note that the hypotheses of 3.23 hold in $\mathfrak{N}$ with respect to $\mathcal{P}, \gamma = \delta, \Sigma{\restriction}\mathfrak{N}, \mathbb{C}$.

Now there is $\alpha < \delta$ such that 3.23(ii) attains. For in $\mathfrak{N}$, δ is Woodin, and $\mathcal{P}$ is super-small, so we can apply the universality of $\mathcal{N}_\delta$ (see [17, Lemma 11.1]). Note that $\alpha < \mu$ where μ is the least strong of $\mathfrak{N}$. Let γ be a cutpoint of $\mathfrak{N}$ such that $\alpha < \gamma < \mu$, and let $\theta = (\gamma^{++})^{\mathfrak{N}}$. Then via copying/resurrection, $\mathcal{N}_\alpha$, and therefore also $\mathcal{P}$, are normally iterable in V via lifting to nowhere-dropping normal trees on $\mathfrak{N}$, via $\Lambda_{\mathfrak{N}}$, based on $\mathfrak{N}|\theta$. Let $\Sigma_{\mathcal{P}}$ be the resulting strategy for $\mathcal{P}$. By the uniqueness of Σ we have $\Sigma_{\mathcal{P}} = \Sigma$. Note that $\theta \in \mathrm{rg}(i^{\mathcal{X}})$.

Now consider case (ii). So κ is regular. Let $\Lambda_{\mathfrak{M}}$ be the Σ-$(0, \kappa, \kappa+1)$-maximal strategy for $\mathfrak{M}$ given by 3.5. Let $\Lambda_{\mathfrak{N}}$ be the Σ-$(0, \kappa, \kappa+1)$-maximal strategy for $\mathfrak{N}$ which is the tail of $\Lambda_{\mathfrak{M}}$. Let $\mathbb{D}$ be the $(\mathcal{P}, \Sigma)$-bounded hod pair construction of $\mathfrak{N}$. By 3.28, we have $\alpha < \delta$ and a normal tree $\mathcal{T}$ via Σ with last model $\mathcal{R}$ such that $b^{\mathcal{T}}$ does not drop, $\mathbb{D}$ has length $\beta + 1$ and $\mathcal{R} = \mathcal{R}_\beta^{\mathbb{D}}$, and $\Lambda_\beta^V = \Sigma_{\mathcal{R},\mathcal{T}}$ (where Λ_β^V is as in 3.28; so this is just the strategy for $\mathcal{R}$ which lifts to $\Lambda^V_{\mathfrak{M}_\beta^{\mathbb{D}}}$). By hull condensation, Σ has pullback consistency, so $\Sigma = \Sigma_{\mathcal{P}}$, where $\Sigma_{\mathcal{P}}$ is the pullback of Λ_β^V. Note that $\mathrm{o}(\mathfrak{M}_\beta^{\mathbb{D}}) < \mu$ where μ is the least strong of $\mathfrak{N}$. Let γ be a cutpoint of $\mathfrak{N}$ such that $\mathrm{o}(\mathfrak{M}_\beta^{\mathbb{D}}) < \gamma < \mu$ and let $\theta = (\gamma^{++})^{\mathfrak{N}}$. And $\Sigma_{\mathcal{P}}$ is again computed by lifting to nowhere-dropping

[30] In case (i), we use the fact here that Σ is only an (n, κ)-strategy. If κ is singular then it seems difficult to deal with trees of length $(\kappa + 1)$.

trees on $\mathfrak{N}$, based on $\mathfrak{N}|\theta$ (this time stacks of normal such trees). Again $\theta \in \mathrm{rg}(i^{\mathcal{X}})$.

We now continue with both cases. It suffices to see that $\Lambda_{\mathfrak{N}}{\upharpoonright}X$ is sufficiently definable over $\mathfrak{N}[g]$, where X is the class of trees $\mathcal{T} \in \mathfrak{N}[g]$ such that $\mathcal{T}$ is based on $\mathfrak{N}|\theta$ and is nowhere-dropping. Iterating $\mathfrak{N}$ for $\mathfrak{N}|\theta$-based trees just requires computing the correct Q-structures, which requires sufficient ordinals and knowledge of Σ. But we don't yet know that $\Sigma``\mathfrak{N}[g] \subseteq \mathfrak{N}[g]$. We will computed the Q-structures by reducing such trees $\mathcal{T}$ to trees in $\mathfrak{N}$.

Let $\mathbb{P}, \dot{\mathcal{T}} \in \mathfrak{N}|\mathrm{crit}(F^{\mathfrak{N}})$ be a poset and a $\mathbb{P}$-name such that $\mathbb{P}$ forces that $\dot{\mathcal{T}}$ is a nowhere dropping, $\mathfrak{N}|\theta$-based tree on $\mathfrak{N}$, of limit length, via the strategy to be described; it will follow that $\dot{\mathcal{T}}^g$ is a correct tree on $\mathfrak{N}$ for any $\mathfrak{N}$-generic $g \subseteq \mathbb{P}$.

Claim 3.30. *Let g be $\mathbb{P}$-generic over $\mathfrak{N}$. Let $Q = Q(\dot{\mathcal{T}}^g)$. Then $Q \in \mathfrak{N}[g]$.*

In fact, let λ be the maximum of δ, $(\mathrm{lh}(\dot{\mathcal{T}}^g)^{++})^{\mathfrak{N}[g]}$, and $(\mathrm{card}(\mathbb{P})^{++})^{\mathfrak{N}}$. Then there is a short tree $\mathcal{V} \in \mathfrak{N}|\lambda$, $\mathcal{V}$ on $\mathfrak{N}$, according to $\Lambda_{\mathfrak{N}}$, of successor length, such that for some $\alpha < \mathrm{crit}(F^{\mathfrak{N}})$, if G is $\mathrm{Col}(\omega, \lambda)$ generic over $\mathfrak{N}[g]$, then in $\mathfrak{N}[g][G]$, there is an spm Q which is a Q-structure for $\mathcal{M}(\dot{\mathcal{T}}^g)$, and a Σ_1-elementary embedding $\pi : Q \to M^{\mathcal{V}}_\infty|\alpha$. So Q is unique with these properties and $Q(\dot{\mathcal{T}}^g) = Q \in \mathfrak{N}[g]$.

Proof. Suppose not and assume that $\mathbb{P}$ forces the failure. In $\mathfrak{N}$, we first form a Boolean valued comparison of $M(\dot{\mathcal{T}})$ with $\mathfrak{N}$, forming a $\mathbb{P}$-name for a tree $\dot{\mathcal{U}}$ on $M(\dot{\mathcal{T}})$ and a tree $\mathcal{V}$ on $\mathfrak{N}$. Note that $\mathfrak{N}$ correctly computes Q-structures as far as they exist during this comparison. Consider a limit stage $(\mathcal{V}, \dot{\mathcal{U}}){\upharpoonright}\lambda$ of the comparison. If a condition q forces that $\dot{\mathcal{U}}{\upharpoonright}\lambda$ is eventually only padding then below q, nothing need be done for $\dot{\mathcal{U}}$ at stage λ. Now suppose q forces otherwise. Suppose $p \leq q$ forces that here is a cofinal branch b of $\dot{\mathcal{U}}$ such that $Q(M(\mathcal{V}{\upharpoonright}\lambda)) \trianglelefteq M^{\dot{\mathcal{U}}}_b$. Then below p, we set $[0, \lambda]_{\dot{\mathcal{U}}} = b$. If $p \leq q$ forces otherwise, then below p, we declare that $\dot{\mathcal{U}}$ is **uncontinuable**, and terminate the comparison. (In the latter case p forces that $\dot{\mathcal{U}}$ has limit length; we deal with this later.) For each stage α of the comparison, let lh_α be the index of any extender (forced by some p to be) used at that stage. For limit λ, let $M((\mathcal{V}, \dot{\mathcal{U}}){\upharpoonright}\lambda)$ be the lined up part of that stage, of height $\sup_{\alpha<\lambda} \mathrm{lh}_\alpha$.

Subclaim 3.31. *We have:*

(*a*) *$\mathcal{V}$ is based on $\mathfrak{N}|\theta$;*

(*b*) *if α is such that $[0,\alpha]_{\mathcal{V}}$ does not drop and $\mathbb{P}$ forces that $M^{\dot{\mathcal{U}}}_\alpha|\theta' = M^{\mathcal{V}}_\alpha|\theta'$, where $\theta' = i^{\mathcal{V}}_{0,\alpha}(\theta)$, then the comparison terminates at stage α, and in fact, $\mathbb{P}$ forces that $M^{\dot{\mathcal{U}}}_\alpha \trianglelefteq M^{\mathcal{V}}_\alpha|\theta'$;*

(*c*) *at every limit stage λ, a Q-structure for $M((\mathcal{V},\dot{\mathcal{U}})\restriction\lambda)$ exists;*

(*d*) *the comparison terminates (i.e. there is α such that $\mathbb{P}$ forces that either $\dot{\mathcal{U}}$ is uncontinuable, or $M^{\mathcal{V}}_\alpha \trianglelefteq M^{\dot{\mathcal{U}}}_\alpha$, or $M^{\dot{\mathcal{U}}}_\alpha \trianglelefteq M^{\mathcal{V}}_\alpha$);*

(*e*) *there is $p \in \mathbb{P}$ forcing that if $\dot{\mathcal{U}}$ has a final model, then $M^{\dot{\mathcal{U}}}_\infty \triangleleft M^{\mathcal{V}}_\infty$.*

Proof. Part (b) implies (a) and (c). Suppose (b) fails. Let α be the least failure, and let p be a condition forcing this failure. Let $g \subseteq \mathbb{P}$ be generic with $p \in g$. Let $\mathcal{T}'$ be the tree on $\mathfrak{N}$ which uses the same extenders as does $\mathcal{T} = \dot{\mathcal{T}}^g$, followed by $\Lambda_{\mathfrak{N}}(\mathcal{T})$, and let $W_0 = M^{\mathcal{T}'}_\infty$. So $b^{\mathcal{T}'}$ is non-dropping (as $\mathcal{T}$ was nowhere dropping). Let $\mathcal{U}'$ be the tree on W_0 using the same extenders as $\mathcal{U}^g$. Let $W = M^{\mathcal{U}'}_\alpha$. So $\theta' < \mathrm{o}(W)$. We can compare $(M^{\mathcal{V}}_\alpha, W)$, producing trees $(\mathcal{T}_1,\mathcal{T}_2)$. The comparison begins above θ', a cardinal of $M^{\mathcal{V}}_\alpha$. Note that by choice of θ, all extenders used in the comparison have critical point $> \theta'$. Suppose $b^{\mathcal{U}'}$ drops. Then $\rho^{\mathcal{W}}_{n+1} < \theta'$, where $n = \deg^{\mathcal{U}'}(\alpha)$. Also then, $b^{\mathcal{T}_1}$ drops, whereas $b^{\mathcal{T}_2}$ does not, and $\mathcal{T}_1,\mathcal{T}_2$ have the same last model. But the last model Z of $\mathcal{T}_1$ has $\rho_\omega(Z) \geq \theta'$, contradiction. So $b^{\mathcal{U}'}$ does not drop, and so neither do $b^{\mathcal{T}_1}, b^{\mathcal{T}_2}$, and $j = k$ where $j = i^{(\mathcal{X},\mathcal{V},\mathcal{T}_1)}$ and $k = i^{(\mathcal{X},\mathcal{T}',\mathcal{U}',\mathcal{T}_2)}$. But $j(\theta) = \theta'$ and $k(\theta) > \theta'$, contradiction. This gives (b).

The usual proof that boolean-valued comparisons terminate gives (d).

So if (e) fails, then $b^{\mathcal{V}}$ drops, so $M^{\mathcal{V}}_\infty$ is unsound, and $\mathbb{P}$ forces that $M^{\dot{\mathcal{U}}}_\infty = M^{\mathcal{V}}_\infty$. But then again the usual methods yield a contradiction. □

Now let p be as in part (e), and let $g \subseteq \mathbb{P}$ be $\mathfrak{N}$-generic, with $p \in g$. Let $\mathcal{T} = \dot{\mathcal{T}}^g$ and $\mathcal{U} = \dot{\mathcal{U}}^g$. Let $Q = Q(M(\mathcal{T}))$. Let $W_0, \mathcal{U}'$ be as before, and let $\mathcal{U}_Q$ be the 0-maximal tree on Q given by $\mathcal{U}$ (with the same extenders and branches).

Suppose that $\mathcal{U}$ has a last model R. So we have $R \triangleleft M^{\mathcal{V}}_\infty$ and $b^{\mathcal{U}}$ does not drop, and so neither do $b^{\mathcal{U}'}$ or $b^{\mathcal{U}_Q}$. Let $\pi : M^{\mathcal{U}_Q}_\infty \to i^{\mathcal{U}'}(Q)$ be the factor map. Then π is a weak 0-embedding. So by 2.36, $M^{\mathcal{U}_Q}_\infty$ is a Σ-premouse. Also, $i^{\mathcal{U}_Q} : Q \to M^{\mathcal{U}_Q}_\infty$ is continuous at $\delta = \delta(\dot{\mathcal{T}}^g)$, and $M^{\mathcal{U}_Q}_\infty$ has no E-active levels above $i^{\mathcal{U}_Q}(\delta)$ and $i^{\mathcal{U}_Q}(\delta)$ is Woodin in $M^{\mathcal{U}_Q}_\infty$. It follows that $M^{\mathcal{U}_Q}_\infty \trianglelefteq M^{\mathcal{V}}_\infty$.

Also, $i^{\mathcal{U}_Q}$ is Σ_1-elementary. So Q, $\mathcal{V}$, $M_\infty^{\mathcal{U}_Q}$ and $i^{\mathcal{U}_Q}$ witness the truth of the claim, a contradiction.[31]

Suppose now that $\mathcal{U}$ is uncontinuable, so has limit length. Let b be the $\mathcal{U}$-cofinal branch determined by $\Lambda_{\mathfrak{N}}$. Note that b does not drop, and $M(\mathcal{U}) = M_\infty^{\mathcal{U}}$. But this leads to the same contradiction as in the previous paragraph. □

This completes the proof that $\mathfrak{N}[g]$ computes $\Sigma{\restriction}\mathfrak{N}[g]$. Now let Φ be the formula "There is no largest cardinal, there is a Woodin cardinal δ, in case (i) the $L[\mathbb{E}]$-construction reaches a non-dropping Σ-iterate of $\mathcal{P}$, and in case (ii) the $(\mathcal{P}, \Sigma)$-bounded hod pair construction is successful at some stage $< \delta$, and every partial order $\mathbb{P}$ forces that the process described above always succeeds". Let Ψ be the formula defining $\Sigma{\restriction}\mathfrak{N}[g]$ through the above process. Note that if $\mathfrak{N}' \trianglelefteq \mathfrak{N}$ and $\mathfrak{N}' \vDash \Phi$ and g is set generic over $\mathfrak{N}'$, then $\mathfrak{N}'[g]$ is indeed closed under Σ, and $\Sigma{\restriction}\mathfrak{N}'[g]$ is defined over $\mathfrak{N}'[g]$ by Ψ. So (Φ, Ψ) generically determines $t_{(\Sigma,a)}$, as required. (We don't actually need that the Woodin of $\mathfrak{N}'$ is a cardinal of $\mathfrak{N}$.) □

Notation 3.32. Let (Ω, A_0) be nice, $t_0 = t_{\Omega,A_0}$ and $\kappa_0 = \kappa_t$. Let $\mathfrak{M} = \mathcal{M}_1^{\Omega,\#}(A_0)$ and $\Lambda_{\mathfrak{M}} = \Lambda_{\mathfrak{M}}^{\Omega,\kappa_0}$. Let (Φ_0, Ψ_0) be a pair that generically determines (Ω, A_0). Let $a_0 \in \mathbb{R}$ code A_0 in a canonical way.[32] These objects are fixed for the remainder of the paper.

Definition 3.33. An hpm $\mathcal{N}$ is **$\mathfrak{M}$-like**[33] iff $\mathcal{N}$ is non-1-small, all proper segments of $\mathcal{N}$ are 1-small, and $\exists \gamma \in (\delta^{\mathcal{N}}, l(\mathcal{N}))$ such that $\mathcal{N}|\gamma \vDash \Phi_0$. ⊣

Remark 3.34. G-organization will use an initial segment of the tree for making a structure *generically generic*, due to Sargsyan [5]. We recall this notion and define some related notation and terminology now.

Let $\mathcal{N}, \mathcal{P}$ be transitive structures, where $\mathcal{P}$ is $\mathfrak{M}$-like. Let $\mathbb{Q} = \mathrm{Col}(\omega, \mathcal{N})$. Let $\dot{x}_{\mathcal{N}}$ be the canonical $\mathbb{Q}$-name for the real coding $\mathcal{N}$ determined by a $\mathbb{Q}$-generic filter. Let $\mathcal{T}$ be a normal iteration tree on $\mathcal{P}$. We say that $\mathcal{T}$ is **making $\mathcal{N}$ generically generic** iff:

– $\mathcal{T}{\restriction}\mathrm{o}(\mathcal{N}) + 1$ is a linear iteration at the least measurable of $\mathcal{P}$.

[31] Ostensibly $M_\infty^{\mathcal{U}_Q}$ might be a strict segment of the Q-structure for $M_\infty^{\mathcal{V}}|i^{\mathcal{U}_Q}(\delta)$, but this is not relevant. If one chooses $n < \omega$ appropriately, and takes $\mathcal{U}_Q$ to be n-maximal instead of 0-maximal, then one can arrange that $M_\infty^{\mathcal{U}_Q}$ is the Q-structure.

[32] If a_0 can be chosen such that $\mathfrak{M}$ codes a_0 then we do so, and a_0 is redundant.

[33] The "$\mathfrak{M}$" in "$\mathfrak{M}$-like" is just a symbol; it does not refer to the fixed structure $\mathfrak{M}$.

– Suppose $\mathrm{lh}(\mathcal{T}) \geq \mathrm{o}(\mathcal{N}) + 2$ and let $\alpha + 1 \in (\mathrm{o}(\mathcal{N}), \mathrm{lh}(\mathcal{T}))$. Let $\delta = \delta(M^{\mathcal{T}}_\alpha)$ and let $\mathbb{B} = \mathbb{B}(M^{\mathcal{T}}_\alpha)$. Then $E^{\mathcal{T}}_\alpha$ is the extender $E \in \mathbb{E}_+(M^{\mathcal{T}}_\alpha)$ with least index such that some $p \in \mathbb{Q}$ forces "There is a $\mathbb{B}$-axiom induced by E which fails for $\dot{x}_{\mathcal{N}}$".

Given a putative strategy Σ for $\mathcal{P}$, let $\mathcal{T}^{*\Sigma}_{\mathcal{N}}$ denote the longest putative tree $\mathcal{T}$ via Σ which is making $\mathcal{N}$ generically generic. Clearly if Σ is a normal κ-strategy for a large enough κ then $\mathcal{T} =_{\mathrm{def}} \mathcal{T}^{*\Sigma}_{\mathcal{N}}$ has successor length and $\mathbb{Q}$ forces that $\dot{x}_{\mathcal{N}}$ is generic for $\mathbb{B}(M^{\mathcal{T}}_\infty)$.

Let $\mathcal{T}^*_{\mathcal{N}}$ denote $\mathcal{T}^{*\Lambda_{\mathfrak{M}}}_{\mathcal{N}}$.

Sargsyan noticed (see [5, Definition 3.37]) that one can feed Ω into a strategy mouse $\mathcal{N}$ indirectly, by feeding in the branches for something like $\mathcal{T}^*_{\mathcal{M}}$, for various $\mathcal{M} \trianglelefteq \mathcal{N}$. The key notion of g-*organized* Ω-*premice*, to come, uses this idea, and the main point of it is due to Sargsyan. We will only actually use a certain initial segment $\mathcal{T}^{\Sigma}_{\mathcal{N}}$ of $\mathcal{T}^{*\Sigma}_{\mathcal{N}}$:

Definition 3.35. Let $\mathcal{P}$ be $\mathfrak{M}$-like. Then $\mathcal{P}_{\Phi_0}$ denotes the least $\mathcal{P}' \triangleleft \mathcal{P}$ such that for some cardinal δ' of $\mathcal{P}$, $\mathcal{P}' \vDash \Phi_0 +$"$\delta'$ is Woodin". Note that $\mathcal{P}_{\Phi_0}$ is a strong cutpoint of $\mathcal{P}$. Given a transitive structure $\mathcal{N}$ and a putative strategy Σ for $\mathcal{P}$, $\mathcal{T}^{\Sigma}_{\mathcal{N}}$ denotes the initial segment of $\mathcal{T}^{*\Sigma}_{\mathcal{N}}$ based on $\mathcal{P}_{\Phi_0}$. Let $\mathcal{T}_{\mathcal{N}}$ denote $\mathcal{T}^{\Lambda_{\mathfrak{M}}}_{\mathcal{N}}$. ⊣

To ensure the absoluteness of iterations making structures generically generic, we will require our models to add branches to iteration trees sufficiently slowly:

Definition 3.36. Let $\mathcal{M}$ be an aspm and $\eta < \mathrm{o}(\mathcal{M})$. Let $\mathcal{T} \in \mathcal{M}$ be a putative tree via $\Sigma^{\mathcal{M}}$. Then $\mathcal{S}^{\mathcal{M}}_{\mathcal{T}}$ denotes the least $\mathcal{S} \trianglelefteq \mathcal{M}$ such that $\mathcal{T}$ is via $\Sigma^{\mathcal{S}}$. We say that $\mathcal{T}$ is **$\mathcal{M}$-reckonable above** η iff for every limit $\alpha \in [\eta, \mathrm{lh}(\mathcal{T}))$ we have the following. Let $\zeta = \sup_{n<\omega} \mathrm{wfp}(\mathrm{o}(M^{\mathcal{T}}_{\alpha+n}))$. Then:

– if $\alpha + \omega < \mathrm{lh}(\mathcal{T})$ then $\mathrm{o}(\mathcal{S}^{\mathcal{M}}_{\mathcal{T}\restriction\alpha+1}) + \zeta \leq \mathrm{o}(\mathcal{S}^{\mathcal{M}}_{\mathcal{T}\restriction\alpha+\omega})$,

– if $\mathrm{lh}(\mathcal{T}) \leq \alpha + \omega$ then $\mathrm{o}(\mathcal{S}^{\mathcal{M}}_{\mathcal{T}\restriction\alpha+1}) + \zeta \leq \mathrm{o}(\mathcal{M})$, and

– if $\mathrm{lh}(\mathcal{T}) < \alpha+\omega$ and $\mathcal{M} \vDash$"$M^{\mathcal{T}}_\infty$ is wellfounded" then $M^{\mathcal{T}}_\infty$ is wellfounded (equivalently, $M^{\mathcal{T}}_\infty \vDash$"$\mathrm{o}(\mathcal{S}^{\mathcal{M}}_{\mathcal{T}\restriction\alpha+1}) + \mathrm{o}(M^{\mathcal{T}}_\infty) \leq \mathrm{o}(V)$"). ⊣

Remark 3.37. Let $\mathcal{M}$ be an aspm such that $\mathrm{cp}^{\mathcal{M}}$ is $\mathfrak{M}$-like. Let $\mathcal{N} \triangleleft \mathcal{M}$ satisfy ZF. Let $\mathcal{T} \in \mathcal{M}$ be a putative tree via $\Sigma^{\mathcal{M}}$ (on $\mathrm{cp}^{\mathcal{M}}$), based on $\mathrm{cp}^{\mathcal{M}}_{\Phi_0}$, such that $\mathcal{T}$ is $\mathcal{M}$-reckonable above $\mathrm{o}(\mathcal{N})$. Then $\mathcal{T}$ is making $\mathcal{N}$ generically generic (in V) iff $\mathcal{M} \vDash$"$\mathcal{T}$ is making $\mathcal{N}$ generically generic". Moreover, let $\mathcal{U}' = \mathcal{T}^{\Sigma^{\mathcal{M}}}_{\mathcal{N}}$ (as computed in V) and $\mathcal{U} = \mathcal{U}' \restriction \lambda$ where λ is largest such that $\mathcal{U} \restriction \alpha + 1$ is $\mathcal{M}$-reckonable above $\mathrm{o}(\mathcal{N})$ for all $\alpha < \lambda$. Given $\alpha + 1 < \mathrm{lh}(\mathcal{U}')$ let $e_\alpha = E^{\mathcal{U}'}_\alpha$, and given $\alpha + 1 = \mathrm{lh}(\mathcal{U}')$, if $M^{\mathcal{U}}_\alpha$ is illfounded then let $e_\alpha = 0$, and otherwise let $e_\alpha = 1$. Then the map $\alpha \mapsto (\mathcal{U} \restriction \alpha + 1, e_\alpha)$, with domain λ, is $\mathrm{r}\Pi^{\mathcal{M}}_2(\mathcal{L}^-, \{\mathcal{N}\})$, uniformly in $\mathcal{M}, \mathcal{N}$.[34] Further, suppose that $\mathcal{T} =_{\mathrm{def}} \mathcal{T}^{\Sigma^{\mathcal{M}}}_{\mathcal{N}}$ exists, is in $\mathcal{M}$, and is $\mathcal{M}$-reckonable above $\mathrm{o}(\mathcal{N})$. Then $\{(\mathcal{T}, (M^{\mathcal{T}}_\infty)_{\Phi_0})\}$ is $\Sigma^{\mathcal{M}}_1(\mathcal{L}^-, \{\mathcal{N}\})$, uniformly in $\mathcal{M}, \mathcal{N}$.

These facts use the local definability of the $\mathrm{Col}(\omega, \mathcal{N})$ forcing relation. Given $p \in \mathrm{Col}(\omega, \mathcal{N})$, $n < \omega$, a limit ordinal $\alpha < \lambda$ and $E \in \mathbb{E}(M^{\mathcal{T}}_{\alpha+n})$ such that $\nu(E)$ is inaccessible in $M^{\mathcal{T}}_{\alpha+n}$, the question of whether $p \Vdash$"E induces an extender algebra axiom not satisfied by $\dot{x}_{\mathcal{N}}$" is computed over $\mathcal{J}^{\mathrm{hpm}}_{\nu(E)}(\mathcal{S}^{\mathcal{M}}_{\mathcal{T} \restriction \alpha+1})$. (Such an axiom has the form

$$\bigvee_{\gamma < \mathrm{crit}(E)} \varphi_\gamma \iff \bigvee_{\gamma < \nu(E)} \varphi_\gamma,$$

where for each $\gamma < \nu(E)$, $\varphi_\gamma \in M^{\mathcal{T}}_{\alpha+n} | \nu(E)$, so the forcing relation below p regarding the truth of φ_γ is computed over some proper segment of $\mathcal{J}^{\mathrm{hpm}}_{\nu(E)}(\mathcal{S}^{\mathcal{M}}_{\mathcal{T} \restriction \alpha+1})$.)

Definition 3.38. Let $\mathcal{R}$ be an aspm such that $\mathrm{cp}^{\mathcal{R}}$ is $\mathfrak{M}$-like. Let $\psi \in \mathcal{L}$. The **(g, ψ)-hierarchy** of $\mathcal{M}$ is the pair $(\langle \mathcal{M}_\alpha \rangle_{\alpha \leq \gamma}, \langle \mathcal{N}_{\alpha+1} \rangle_{\alpha < \gamma'})$ with $\gamma, \gamma' \in \mathrm{Ord}$ both as large as possible such that $\gamma \leq \gamma' \leq \gamma + 1$ and:

1. $\mathcal{N}_{\alpha+1} \trianglelefteq \mathcal{R}$ for each $\alpha < \gamma'$ and $\mathcal{M}_\alpha \trianglelefteq \mathcal{R}$ for each $\alpha \leq \gamma$.
2. $\mathcal{M}_0 = \mathcal{M}|1$ and $\mathrm{o}(\mathcal{M}_\lambda) = \lim_{\alpha < \lambda} \mathrm{o}(\mathcal{M}_\alpha)$ for limit λ.
3. For $\alpha < \gamma'$, $\mathcal{N}_{\alpha+1}$ is the least $\mathcal{N} \trianglelefteq \mathcal{R}$ such that $\mathcal{M}_\alpha \triangleleft \mathcal{N}$ and $\mathcal{N} \vDash \mathsf{ZF}$.
4. For $\alpha < \gamma$, $\mathcal{M}_{\alpha+1}$ is the least $\mathcal{M} \trianglelefteq \mathcal{R}$ such that $\mathcal{N}_{\alpha+1} \triangleleft \mathcal{M}$ and for some $\mathcal{S}$ with $\mathcal{N}_{\alpha+1} \trianglelefteq \mathcal{S} \trianglelefteq \mathcal{M}$ we have either:
    - $\mathcal{S} \vDash \neg\psi$, or

[34] There is a natural Σ_1 formula which attempts to define this function, which computes the correct values on the domain of the function, but might give a larger domain.

– $\mathcal{T}' =_{\text{def}} \mathcal{T}^{\Sigma^{\mathcal{S}}}_{\mathcal{N}_{\alpha+1}}$ exists, is in $\mathcal{M}$ and is $\mathcal{M}$-reckonable above $\mathrm{o}(\mathcal{N}_{\alpha+1})$.

For $\mathcal{N} \trianglelefteq \mathcal{M}$, we say that $\mathcal{N}$ is a (g, ψ)**-tree activation level** of $\mathcal{M}$ iff $\mathcal{N} = \mathcal{N}_{\alpha+1}$ for some α. We say that $\mathcal{M}$ is (g, ψ)**-whole** iff $\mathcal{M} = \mathcal{M}_\gamma$, and say that $\mathcal{M}$ is (g, ψ)**-closed** iff $\mathcal{M}$ is (g, ψ)-whole and γ is a limit.

We abbreviate $(\mathrm{g}, \mathrm{true})$ with g (for example in the g-*hierarchy* of $\mathcal{R}$, etc). We abbreviate (g, "Θ exists") with both (g, Θ) and Θ-g. ⊣

Remark 3.39. Let $\mathcal{M}$ be an aspm such that $\mathrm{cp}^{\mathcal{M}}$ is $\mathfrak{M}$-like, and $\psi \in \mathcal{L}$. Let the (g, ψ)-hierarchy of $\mathcal{M}$ be $(\langle \mathcal{M}_\alpha \rangle_{\alpha \leq \gamma}, \langle \mathcal{N}_{\alpha+1} \rangle_{\alpha < \gamma'})$. Then $\langle \mathcal{M}_\alpha \rangle_{\alpha \leq \gamma} \restriction \mathcal{M}$ is $\Sigma_1^{\mathcal{M}}(\mathcal{L}^-)$, and $\langle \mathcal{N}_{\alpha+1} \rangle_{\alpha < \gamma'} \restriction \mathcal{M}$ is $\Delta_2^{\mathcal{M}}(\mathcal{L}^-)$, uniformly in $\mathcal{M}$; this follows easily from 3.37.[35] Similarly, there is $\varrho_\psi \in \mathcal{L}$ such that $\mathcal{M} \vDash \varrho_\psi$ iff $\mathcal{M}$ is (g, ψ)-whole, uniformly in $\mathcal{M}$.

Definition 3.40. Let "V is an aspm" be the natural formula $\psi \in \mathcal{L}$ such that for any transitive $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \vDash \psi$ iff $\mathcal{M}$ is an aspm. ⊣

Definition 3.41 ($\varphi_{(\mathrm{g},\psi)}$)**.** For $\psi \in \mathcal{L}$, $\varphi_{(\mathrm{g},\psi)}$ denotes the $\mathcal{L}$-formula of one free variable $\mathcal{T}$ asserting (when interpreted over transitive $\mathcal{L}$-structures) "V is an aspm, cp is an $\mathfrak{M}$-like hpm, the (g, ψ)-hierachy of V has the form

$$(\langle \mathcal{M}_\alpha \rangle_{\alpha \leq \gamma}, \langle \mathcal{N}_{\alpha+1} \rangle_{\alpha < \gamma+1})$$

with $\mathcal{N} =_{\text{def}} \mathcal{N}_{\gamma+1} \triangleleft V$, $\mathcal{T}$ is a limit length iteration tree via Σ^V (on cp), based on cp_{Φ_0}, making $\mathcal{N}$ generically generic, $\mathcal{T}$ is V-reckonable above $\mathrm{o}(\mathcal{N})$, and $\Sigma^V(\mathcal{T})$ is undefined."

We have $\varphi_{\mathrm{g}} = \varphi_{(\mathrm{g},\mathrm{true})}$; let $\varphi_{\mathrm{G}} = \varphi_{(\mathrm{g},\Theta)}$. ⊣

The notion g-*organized* Ω-*premouse* below is a variant of Sargsyan's reorganized hybrid strategy premouse, [5, Definition 3.37]:

Definition 3.42. ${}^{\mathrm{g}}\Omega = (\Lambda_{\mathfrak{M}}, \varphi_{\mathrm{g}})$ and ${}^{\mathrm{G}}\Omega = (\Lambda_{\mathfrak{M}}, \varphi_{\mathrm{G}})$. For example, a ${}^{\mathrm{g}}\Omega$-premouse is a $(\Lambda_{\mathfrak{M}}, \varphi_{\mathrm{g}})$-premouse and $\mathrm{Lp}^{{}^{\mathrm{g}}\Omega}(x) = \mathrm{Lp}^{(\Lambda_{\mathfrak{M}}, \varphi_{\mathrm{g}})}(x)$, etc. A **g-organized Ω-premouse** is a ${}^{\mathrm{g}}\Omega$-premouse. ⊣

So a g-organized Ω-pm is over A for some $A \in \widehat{V}$ where $\mathfrak{M} \in \mathcal{J}_1(A)$.

Lemma 3.43. *The class of* g*-organized* Ω*-pms* $\mathcal{M}$ *such that* $\Psi^{\mathcal{M}} = \emptyset$ *is very condensing. For any* g*-organized* Ω*-pm* $\mathcal{M}$ *not of type 3, and any* $\pi : \mathcal{R} \to \mathcal{M}$ *a weak* 0*-embedding,* $\mathcal{R}$ *is a* g*-organized* Ω*-pm.*

[35] If $\gamma' = \alpha + 1$ and $l(\mathcal{M}) = \gamma' + 1$ then $\langle \mathcal{N}_{\alpha+1} \rangle_{\alpha < \gamma'}$ is not $\Sigma_1^{\mathcal{M}}(\mathcal{L}^-)$ because $\mathcal{N}_{\gamma'} \vDash \mathsf{ZF}$.

Proof. These facts follow from 2.37 and 2.36 respectively. □

As in [5, Lemma 3.38], the first consequence of g-organization is the following. Because t_{Ω,A_0} determines itself on generic extensions, g-closure ensures closure under Ω:

Lemma 3.44. *Let $\mathcal{M}$ be a* g*-closed* g*-organized Ω-pm. Then $\mathcal{M}$ is closed under Ω. In fact, for any set generic extension $\mathcal{M}[g]$ of $\mathcal{M}$, with $g \in V$*[36]*, $\mathcal{M}[g]$ is closed under Ω and $\Omega{\restriction}\mathcal{M}[g]$ is $\mathcal{L}^-$-definable over $\mathcal{M}[g]$, uniformly in $\mathcal{M}, g$.*

Proof sketch. We show that $\mathcal{M}$ is closed under Ω; the generalization to generic extensions of $\mathcal{M}$ and the definability of Ω is similar. We assume that Ω is an operator; the strategy case is similar.

Let $z \in \lfloor\mathcal{M}\rfloor \cap \mathrm{dom}(\Omega)$; we want to see that $\Omega(z) \in \lfloor\mathcal{M}\rfloor$. Let $t = \mathrm{Th}_\omega^{\Omega(z)}(z)$; it suffices to see that $t \in \mathcal{M}$. Let $\mathcal{N}, \mathcal{N}' \triangleleft \mathcal{M}$ be tree activation levels of $\mathcal{M}$ with $z \in \mathcal{N} \triangleleft \mathcal{N}'$. Then $\mathcal{T} =_{\mathrm{def}} \mathcal{T}_\mathcal{N} \in \mathcal{N}'$. Let $\mathfrak{M}^* = \mathfrak{M}^{\mathcal{T}}_{\Phi_0}$ and $\mathbb{Q} = \mathrm{Col}(\omega, \mathcal{N})$. Then in $\mathcal{N}'$, $\mathbb{Q}$ forces that $\dot{x}_\mathcal{N}$ is extender algebra generic over $\mathfrak{M}^*$. So by 3.9, for $w \in z^{<\omega}$ and any formula φ, $\varphi(w) \in t$ iff in $\mathcal{N}'$, $\mathbb{Q}$ forces that $\check{\mathfrak{M}}^*[\dot{x}_\mathcal{N}] \models$"There is y such that $\Psi_0(\check{z}, y)$ and $y \models \varphi(\check{w})$". □

The analysis of scales in $\mathrm{Lp}^{\mathrm{g}\Omega}(\mathbb{R})$ runs into some problems (see footnotes 49 and 68). So we will analyze scales in a slightly different hierarchy, which we now describe.

Definition 3.45. Fix a natural coding of elements of HC by reals. Let $\Upsilon \subseteq \mathrm{HC}$. Given a set $\Upsilon \subseteq \mathrm{HC}$, Υ^{cd} denotes the set of codes for elements of Υ in this coding.[37] We say that Υ is **self-scaled** iff there are scales on Υ^{cd} and $\mathbb{R}\backslash\Upsilon^{\mathrm{cd}}$ which are analytical in Υ^{cd} (i.e. $\Sigma^1_n(\Upsilon^{\mathrm{cd}})$ for some $n < \omega$). ⊣

Definition 3.46. An aspm $\mathcal{M}$ is **suitably based** iff $\mathrm{cp}^\mathcal{M} \in \mathrm{HC}^\mathcal{M}$ is $\mathfrak{M}$-like, $cb^\mathcal{M} = \hat{x}$ where $x = (\mathrm{HC}^\mathcal{M}, \Upsilon)$ for some $\Upsilon \subseteq \mathrm{HC}^\mathcal{M}$ such that $\mathcal{M} \models$"Υ is self-scaled", and $\Psi^\mathcal{M} = \emptyset$. Abusing terminology, we say that $\mathcal{M}$ is **over** Υ and write $\Upsilon^\mathcal{M} = \Upsilon$. Let $\mathcal{M}$ be a suitably based aspm over Υ. Let $\vec{\leq}^\mathcal{M}, \vec{\leq'}^\mathcal{M}$ denote what are, in $\mathcal{M}$, the least analytical-in-Υ^{cd} scales on $\Upsilon^{\mathrm{cd}}, \mathbb{R}\backslash\Upsilon^{\mathrm{cd}}$. If

[36]Without the assumption that $g \in V$, it seems that the domain of $\Omega{\restriction}\mathcal{M}[g]$ might not be definable over $\mathcal{M}[g]$.

[37]Note that for any $\mathcal{J}$-structure $\mathcal{M}$ such that $\mathrm{HC}^\mathcal{M} \in \mathcal{M}$, the decoding function (for the above codes), restricted to $\mathbb{R}^\mathcal{M}$, is definable over $\mathrm{HC}^\mathcal{M}$, so $(\Upsilon \cap \mathrm{HC}^\mathcal{M})^{\mathrm{cd}} = \Upsilon^{\mathrm{cd}} \cap \mathcal{M}$.

there is some $\mathcal{N} \trianglelefteq \mathcal{M}$ which is admissible, then working in $\mathcal{M}$ (or $\mathcal{N}$) let $U^{\mathcal{M}}, U'^{\mathcal{M}}$ denote the trees of these scales, respectively.

A Θ-g-**spm** is a suitably based φ_{G}-indexed spm.

A **Θ-g-organized Ω-premouse** is a Θ-g-spm which is a $(\Lambda_{\mathfrak{M}}, \varphi_{\mathrm{G}})$-pm. $\dashv$

In our application to core model induction, we will be most interested in the cases that either $\Upsilon^{\mathcal{M}} = \emptyset$ or $\Upsilon^{\mathcal{M}} = \Omega{\restriction}\mathrm{HC}^{\mathcal{M}}$.

Definition 3.47. Let "V is a Θ-g-spm" be the natural formula $\psi \in \mathcal{L}$ such that for all transitive $\mathcal{L}$-structures $\mathcal{M}$, $\mathcal{M} \vDash \psi$ iff $\mathcal{M}$ is a Θ-g-spm. $\dashv$

Definition 3.48. Let $\mathcal{M}$ be an aspm and let $\mathcal{P} \triangleleft \mathcal{J}^{\mathrm{hpm}}(\mathcal{P}) \trianglelefteq \mathcal{M}$ with $\mathcal{P}$ a strong cutpoint of $\mathcal{M}$. Then $\mathcal{M}{\downarrow}\mathcal{P}$ denotes the aspm $\mathcal{M}'$ defined by induction on $\mathcal{M}$ as follows: $\lfloor \mathcal{M}' \rfloor = \lfloor \mathcal{M} \rfloor$, $cb^{\mathcal{M}'} = \hat{\mathcal{P}}$, $cp^{\mathcal{M}'} = cp^{\mathcal{M}}$, $\Psi^{\mathcal{M}'} = \Sigma^{\mathcal{P}}$, $P^{\mathcal{M}'} = P^{\mathcal{M}}$, $E^{\mathcal{M}'} = E^{\mathcal{M}}$, $l(\mathcal{P}) + l(\mathcal{M}') = l(\mathcal{M})$ and $\mathcal{N}{\downarrow}\mathcal{P} \triangleleft \mathcal{M}'$ for all $\mathcal{N}$ such that $\mathcal{P} \triangleleft \mathcal{N} \triangleleft \mathcal{M}$ (this determines $\vec{P}^{\mathcal{M}'}$). $\dashv$

Lemma 3.49. *Let $\mathcal{M}$ be an hpm. Then the following are equivalent: (i) $\mathcal{M}$ is a Θ-g-organized Ω-pm; (ii) $\mathcal{M} \vDash$"V is a Θ-g-spm" and $cp^{\mathcal{M}} = \mathfrak{M}$ and $\Sigma^{\mathcal{M}}_{\varphi_{\mathrm{G}}} \subseteq \Lambda_{\mathfrak{M}}$; (iii) $\mathcal{M}$ is a suitably based aspm and $cp^{\mathcal{M}} = \mathfrak{M}$ and for all $\mathcal{N} \trianglelefteq \mathcal{M}$:*

- *if $\mathcal{P} \triangleleft \mathcal{J}^{\mathrm{hpm}}(\mathcal{P}) \trianglelefteq \mathcal{N}$ and $\mathcal{P} \vDash \mathsf{ZF}^-$ and every $\mathcal{R}$ such that $\mathcal{P} \trianglelefteq \mathcal{R} \triangleleft \mathcal{N}$ has $\Theta^{\mathcal{R}} = \mathrm{o}(\mathcal{P})$ (possibly $\mathcal{R} = \mathcal{P}$; therefore $\mathcal{P}$ is a strong cutpoint of $\mathcal{N}$) then $\mathcal{N}{\downarrow}\mathcal{P}$ is a g-organized Ω-pm, and*
- *if there are arbitrarily large $\mathcal{R} \triangleleft \mathcal{N}$ satisfying "Θ does not exist" then $\mathcal{N}$ is passive.*

Lemma 3.50. *The class of* Θ-g-*organized Ω-premice is very condensing.*

Proof. By 2.37. $\square$

Corollary 3.51. *Let $\mathcal{M}$ be an n-sound* Θ-g-*organized Ω-premouse and let $\pi : \mathcal{N} \to \mathcal{M}$ be a weak n-embedding. If $\mathcal{M}$ is n-maximally iterable then so is $\mathcal{N}$.*

Remark 3.52. It seems that one might try to define strategy premice over non-wellordered sets A by feeding in branches b_x for multiple trees $\mathcal{T}_x$ simultaneously, thus avoiding the need to select a single tree $\mathcal{T}$. However, we do not see how to arrange this in such a manner that the branch predicate B is always amenable. For example, suppose $A=\mathbb{R}$, and $\mathcal{N}|\eta$ is given, and we have identified, for each $x\in\mathbb{R}$, a tree $\mathcal{T}_x\in\mathcal{N}|\eta$, and now we want to feed in $b_x=\Sigma(\mathcal{T}_x)$, simultaneously. Let's say we have arranged that $\lambda=\mathrm{lh}(\mathcal{T}_x)$ is independent of x. Then we can easily knit together the predicates used to define $\mathfrak{B}(\mathcal{N}|\eta,\mathcal{T}_x,b_x)$, as x ranges over $\mathbb{R}$. Let $\mathcal{M}$ be the resulting structure and let $B=B^{\mathcal{M}}$. For B to be amenable, for each $\alpha<\lambda$, we must have that the function B_α is in $\mathcal{M}$, where $B_\alpha(x)=b_x\cap\alpha$. But it seems that even B_2 could contain non-trivial information, and maybe $B_2\notin\mathcal{M}$; note that essentially, $B_2\subseteq\mathbb{R}$. Maybe one could first add the sets B_α (amenably). But even if one achieved this, it seems that the first problem described in 2.47 would be an obstacle to proving that the resulting hierarchy has nice condensation.

4 $\mathcal{H}^{\mathcal{M}}$, the local $\mathrm{HOD}^{\mathcal{M}}_{a_0}$

Lemma 4.1. *Let $\mathcal{M}$ be a Θ-g-organized Ω-pm such that $\mathcal{M}\models$ "Θ exists". Let $\theta=\Theta^{\mathcal{M}}$. Let $n_0\leq\omega$ be such that $\mathcal{M}$ is n_0-sound and $\rho^{\mathcal{M}}_{n_0}\geq\theta$. Let $\gamma_0=l(\mathcal{M})$. Assume that for all $(\xi,k)<_{\mathrm{lex}}(\gamma_0,n_0)$, $\mathcal{M}|\xi$ is countably ${}^{\mathrm{G}}\Omega$-(k,ω_1+1)-iterable. Assume $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$. Then (i)*

$$\mathcal{M}|\theta\preccurlyeq_{\Sigma_1(\mathcal{L}^-)}\mathcal{M}$$

and (ii) for any $(\xi,k)<_{\mathrm{lex}}(\gamma_0,n_0)$ with $\theta\leq\xi$, and any $a\in\mathcal{M}|\theta$,

$$\mathrm{cHull}^{\mathcal{M}|\xi}_{n+1}(\mathbb{R}^{\mathcal{M}}\cup\{a\})\trianglelefteq\mathcal{M}|\theta.$$

Proof. (i) from (ii): Let $\varphi\in\mathcal{L}^-$ be Σ_1 and $a\in\mathcal{M}|\theta$. Suppose $\mathcal{M}\models\varphi(a)$. We must show that $\mathcal{M}|\theta\models\varphi(a)$. Let $\xi<\gamma_0$ be least such that $\mathcal{M}|(\xi+1)\models\varphi(a)$. We need to see that $\xi<\theta$. Assume $\theta\leq\xi$. Fix $n<\omega$ and an $\mathrm{r}\Sigma_{n+1}$ formula $\psi\in\mathcal{L}$ such that $\mathcal{M}|\xi\models\psi(a)$, and for any hpm $\mathcal{N}$ and $a'\in\mathcal{N}$, if $\mathcal{N}\models\psi(a')$ then $\mathcal{J}^{\mathrm{hpm}}(\mathcal{N})\models\varphi(a')$. Let

$$\mathcal{H}=\mathrm{cHull}^{\mathcal{M}|\xi}_{n+1}(\mathbb{R}^{\mathcal{M}}\cup\{a\}).$$

Then $a\in\mathcal{H}$ and $\mathcal{J}_1(\mathcal{H})\models\varphi(a)$. But by (ii), $\mathcal{H}\trianglelefteq\mathcal{M}|\theta$, a contradiction.

(ii): For $\eta<\theta$, let $\mathcal{H}_\eta=\mathrm{cHull}^{\mathcal{M}|\xi}_{n+1}(\mathbb{R}^{\mathcal{M}}\cup\eta)$, and $\pi_\eta:\mathcal{H}_\eta\to\mathcal{M}|\xi$ be the uncollapse. Note that $\mathrm{crit}(\pi_\eta)$ exists iff $\mathcal{H}_\eta\models$"Θ exists", and $\mathrm{crit}(\pi_\eta)=\Theta^{\mathcal{H}_\eta}$ when they exist. Let $\theta_\eta=\Theta^{\mathcal{H}_\eta}$ (where $\Theta^{\mathcal{H}_\eta}=\mathrm{o}(\mathcal{H}_\eta)$ if $\mathcal{H}_\eta\models$"Θ does not exist”). Then $\mathcal{H}_\eta \in \mathcal{M}|\theta$ and $\theta_\eta < \theta$, since $\rho_{n+1}^{\mathcal{M}|\xi} \neq \omega$. We say η is a **generator** iff $\eta = \theta_\eta$. The generators are club in θ. Let $\mathcal{H}'_\eta$ be the least $\mathcal{H} \lhd \mathcal{M}|\theta$ such that $\eta \leq \mathrm{o}(\mathcal{H})$ and $\rho_\omega^{\mathcal{H}} = \omega$. Now $\mathrm{cHull}_{n+1}^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup \{a\}) = \mathcal{H}_\eta$ for some generator η. So the following claim finishes the proof:

Claim 4.2. *Let $\eta < \theta$ be a generator. Then:*

- $\mathcal{H}_\eta \trianglelefteq \mathcal{H}'_\eta \lhd \mathcal{M}|\theta$.
- *If η is the least generator then $\rho_{n+1}^{\mathcal{H}_\eta} = \omega$ and $p_{n+1}^{\mathcal{H}_\eta} = \emptyset$.*
- *If $\zeta < \eta$ is the largest generator $< \eta$, then $\rho_{n+1}^{\mathcal{H}_\eta} = \omega$ and $p_{n+1}^{\mathcal{H}_\eta} = \{\zeta\}$.*
- *If η is a limit of generators then $\rho_{n+1}^{\mathcal{H}_\eta} = \eta$ and $p_{n+1}^{\mathcal{H}_\eta} = \emptyset$.*

Proof. The proof is by induction on η.

Suppose η is the least generator. Clearly $cb^{\mathcal{H}_\eta} = cb^{\mathcal{M}}$ and $\omega_1^{\mathcal{M}} < \eta$ and $\mathcal{H}_\eta = \mathrm{cHull}_{n+1}^{\mathcal{H}_\eta}(\mathbb{R}^{\mathcal{M}})$, which gives that $\rho_{n+1}^{\mathcal{H}_\eta} = \omega$ and $p_{n+1}^{\mathcal{H}_\eta} = \emptyset$ and $\mathcal{H}_\eta$ is a fully sound Θ-g-organized Ω-pm. So by $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$, countable iterability and 3.51, we have $\mathcal{H}_\eta \lhd \mathcal{M}|\theta$, and $\mathcal{H}_\eta = \mathcal{H}'_\eta$ since $\eta = \Theta^{\mathcal{H}_\eta}$.

Now suppose ζ is the largest generator $< \eta$. Then

$$\eta \subseteq Y =_{\mathrm{def}} \mathrm{Hull}_{n+1}^{\mathcal{M}|\xi}(\mathbb{R}^{\mathcal{M}} \cup \{\zeta\}),$$

so $\rho_{n+1}^{\mathcal{H}_\eta} = \omega$ and $p_{n+1}^{\mathcal{H}_\eta} \leq \{\zeta\}$. But $\mathcal{H}'_\zeta \in Y$, so $\mathcal{H}'_\zeta \subseteq Y$ and $\mathcal{H}_\zeta \in Y$. Therefore $p_{n+1}^{\mathcal{H}_\eta} = \{\zeta\}$ and $\mathcal{H}_\eta$ is $(n+1)$-solid, and $(n+1)$-sound, so fully sound. The rest is as in the previous case; again we get $\mathcal{H}'_\eta = \mathcal{H}_\eta$.

Suppose η is a limit of generators. The $\mathrm{r}\Sigma_{n+1}$ facts about $\mathcal{H}_\eta$ follow readily by induction. Since $\rho_{n+1}^{\mathcal{H}_\eta} = \eta = \Theta^{\mathcal{H}_\eta}$ and $\mathcal{H}_\eta$ is $(n+1)$-sound, and $\mathcal{H}_\eta$ cannot have extenders overlapping η, comparison gives $\mathcal{H}_\eta \trianglelefteq \mathcal{H}'_\eta$, as required. □

Definition 4.3. Let $\mathcal{M}$ be a Θ-g-organized Ω-pm satisfying “Θ exists” and $\theta = \Theta^{\mathcal{M}}$. Let

$$\tilde{T}^{\mathcal{M}} =_{\mathrm{def}} \mathrm{Th}_{\Sigma_1(\mathcal{L}^-)}^{\mathcal{M}|\theta}(\theta \cup \{a_0\}).$$

Let $W^{\mathcal{M}} = \mathcal{J}_\theta[\tilde{T}^{\mathcal{M}}]$ and $T^{\mathcal{M}} = (W^{\mathcal{M}}, \tilde{T}^{\mathcal{M}})$. We say that a set of ordinals A is $\mathrm{OD}_{a_0}^{\mathcal{M}}$ iff $A \in \mathcal{M}$ and there is $\xi < l(\mathcal{M})$ such that A is definable from a_0 and ordinal parameters over $\mathcal{M}|\xi$.[38] ⊣

[38]Note that this provides much more expressive power than $\mathrm{OD}_{a_0}^{\lfloor \mathcal{M} \rfloor}$.

Remark 4.4. With $\mathcal{M}$ as above, note that $\mathfrak{M}, U^{\mathcal{M}}, U'^{\mathcal{M}} \in W^{\mathcal{M}}$ (for $\mathfrak{M}$, this uses the parameter a_0) and $\Sigma^{\mathcal{M}|\theta} \in \mathcal{J}(W^{\mathcal{M}})$. Let $\mathfrak{P}_{<\theta}$ denote the bounded subsets of θ. By 4.1, if the hypotheses of 4.1 hold, then

$$\tilde{T}^{\mathcal{M}} = \mathrm{Th}^{\mathcal{M}}_{\Sigma_1(\mathcal{L}^-)}(\theta \cup \{a_0\})$$

and $\mathfrak{P}_{<\theta} \cap \mathrm{OD}^{\mathcal{M}}_{a_0} = \mathfrak{P}_{<\theta} \cap W^{\mathcal{M}}$.

Definition 4.5. A Θ-g-organized Ω-pm is **relevant** iff $\mathcal{M} \models$"Θ exists" and $\exists \mathcal{N} \lhd \mathcal{M}[\Theta^{\mathcal{M}} < \mathrm{o}(\mathcal{N})$ and $\mathcal{N} \models \mathsf{ZF}]$. ⊣

Definition 4.6. Adopt the hypotheses of 4.1, and suppose $\mathcal{M}$ is relevant. We define a g-organized Ω-pm $\mathcal{H} =_{\mathrm{def}} \mathcal{H}^{\mathcal{M}}$ over $\widehat{T^{\mathcal{M}}}$, with $\mathrm{o}(\mathcal{H}) = \mathrm{o}(\mathcal{M})$, much as in [18]. (We show in 4.8 that $\mathcal{H}$ is indeed a g-organized Ω-pm. It is natural to consider $\mathcal{H}^{\mathcal{M}}$ as a locally defined $\mathrm{HOD}^{\mathcal{M}}_{a_0}$.)

Let $\theta = \Theta^{\mathcal{M}}$. Set $cb^{\mathcal{H}} = \widehat{T^{\mathcal{M}}}$, $cp^{\mathcal{H}} = \mathfrak{M}$ and $\Psi^{\mathcal{H}} = \Sigma^{\mathcal{M}|\theta}$. For $\alpha \geq 1$ define the predicates of $\mathcal{H}|\alpha$ by restricting those of $\mathcal{M}|\theta + \alpha$, setting (i) $P^{\mathcal{H}|\alpha} = P^{\mathcal{M}|\theta+\alpha}$ and (ii) $E^{\mathcal{H}|\alpha} = E^{\mathcal{M}|\theta+\alpha} \cap \mathcal{H}|\alpha$. ⊣

Continue with the notation above. Note that $\mathbb{P} \in \mathcal{H}|2$, where $\mathbb{P}$ is the Vopenka algebra defined over $\mathcal{M}|\theta$ as in [18]. Let $\zeta > \theta$ be least such that $\mathcal{M}|\zeta \models \mathsf{ZF}$. Note that $\mathcal{M}|\alpha$ is passive for all $\alpha \leq \zeta$, because $\mathcal{M}|\theta$ is (g, Θ)-whole. For $\alpha \geq \zeta$ we have $\theta + \alpha = \alpha$, and to see that $\mathcal{H}$ is indeed a g-organized Ω-pm we will need to consider how $\mathcal{M}|\alpha = \mathcal{M}|(\theta + \alpha)$ relates to $\mathcal{H}|\alpha$. We will observe that for $\alpha \geq \zeta$, $\mathcal{H}|\alpha$ is a g-organized Ω-pm, $\mathcal{M}|\alpha$ is a symmetric submodel of a generic extension of $\mathcal{H}|\alpha$ (via $\mathbb{P}$), $\Sigma^{\mathcal{H}|\alpha} = \Sigma^{\mathcal{M}|\alpha}$, that $\mathcal{M}|\alpha$ is a (g, Θ)-activation level of $\mathcal{M}$ iff $\mathcal{H}|\alpha$ is a g-activation level of $\mathcal{H}$, and $\mathcal{T}_{\mathcal{M}|\alpha} \restriction \gamma = \mathcal{T}_{\mathcal{H}|\alpha} \restriction \gamma$ for enough γ that condition (i) above will be appropriate. We will also need to see that the fine structures of $\mathcal{H}|\alpha$ and $\mathcal{M}|\alpha$ correspond appropriately. The fine structural correspondence is mostly as in [18], so we omit most of the details, but give a summary.

Definition 4.7. Adopt the hypotheses of 4.6 and the notation above. For $\alpha \geq \zeta$ and $\mathcal{I} = \mathcal{H}|\alpha$ we define the $\mathcal{L}$-structure

$$\mathcal{H}_\alpha(\mathbb{R}^{\mathcal{M}}) = \mathcal{I}(\mathbb{R}^{\mathcal{M}}) = (\mathcal{J}_\alpha^{\vec{P}^{\mathcal{H}}}(\widehat{T^{\mathcal{M}}} \cup \mathrm{HC}^{\mathcal{M}}), \vec{P}^{\mathcal{I}}, \widehat{T^{\mathcal{M}}}, E^{\mathcal{I}}, P^{\mathcal{I}}; \mathfrak{M}, \Sigma^{\mathcal{M}|\theta}). \quad \dashv$$

Truth in $\mathcal{I}(\mathbb{R}^{\mathcal{M}})$ can be reduced to truth in $\mathcal{I}$ via forcing with $\mathbb{P}$. And $\mathcal{I}(\mathbb{R}^{\mathcal{M}})$ determines $\mathcal{M}|\alpha$: if $\mathcal{M}|\theta \in \mathcal{H}_\zeta(\mathbb{R}^{\mathcal{M}})$ then $\mathbb{E}^{\mathcal{I}}_+$ determines $\mathbb{E}^{\mathcal{M}|\alpha}_+ \restriction [\theta, \alpha]$ by the local definability of the forcing; because $\mathfrak{M}, U, U' \in \mathcal{H}|1$ and by induction applied to relevant initial segments of $\mathcal{M}|\theta$, we do have $\mathcal{M}|\theta \in \mathcal{H}_\zeta(\mathbb{R}^{\mathcal{M}})$. The main facts, which generalize [18, 3.9], are summarized as follows:

Lemma 4.8. *Under the hypotheses of 4.6 and with ζ as above, we have:*

(*1*) *For relevant $\mathcal{N} \trianglelefteq \mathcal{M}$, $\mathcal{N}||\mathrm{o}(\mathcal{N})$ is $\Sigma_1(\mathcal{L}^-)$ over $\mathcal{H}^{\mathcal{N}}(\mathbb{R}^{\mathcal{M}})$, and $\mathcal{N}$ is $\Sigma_1(\mathcal{L})$ over $\mathcal{H}^{\mathcal{N}}(\mathbb{R}^{\mathcal{M}})$, uniformly in $\mathcal{N}$.*

(*2*) *$\mathcal{H}$ is an n_0-sound g-organized Ω-pm (over $\widehat{T^{\mathcal{M}}}$, with $\Psi^{\mathcal{H}} = \Sigma^{\mathcal{M}|\theta}$), θ is a cardinal of $\mathcal{H}$, and ζ is least such that $\mathcal{H}|\zeta \vDash \mathsf{ZF}$.*

(*3*) *For all $(\beta, k) \leq_{\mathrm{lex}} (l(\mathcal{M}), n_0)$ with $\zeta \leq \beta$, we have $\rho_k(\mathcal{H}|\beta) = \rho_k(\mathcal{M}|\beta)$ and $p_k(\mathcal{H}|\beta) = p_k(\mathcal{M}|\beta)\backslash\{\theta\}$.*

(*4*) *For all $\beta \in [\zeta, l(\mathcal{M})]$, for any $p \in \mathbb{P}$, $\mathcal{H}_\beta(\mathbb{R}^{\mathcal{M}})$ is a symmetric inner model of a $\mathbb{P}$-forcing extension of $\mathcal{H}|\beta$.*

(*5*) *For all $\beta \in [\zeta, l(\mathcal{M})]$, $\mathcal{M}|\beta$ is determined by $\mathcal{H}_\beta(\mathbb{R}^{\mathcal{M}})$ as described above.*

(*6*) *Let $\beta \in [\theta, l(\mathcal{M})]$. Then $\mathcal{M}|\beta$ is (g, Θ)-whole iff either $\beta = \theta$, or $\beta > \zeta$ and $\mathcal{H}|\beta$ is g-whole. Similarly, $\mathcal{M}|\beta$ is a (g, Θ)-activation level of $\mathcal{M}$ iff $\mathcal{H}|\beta$ is a g-activation level of $\mathcal{H}$.*

Proof sketch. For most of the details, see the proof of [18, 3.9]. We just give enough of a sketch to describe the new features.

As usual, (1) will follow from the proof, and by induction, we may assume that (1) holds for $\mathcal{N} \trianglelefteq \mathcal{M}|\theta$. This implies $\mathcal{M}|\theta \in \mathcal{H}_\zeta(\mathbb{R}^{\mathcal{M}})$, unless there is no relevant $\xi < \theta$ (a fact regarding which $T^{\mathcal{M}}$ informs us). In the latter case, $\mathcal{M}|\theta = \mathcal{J}_\theta^{\mathrm{hpm}}(cb^{\mathcal{M}}; \mathfrak{M}, \emptyset)$. But $U^{\mathcal{M}} \in W^{\mathcal{M}}$, so $\Upsilon^{\mathcal{M}}, cb^{\mathcal{M}} \in \mathcal{H}_\zeta(\mathbb{R}^{\mathcal{M}})$, which suffices.

Let $\eta \in [\zeta, l(\mathcal{M})]$. We say that $\mathcal{M}|\eta, \mathcal{H}|\eta$ are **fine structurally related** iff (3), (4) and (5) hold for $\beta \leq \eta$. We say that $\mathcal{M}|\eta$, $\mathcal{H}|\eta$ are g-**related** iff (6) holds for $\beta \leq \eta$. We say that $\mathcal{M}|\eta, \mathcal{H}|\eta$ are **related** iff they are both fine structurally related and g-related.

Claim 4.9. *For $\eta \in [\zeta, l(\mathcal{M})]$, $\mathcal{H}|\eta$ is a g-organized Ω-pm over $\widehat{T^{\mathcal{M}}}$, and the models $\mathcal{M}|\eta, \mathcal{H}|\eta$ are related, and uniformly so in η.*

Proof. By induction on η; the uniformity follows from the proof. Let $\mathcal{M}_0 \triangleleft \mathcal{M}$ be (g, Θ)-whole, with $\theta \leq \beta_0 =_{\mathrm{def}} l(\mathcal{M}_0)$, and suppose that if $\theta < \beta_0$ then the claim holds for all $\eta \in [\zeta, \beta_0]$. (By (g, Θ)-wholeness, either $\beta_0 = \theta$ or $\zeta < \beta_0$.) For simplicity, suppose that $(*)$ there is a (g, Θ)-whole $\mathcal{M}_1 \trianglelefteq \mathcal{M}$ such that $\mathcal{M}_0 \triangleleft \mathcal{M}_1$. Let $\mathcal{M}_1$ be least such and $\beta_1 = l(\mathcal{M}_1)$. We will prove the claim for $\eta \in (\beta_0, \beta_1]$. The fact that $\mathcal{M}_1$ and $\mathcal{H}^{\mathcal{M}_1}$ are fine structurally related is proved as in [18, 3.9] (this is actually easier than in [18], as we have $P^{\mathcal{H}|\eta} = P^{\mathcal{M}|\eta}$ and $E^{\mathcal{H}|\eta} = \emptyset = E^{\mathcal{M}|\eta}$ for all $\eta \in (\beta_0, \beta_1]$). It remains to see that they are g-related. For this we need to see that

- α is least such that $\alpha > \beta_0$ and $\mathcal{H}|\alpha \vDash \mathsf{ZF}$, and
- $\mathcal{T} =_{\text{def}} \mathcal{T}_{\mathcal{H}|\alpha} = \mathcal{U} =_{\text{def}} \mathcal{U}_{\mathcal{M}|\alpha}$.

The former is straightforward, using forcing as in [18, 3.9]. So $\mathcal{H}|\alpha$ is the next g-activation level of $\mathcal{H}$, beyond $\mathcal{H}|\beta_0$ if $\beta_0 > \theta$, or at all if $\beta_0 = \theta$. We now prove by induction on γ that $\mathcal{T}\restriction\gamma + 1 = \mathcal{U}\restriction\gamma + 1$ for all $\gamma + 1 \leq \epsilon = \max(\text{lh}(\mathcal{T}), \text{lh}(\mathcal{U}))$. But then $\mathcal{T} = \mathcal{U}$ as required.

We have $\mathcal{T}\restriction\alpha + 1 = \mathcal{U}\restriction\alpha + 1$ (this part is linear iteration). So let $\gamma \geq \alpha$ and suppose that $\mathcal{T}\restriction\gamma + 1 = \mathcal{U}\restriction\gamma + 1$ and $\gamma + 1 < \epsilon$; we just need to see that $E^{\mathcal{T}}_\gamma = E^{\mathcal{U}}_\gamma$ (and in particular, both are defined). Let ξ be the largest limit ordinal such that $\xi \leq \gamma$. Let $\mathcal{S} = \mathcal{S}^{\mathcal{M}}_{\mathcal{U}\restriction\xi+1}$. Let $\delta = l(\mathcal{S}) + \text{o}(M^{\mathcal{U}}_\gamma)$. So $\delta \leq l(\mathcal{M}_1)$.

Suppose that $E^{\mathcal{T}}_\gamma \neq \emptyset$. Let $p \in \text{Col}(\omega, \mathcal{H}|\alpha)$ be such that p forces, over[39] $\mathcal{H}|\delta$, that $E^{\mathcal{T}}_\gamma$ induces an axiom which fails for $\dot{x}_{\mathcal{H}|\alpha}$. Now in $\mathcal{M}|\delta$, $\mathbb{Q} =_{\text{def}} \text{Col}(\omega, \mathcal{M}|\alpha)$ factors naturally as $\mathbb{Q}_0 \times \mathbb{Q}$ where $\mathbb{Q}_0 = \text{Col}(\omega, \mathcal{H}|\alpha)$. Let $\dot{G}_0, \dot{G}_1$ be the resulting $\mathbb{Q}$-names for the factor generics (so under the factoring just mentioned, $\dot{G}_0 \times \dot{G}_1$ corresponds to $\dot{G}$, the standard $\mathbb{Q}$-name for the $\mathbb{Q}$-generic). Let $\dot{x}_{0,\mathcal{M}|\alpha}$ and $\dot{x}_{1,\mathcal{M}|\alpha}$ be the $\mathbb{Q}$-names for the generic reals determined by $\dot{G}_0$ and $\dot{G}_1$. Let $p' \in \text{Col}(\omega, \mathcal{M}|\alpha)$ force that $p \in \dot{G}_0$. we have that p' forces that $E^{\mathcal{T}}_\gamma$ induces an axiom which fails for $\dot{x}_{0,\mathcal{M}|\alpha}$. But assuming we have used the natural definitions, $\dot{x}_{0,\mathcal{M}|\alpha}$ is arithmetic in $\dot{x}_{\mathcal{M}|\alpha}$, and so it is easy to see that p' forces that $E^{\mathcal{T}}_\gamma$ induces an axiom which fails for $\dot{x}_{\mathcal{M}|\alpha}$, as required.

The case that $E^{\mathcal{U}}_\gamma \neq \emptyset$ is similar, but we need to use the fact that $\mathcal{M}|\delta$ can be realized as a symmetric submodel of a $\mathbb{P}$-generic extension of $\mathcal{H}|\delta$. (It doesn't suffice that this holds for $\mathcal{M}|\alpha$ and $\mathcal{H}|\alpha$, since the forcing relation which demonstrates the fact that $E^{\mathcal{U}}_\gamma$ induces a bad axiom need not be in $\mathcal{M}|\alpha$.) We omit further detail.

If $(*)$ fails then it is almost the same. However, suppose there is a (g, Θ)-activation level $\trianglelefteq \mathcal{M}$ beyond $\mathcal{M}_0$. Then it can be that $\mathcal{T} \neq \mathcal{U}$, where $\mathcal{T}, \mathcal{U}$ are as before. However, the preceding argument still shows that $\mathcal{T}\restriction\gamma{+}1 = \mathcal{U}\restriction\gamma{+}1$ for enough ordinals γ that the proof goes through.

[39] This forcing is absolute, but the point is that the relevant forcing relation is in $\mathcal{H}|\delta$.

The remaining details (in particular the fact that $E^{\mathcal{H}_\alpha(\mathbb{R}^\mathcal{M})}$ determines $E^{\mathcal{M}|\alpha}$) are as in [18]. This completes the sketch of the proof of the lemma. $\square$

The next theorem relates the iterability of $\mathcal{H}$ and $\mathcal{M}$. The proof of 4.10 uses 4.8 and is just like that in [18, 3.18].

Theorem 4.10. *Assume the hypotheses of 4.6. Let $\gamma \in \mathrm{Ord}$. Then $\mathcal{H}^\mathcal{M}$ is (countably) (n_0, γ)-iterable iff $\mathcal{M}$ is (countably) above-$\Theta^\mathcal{M}$ (n_0, γ)-iterable.*

Remark 4.11. Constructions having the flavor of 4.6, as well as their inverses, are referred to as **S-constructions**. In the sequel, we will also need S-construction, performed mostly as in 4.6, for example, in the following context. Let $\mathcal{M}$ be a g-organized Ω-pm. Let $\mathcal{N} \triangleleft \mathcal{M}$ be a g-whole strong cutpoint of $\mathcal{M}$. Let $g \subseteq \mathrm{Col}(\omega, \mathcal{N})$ be $\mathcal{M}$-generic. Then $\mathcal{M}[g]$ can be reorganized as a g-organized Ω-pm $\mathcal{M}[g]^*$ over $\hat{x}$ where $x = (\mathcal{N}, g)$, with $\Psi^{\mathcal{M}[g]^*} = \Sigma^\mathcal{N}$. Moreover, the fine structure and iterability of $\mathcal{M}[g]^*$ corresponds to the fine structure and iterability of $\mathcal{M}$ above η, in a manner similar to 4.8 and 4.10. We leave the precise formulation and proofs of these facts to the reader.

Assume $\mathsf{DC}_\mathbb{R}$ and suppose that $\kappa_0 \geq \Theta$ (see 3.32). Using similar arguments, we also get that $\mathcal{M} =_{\mathrm{def}} \mathrm{Lp}^{{}^{\mathrm{g}}\Omega}(\mathbb{R})$ and $\mathcal{N} =_{\mathrm{def}} \mathrm{Lp}^{{}^{\mathrm{G}}\Omega}(\mathbb{R})$ and $\mathcal{P} =_{\mathrm{def}} \mathrm{Lp}^{{}^{\mathrm{G}}\Omega}(\mathrm{HC}, \Omega{\restriction}\mathrm{HC})$ have the same $\mathfrak{P}(\mathbb{R})$ (we have $\Omega{\restriction}\mathrm{HC} \in \mathcal{M} \cap \mathcal{N}$ by 3.44 and 3.49). Moreover, if $\mathcal{Q} = \mathrm{Lp}^\Omega(\mathbb{R})$ is well-defined and Ω has a property along the lines of *relativizes well* (see [15, Definition 1.3.21(?)]) then the same holds of $\mathcal{Q}$. In fact, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ (and $\mathcal{Q}$) have literally the same extender sequences and for all α such that $\mathcal{M}|\alpha$ is E-active, there is a straightforward translation between $\mathcal{M}|\alpha$, $\mathcal{N}|\alpha$, $\mathcal{P}|\alpha$ (and $\mathcal{Q}|\alpha$). (To see that $\mathcal{Q}|\alpha$ computes the others, note that the P-predicates of the others are determined by Q-structures for trees $\mathcal{T}$, where the Q-structures are in $L^\Omega(M(\mathcal{T}))$.)

5 Scales

We now begin the main project of the paper: the analysis of scales in Θ-g-organized Ω-premice.[40] In our application to the core model induction, the

[40] Let $\mathcal{M}$ be an hpm. When we say that $\mathcal{M} \models$"rΣ_n has the scale property", recall that rΣ_n uses the language $\mathcal{L}^+$ and rΣ_n formulas are interpreted over $\mathfrak{C}_0(\mathcal{M})$, so the statement literally means that $\mathfrak{C}_0(\mathcal{M}) \models$"r$\Sigma_n$ has the scale property". Moreover, since it is a statement satisfied by $\mathfrak{C}_0(\mathcal{M})$, it is interpreted with respect to sequences of reals

analysis proceeds from optimal determinacy hypotheses; cf. [19].[41]

5.1 Scales on $\Sigma_1^{\mathcal{M}}$ sets for passive $\mathcal{M}$

Theorem 5.1. *Let $\mathcal{M}$ be a passive Θ-g-organized Ω-pm satisfying* AD*. Assume* $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$*. Suppose that every proper segment of $\mathcal{M}$ is countably ${}^{\mathrm{G}}\Omega$-(ω, ω_1+1)-iterable. Then $\mathcal{M} \vDash$ "$\mathrm{r}\Sigma_1(a_0)$ has the scale property".*

Proof. By $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$, 3.50 and 3.51 we may assume that $\mathcal{M}$ is countable. For simplicity we assume that $l(\mathcal{M})$ is a limit ordinal; for the contrary case make the usual modifications using the $\mathcal{S}$-hierarchy as in 5.9 below. For this proof we abbreviate $\mathbb{R}^{\mathcal{M}}$ with $\mathbb{R}$, and likewise interpret HC and terms like *real, analytical*, etc, over $\mathcal{M}$.

Let $\Phi \in \mathcal{L}^-$ be Σ_1. For $x \in \mathbb{R}$, let $A(x) \Leftrightarrow \mathcal{M} \vDash \Phi(x)$. We will define a $\Sigma_1^{\mathcal{M}}(a_0)$-scale on A. For $x \in \mathbb{R}$ and $1 \leq \beta < l(\mathcal{M})$ let $A^\beta(x) \Leftrightarrow \mathcal{M}|\beta \vDash \Phi(x)$. Then $A = \bigcup_{\beta < l(\mathcal{M})} A^\beta$. We will construct a closed game representation $x \mapsto G_x^\beta$ for A^β, with G_x^β continuously associated to x. For u a partial play of G_x^β, let $G_{x,u}^\beta$ be the game in which the players continue the play of G_x^β from u. Let A_k^β be the set of pairs (x, u) such that $\mathcal{M} \vDash$"u is a partial play of G_x^β of length k, and player I has a winning quasi-strategy in $G_{x,u}^\beta$".[42] We will eventually show that $A_k^\beta \in \mathcal{M}$ and the map $(\beta, k) \mapsto A_k^\beta$ is $\Sigma_1^{\mathcal{M}}(a_0)$.[43]

The foregoing yields a $\Sigma_1^{\mathcal{M}}(a_0)$ scale essentially as in [16]. However, let us mention two small differences.

Firstly, there will be certain moves in G_x^β which are, in the sense of $\mathcal{M}$, prewellordering equivalence classes of reals. In the scale computation, these are simply treated in the same manner that ordinals are treated in [16]. Let

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$\langle x_n \rangle_{n<\omega} \in \mathcal{M}$. Likewise when we say that $\mathcal{M} \vDash$"$\mathrm{r}\Sigma_n(\mathbb{R})$ has the scale property", and here any parameters in $\mathbb{R}^{\mathcal{M}}$ are allowed; for $\mathrm{r}\Sigma_n$, any parameters in $\mathfrak{C}_0(\mathcal{M})$ are allowed, as usual.

[41] Let Σ be the unique iteration strategy for $\mathcal{M}_1^\sharp$. Suppose $\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{MC}$. Then in fact $\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R}) \cap \mathfrak{P}(\mathbb{R}) = \mathrm{Lp}(\mathbb{R}) \cap \mathfrak{P}(\mathbb{R})$. This is because in $L(\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R}))$, $L(\mathfrak{P}(\mathbb{R})) \vDash \mathsf{AD}^+ + \Theta = \theta_0 + \mathsf{MC}$ and hence by [6], in $L(\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R}))$, $\mathfrak{P}(\mathbb{R}) \subseteq \mathrm{Lp}(\mathbb{R})$. Therefore, even though the hierarchies $\mathrm{Lp}(\mathbb{R})$ and $\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R})$ are different, as far as sets of reals are concerned, we don't lose any information by analyzing the scales pattern in $\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R})$ instead of that in $\mathrm{Lp}(\mathbb{R})$.

[42] Note that we require the winning quasi-strategy to be in $\mathcal{M}$, unlike in Steel's arguments.

[43] Because we required that the winning quasi-strategy be in $\mathcal{M}$, we already know that A_k^β is definable over $\mathcal{M}$, but this does not particularly help us prove that $A_k^\beta \in \mathcal{M}$.

us set up some notation for these moves. Let $\Upsilon = \Upsilon^{\mathcal{M}}$ and $\vec{\leq} = \langle \leq_n \rangle_{n<\omega} =_{\text{def}} \vec{\leq}^{\mathcal{M}}$ (see 3.46). For $n < \omega$ let e_n be the set of $\leq_n$-equivalence classes of reals. Let $e = \bigcup_{n<\omega} e_n$. Let W be the tree of the scale in the codes in e; so W is a tree on $\omega \times e$ and $p[W] = \Upsilon^{\text{cd}}$ (in $\mathcal{M}$). Let W' be likewise for $\mathbb{R}\backslash\Upsilon$. (If $\mathcal{M}$ has an admissible initial segment then we could just use $U^{\mathcal{M}}, U'^{\mathcal{M}}$ instead of W, W'.) Then $\{(\Upsilon, \Upsilon^{\text{cd}}, W, W')\}$ is $\Delta_1^{\mathcal{M}}$.

Now secondly, because the payoff is closed for player I, if $\mathcal{M} \models$"Σ is a winning quasi-strategy for player I for $G^{\beta}_{x,u}$", then V satisfies the same. However, $\mathcal{M}$ might not have a winning quasi-strategy for player I for $G^{\beta}_{x,u}$, although V does. But this does not cause problems for the computation of a $\Sigma_1^{\mathcal{M}}(a_0)$ scale. For the fact that each $A^{\beta}_k \in \mathcal{M}$ ensures that A^{β}_k is related to A^{β}_{k+1} in essentially the usual manner. That is, A^{β}_k is either of the form $\exists^{\mathbb{R}} A^{\beta}_{k+1}$, or $\exists \alpha < \beta[A^{\beta}_{k+1}]$, or $\exists n \leq k \exists X \in e_n[A^{\beta}_{k+1}]$, or $\forall^{\mathbb{R}} A^{\beta}_{k+1}$. Because the relevant computations propagating norms are made inside $\mathcal{M}$ – where, in particular, $\leq_n$ and $\leq'_n$ are wellfounded – this is enough for the scale computation.

Before defining G^{β}_x we give an outline. Player II will play reals. Player I will (attempt to) build a countable, iterable, passive, Θ-g-organized Ω-pm $\mathcal{P}$ over $\Upsilon \cap \mathcal{P}$, containing all reals played by player II, such that $\mathcal{P} \models \Phi(x)$, but for all $\gamma < l(\mathcal{P})$, $\mathcal{P}|\gamma \models \neg\Phi(x)$. To ensure that player I indeed plays an iterable Θ-g-organized Ω-pm, he must simultaneously build a (cofinal) very weak 0-embedding $\pi : \mathcal{P} \to \mathcal{M}|\gamma$ for some $\gamma \leq \beta$ [44]. To ensure that $\mathcal{P}$ is over $\Upsilon \cap \mathcal{P}$, he must also build various branches through W and W'. (Here we will be interested in the case that those branches appear in generic extensions of $\mathcal{M}$, which will ensure that really prove that a given real of $\mathcal{M}$ is in the set it is claimed to be in.)

We now proceed to the details. Player I will describe his model using the language

$$\mathcal{L}^* =_{\text{def}} \mathcal{L} \cup \{\dot{x}_i \mid i < \omega\} \cup \{\dot{\Upsilon}\}.$$

Here $\dot{x}_i$ and $\dot{\Upsilon}$ are constants; $\dot{x}_i$ will denote the i^{th} real played in the game. Fix recursive maps

$$m, n : \{\sigma \mid \sigma \text{ is an } \mathcal{L}^*\text{-formula}\} \to \{2n \mid 2 \leq n < \omega\}$$

which are one-to-one, have disjoint recursive ranges, and are such that whenever $\dot{x}_i$ occurs in σ, then $i < \min(m(\sigma), n(\sigma))$.

[44] One could have instead used an approach more like that used in [18].

Fix a $\Sigma_1(\mathcal{L}^-)$ formula $\sigma_0(v_0, v_1, v_2)$ that defines over each $\mathcal{M}|\gamma$, a map

$$h_\gamma : \gamma^{<\omega} \times \mathbb{R} \overset{\text{onto}}{\to} \mathcal{M}|\gamma.$$

Let T be the following $\mathcal{L}^*$ theory:

(1) Extensionality

(2) "V is a Θ-g-spm"

$(3)_i$ $\dot{x}_i \in \mathbb{R}$

(4) $\Phi(\dot{x}_2)$ & $\neg\exists\mathcal{N} \lhd V\, [\mathcal{N} \vDash \Phi(\dot{x}_2)]$

(5) $\forall u, v, y, z\, [\sigma_0(u, v, y) \wedge \sigma_0(u, v, z) \implies y = z]$

$(6)_\varphi$ $[\exists v \varphi(v)] \implies \exists v \exists F \in l(V)^{<\omega} \left[\varphi(v) \wedge \sigma_0(F, \dot{x}_{m(\varphi)}, v)\right]$

$(7)_\varphi$ $\exists v\, [\varphi(v) \wedge v \in \mathbb{R}] \implies \varphi(\dot{x}_{n(\varphi)})$

(8) $\dot{cb} = \widehat{x}$ where $x = (\mathrm{HC}, \dot{\Upsilon})$

(9) $\dot{cp}$ is an hpm over the transitive set coded by $\dot{x}_0$

A run of the game G^β_x has ω rounds. In round n, player I first plays $i_n, x_{2n}, \eta_n, \Lambda_n$ where $i_n \in \{0, 1\}$, $x_{2n} \in \mathbb{R}$, $\eta_0 \leq \beta$ and $\eta_{n+1} < \mathrm{o}(\mathcal{M}|\eta_0)$, and $\Lambda_n \in (W \cup W')^n$; player II plays then $x_{2n+1} \in \mathbb{R}$.

The payoff for player I is mostly analogous to that in [18]. Conditions (f) and (g) are new, and they ensure that for each $i < \omega$, if player I asserts, for example, that "$\dot{x}_i \in \dot{\Upsilon}^{\mathrm{cd}}$" then $\langle \Lambda_{n,i} \rangle_{n \in (i,\omega)}$ is an infinite branch through W witnessing that $x_i \in \Upsilon^{\mathrm{cd}}$.

If $u = \langle (i_k, x_{2k}, \eta_k, x_{2k+1}) \mid k < n \rangle$ is a partial play of G^β_x, let

$$T^*(u) = \{(\neg)^i \sigma \mid \sigma \text{ is an } \mathcal{L}^*\text{-sentence} \wedge n(\sigma) < n \wedge i = i_{n(\sigma)}\},$$

where $(\neg)^0\sigma = \sigma$ and $(\neg)^1\sigma = \neg\sigma$. If p is a full run of G^β_x, let $T^*(p)$ be the union of all $T^*(p{\restriction}n)$, for $n < \omega$. We write "$\iota v \varphi(v)$" for "the unique v such that $\varphi(v)$". For $\sigma = ((a_0, b_0), \ldots, (a_{n-1}, b_{n-1}))$ let $p_0[\sigma] = (a_0, \ldots, a_{n-1})$ and $p_1[\sigma] = (b_0, \ldots, b_{n-1})$.

A run $p = \langle (i_k, x_{2k}, \eta_k, \Lambda_k, x_{2k+1}) \mid k < \omega \rangle$ of G^β_x is a win for player I iff

(a) $T^*(p)$ is a consistent extension of T,

(b) $x_0 = a_0$ and $x_2 = x$,

(c) for all $i, m, n < \omega$, "$\dot{x}_i(n) = m$" $\in T^*(p)$ iff $x_i(n) = m$,

(d) if φ and ψ are $\mathcal{L}^*$-formulae with one free variable and

$$\text{“}\iota v\varphi(v) \in \mathrm{Ord} \ \&\ \iota v\psi(v) \in \mathrm{Ord}\text{”} \in T^*(p),$$

then “$\iota v\varphi(v) \leq \iota v\psi(v)$”$\in T^*(p)$ iff $\eta_{n(\varphi)} \leq \eta_{n(\psi)}$,

(e) if $\sigma_0, \ldots, \sigma_{n-1}$ are $\mathcal{L}^*$-formulas with one free variable and

$$\text{“}\iota v\sigma_k(v) \in \mathrm{Ord}\text{”} \in T^*(p)$$

for all $k < n$, then for any rΣ_1-formula $\theta(v_0, \ldots, v_{n-1}, v)$,

$$\theta(\iota v\sigma_0(v), \ldots, \iota v\sigma_{n-1}(v), \dot{x}_0) \in T^*(p)$$

if and only if

$$\mathcal{M}|\eta_0 \vDash \theta(\eta_{n(\sigma_0)}, \ldots, \eta_{n(\sigma_{n-1})}, a_0),$$

(f) for all $i < m \leq n < \omega$, $\Lambda_{m,i} \trianglelefteq \Lambda_{n,i}$ and $p_0[\Lambda_{n,i}] = x_i{\restriction}n$,

(g) for all $i < m < \omega$, if “$\dot{x}_i \in \dot{\Upsilon}^{\mathrm{cd}}$” $\in T^*(p)$ then $\Lambda_{m,i} \in W$, and otherwise $\Lambda_{m,i} \in W'$.

Because of the payoff conditions, we could have added a sentence like “$\dot{c}p$ is $\mathfrak{M}$-like” to T (or any other sentences satisfied by all initial segments of $\mathcal{M}$), without any significant effect.

We next define the notion of *honesty* and show that the only winning strategy for player I is to be honest. A partial play

$$u = \langle (i_k, x_{2k}, \eta_k, \Lambda_k, x_{2k+1}) \mid k < n \rangle$$

is (β, x)-**honest** iff $\mathcal{M}|\beta \vDash \Phi(x)$ and if $n > 0$ then letting η be least such that $\mathcal{M}|\eta \vDash \Phi(x)$, we have:

(i) $x_0 = a_0$ and if $n > 1$ then $x_2 = x$.

(ii) Let I_u be any interpretation of $\mathcal{L}^*$ in which $\dot{x}_i^{I_u} = x_i$ for $0 < i < 2n$ and $\dot{\Upsilon}^{I_u} = \Upsilon$. Then $(\mathcal{M}|\eta, I_u) \vDash T^*(u)$.

(iii) Let $\langle \sigma_i \rangle_{i<m}$ enumerate all formulas $\sigma \in \mathcal{L}^*$ of one free variable such that $n(\sigma) < n$ and $(\mathcal{M}|\eta, I_u) \vDash$ “$\iota v\sigma(v) \in \mathrm{Ord}$”. For $k < m$, let $\delta_k \in \mathcal{M}|\eta$ be such that $(\mathcal{M}|\eta, I_u) \vDash \sigma_k(\delta_k)$. Then in $\mathcal{M}$, $\mathrm{Col}(\omega, \mathbb{R})$ forces the existence of a partial embedding

$$\pi : \mathcal{M}|\eta \dashrightarrow \mathcal{M}|\eta_0$$

such that $\mathrm{o}(\mathcal{M}|\eta) \cup \{a_0\} \subseteq \mathrm{dom}(\pi)$, $\pi(\delta_k) = \eta_{n(\sigma_k)}$ for each $k < m$, and π is rΣ_1-elementary on its domain.

(iv) For each $i < m < n$, $\Lambda_{m,i} \trianglelefteq \Lambda_{n-1,i}$ and $x_i \restriction m = p_0[\Lambda_{m,i}]$, and if $x_i \in \Upsilon^{\mathrm{cd}}$ (if $x_i \notin \Upsilon^{\mathrm{cd}}$) then there is $f \in \mathcal{M} \cap [W_{x_i}]$ ($f \in \mathcal{M} \cap [W'_{x_i}]$) such that $f \restriction m = p_1[\Lambda_{m,i}]$".

Let $Q^\beta_k(x,u)$ iff u is a (β,x)-honest position of length k.

The following two claims complete our proof of Theorem 5.1. Their proofs are similar to those of [18, Claims 4.2, 4.3].

Claim 5.2. *$Q^\beta_k \in \mathcal{M}$ for all β, k, and the map $(\beta,k) \mapsto Q^\beta_k$ is $\Sigma_1^{\mathcal{M}}(a_0)$.*

Proof Sketch. For condition (iv), observe that there is $k < \omega$ such that every infinite branch $b \in \mathcal{M}$ through W or W', is in fact in $\mathcal{S}_k((\mathrm{HC}, \Upsilon, \mathfrak{M}))$. For let $b = (x,f) \in \mathcal{M}$ be a branch through, say, W. Because W is the tree of $\vec{\leq}$, by $\mathsf{AC}_{\omega,\mathbb{R}}$ in $\mathcal{M}$ (where AD holds), there is $\langle x_n \rangle_{n<\omega} \in \mathcal{M}$ such that for each n, $x_m \restriction n = x \restriction n$ and $x_m \leq_i x_n \leq_i x_m$ for each $i < n \leq m$. But $\langle x_n \rangle_{n<\omega}$ determines b, and gives the observation.

Regarding the other conditions, the proof is mostly like that of [18, Claim 4.2], but we modify some details and give a complete proof of some points only hinted at in [18]. Let $\gamma = \mathrm{o}(\mathcal{M}|\beta)$, $A = \mathrm{Th}_1^{\mathcal{M}|\beta}(\gamma \cup \{a_0\})$ and $A' = \gamma \cup \{A\}$. Let $\lambda \in \mathrm{Ord}$ be least such that $\mathcal{J}_\lambda(A')$ is admissible. The "embedding game" $\mathcal{G}$ (see [18, Claim 4.2]) is definable from A and is fully analysed in $\mathcal{J}_\alpha(A')$ for some $\alpha < \lambda$. Now we claim that for each $\alpha < \lambda$,

$$t_\alpha = \mathrm{Th}_1^{\mathcal{J}_\alpha(A')}(A') \in \mathcal{M}.$$

This suffices. For if N is any structure with $A' \subseteq N$ and satisfying "$V = L[A']$, I see a full analysis of $\mathcal{G}$ but no proper segment of me does", then N is wellfounded and so $N = \mathcal{J}_\alpha(A')$ for some α (since otherwise the wellfounded part of N is admissible, contradicting the minimality of N). Therefore $\mathcal{M}$ can identify the theory of the unique such N, allowing the rest of the proof of [18, Claim 4.2] to go through.

So we show that $t_\alpha \in \mathcal{M}$. Let $\leq$ be a prewellorder of $\mathbb{R}^{\mathcal{M}}$ of length $\geq \gamma$, with $\leq$ in $\mathcal{M}$. Say that a structure N (possibly illfounded) is **good** iff N extends A' and $N \vDash$"$V = L[A']$" and $N = \mathrm{Hull}_1^N(A')$ and $\mathrm{Th}_1^N(A')$ is $(\Sigma^1_1(\leq))^{\mathcal{M}}$ (in the codes given by $\leq$). We claim that for every $\alpha < \lambda$, $\mathcal{J}_\alpha(A')$ is good (and therefore $t_\alpha \in \mathcal{M}$). All requirements are clear other than the fact that t_α is $(\Sigma^1_1(\leq))^{\mathcal{M}}$.

Now if there is any illfounded good N, then the wellfounded part of N is admissible, and therefore $\mathcal{J}_\alpha(A') \triangleleft N$ for each $\alpha < \lambda$, which easily gives the claim. So suppose all good structures are wellfounded.

We claim that there is a largest good structure. For suppose not. Let S be the set of all Σ_1 theories of good structures. Clearly $S \in \mathcal{M}$. Now for each $N \in S$ let $t_N = \mathrm{Th}_1^N(A')$. Let $t = \bigcup S$. Then $t \in \mathcal{M}$, and $t = \mathrm{Th}_1^N(A')$ for $N = \mathcal{J}_\xi(A')$, for some ordinal ξ. Moreover, $N = \mathrm{Hull}_1^N(A')$. But then by the coding lemma applied in $\mathcal{M}$, N is good, contradiction.

So let N be the largest good structure. Let $N = \mathcal{J}_\xi(A')$ and $N' = \mathcal{J}_{\xi+1}(A')$. We claim that $N \preccurlyeq_1 N'$, and therefore that N is admissible, completing the proof. So suppose otherwise. We claim that N' is good, for a contradiction. Clearly $N' = \mathrm{Hull}_1^{N'}(A')$, so we just need to see that $t' = \mathrm{Th}_1^{N'}(A')$ is $(\Sigma_1^1(\leq))^{\mathcal{M}}$. By the coding lemma, it suffices to see that $t' \in \mathcal{M}$. Now t' is recursively equivalent to $\oplus_{n<\omega} T_n$ where $T_n = \mathrm{Th}_n^N(A')$. But each of these theories are in $\mathcal{M}$ since $T_1 = t_N \in \mathcal{M}$. Therefore, by the coding lemma, each T_n is $(\Sigma_1^1(\leq))^{\mathcal{M}}$. Let T be the set of parameters $x \in \mathbb{R}$ coding (relative to $(\Sigma_1^1(\leq))^{\mathcal{M}}$) one of the theories T_n, for some $n < \omega$. Then $T \in \mathcal{M}$ because in fact, T is $(\Sigma_{10}^1(\leq))^{\mathcal{M}}$. Therefore $\oplus_{n<\omega} T_n \in \mathcal{M}$, as required. □

Because $\mathcal{G}$ is fully analysed inside $\mathcal{M}$, the existence of the embedding in condition (iii) of (β, x)-honesty is actually absolute between $\mathcal{M}^{\mathrm{Col}(\omega,\mathbb{R})}$ and $V^{\mathrm{Col}(\omega,\mathbb{R})}$.

Claim 5.3. $A_k^\beta = Q_k^\beta$.

Proof Sketch. Let $u \in Q_k^\beta$. Then as in [18] there is $\Sigma \in \mathcal{M}^{\mathrm{Col}(\omega,\mathbb{R})}$ which is a winning quasi-strategy for player I in $G_{x,u}^\beta$. For every $\alpha < l(\mathcal{M})$ and $n < \omega$, the Σ_0 forcing relation for $\mathcal{S}_n(\mathcal{M}|\alpha)$ is in $\mathcal{M}$. (Note here that for $x \in \mathrm{HC}$, the Σ_0 forcing relation restricted to elements of $\mathrm{trancl}(x)$ is essentially in HC, as it is trivial on conditions $p \notin \mathrm{trancl}(x)$.) Let $\tilde{\Sigma} \in \mathcal{M}$ and $p \in \mathrm{Col}(\omega, \mathbb{R})$ be such that in $\mathcal{M}$, $p \Vdash$"$\tilde{\Sigma}$ is a winning quasi-strategy". Let Σ' be the set of all partial plays v extending u such that for some $q \leq p$, in $\mathcal{M}$, $q \Vdash$"v is according to $\tilde{\Sigma}$". Then $\Sigma' \in \mathcal{M}$, and it is easy to see that Σ' is a winning quasi-strategy, so $u \in A_k^\beta$ as required.

Now consider the converse. Let $(u, x) \in A_k^\beta$ and let $\Sigma \in \mathcal{M}$ be a winning quasi-strategy witnessing this. Let G be $(\mathcal{M}, \mathrm{Col}(\omega, \mathbb{R}^{\mathcal{M}}))$-generic (recall we have reduced to the case that $\mathcal{M}$ is countable). As in the proof of [18, Claim 4.3], but working in $\mathcal{M}[G]$ (where we have Σ), let let $\mathcal{N}$ be a model produced by playing $G_{x,u}^\beta$ according to Σ and having player II play out all reals in $\mathbb{R}^{\mathcal{M}}$. Let $\pi' : \mathcal{N} \dashrightarrow \mathcal{M}|\eta_0$ be the partial embedding, with $\mathrm{dom}(\pi) = \mathrm{o}(\mathcal{N}) \cup \{a_0\}$, provided by payoff condition (e); so π' is Σ_1-elementary on its domain. Now

$cp^{\mathcal{N}}$ is an hpm over A_0 (as $a_0 = \dot{x}_0^{\mathcal{N}}$). Using π' and since $\dot{cp} \in \mathcal{L}$, it easily follows that $cp^{\mathcal{N}} = \mathfrak{M}$, and that π extends uniquely to very weak 0-embedding $\pi : \mathcal{N} \dashrightarrow \mathcal{M}|\eta_0$ which is Σ_1-elementary on its domain. It follows that $\mathcal{N}$ is a Θ-g-spm with $\mathbb{R}^{\mathcal{N}} = \mathbb{R}^{\mathcal{M}}$, and in fact, $\mathcal{N}$ is a Θ-g-organized Ω-pm over some Υ', by 3.50.

Actually, $\Upsilon' = \Upsilon$. This is because because player I built witnessing branches through W, W', and because if $x \in \mathbb{R}^{\mathcal{M}}$ and $\mathcal{M}[G] \models$"$x \in p[W]$", for example, then $\mathcal{M} \models$"$x \in p[W]$". The latter is because the relevant forcing relations are in $\mathcal{M}$, and so, if $p \Vdash$"$b \in [W_x]$" then $\mathcal{M}$ can compute the left-most branch $b' \in [W_x]$ such that for all $n < \omega$, there is some $q \leq p$ forcing "$b{\restriction}n = b'{\restriction}n$". Similar considerations also give condition (iv) of (β, x)-honesty (the relevant branches are in $\mathcal{M}$, not just $\mathcal{M}[G]$).

Now $\mathcal{N} \models \Phi(x)$ but no $\mathcal{N}' \triangleleft \mathcal{N}$ satisfies $\Phi(x)$, so $l(\mathcal{N}) = \alpha + 1$ for some α, and $\mathcal{N}|\alpha$ projects to ω. But $\mathcal{N}|\alpha$ is ${}^{\mathrm{G}}\Omega$-$(\omega, \omega_1 + 1)$-iterable, by 4.10 and using π as in [18, Claim 4.3]. The rest is as in [18]. □

This completes our sketch of the proof. □

Remark 5.4. In the circumstances of the preceding theorem, if $\mathcal{M}$ has no admissible proper segment, then there is an alternate scale construction. We include this also, as it yields some extra information. It is related to Moschovakis' construction of inductive scales on inductive sets.

Let $Q \subseteq \mathbb{R} \times \mathbb{R}^{<\omega}$. We say that Q is **open** iff $(v, \vec{w} \,\widehat{}\, (x)) \in Q$ for all $(v, \vec{w}) \in Q$ and $x \in \mathbb{R}$. We say that Q is a **basic payoff** iff Q is open, and definable over $(\mathrm{HC}, \Upsilon^{\mathcal{M}}, \mathfrak{M})$.

Let Q be a basic payoff. For $\alpha \leq \omega \cdot l(\mathcal{M})$ let $Q_{<\alpha} = \bigcup_{\beta<\alpha} Q_\beta$, where $Q_0 = Q$ and for $1 \leq \alpha < \omega \cdot l(\mathcal{M})$ and $\vec{w} \in \mathbb{R}^{<\omega}$,

$$(v, \vec{w}) \in Q_\alpha \iff \mathsf{q}^{\mathbb{R}} x[(v, \vec{w} \,\widehat{}\, (x)) \in Q_{<\alpha}],$$

where if $\mathrm{lh}(\vec{w})$ is even then $\mathsf{q}^{\mathbb{R}} = \forall^{\mathbb{R}}$, and otherwise $\mathsf{q}^{\mathbb{R}} = \exists^{\mathbb{R}}$. Let $v \in Q'_\alpha$ iff $(v, \emptyset) \in Q_\alpha$, and likewise $Q'_{<\alpha}$. Let $v \in \mathbb{R}$. The game $\mathcal{G}^Q_v$ is that where players I and II alternate playing reals $x_0, x_1, \ldots$ (player I moving first), and player II wins iff there is $n < \omega$ such that $(v, (x_0, \ldots, x_{n-1})) \in Q$. For $\vec{w} \in \mathbb{R}^{<\omega}$, let $\mathcal{G}^Q_{v,\vec{w}}$ be the game like $\mathcal{G}^Q_v$, except that we interpret $\vec{w}$ as the first $\mathrm{lh}(\vec{w})$ moves. Clearly if $(v, \vec{w}) \in Q_\alpha$ then II has a winning quasi-strategy $\mathcal{G}^Q_{v,\vec{w}}$.

We say $P \subseteq \mathbb{R}$ is $\mathrm{IND}^{\mathcal{M}}$ iff $P = Q'_{<\omega \cdot l(\mathcal{M})}$ for some basic payoff Q.

Claim 5.5. $\mathfrak{P}(\mathbb{R}) \cap \Sigma_1^{\mathcal{M}} = \mathrm{IND}^{\mathcal{M}}$.

Proof Sketch. The fact that $\mathrm{IND}^{\mathcal{M}} \subseteq \mathfrak{P}(\mathbb{R}) \cap \Sigma_1^{\mathcal{M}}$ is routine. We now sketch a proof that $\Sigma_1^{\mathcal{M}} \subseteq \mathrm{IND}^{\mathcal{M}}$. Fix a $\Sigma_1(\mathcal{L}^-)$-formula Φ. We define a basic payoff Q, implicitly, by directly defining the corresponding games $\mathcal{G}_v^Q$. In the definition of the game, some moves are specified as integers (or formulas, etc), but we take all moves to literally be reals. In some places, one player will play several items consecutively, or in a block, but for convenience, we also assume that literally the other player plays a dummy real between each consecutive pair of such items.[45] At certain points, given $d < \omega$, we will have a **delay of length** d, which is just a string of d alternating moves, whose values will be ignored.[46] We refer to player II as "player $\exists$" and player I as "player $\forall$". In $\mathcal{G}_v^Q$, player $\exists$ attempts to prove that $\mathcal{M} \vDash \Phi(v)$, roughly by describing a strictly descending sequence $\langle \mathcal{M}_{n+1} \rangle_{n<N}$ of (putative) proper segments of $\mathcal{M}$ and making claims about formulas they satisfy.[47] Player $\forall$ keeps player $\exists$ honest, by playing reals for which player $\exists$ must furnish witnesses to his assertions. For $\alpha \in [1, \mathrm{lh}(\mathcal{M})]$, we will get $v \in Q'_{<\omega\alpha}$ iff $\mathcal{M}|\alpha \vDash \Phi(v)$, thereby proving the claim.

$\mathcal{G}_v^Q$ will be broken into rounds, each of which consists of a finite sequence of real moves. Suppose we have a partial play p consisting of n complete rounds, after which neither player has already won the game. Then p will determine a $\Sigma_1(\mathcal{L}^-)$-formula $\varphi^n = \varphi^n(p)$ and $\vec{w}_n = \vec{w}_n(p) \in \mathbb{R}^{<\omega}$, where $\varphi^0 = \Phi$ and $\vec{w}_0 = (v)$. Player $\exists$ will have claimed that $\mathcal{M} \vDash \varphi^n(\vec{w}_n)$.

Let $\varphi \mapsto \langle \varphi_m \rangle_{m<\omega}$ be the natural recursive function sending $\Sigma_1(\mathcal{L}^-)$ formulas φ to sequences $\langle \varphi_m \rangle_{m<\omega}$ with $\varphi_m \in \mathcal{L}^-$, such that for all $\alpha < l(\mathcal{M})$ and $\vec{x} \in \mathbb{R}^{<\omega}$, we have $\mathcal{M}|(\alpha+1) \vDash \varphi(\vec{x})$ iff there is $m < \omega$ such that either $\alpha > 0$ and $\mathcal{M}|\alpha \vDash \varphi_m(\vec{x})$, or $\alpha = 0$ and $(\mathrm{HC}, \Upsilon^{\mathcal{M}}, \mathfrak{M}) \vDash \varphi_m(\vec{x})$.

Round n proceeds as follows. Player $\exists$ first plays a code (m, ψ, z) for a witness to the claim that $\mathcal{M} \vDash \varphi^n(\vec{w}_n)$, where $m < \omega$ and $\psi \in \Sigma_1(\mathcal{L}^-) \cup \{\emptyset\}$ and $z \in \mathbb{R}$, claiming that $\mathcal{N} \vDash \varphi_m^n(\vec{w}_n)$ where:

– if $\psi = \emptyset$ then $\mathcal{N} = (\mathrm{HC}, \Upsilon^{\mathcal{M}}, \mathfrak{M})$, and

[45] We supress these dummy reals from the definition of the game as we ignore their values. Their point is that they allow us to use the notation $\mathcal{G}_{v,\vec{x}}^Q$ even when $\vec{x}$ is a partial play stopping in the middle of some block of items played consecutively by a single player.

[46] These moves help calibrate the length of inductive computations of winning quasi-strategies, as explained later.

[47] In what follows, the (putative) model $\mathcal{M}_{n+1}$ is described in round n and is denoted $\mathcal{N}$ in our discussion. If player $\exists$ plays according to a winning strategy of simple enough complexity then the models $\mathcal{M}_{n+1}$ exist and $\mathcal{M}_{n+1} \triangleleft \mathcal{M}_n$, where $\mathcal{M}_0 = \mathcal{M}$.

– if $\psi \neq \emptyset$ then there is $\mathcal{N}' \triangleleft \mathcal{M}$ satisfying $\forall^{\mathbb{R}} x[\psi(x,z)]$, and $\mathcal{N}$ is the least such $\mathcal{N}'$.

Next, player $\forall$ can either *dispute* or *accept* the existence of $\mathcal{N}$, where she must accept if $\psi = \emptyset$.

Suppose $\forall$ disputes. Then $\forall$ plays $x \in \mathbb{R}$, then $d < \omega$, which is followed by a delay of length d; neither player has yet won. Set $\vec{w}_{n+1} = (x,z)$ and $\varphi^{n+1} = \psi$.

Now suppose $\forall$ accepts and $\psi \neq \emptyset$. Let

$$\forall X_0 \exists X_1 \ldots \forall X_{2k} \exists X_{2k+1}[\varphi^*(\ \cdot\ , X_0, \ldots, X_{2k+1})]$$

be the prenex normal form of $\varphi^n_m(\ \cdot\ )$, with $\varphi^* \in \Sigma_1(\mathcal{L}^-)$ (here " $\cdot$ " represents free variables), and then pass in the natural way from (k, φ^*, ψ) to a $\Sigma_1(\mathcal{L}^-)$ formula $\tilde{\varphi}$ such that if $\mathcal{M} \models \forall^{\mathbb{R}} x \psi(x,z)$, then letting $\mathcal{N} \trianglelefteq \mathcal{M}$ be least satisfying $\forall^{\mathbb{R}} x \psi(x,z)$, we have

$$\mathcal{N} \models \varphi^n_m(\vec{w}_n) \iff \mathcal{N} \models \varrho(\vec{w}_n, z) \iff \mathcal{M} \models \varrho(\vec{w}_n, z)$$

where $\varrho(\ \cdot\ )$ is the formula

$$\forall^{\mathbb{R}} x_0 \exists^{\mathbb{R}} x_1 \ldots \forall^{\mathbb{R}} x_{2k} \exists^{\mathbb{R}} x_{2k+1}[\tilde{\varphi}(\ \cdot\ , x_0, \ldots, x_{2k+1})].$$

Then $\forall$ plays $x_0 \in \mathbb{R}$, $\exists$ plays $x_1 \in \mathbb{R}$, etc, producing $\vec{x} = (x_0, \ldots, x_{2k+1})$. Then $\forall$ plays $d < \omega$, which is followed by a delay of length d. This completes the round; neither player has yet won. Set $\vec{w}_{n+1} = (\vec{w}_n, z, \vec{x})$ and $\varphi^{n+1} = \tilde{\varphi}$.

Finally suppose that $\psi = \emptyset$, so $\forall$ accepts. Pass in the natural way from φ^n_m to a $\Sigma_1(\mathcal{L}^-)$ formula φ^* such that $(\mathrm{HC}, \Upsilon^{\mathcal{M}}, \mathfrak{M}) \models \varphi^n_m(\vec{w}_n)$ iff

$$(\mathrm{HC}, \Upsilon^{\mathcal{M}}, \mathfrak{M}) \models \forall^{\mathbb{R}} x_0 \exists^{\mathbb{R}} x_1 \ldots \forall^{\mathbb{R}} x_{2k} \exists^{\mathbb{R}} x_{2k+1}[\varphi^*(\vec{w}_n, x_0, \ldots, x_{2k+1})].$$

Then $\vec{x} = (x_0, \ldots, x_{2k+1})$ is played out in the obvious manner. This finishes the game; $\exists$ wins iff $(\mathrm{HC}, \Upsilon^{\mathcal{M}}, \mathfrak{M}) \models \varphi^*(\vec{w}_n, \vec{x})$.

This completes the description of round n. We declare $\exists$ the winner iff he wins at some finite stage (in the situation of the previous paragraph). This completes the definition of $\mathcal{G}^Q_v$, and hence the implicit definition of Q.

Subclaim 5.6. *Let p be a partial play of $\mathcal{G}^Q_v$ consisting of n full rounds, after which neither player has yet won. Let $\varphi^n = \varphi^n(p)$ and $\vec{w}_n = \vec{w}_n(p)$. Let $\alpha \in [1, l(\mathcal{M})]$. Then:*

– $(v,p) \in Q_{<\omega\alpha}$ *iff* $\mathcal{M}|\alpha \vDash \varphi^n(\vec{w}_n)$.

– *Let* $w = (m, \psi, z)$ *be a valid move for player* $\exists$ *in* $\mathcal{G}_v^Q$*, following* p*, and* $p' = p^\frown w$*. Then* $(v, p') \in Q_{<\omega\alpha}$ *iff either:*

  – $\psi = \emptyset$ *and* $(\mathrm{HC}, \Upsilon^{\mathcal{M}}, \mathfrak{M}) \vDash \varphi^n_m(\vec{w}_n)$*, or*

  – $\psi \neq \emptyset$ *and there is* $\mathcal{N} \triangleleft \mathcal{M}|\alpha$ *satisfying* $\forall^{\mathbb{R}} x \psi(x,z)$*, and the least such* $\mathcal{N}$ *satisfies* $\varphi^n_m(\vec{w}_n)$*.*

Proof. This is a straightforward induction on α, which we omit.[48] □

Applying the first conclusion of the subclaim to the case that $\alpha = l(\mathcal{M})$ and $p = \emptyset$ (so $n = 0$), we have proved the claim. □

Because of the preceding claim, we just need to prove the next one:

Claim 5.7. $\mathcal{M} \vDash$ *"Every* $\mathrm{IND}^{\mathcal{M}}$ *set has a* $\Sigma_1^{\mathcal{M}}(a_0)$ *scale".*

Proof. This is a standard calculation, but here is a sketch. Fix a basic payoff Q. We define a scale on $Q_{<\omega \cdot l(\mathcal{M})}$ which is $\Sigma_1^{\mathcal{M}}(a_0)$.

Using the periodicity theorems and determinacy, over $(\mathrm{HC}, \Upsilon^{\mathcal{M}}, \mathfrak{M})$ we can define from the parameter a_0 a very good scale $\vec{\leq}^0$ on Q. (Use $(a_0, \mathfrak{M})$ to determine the code a_0' for $\mathfrak{M}$ relative to a_0, and from a_0', define a scale on the set C of all codes for $\mathfrak{M}$, and on $\mathbb{R}\backslash C$. Then produce scales on Boolean combinations of $\Upsilon^{\mathcal{M}}$, C and projective sets first by reducing to the case of *disjoint* unions of intersections of $\Upsilon^{\mathcal{M}}$, $\mathbb{R}\backslash\Upsilon^{\mathcal{M}}$, C, $\mathbb{R}\backslash C$ and projective sets.)

Now propagate $\vec{\leq}^0$ to scales on $Q^{<\beta}$, for $\beta \leq \omega \cdot l(\mathcal{M})$, in the usual manner. (For limit β, $\vec{\leq}^{<\beta} = \bigcup_{\gamma<\beta} \vec{\leq}^{<\gamma}$. For each β, $\leq_0^{<\beta}$ is the prewellorder of the norm on $Q^{<\beta}$ given by $x \mapsto \gamma$ where γ is least such that $x \in Q^{\gamma}$. For successor β, the remaining norms are given by propagating $\vec{\leq}^{<\beta-1}$ using the periodicity theorems, interleaving integer norms in the usual way to yield a very good scale.) The propagation process is $\Sigma_1^{\mathcal{M}}$, so the scale is $\Sigma_1^{\mathcal{M}}(a_0)$. □

[48] Let us just illustrate how delays help to calibrate the ranks of winning strategies for $\exists$. Let p' be as above, and adopt the notation there. Suppose that $\mathcal{M}|\alpha \vDash \forall^{\mathbb{R}} x \psi(x,z)$, and $\mathcal{M}|\alpha$ is least such. Let $p^* = p'^\frown$"dispute". Since the putative $\mathcal{N}$ does not exist (from the perspective of $\mathcal{M}|\alpha$) we want to know that $p^* \notin Q_{<\omega\alpha}$. For $x \in \mathbb{R}$ and $d < \omega$ let $\beta_{x,d}$ be the least β such that $p^{*\frown}(x,d) \in Q_\beta$, if such β exists. Then $\beta_{x,d}$ does exist, and $\beta < \omega\alpha$, since $\mathcal{M}|\alpha \vDash \psi(x,z)$. Moreover, for each $\gamma < \alpha$, there is x such that $p^{*\frown}(0,x) \notin Q_{<\omega\gamma}$ (by the minimality of $\mathcal{M}|\alpha$). But $\sup_{x,d} \beta_{x,d}$ is a limit because the arbitrary $d < \omega$ is followed by a delay of length d (after which the next round starts), so $\sup_{d,x} \beta_{d,x} = \omega\alpha$.

We now proceed to a variant of 5.1 we will need, in which $\mathcal{M}$ is P-active but satisfies "Θ does not exist". (Because $\mathcal{M}$ is a Θ-g-spm, this can only happen if $l(\mathcal{M}) = \alpha + 1$ for some α where $\mathcal{M}|\alpha \vDash$"$\Theta$ exists".)

Definition 5.8. Let $\mathcal{R} = \mathfrak{C}_0(\mathcal{M})$ where $\mathcal{M}$ is an hpm over A. Let $\beta < l(\mathcal{R})$ and $n < \omega$ and $H = \mathcal{S}^{\vec{P}^{\mathcal{R}}}_{\omega\beta+n}(A)$ be such that

$$cp^{\mathcal{R}}, \Psi^{\mathcal{R}}, \mu^{\mathcal{R}}, e^{\mathcal{R}} \in H.$$

Define the $\mathcal{L}^+$-structure

$$\mathcal{R} \wr (\beta, n) = (H, \vec{P}, A; E, P; cp^{\mathcal{R}}, \Psi^{\mathcal{R}}, \mu^{\mathcal{R}}, e^{\mathcal{R}}),$$

where $\vec{P} = \vec{P}^{\mathcal{R}} \cap H$, $E = E^{\mathcal{R}} \cap H$ and $P = P^{\mathcal{R}} \cap H$. ⊣

Note that $\lfloor \mathcal{R}|\beta \rfloor = \lfloor \mathcal{R} \wr (\beta, 0) \rfloor$ and $\vec{P}^{\mathcal{R}|\beta} = \vec{P}^{\mathcal{R}\wr(\beta,0)}$, but the E and P predicates of $\mathcal{R}|\beta$ and $\mathcal{R} \wr (\beta, 0)$ can differ.

Theorem 5.9. *Let $\mathcal{M}$ be a countably iterable Θ-g-organized Ω-pm satisfying* AD. *Assume* $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$. *Suppose $l(\mathcal{M}) = \beta_0 + 1$, and if $\beta_0 > 0$ then $\mathcal{M}|\beta_0 \not\preccurlyeq_{\mathrm{r}\Sigma_1(\mathcal{L}^-,\mathbb{R})} \mathcal{M}$. Then $\mathcal{M} \vDash$"$\mathrm{r}\Sigma_1^{\mathcal{M}}(\mathbb{R})$ has the scale property".*

Proof. By 5.1 we may assume that $\mathcal{M}$ is P-active. The proof is given by modifying that of 5.1 as follows. We again work with $\mathrm{HC} = \mathrm{HC}^{\mathcal{M}}$. We assume for simplicity that $\Upsilon^{\mathcal{M}} = \emptyset$; otherwise make adaptations as in 5.1.

We have $\mathcal{M}^- \trianglelefteq \mathcal{M}_0 =_{\mathrm{def}} \mathcal{M}|\beta_0$. Let $\mathrm{o}(\mathcal{M}_0) = \mathrm{o}(\mathcal{M}^-) + \lambda_0$ and $b_0 = b^{\mathcal{M}} \cap \lambda_0$ and $b^{\mathcal{M}} = b_0 \cup (\lambda_0 + b_1)$. (So $b_1 \subseteq \omega$. Note that $b_0 \in \mathcal{M}_0$.) Let $z_0 \in \mathbb{R}$, $d < \omega$, $k_0 \in [1, \omega)$, $\varrho_0 \in \mathcal{L}$, $\psi_0 \in \Sigma_1(\mathcal{L}^-)$, $\Psi_0 \in \mathcal{L}$ be such that:

- $z_0 \geq_T a_0$; a_0 is computed by the d^{th} Turing machine $\Phi^{z_0}_d$ with oracle z_0,
- $\mathcal{M} \vDash \psi_0(z_0)$ and $\mathcal{M}_0 \vDash \Psi_0(z_0) \& \neg\psi_0(z_0)$, and for all hpms $\mathcal{N}$, if $z \in \mathcal{N}$ and $\mathcal{N} \vDash \Psi_0(z)$ then $\mathcal{J}^{\mathrm{hpm}}(\mathcal{N}) \vDash \psi_0(z)$,
- $\mathcal{M}_0 = \mathrm{Hull}^{\mathcal{M}_0}_{k_0}(\mathbb{R})$ and $\mathcal{M}_0 \vDash b_0 = \iota b \varrho_0(z_0, b)$,
- ϱ_0, Ψ_0 are $\mathrm{r}\Sigma_{k_0}$ formulas.

Let $\Phi \in \mathcal{L}$ be Σ_1. For $x \in \mathbb{R}$, let $A(x) \iff \mathcal{M} \vDash \Phi(x)$. We will show that $\mathcal{M} \vDash$"There is a $\Sigma_1^{\mathcal{M}}(z_0)$-scale on A". For $x \in \mathbb{R}$ and $k \in [k_0, \omega)$ let

$$A^k(x) \Leftrightarrow \mathcal{M} \wr (\beta_0, k) \vDash \Phi(x).$$

Then $A = \bigcup_{k \geq k_0} A^k$. We will a construct closed game representation $x \mapsto G^k_x$ for A^k, and define A^k_l, much as before; player I will essentially be attempting to build a structure $\mathcal{R} \vDash \Phi(x)$ and corresponding to $\mathcal{M} \wr (\beta_0, k)$. Literally, he will not build the full $\mathcal{R}$ but just a countably iterable Θ-g-organized Ω-pm $\mathcal{N}$ corresponding to $\mathcal{M}_0$, with $\mathcal{N}$ satisfying a formula which will ensure that an $\mathcal{R}$ as above is given by extending $\mathcal{N}$.

We proceed to the details. Let $\mathcal{L}^*, m, n, \sigma_0$ be as before. Let $(k, c) \mapsto \Phi_{k,c}$ be the natural (and recursive) function with domain $[k_0, \omega) \times 2^{<\omega}$ such that $\Phi_{k,c} \in \mathcal{L}$ is a formula with the following property. Let $\mathcal{N}$ be any ω-sound Θ-g-spm such that $\mathcal{N} \vDash$"Θ exists" and $\mathcal{T} =_{\text{def}} \mathcal{T}^{\mathcal{N}}_{\varphi_{\text{G}}}$ is defined. Let $\lambda = \text{o}(\tilde{\mathcal{N}})$ where $\tilde{\mathcal{N}}$ is the largest φ_{G}-whole initial segment of $\mathcal{N}$. Let $y, z \in \mathbb{R}^{\mathcal{N}}$ and $b \in \mathcal{N} \cap \mathfrak{P}(< \text{o}(\mathcal{N}))$ and suppose $\mathcal{N} \vDash b = \iota b' \varrho_0(z, b')$. Let

$$\vec{P} = \vec{P}^{\mathcal{N}} \frown (E^{\mathcal{N}}, P^{\mathcal{N}}),$$

$$b^* = (\lambda + b) \cup (\text{o}(\mathcal{N}) + c),$$

$$P = (\{\mathcal{T}\} \times b^*) \cap \mathcal{S}^{\vec{P}}_k(\mathcal{N}),$$

and $\mathcal{R}$ be the $\mathcal{L}$-structure

$$\mathcal{R} = (\mathcal{S}^{\vec{P}}_k(\mathcal{N}), \vec{P}, A^{\mathcal{N}}; \emptyset, P, cp^{\mathcal{N}}, \Psi^{\mathcal{N}}).$$

Then $\mathcal{R} \vDash \Phi(x)$ iff $\mathcal{N} \vDash \Phi_{k,c}(x, z)$.

For $k \geq k_0$ and $c \in 2^{<\omega}$ let $T'_{k,c}$ be the theory given by modifying the theory T of 5.1 by replacing formulas (4) and (9) respectively with (4') and (9') below, and adding (10'):

$$
\begin{aligned}
&(4') && \Phi_{k,c}(\dot{x}_2, \dot{x}_0) \ \&\ \Psi_0(\dot{x}_0) \ \&\ \neg\psi_0(\dot{x}_0) \\
&(9') && \dot{cp} \text{ is an hpm over the transitive set coded by } \Phi^{\dot{x}_0}_d \\
&(10') && \dot{\Upsilon} = \emptyset \ \&\ V \text{ is } \omega\text{-sound} \ \&\ V = \text{Hull}^V_{k_0}(\mathbb{R})
\end{aligned}
$$

In round n of G^k_x, player I first plays i_n, x_{2n}, η_n where $i_n \in \{0, 1\}$, $x_{2n} \in \mathbb{R}$, $\eta_n < \text{o}(\mathcal{M}_0)$; player II plays then $x_{2n+1} \in \mathbb{R}$. Define $T^*(u)$, etc, as before. Let $s < \omega$ be such that for any transitive structure $\mathcal{N}$, $\text{o}(\mathcal{S}(\mathcal{N})) = \text{o}(\mathcal{N}) + s$. The payoff for player I is given by modifying that of 5.1 as follows. Drop conditions (f) and (g), replace condition (a) with

(a') $T^*(p)$ is a consistent extension of $T'_{k,c}$, where $c = b_1 \cap s \cdot k$, modify conditions (b), (e) by replacing "a_0", "$\mathrm{r}\Sigma_1$" and "$\mathcal{M}|\eta_0$" respectively with "z_0", "$\mathrm{r}\Sigma_{k_0+5}$" and "$\mathcal{M}_0$", and retain the remaining conditions unmodified.

We say that a partial play u of G^k_x is (k,x)-**honest** iff $\mathcal{M} \wr (\beta_0, k) \vDash \Phi(x)$ and if $n > 0$ then the modifications of properties (i)–(iii) of (β,x)-honesty of 5.1 hold, given by replacing "a_0", "$\mathcal{M}|\eta$", "$\mathcal{M}|\eta_0$", "Υ" and "$\mathrm{r}\Sigma_1$" respectively with "z_0", "$\mathcal{M}_0$", "$\mathcal{M}_0$", "$\emptyset$" and "$\mathrm{r}\Sigma_{k_0+5}$". Let $Q^k_l(x,u)$ iff u is a (k,x)-honest position of length l.

Claim 5.10. *$Q^k_l \in \mathcal{M}$ and the map $(k,l) \mapsto Q^k_l$ is $\Sigma_1^{\mathcal{M}}(z_0)$.*

Proof. As before, using that b_1 is $\Sigma_1^{\mathcal{M}}(\{\beta_0\})$ to compute $c = b_1 \cap s \cdot k$. □

Claim 5.11. *$A^k_l = Q^k_l$.*

Proof. $Q^k_l \subseteq A^k_l$ as before. For the converse, let $\mathcal{N}$ and $\pi : \mathcal{N} \dashrightarrow \mathcal{M}_0$ be produced as before. We get $\mathcal{N} = \mathcal{M}_0$ because $\mathcal{N}$ is sound, $\rho^{\mathcal{N}}_\omega = \omega$ and $\mathcal{N} \vDash \Psi(z_0)\&\neg\psi(z_0)$, and $\mathcal{N}$ is sufficiently iterable above $\Theta^{\mathcal{N}}$ as $\mathcal{N} = \mathrm{Hull}^{\mathcal{N}}_{k_0}(\mathbb{R})$ and if $\mathcal{N}$ is relevant then π induces a near k_0-embedding $\mathcal{H}^{\mathcal{N}} \to \mathcal{H}^{\mathcal{M}_0}$. Therefore b_0 is the unique $b' \in \mathcal{N}$ such that $\mathcal{N} \vDash \varrho_0(b', z_0)$. Since $\mathcal{N} \vDash \Phi_{k,c}(x, z_0)$ where $c = b_1 \cap k \cdot s$, it follows that $\mathcal{M} \wr (\beta_0, k) \vDash \Phi(x)$. □

This completes the proof. □

5.2 Σ_1 gaps

Definition 5.12. Let $\preccurlyeq^-_{\mathbb{R}}$ abbreviate $\preccurlyeq_{\mathrm{r}\Sigma_1(\mathcal{L}^-,\mathbb{R})}$. Let $\mathcal{M}$ be an hpm with $\mathrm{HC}^{\mathcal{M}} \in \mathcal{M}|1$. Let $\alpha \leq \beta \leq l(\mathcal{M})$. The interval $[\alpha, \beta]$ is a Σ_1 **gap** of $\mathcal{M}$ iff:

– $\mathcal{M}|\alpha \preccurlyeq^-_{\mathbb{R}} \mathcal{M}|\beta$,

– $\forall \alpha' \in [1, \alpha)$, $\mathcal{M}|\alpha' \not\preccurlyeq^-_{\mathbb{R}} \mathcal{M}|\alpha$, and $\forall \beta' \in (\beta, l(\mathcal{M})]$, $\mathcal{M}|\beta \not\preccurlyeq^-_{\mathbb{R}} \mathcal{M}|\beta'$,

– if $\beta = l(\mathcal{M})$ then $\mathcal{M}' =_{\mathrm{def}} \mathcal{J}^{\mathrm{hpm}}(\mathcal{M})$ is an hpm (i.e. $\mathcal{M}$ is ω-sound and $< \omega$-condensing), $\mathrm{HC}^{\mathcal{M}'} = \mathrm{HC}^{\mathcal{M}}$ and $\mathcal{M} \not\preccurlyeq^-_{\mathbb{R}} \mathcal{M}'$. ⊣

Definition 5.13. Let $0 < n < \omega$. Let $\mathcal{M}$ be an hpm with $\mathrm{HC}^{\mathcal{M}} \in \mathcal{M}|1$ and $b \in \mathfrak{C}_0(\mathcal{M})$. The $\mathrm{r}\Sigma_n$ **type realized by** b **over** $\mathcal{M}$ is

$$\mathrm{r}\Sigma^{\mathcal{M}}_{n,b} =_{\mathrm{def}} \{\varphi(v) \in \mathcal{L}^+ \mid \varphi \text{ is either } \mathrm{r}\Sigma_n \text{ or } \mathrm{r}\Pi_n \text{ and } \mathfrak{C}_0(\mathcal{M}) \vDash \varphi(b)\}.$$

Let $[\alpha, \beta]$ be a Σ_1 gap of $\mathcal{M}$. The gap is **admissible** iff $\mathcal{M}|\alpha$ is admissible. The gap is **strong** iff it is admissible and letting $n < \omega$ be least such that $\rho_n^{\mathcal{M}|\beta} = \omega$, every $\mathrm{r}\Sigma_n$ type realized over $\mathcal{M}|\beta$ is realized over $\mathcal{M}|\gamma$ for some $\gamma < \beta$. The gap is **weak** iff it is admissible but not strong. $\dashv$

There are no new scales inside the Σ_1 gaps in which we are interested. The proof of the following theorems are routine generalizations of the corresponding proofs in [16].

Theorem 5.14 (Kechris-Solovay)**.** *Let $\mathcal{M}$ be a Θ-g-organized Ω-pm satisfying AD. Assume $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$ and that $\mathcal{M}$ is countably $(0, \omega_1 + 1)$-iterable. Let $[\alpha, \beta]$ be a Σ_1 gap of $\mathcal{M}$. Then:*

1. *There is a $\Pi_1^{\mathcal{M}|\alpha}$ subset of $\mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}$ not uniformized in $\mathcal{M}|\beta$.*

2. *Let $\alpha \leq \gamma < \beta$ and $1 \leq n < \omega$, and either let $\Gamma = \mathrm{r}\Pi_n^{\mathcal{M}|\gamma}$ or suppose $(\alpha, 1) <_{\mathrm{lex}} (\gamma, n)$ and let $\Gamma = \mathrm{r}\Sigma_n^{\mathcal{M}|\gamma}$. Then $\mathcal{M} \vDash$"Γ does not have the scale property".*

A relation witnesing 5.14(1) is $(\mathbb{R}^{\mathcal{M}})^2 \backslash \mathcal{C}^{\mathcal{M}|\alpha}$ where $\mathcal{C}^{\mathcal{M}|\alpha}(x, y)$ iff $x, y \in \mathbb{R}^{\mathcal{M}}$ and there is $\gamma < \alpha$ such that y is definable over $\mathcal{M}|\gamma$ from parameters in $\mathrm{Ord} \cup \{x\}$. The same relation witnesses that there is no new scale definable over the end of a strong gap:

Theorem 5.15 (Martin)**.** *Let $\mathcal{M}, [\alpha, \beta]$ be as in 5.14. Suppose that $\beta < l(\mathcal{M})$ and $[\alpha, \beta]$ is a strong gap of $\mathcal{M}$. Then:*

1. *There is a $\Pi_1^{\mathcal{M}|\alpha}$ subset of $\mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}$ not uniformized in $\mathcal{M}|\beta + 1$.*

2. *Let $n < \omega$, and either let $\Gamma = \mathrm{r}\Pi_n^{\mathcal{M}|\beta}$ or suppose $(\alpha, 1) <_{\mathrm{lex}} (\beta, n)$ and let $\Gamma = \mathrm{r}\Sigma_n^{\mathcal{M}|\beta}$. Then $\mathcal{M} \vDash$"Γ does not have the scale property".*

Remark 5.16. The only case remaining in the analysis of scales in $\mathrm{Lp}^{{}^{\mathrm{G}}\Omega}(\mathbb{R}, \Upsilon)$, where Υ is self-scaled, is at the end of a weak gap. For let $\mathcal{M}$ be a Θ-g-organized Ω-pm and let $[\alpha, \beta]$ be a gap of $\mathcal{M}$. Suppose $[\alpha, \beta]$ is inadmissible. Then $\alpha = \beta$ and $\mathcal{M}|\alpha \vDash$"$\Theta$ does not exist". Note then $\mathcal{M} \vDash$"$\mathrm{r}\Sigma_1^{\mathcal{M}|\alpha}$ has the scale property", by 5.1 and 5.9.[49] Combined with the argument in [16],

[49] It is important here that our structure is Θ-g-organized, as opposed to g-organized, since g-organized structures can satisfy "Θ does not exist", be of limit length, and be P-active. We do not see how to generalize the proof of 5.9 to deal with this case.

this ensures that $\mathcal{J}(\mathcal{M}|\alpha) \models$"Every set of reals has a scale", assuming that $\mathbb{R}^{\mathcal{J}(\mathcal{M}|\alpha)} = \mathbb{R}^{\mathcal{M}}$ and $\mathcal{J}(\mathcal{M}|\alpha) \models \mathsf{AD}$. The ends of strong gaps have just been dealt with, so we are left with weak gaps. We deal with weak gaps in three cases, as described in the introduction.

5.3 Scales at the end of a weak gap from strong determinacy

The first scale construction for weak gaps proceeds from a strong determinacy assumption. It is most useful for weak gaps $[\alpha, \beta]$ of $\mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R}, \Upsilon)$ where $\Omega{\restriction}\mathrm{HC} \notin \mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R}, \Upsilon)|\alpha$.

Theorem 5.17. *Let $\mathcal{R}$ be a Θ-g-organized Ω-pm satisfying* AD*. Assume* $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$ *and that $\mathcal{R}$ is countably ${}^{\mathrm{G}}\Omega$-$(0, \omega_1 + 1)$-iterable. Let $[\alpha, \beta]$ be a weak gap of $\mathcal{R}$ with $\beta < l(\mathcal{R})$. Let $n + 1 < \omega$ be least such that $\rho_{n+1}^{\mathcal{R}|\beta} = \omega$. Then $\mathcal{R} \models$ "$\Upsilon\Sigma_{n+1}^{\mathcal{R}|\beta}$ has the scale property".*

Proof Sketch. The proof is almost that of [18, Theorem 4.16], so we only sketch it. However, our approach is a little different from that used in [18].[50] For simplicity, we assume that $\Upsilon^{\mathcal{R}} = \emptyset$ and $n = 0$ and β is a limit ordinal. (If $\Upsilon^{\mathcal{R}} \neq \emptyset$ make changes as in the proof of 5.1.) Let $\mathcal{M} = \mathcal{R}|\beta$.

Let $p = p_1^{\mathcal{M}}$ and let $w_1 \in \mathbb{R}^{\mathcal{M}}$ be such that $w_1 \geq_T a_0$ and the solidity witness(es) W for p is in $\mathrm{Hull}_1^{\mathcal{M}}(p, w_1)$ and $\Sigma =_{\mathrm{def}} \mathrm{r}\Sigma_{1,(p,w_1)}^{\mathcal{M}}$ is a non-reflecting type. Let $\mathcal{M}_\gamma^\wr$ denote $\mathcal{M}\wr(\gamma, 0)$.[51]. We now define a sequence $\langle \beta_i, Y_i, \psi_i, \xi_i \rangle_{i<\omega}$ by recursion on i, as follows:

$$\begin{aligned}
\beta_0 &= \text{least } \gamma > \mu^{\mathcal{M}} \text{ such that } \max(p) < \mathrm{o}(\mathcal{M}_\gamma^\wr), \\
Y_i &= \mathrm{Hull}_\omega^{\mathcal{M}_{\beta_i}^\wr}(\mathbb{R}^{\mathcal{M}} \cup \{p\}), \\
\psi_i &= \text{least } \psi \in \Sigma \text{ such that } \mathcal{M}_{\beta_i}^\wr \models \neg\psi((p, w_1)),
\end{aligned}$$

and then if $\mathcal{M}$ is either E-passive or E-active type 3, let $\xi_i = 0$ and

$$\beta_{i+1} = \text{least } \gamma \text{ such that } \mathcal{M}_\gamma^\wr \models \psi_i((p, w_1)),$$

[50] This is because the authors do not see, in the proof of [18, Claim 4.18], and in the notation of that proof, why $\mathcal{N} = \mathcal{M}$, because it is not clear that $\mathcal{N}$ is sound. Our approach gets around this problem, and also simplifies the proof, because it eliminates the need for the "bounding integers" m_k and n_k played by player I in the game G_x^i of [18].

[51] In [18], this is denoted $\mathcal{M}||\gamma$.

and otherwise ($\mathcal{M}$ is E-active type 1 or 2), let[52]

$$\begin{aligned}\xi_i &= \sup(Y_i \cap ((\mu^{\mathcal{M}})^+)^{\mathcal{M}}),\\ \beta_{i+1} &= \text{ least } \gamma \text{ such that } \mathcal{M}^{\wr}_{\gamma} \vDash \psi_i((p, w_1)) \text{ and}\\ &\quad\ E^{\mathcal{M}} \cap \mathcal{M}^{\wr}_{\gamma} \text{ measures all sets in } \mathcal{M}|\xi_i.\end{aligned}$$

Claim 5.18. *$\bigcup_{i<\omega} Y_i = \mathfrak{C}_0(\mathcal{M})$. In particular, $l(\mathfrak{C}_0(\mathcal{M})) = \lim_{i<\omega} \beta_i$.*

Proof. Let $\mathcal{N}$ be the transitive collapse of $\bigcup_{i<\omega} Y_i$ and let $\pi\colon \mathcal{N} \to \bigcup_{i<\omega} Y_i$ be the uncollapse map. Let $\beta_\omega = \sup_{i<\omega} \beta_i$. Note that $\mathcal{M}^{\wr}_{\beta_\omega} \vDash \Sigma$ and $H \subseteq \mathrm{rg}(\pi)$ where $H = \mathrm{Hull}_1^{\mathcal{M}}(\{p, w_1\})$, $\beta_i \in H$, π is Σ_1-elementary on $\pi^{-1}``H$, and the latter is $\in$-cofinal in $\mathcal{N}$.[53] In particular, π is a weak 0-embedding. So essentially by 3.50, $\mathcal{N}$ is a Θ-g-organized Ω-pm, and clearly $\mathrm{HC}^{\mathcal{N}} = \mathrm{HC}^{\mathcal{M}}$.

Let $\pi(p^*) = p$. It is easy to see that $\mathcal{N} = \mathrm{Hull}_1^{\mathcal{N}}(\mathbb{R}^{\mathcal{N}} \cup \{p^*\})$. But $\pi^{-1}(W)$ is a generalized solidity witness for p^*.[54] So $\mathcal{N}$ is $(1, p^*)$-solid. Therefore $\mathcal{N}$ is 1-sound and $p_1^{\mathcal{N}} = p^*$. Since trees on $\mathcal{N}$ can be lifted to trees on $\mathcal{M}$ via π, $\mathcal{N}$ is countably ${}^{\mathrm{G}}\Omega$-$(0, \omega_1 + 1)$–iterable. Since $\mathcal{N}$ is also minimal realizing Σ, therefore $\mathcal{N} = \mathcal{M}$.

The fact that $\pi = \mathrm{id}$ now follows as usual, using the fact that $p^* = p$. □

Using notation mostly as in the proof of [18, Theorem 4.16], we define the game G^k_x mostly as there, with some modifications. Player I describes his model using the language $\mathcal{L}^* = \mathcal{L}^+ \cup \{\dot{x}_i, \dot{\beta}_i, \dot{\mathcal{M}}_i\}_{i<\omega} \cup \{\dot{G}, \dot{p}, \dot{W}\}$; the symbols in $\mathcal{L}^* \backslash \mathcal{L}^+$ are constants. Let B_0 be defined from $\mathcal{L}^*$ as in [18].[55] Let S_0 be the set of sentences $\varphi \in B_0$ such that $i \in \{1, 2\}$ whenever $\dot{x}_i$ appears in φ, and $(\mathfrak{C}_0(\mathcal{M}), I) \vDash \varphi$ where I is the assignment

$$(\dot{x}_1, \dot{x}_2, \dot{G}, \dot{p}, \dot{W}, \langle \dot{\beta}_i, \dot{\mathcal{M}}_i \rangle_{i<\omega})^I = (w_1, w_2, p, p, W, \langle \beta_i, \mathcal{M}^{\wr}_{\beta_i} \rangle_{i<\omega}).$$

A run of G^k_x has the form

$$\begin{array}{llllll} \mathrm{I} & T_0, s_0, \eta_0 & & T_1, s_1, \eta_1 & & \cdots \\ \mathrm{II} & & s_1 & & s_3 & \cdots \end{array}$$

[52] Recall that E is the $\mathcal{M}$-amenable predicate coding the active extender of $\mathcal{M}$.

[53] So $\mathrm{Th}_1^{\mathcal{M}}(\{\beta_0, \beta_1, \ldots\})$ is recorded in Σ; it would not have made any difference to add the parameter β_i to Y_{i+1}.

[54] This only uses the Σ_0-elementarity of π. Actually $W \in H$, so π is even Σ_1-elementary on $\pi^{-1}(W)$. But we would in general need Σ_2-elementarity to infer already that $\pi^{-1}(W)$ is the *standard* solidity witness for p^*.

[55] That is, in the manner that B_0 is defined from the $\mathcal{L}$ of [18]. The symbols $\mathcal{L}$ and $\mathcal{L}^*$ have had their roles interchanged from [18].

where T_i, s_i are as in [18] and $\eta_i \in \mathrm{o}(\mathcal{M})$. The winning conditions for player I are the winning conditions (1)–(6)[56] of [18] verbatim (other than a small notational difference), and (k,x)**-honesty** is as in [18] except that we drop condition (iv) from there. Define A^k_l (strategic) and Q^k_l (honest) in the obvious manner (the analogue of A^k_l was denoted P^k_l in [18]).

Claim 5.19. *$A^k_l = Q^k_l$.*

Proof Sketch. Consider the proof that every strategic position is honest. We use notation mostly as in the proof of [18, Claim 4.19], with a couple of changes. Let $\mathcal{N}$ be the reduct of $\mathcal{A}$ to an $\mathcal{L}^+$-structure. Let $\mathcal{N}_i$ be (the $\mathcal{L}^+$-structure) $\dot{\mathcal{M}}_i^{\mathcal{A}}$. Because $\mathcal{A} \models S_0$, $\mathcal{N}_i = \mathcal{N}^{\wr}_{\beta^*_i}$ and $\mathcal{N}$ is the "union" of the $\mathcal{N}_i$. Let $p^* = \dot{p}^{\mathcal{A}} = G^*$. As in the proof of [18, Claim 4.19] we get that $\mathcal{N}$ is a countably ${}^{\mathrm{G}}\Omega$-$(0,\omega_1+1)$-iterable Θ-g-organized Ω-pm which is minimal for realizing Σ. Clearly $\Upsilon^{\mathcal{N}} = \emptyset = \Upsilon^{\mathcal{M}}$. Also, $\mathcal{N}$ is sound with $\rho_1^{\mathcal{N}} = \mathbb{R}^{\mathcal{N}}$ and $p_1^{\mathcal{N}} = p^*$. For let $H = \mathrm{Hull}_1^{\mathcal{N}}(\mathbb{R}^{\mathcal{N}} \cup p^*)$. Then because $\mathcal{A} \models S_0$, we have:

– $\mathcal{N}_i \in H$ for each i (it follows that $H = \lfloor \mathcal{N} \rfloor$),

– W^* is a generalized solidity witness for p^* (so $\mathcal{N} = \mathcal{M}$ and $p^* = p$),

– $W^* = W$, $\beta^*_i = \beta_i$ and $\mathcal{N}_i = \mathcal{M}^{\wr}_{\beta_i}$ for all i. □

Claim 5.20. *$Q^k_l \in \mathcal{M}$ for all k,l, and the map $(k,l) \mapsto Q^k_l$ is $\mathrm{r}\Sigma_1^{\mathcal{M}}(p, w_1, w_2)$.*

Proof sketch. The proof is the same as that of [18, Claim 4.20] (except that condition (iv) of [18] is not involved, so the use of the Coding Lemma regarding this condition is avoided). In the computation of the definability of (v) we still use the Coding Lemma; it is here that we use our assumption that $\mathcal{J}_1(\mathcal{M}) \models \mathsf{AD}$ (beyond that $\mathcal{M} \models \mathsf{AD}$). □

The remaining details are as in [18]. □

5.4 Scales at the end of a weak gap from optimal determinacy

As described in [19], typically in the core model induction, one does not have the stronger determinacy hypothesis at the stage required to apply 5.17. So

[56] We have no need for the integer moves m_k, nor any version of condition (8) used in [18].

we need generalizations of [18, Theorem 4.17] and [19, Theorem 0.1], which are the second and third cases of our scale constructions for weak gaps, respectively.

Definition 5.21. Let $\mathcal{M}$ be a Θ-g-organized Ω-pm. We say $\mathcal{M}$ is **subtle** iff $\mathcal{M} \vDash$"Θ exists" and either $\mathcal{M}$ is P-active or there is an $\mathcal{M}$-total $E \in \mathbb{E}_+^{\mathcal{M}}$. We say $\mathcal{M}$ is **self-analysed** iff for every subtle $\mathcal{N} \trianglelefteq \mathcal{M}$ there is $\mathcal{P} \trianglelefteq \mathcal{M}$ such that $\mathcal{N} \triangleleft \mathcal{P}$ and $\mathcal{P}$ is admissible. We say $\mathcal{M}$ is **self-coded** iff for every subtle $\mathcal{N} \trianglelefteq \mathcal{M}$ there is $\mathcal{P} \triangleleft \mathcal{M}$ such that $\mathcal{N} \trianglelefteq \mathcal{P}$ and $\rho_\omega^{\mathcal{P}} = \omega$. ⊣

Note that if $\mathcal{M} \vDash$"Θ does not exist" or $\mathcal{M}$ has no active segment above $\Theta^{\mathcal{M}}$ then $\mathcal{M}$ is self-coded.

Theorem 5.22. *Let $\mathcal{M}$ be a Θ-g-organized Ω-pm satisfying* AD. *Assume* $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$ *and that every proper segment of $\mathcal{M}$ is countably ${}^{\mathrm{G}}\Omega$-$(\omega, \omega_1 + 1)$-iterable. Suppose that $\mathcal{M}$ ends a weak gap of $\mathcal{M}$, and $\mathcal{M}$ is either self-analysed or self-coded. Let $n < \omega$ be least such that $\rho_{n+1}^{\mathcal{M}} = \omega$. Then $\mathcal{M} \vDash$"$\Upsilon\Sigma_{n+1}^{\mathcal{M}}$ has the scale property".*

Proof Sketch. The proof is similar to that of 5.17, but we use the fact that $\mathcal{M}$ is either self-analysed or self-coded to reduce the reliance on determinacy.[57]

Suppose first that $\mathcal{M}$ is passive. We assume for simplicity that $\Upsilon^{\mathcal{M}} = \emptyset$, $l(\mathcal{M})$ is a limit and $n = 0$. We define most things, including Y_k and B_k, as in the proof of 5.17. Fix $x \in \mathbb{R}$ and $i < \omega$; we want to define the game G_x^i. Let $m : B_0 \times B_0 \to \omega$ and $n : B_0 \to \omega$ be recursive and injective with disjoint ranges, and such that for all $\varphi, \psi \in B_0$, φ, ψ have support $m(\varphi, \psi)$ and φ has support $n(\varphi)$ and if $\varphi \neq \psi$ then $m(\varphi, \varphi) < m(\varphi, \psi)$. A run of G_x^i consists of the same types of objects as in the proof of 5.17, except that we also require that $\eta_k \in Y_k$. The rules of G_x^i are (1)–(5) as stated in [18], along with rule (6) below, which requires player I to play a wellfounded model, and rule (7) below, which requires player I to build, for each subtle initial segment $\mathcal{P}$ of his model, a partial embedding $\mathcal{P} \to \mathcal{R}$ for some $\mathcal{R} \trianglelefteq \mathcal{M}$, which is elementary on ordinal parameters (but these embeddings need not agree with one another):

(6) if $\varphi, \psi \in B_0$ each have one free variable and

$$\text{"}\iota v \varphi(v) \in \mathrm{Ord} \ \& \ \iota v \psi(v) \in \mathrm{Ord}\text{"} \in T^*,$$

[57] Of course determinacy is still required in the, supressed, norm propagation part of the argument.

then "$\iota v\varphi(v) \leq \iota v\psi(v)$"$\in T^*$ iff $\eta_{n(\varphi)} \leq \eta_{n(\psi)}$,

(7) if $\psi, \sigma_0, \ldots, \sigma_{j-1} \in B_0$ each have one free variable and $k < \omega$ and

$$\text{“}\iota v\psi(v) < l(\dot{\mathcal{M}}_k) \ \&\ \dot{\mathcal{M}}_k|(\iota v\psi(v)) \text{ is subtle”} \in T^*$$

and for all $i < j$,

$$\text{“}\iota v\sigma_i(v) \in \mathrm{o}(\dot{\mathcal{M}}_k|(\iota v\psi(v)))\text{”} \in T^*$$

then $\eta_{m(\psi,\psi)} < l(\mathcal{M}_k)$ and for any $\mathcal{L}$-formula $\theta(v_0, \ldots, v_{j-1}, u)$,

$$\text{“}\dot{\mathcal{M}}_k|(\iota v\psi(v)) \vDash \theta(\iota v\sigma_0(v), \ldots, \iota v\sigma_{j-1}(v), \dot{x}_1)\text{”} \in T^*$$

if and only if

$$\mathcal{M}|\eta_{m(\psi,\psi)} \vDash \theta(\eta_{m(\psi,\sigma_0)}, \ldots, \eta_{m(\psi,\sigma_{j-1})}, w_1).$$

We omit most of the remaining details, including the precise formulation of *x-honesty* (of a position in G^i_x). The analysis of commitments made pertaining to rule (6) are dealt with as in [16]. Consider rule (7). If $\mathcal{M}$ is self-analysed then the analogue of condition (v) of *x-honest* from [18] can be computed in some admissible proper segment of $\mathcal{M}$ (without the Coding Lemma). Suppose $\mathcal{M}$ is self-coded but not self-analysed. Then there is $\mathcal{R} \triangleleft \mathcal{M}$ such that $\rho^{\mathcal{R}}_\omega = \omega$ and every subtle initial segment of $\mathcal{M}$ is a segment of $\mathcal{R}$. One can therefore use the Coding Lemma as in the proof of Claim 5.2 to compute the analogue of condition (v) over $\mathcal{R}$. In rule (7) we have required elementarity with respect to w_1 (and ordinals) just to ensure elementarity with respect to a_0 (and ordinals).

This completes a sketch of the proof in the passive case. Now suppose that $\mathcal{M}$ is active. The scale construction in this case combines elements of 5.9 and 5.17, and we just outline what is new. Since $\mathcal{M}$ is not subtle, $\mathcal{M} \vDash$"Θ does not exist" and $\mathcal{M}$ is P-active, so because $\mathcal{M}$ is Θ-g-organized, $l(\mathcal{M}) = \beta_0 + 1$ for some $\beta_0 > 0$, and $\mathcal{M}|\beta_0 \vDash$"$\Theta$ exists" and $\rho^{\mathcal{M}|\beta_0}_\omega = \omega$. Therefore $n = 0$. Let $\mathcal{T} = \mathcal{T}^{\mathcal{M}}$. Assume $\Upsilon^{\mathcal{M}} = \emptyset$, and also that $\mathrm{lh}(\mathcal{T}^{\mathcal{M}}) > \omega$; the case that $\mathrm{lh}(\mathcal{T}) = \omega$ is simpler, partly because then $\mathcal{T}$ is linear, as $\mathcal{M}$ is Θ-g-organized. Let $\mathcal{M}_0 = \mathcal{M}|\beta_0$. Note that $\mathcal{M}_0, \beta_0 \in \mathrm{Hull}^{\mathcal{M}}_1(\emptyset)$. Let $k_0 \in [1, \omega)$ be such that $\mathcal{M}_0 = \mathrm{Hull}^{\mathcal{M}_0}_{k_0}(\mathbb{R})$. Let $\lambda^{\mathcal{M}}_0$ be the limit ordinal such that $\mathrm{lh}(\mathcal{T}^{\mathcal{M}}) = \lambda^{\mathcal{M}}_0 + \omega$. Let $b^{\mathcal{M}}_0 = b^{\mathcal{M}} \cap \lambda^{\mathcal{M}}_0$. Let p, w_1, Σ be as usual, except with the added requirement that $b_0^{\mathcal{M}} \in \mathrm{Hull}_{k_0}^{\mathcal{M}_0}(\{w_1\})$. In G_x^i, player I is required to build a Θ-g-spm $\mathcal{N}$ with $w_1 \in \mathbb{R}^{\mathcal{N}}$ and with $p^* \in \mathcal{N}$ such that $\mathrm{r}\Sigma_{1,(p^*,w_1)}^{\mathcal{N}} = \Sigma$ and $\mathcal{N} \wr (\beta_0^*, i) \vDash \Phi(x)$, where $l(\mathcal{N}) = \beta_0^* + 1$, and letting $\mathcal{N}_0 = \mathcal{N}|\beta_0^*$, is also required to build a partial embedding $\pi : \mathcal{N}_0 \dashrightarrow \mathcal{M}_0$, with domain $\mathrm{o}(\mathcal{N}_0) \cup \{w_1\}$, such that π is Σ_{k_0+5}-elementary on its domain. We leave to the reader the precise formulation of G_x^i, and of honesty.

Because player I is only required to embed $\mathcal{N}_0 \dashrightarrow \mathcal{M}_0$, and $\rho_{k_0}^{\mathcal{M}_0} = \omega$, the Coding Lemma argument shows that honesty is sufficiently simply computable. The fact that "strategic (for player I) implies honest" is as follows. Let $\mathcal{N}$ and $\pi : \mathcal{N}_0 \dashrightarrow \mathcal{M}_0$ be produced by a generic run against a winning strategy for player I, as usual. Then $\mathcal{N} = \mathcal{M}$. For we have $\mathcal{N}_0 \triangleleft \mathcal{M}$ as usual. Since $\mathrm{r}\Sigma_{1,(p^*,w_1)}^{\mathcal{N}} = \Sigma$, it therefore suffices to see that $b^{\mathcal{N}} = \Lambda_{\mathfrak{M}}(\mathcal{T}^{\mathcal{N}})$.

We claim that π induces a hull embedding $(\mathcal{T}^{\mathcal{N}} \frown b^{\mathcal{N}}) \to (\mathcal{T}^{\mathcal{M}} \frown b^{\mathcal{M}})$, which suffices. For clearly π induces a hull embedding $\mathcal{T}^{\mathcal{N}} \to \mathcal{T}^{\mathcal{M}}$. Let $\lambda_0^{\mathcal{N}}, b_0^{\mathcal{N}}$ be defined over $\mathcal{N}$, analogously to $\lambda_0^{\mathcal{M}}, b_0^{\mathcal{M}}$ over $\mathcal{M}$. Let ϱ_0 be an $\mathrm{r}\Sigma_{k_0}$ formula such that $b_0^{\mathcal{M}} = (\iota b \varrho_0(b, w_1))^{\mathcal{M}}$. Since $\mathrm{r}\Sigma_{1,(p^*,w_1)}^{\mathcal{N}} = \Sigma$, then $b_0^{\mathcal{N}} = (\iota b \varrho_0(b, w_1))^{\mathcal{N}}$. But let $c^{\mathcal{M}} = b^{\mathcal{M}} \cap [\lambda_0^{\mathcal{M}}, \mathrm{lh}(\mathcal{T}^{\mathcal{M}}))$ and $c^{\mathcal{N}}$ likewise for $\mathcal{N}$. Then $c^{\mathcal{M}} = c^{\mathcal{N}}$ because of how they are determined by Σ. The claim easily follows. $\square$

We now proceed to the generalization of [19, Theorem 0.1], the final scale construction of the paper. While it uses only the weaker determinacy assumption, it requires a mouse capturing hypothesis, as in [19].

Definition 5.23. Suppose V is an hpm and HC exists. Let Γ be a pointclass of the form $\mathrm{r}\Sigma_1^{V|\alpha} \cap \mathfrak{P}(\mathbb{R})$ for some $\alpha < l(V)$. In this setting, for $x \in \mathbb{R}$, we write $C_\Gamma(x)$ for the set of all $y \in \mathbb{R}$ such that for some ordinal $\gamma < \omega_1$, y (as a subset of ω) is $\Delta_\Gamma(\{\gamma, x\})$. Let $x \in \mathrm{HC}$ be such that x is transitive and $f : \omega \overset{\text{onto}}{\to} x$. Then $c_f \in \mathbb{R}$ denotes the code for $(x, \in)$ determined by f. And $C_\Gamma(x)$ denotes the set of all $y \in \mathfrak{P}(x)$ such that for all $f : \omega \overset{\text{onto}}{\to} x$ we have $f^{-1}(y) \in C_\Gamma(c_f)$. $\dashv$

Lemma 5.24. *Let $\mathcal{P}$ be a Θ-g-organized Ω-pm satisfying* AD. *Let $\mathcal{Q} \triangleleft \mathcal{P}$ be such that $\mathcal{Q}$ is passive and admissible. Work in $\mathcal{P}$. Let Γ be the pointclass $\mathrm{r}\Sigma_1^{\mathcal{Q}} \cap \mathfrak{P}(\mathbb{R})$. Let $x \in \mathrm{HC}$ with x transitive and infinite. Then for all $y \in \mathrm{HC}$, the following are equivalent:*

(1) $y \in C_\Gamma(x)$,

(2) *there is* $\mathcal{R} \triangleleft \mathcal{Q}$ *such that* y *is definable over* $\mathcal{R}$ *from parameters in* $\mathrm{Ord} \cup x \cup \{x\}$,

(3) *for comeager many bijections* $f : \omega \to x$, $f^{-1}(y) \in C_\Gamma(c_f)$.

Proof. The proof is mostly like that of [1, Theorem 3.4(?)]; we just mention a couple of points. For $x \in \mathbb{R}$, the equivalence of (1) and (2) follows because $\mathcal{Q} \models \mathsf{AD} + \mathsf{KP}$. Now consider the proof that (3) implies (2). If $\mathcal{P}$ satisfies (3), then we may take the witnessing comeager set C to be a countable intersection of dense sets, and then $C \in \mathcal{Q}$. So by KP there is $\mathcal{R} \triangleleft \mathcal{Q}$ such that for every $f \in C$, $f^{-1}(y)$ is definable over $\mathcal{R}$ from parameters in $\mathrm{Ord} \cup \{c_f\}$. As in [1], there is then some $\alpha < \omega_1^{\mathcal{P}}$ and $n < \omega$ and injection $\sigma : n \to x$ such that for comeager many bijections $f : \omega \to x$ extending σ, $f^{-1}(y)$ is the α^{th} real which is definable over $\mathcal{R}$ from parameters in $\mathrm{Ord} \cup \{c_f\}$, in the natural ordering. Letting $\delta = l(\mathcal{R})$, this defines y over $\mathcal{Q}|(\delta + 2)$ from parameters in $\{\delta, x\} \cup \mathrm{rg}(\sigma)$. □

Definition 5.25. Let $\mathcal{P}, \mathcal{Q}, \Gamma, x$ be as in 5.24. Suppose that $\mathfrak{M} \in \mathcal{J}(\hat{x})$ and $\Omega^* \in \mathcal{Q}$ where $\Omega^* = \Omega \restriction \mathrm{HC}^{\mathcal{P}}$. Work in $\mathcal{P}$. Then $\mathrm{Lp}^{\Gamma, {}^{\mathrm{g}}\Omega^*}(x)$ denotes $(\mathrm{Lp}^{{}^{\mathrm{g}}\Omega^*}(x))^{\mathcal{Q}}$.[58] Similarly for $\mathrm{Lp}_+^{\Gamma, {}^{\mathrm{g}}\Omega^*}(x)$. We say that **super-small Γ-${}^{\mathrm{g}}\Omega^*$-mouse capturing holds on a cone** iff there is $z \in \mathbb{R}$ such that for all transitive $x \in \mathrm{HC}$, if $\mathfrak{M}, z \in \mathcal{J}(\hat{x})$ then $\mathrm{Lp}^{\Gamma, {}^{\mathrm{g}}\Omega^*}(x)$ is super-small and

$$C_\Gamma(x) = \mathrm{Lp}^{\Gamma, {}^{\mathrm{g}}\Omega^*}(x) \cap \mathfrak{P}(x). \qquad \dashv$$

Theorem 5.26. *Let* $\mathcal{M}$ *be a fully sound,* Θ-g-*organized* Ω-*pm satisfying* AD. *Suppose* $[\alpha_0, l(\mathcal{M})]$ *is a weak gap of* $\mathcal{M}$ *and that* $\mathcal{M}$ *is countably* ${}^{\mathrm{g}}\Omega$-$(n, \omega_1 + 1)$*-iterable where* $n < \omega$ *is least such that* $\rho_{n+1}^{\mathcal{M}} = \omega$. *Assume* $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$ *and* $\mathbb{R}^{\mathcal{J}(\mathcal{M})} = \mathbb{R}^{\mathcal{M}}$ *and* $\mathcal{J}(\mathcal{M}) \models \mathsf{DC}_{\mathbb{R}}$.[59] *Suppose that* $\Omega^* \in \mathcal{M}|\alpha_0$ *where* $\Omega^* = \Omega \restriction \mathrm{HC}^{\mathcal{M}}$. *In* $\mathcal{M}$, *let* Γ *be the pointclass* $\mathrm{r}\Sigma_1^{\mathcal{M}|\alpha_0} \cap \mathfrak{P}(\mathbb{R})$, *and assume that super-small* Γ-${}^{\mathrm{g}}\Omega^*$*-mouse capturing holds on a cone. Then* $\mathcal{M} \models$ "$\mathrm{r}\Sigma_{n+1}^{\mathcal{M}}$ *has the scale property".*

[58] So $\mathcal{Q} \models$ "$\mathcal{N}$ is ${}^{\mathrm{g}}\Omega^*$-$(\omega, \omega_1{+}1)$-iterable" for all $\mathcal{N} \triangleleft \mathrm{Lp}^{\Gamma, {}^{\mathrm{g}}\Omega^*}(x)$. Note here that $\mathcal{Q} \models$ "$\mathfrak{P}(\omega_1)$ exists" because $\mathcal{Q} \models \mathsf{AD}$.

[59] $\mathcal{J}(\mathcal{M})$ provides a universe in which we can execute certain arguments in the proof of [19, Theorem 0.1] without introducing new reals. The authors believe that [19, Theorem 0.1] should also have adopted a hypothesis along these lines. Indeed, its proof seems to proceed under the implicit assumption that $\mathbb{R}^{\mathcal{M}} = \mathbb{R}^V$.

Proof. We follow the proof of [19], making some modifications. By $\mathsf{DC}_{\mathbb{R}^{\mathcal{M}}}$ we may assume that $\mathcal{M}$ is countable. By 5.22 we may assume that $\mathcal{M} \models$"Θ exists" and there is some $\xi+1 \in (\Theta^{\mathcal{M}}, l(\mathcal{M}))$ such that $\mathcal{M}|\xi \models \mathsf{ZF}$. Therefore $\mathfrak{P}(\mathbb{R}) \cap \mathcal{M} \in \mathcal{M}|\xi$ and $\mathcal{M}|\xi \models \mathsf{ZF} + \mathsf{AD}$. We work mostly inside $\mathcal{J}(\mathcal{M})$, and so we write $\mathbb{R} = \mathbb{R}^{\mathcal{M}}$, $\mathrm{HC} = \mathrm{HC}^{\mathcal{M}}$, etc. We have $\Omega^* \in \mathcal{M}|\alpha_0$. Let $z_0 \in \mathbb{R}$ be in the mouse capturing cone, with $z_0 \geq_T (a_0, t)$ where t codes $\mathrm{Th}_1^{\mathfrak{M}}$ relative to a_0, and such that $\{\Omega^*\}$ is $\mathrm{r}\Sigma_1^{\mathcal{M}|\alpha_0}(z_0)$. For this proof, except where context dictates otherwise, **premouse** abbreviates g-*organized* Ω^**-pm over* $(\mathcal{N}, x)$ *for some* $x \geq_T z_0$ *and transitive structure* $\mathcal{N}$ *with* $\mathfrak{M} \in \mathcal{J}(\mathcal{N}, x)$; likewise all related terminology (such as **iteration tree**, **iterability**, Lp, etc).

Because $[\alpha_0, l(\mathcal{M})]$ is a Σ_1 gap of $\mathcal{M}$, for $(\mathcal{N}, x)$ as above we have (with terminology as just described above)

$$\mathrm{Lp}^{\Gamma}(\mathcal{N}, x) = \mathrm{Lp}(\mathcal{N}, x)^{\mathcal{M}|\alpha_0} = \mathrm{Lp}(\mathcal{N}, x)^{\mathcal{M}}.$$

Likewise for $\mathrm{Lp}_+^{\Gamma}(\mathcal{N}, x)$.

Remark 5.27. Let $\mathcal{N}$ be a g-whole premouse and $\mathcal{N} \triangleleft \mathcal{Q} \trianglelefteq \mathrm{Lp}_+^{\Gamma}(\mathcal{N})$ with $\mathcal{Q}$ projecting to $\mathcal{N}$. Then $\mathcal{Q}$ translates to some $\mathcal{Q}' \triangleleft \mathrm{Lp}^{\Gamma}(\mathcal{N})$, where $\mathrm{o}(\mathcal{Q}') = \mathrm{o}(\mathcal{Q})$ and $\mathcal{Q}'$ projects to ω, as follows. There is a slight wrinkle in the translation, because we must have $\Psi^{\mathcal{Q}'} = \emptyset$ as $\Psi^{\mathrm{Lp}^{\Gamma}(\mathcal{N})} = \emptyset$, whereas possibly $\Sigma^{\mathcal{N}} \neq \emptyset$. We have $\lfloor \mathcal{Q}' \rfloor = \lfloor \mathcal{Q} \rfloor$ and $cb^{\mathcal{Q}'} = \hat{\mathcal{N}}$. There is $\alpha > 0$ such that $l(\mathcal{N}) + \alpha \leq l(\mathcal{Q})$ and for $\beta > \alpha$, $\mathcal{Q}|(l(\mathcal{N}) + \beta)$ and $\mathcal{Q}'|\beta$ have the same active predicates, and for $\beta \in [1, \alpha]$, $\mathcal{R} =_{\mathrm{def}} \mathcal{Q}|(l(\mathcal{N}) + \beta)$ and $\mathcal{R}' =_{\mathrm{def}} \mathcal{Q}'|\beta$ are both E-passive, and if $\mathcal{R}$ is P-active then $\mathcal{T}^{\mathcal{R}}$ is linear, and if $\mathcal{R}'$ is P-active then $\mathcal{T}^{\mathcal{R}'}$ is linear. These are linear iterations at the least measurable of $\mathfrak{M}$. Because the iterations are linear, the corresponding predicates are trivial, so we can trivially translate between them.[60] It can be that $\mathcal{Q} = \mathrm{Lp}_+^{\Gamma}(\mathcal{N})$, but in this case $\mathcal{Q}$ is not sound, whereas $\mathcal{Q}'$ is sound (recall $cb^{\mathcal{Q}'} = \hat{\mathcal{N}}$), whereas $cb^{\mathcal{Q}} = cb^{\mathcal{N}}$).

[60] $\mathcal{R}$ and $\mathcal{R}'$ can have different predicates, because the definition of spm requires that a particular tree can only have a cofinal branch added at at most one segment of the spm. We must have $\Sigma^{\mathcal{Q}'|1} = \emptyset$ by definition, but possibly $\Sigma^{\mathcal{N}} \neq \emptyset$, in which case there can be conflict between $\mathcal{R}, \mathcal{R}'$ over which tree should have a branch added. But it is easy to see that if $\mathcal{Q}$ is large enough then $\mathcal{Q}$ has a g-closed segment $\mathcal{R}$ such that $\mathcal{R}'$ is also g-closed, and beyond which no disagreements arise. (If $\mathcal{N}_0$ is the least ZF level of $\mathcal{Q}$ such that $\mathcal{N} \triangleleft \mathcal{N}_0$, and if $\mathcal{T}$ is non-linear and via $\Sigma^{\mathcal{N}}$, then $\mathcal{T}$ is not making $\mathcal{N}_0$ generically generic, as its linear initial segment is too short. So $\mathcal{R}, \mathcal{R}'$ never disagree over non-linear trees.)

Definition 5.28. Let $1 \le k \le \omega$. A countable premouse $\mathcal{N}$ over A is **k-suitable** iff there is a strictly increasing sequence $\langle \delta_i \rangle_{i<k}$ such that

(a) For all $\delta \in (\operatorname{rank}(A), \operatorname{o}(\mathcal{N}))$, we have $\mathcal{N} \vDash$"δ is Woodin" if and only if $\delta = \delta_i$ for some $i < k$.

(b) If $k = \omega$ then $\operatorname{o}(\mathcal{N}) = \sup_{i<\omega} \delta_i$, and if $k < \omega$ then $\operatorname{o}(\mathcal{N}) = \sup_{i<\omega} (\delta_{k-1}^{+i})^{\mathcal{N}}$.

(c) If $\mathcal{N}|\eta$ is a strong cutpoint of $\mathcal{N}$ then $\mathcal{N}|(\eta^+)^{\mathcal{N}} = \operatorname{Lp}_+^\Gamma(\mathcal{N}|\eta)$.

(d) Let $\xi \in (\operatorname{rank}(A), \operatorname{o}(\mathcal{N}))$ be such that $\mathcal{N} \vDash$"ξ is not Woodin". Then $C_\Gamma(\mathcal{N}|\xi) \vDash$"$\xi$ is not Woodin".

We write $\delta_i^{\mathcal{N}} = \delta_i$ and $\delta_{-1}^{\mathcal{N}} = 0$. $\dashv$

Let $\mathcal{N}$ be k-suitable over A and let $\xi \in (\operatorname{rank}(A), \operatorname{o}(\mathcal{N}))$ be a limit ordinal such that $\mathcal{N} \vDash$"ξ isn't Woodin". Let $\mathcal{Q} \triangleleft \mathcal{N}$ be the Q-structure for ξ. Let α be such that $\xi = \operatorname{o}(\mathcal{N}|\alpha)$. Suppose that $\mathcal{N}|\alpha \triangleleft \mathcal{Q}$. Then $\alpha = \xi$ and $\mathcal{N}|\xi$ is g-closed. In particular, $\mathcal{N}|\xi$ is g-whole, so $\operatorname{Lp}_+^\Gamma(\mathcal{N}|\xi)$ translates to an initial segment of $\operatorname{Lp}^\Gamma(\mathcal{N}|\xi)$. Assume that $\mathcal{N}$ is reasonably iterable. If ξ is a strong cutpoint of $\mathcal{Q}$, our mouse capturing hypothesis combined with (d) therefore gives that $\mathcal{Q} \triangleleft \operatorname{Lp}_+^\Gamma(\mathcal{N}|\xi)$. Moreover, note that if ξ is a cardinal of $\mathcal{N}$ then $\mathcal{N}|\xi$ *is* a strong cutpoint of $\mathcal{Q}$, since $\mathcal{N}$ has only finitely many Woodins. On the other hand, if ξ is not a (strong) cutpoint of $\mathcal{Q}$, then one can show that $\mathcal{Q} \notin \operatorname{Lp}_+^\Gamma(\mathcal{N}|\xi)$, but $\mathcal{Q}$ is coded over $\operatorname{Lp}_+^\Gamma(\mathcal{N}|\xi)$ (here $\operatorname{Lp}_+^\Gamma(\mathcal{N}|\xi)$ translates to a proper segment of $\operatorname{Lp}^\Gamma(\mathcal{N}|\xi)$).[61]

Definition 5.29 (Γ-guided)**.** Let $\mathcal{P}$ be k-suitable and $\mathcal{T} \in \mathrm{HC}$ be a normal iteration tree on $\mathcal{P}$. We say $\mathcal{T}$ is **Q-guided** iff for each limit $\lambda < \operatorname{lh}(\mathcal{T})$, $\mathcal{Q} = \mathcal{Q}(\mathcal{T}\upharpoonright\lambda, [0,\lambda]_{\mathcal{T}})$ exists and $\Phi(\mathcal{T}\upharpoonright\lambda) \,^\frown\, (\mathcal{Q}, \delta(\mathcal{T}))$ is $(\omega, \omega_1 + 1)$-iterable.

[61]Suppose ξ is not a cutpoint of $\mathcal{Q}$. Then by definition $\mathcal{Q} \ntriangleleft \operatorname{Lp}_+^\Gamma(\mathcal{N}|\xi)$. Let $E \in \mathbb{E}_+^{\mathcal{Q}}$ be least overlapping ξ and $\kappa = \operatorname{crit}(E)$. Since κ is a limit of Woodins in $\mathcal{Q}$, κ is not a cardinal of $\mathcal{N}$. Let $\mathcal{P} \triangleleft \mathcal{N}$ be least such that $\mathcal{Q} \trianglelefteq \mathcal{P}$ and $\rho_\omega^{\mathcal{P}} \le \kappa$, and let $n < \omega$ be such that $\rho_{n+1}^{\mathcal{P}} \le \kappa < \rho_n^{\mathcal{P}}$. We claim that $\operatorname{Lp}_+^\Gamma(\mathcal{N}|\xi) = U$ where $U = \operatorname{Ult}_n(\mathcal{P}, E)$ (and note that $U \vDash$"ξ is Woodin", but $\mathcal{Q}$ is computable from U, as $\mathcal{Q} \trianglelefteq \mathcal{P}$ and $\mathcal{P} = \mathfrak{C}_{n+1}(U)$). For ξ is a strong cutpoint of U, and U is ξ-sound but not fully sound. So it suffices to see that there is an above-κ, (n, ω_1)-iteration strategy for $\mathcal{P}$ in $\mathcal{M}|\alpha_0$. Let $\mathcal{R} \triangleleft \mathcal{N}$ be least such that $\mathcal{Q} \trianglelefteq \mathcal{R}$ and $\rho_\omega^{\mathcal{R}} < \kappa$ (so $\mathcal{P} \trianglelefteq \mathcal{R}$). Note that κ is a limit of strong cutpoints of $\mathcal{R}$ and of Woodins of $\mathcal{R}$. Let $\gamma \in (\rho_\omega^{\mathcal{R}}, \kappa)$ be a strong cutpoint of $\mathcal{R}$, and let η be the least Woodin of $\mathcal{R}$ above γ. Then η is a strong cutpoint of $\mathcal{R}$. Since $C_\Gamma(\mathcal{N}|\eta) \vDash$"$\eta$ is not Woodin", and by our mouse capturing hypothesis, therefore $\mathcal{R} \triangleleft \operatorname{Lp}_+^\Gamma(\mathcal{N}|\eta)$. In particular, there is an above-η iteration strategy for $\mathcal{R}$ in $\mathcal{M}|\alpha_0$, which yields the desired strategy.

We say that $\mathcal{T}$ is **Γ-guided** iff it is Q-guided, as witnessed by iteration strategies in $\mathcal{M}|\alpha_0$. ⊣

Remark 5.30. Let $\mathcal{P}$ be k-suitable. For a normal tree $\mathcal{T}$ on $\mathcal{P}$ of limit length there is at most one $\mathcal{T}$-cofinal branch b such that $\mathcal{T} ^\frown b$ is $\mathcal{Q}$-guided. (Let b_0, b_1 be distinct such branches; we can successfully compare the phalanxes $\Phi(\mathcal{T} ^\frown b_0)$ and $\Phi(\mathcal{T} ^\frown b_1)$. Standard fine structure and the fact that $\mathcal{P}$ has at most ω-many Woodins then leads to contradiction.) Therefore if $\mathcal{T} ^\frown b$ is normal, via an (ω, ω_1+1)-iteration strategy for $\mathcal{P}$, is based on $[\delta^{\mathcal{P}}_{i-1}, \delta^{\mathcal{P}}_i)$ and $\mathcal{Q}(\mathcal{T}, b)$ exists, then $\mathcal{T} ^\frown b$ is $\mathcal{Q}$-guided.

Definition 5.31. Let $\mathcal{N}$ be a g-whole premouse. We write $\mathcal{Q}^{\Gamma}_{\mathrm{t}}(\mathcal{N})$ for the unique $\mathcal{Q} \trianglelefteq \mathrm{Lp}^{\Gamma}_{+}(\mathcal{N})$ such that $\mathcal{Q}$ is a Q-structure for $\mathcal{N}$, if such exists.[62]

Let $k \leq \omega$, $\mathcal{P}$ be k-suitable and $\mathcal{T}$ a normal, limit length, Γ-guided tree on $\mathcal{P}$. We say that $\mathcal{T}$ is **short** iff $\mathcal{Q}^{\Gamma}_{\mathrm{t}}(M(\mathcal{T}))$ exists; otherwise that $\mathcal{T}$ is **maximal**. ⊣

Definition 5.32. Let $\mathcal{P}$ be k-suitable. Let $\mathcal{T}$ be an iteration tree on $\mathcal{P}$. We say that $\mathcal{T}$ is **suitability strict** iff for every $\alpha < \mathrm{lh}(\mathcal{T})$:

(1) If $[0,\alpha]_{\mathcal{T}}$ does not drop then $M^{\mathcal{T}}_{\alpha}$ is k-suitable.

(2) If $[0,\alpha]_{\mathcal{T}}$ drops and there are trees $\mathcal{U}, \mathcal{V}$ such that $\mathcal{T}{\restriction}\alpha+1 = \mathcal{U} ^\frown \mathcal{V}$, where $\mathcal{U}$ has last model $\mathcal{R}$, $b^{\mathcal{U}}$ does not drop, and there is $i \in [0,k)$ such that $\mathcal{V}$ is based on $[\delta^{\mathcal{R}}_{i-1}, (\delta^{+\omega}_i)^{\mathcal{R}})$, then no $\mathcal{Q} \trianglelefteq M^{\mathcal{T}}_{\alpha}$ is $(i+1)$-suitable.

Let Σ be a (partial) iteration strategy for $\mathcal{P}$. We say that Σ is **suitability strict** iff every tree $\mathcal{T}$ via Σ is suitability strict. ⊣

Definition 5.33. Let $\mathcal{P}$ be k-suitable. We say that $\mathcal{P}$ is **short tree iterable** iff for every normal Γ-guided tree $\mathcal{T}$ on $\mathcal{P}$, we have:

(1) $\mathcal{T}$ is suitability strict.

(2) If $\mathcal{T}$ has limit length and is short then there is b such that $\mathcal{T} ^\frown b$ is a Γ-guided tree.

(3) If $\mathcal{T}$ has successor length then every one-step putative normal extension of $\mathcal{T}$ is an iteration tree.

[62]The "t" is for *tame*. While $\mathcal{Q}$ might not be tame, $\mathrm{o}(\mathcal{N})$ is a strong cutpoint of $\mathcal{Q}$.

Let $\mathcal{P}$ be short tree iterable. The **short tree strategy** $\Psi^{\mathrm{sh}}_{\mathcal{P}}$ for $\mathcal{P}$ is the partial iteration strategy Ψ for $\mathcal{P}$, such that $\Psi(\mathcal{T}) = b$ iff $\mathcal{T}$ is normal and short and $\mathcal{T} \frown b$ is Γ-guided. (By 5.30 this specifies $\Psi^{\mathrm{sh}}_{\mathcal{P}}$ uniquely.) $\dashv$

Lemma 5.34. *Let $\mathcal{N}$ be k-suitable.*

(*1*) *The function $\mathcal{N} \mapsto \Psi^{\mathrm{sh}}_{\mathcal{N}}$, where $\mathcal{N}$ is short-tree iterable, is in $\mathcal{M}$; in fact, $\Psi^{\mathrm{sh}}_{\mathcal{N}}$ is $\Gamma(\{\mathcal{N}, z_0\})$-definable, uniformly in $\mathcal{N}$.*[63]

(*2*) *Suppose there is a suitability strict normal $(\omega, \omega_1 + 1)$-strategy Σ for $\mathcal{N}$. Then $\mathcal{N}$ is short tree iterable and $\Psi^{\mathrm{sh}}_{\mathcal{N}} \subseteq \Sigma$. Moreover, for any $\mathcal{T}$ via Σ, $\mathcal{T}$ is via $\Psi^{\mathrm{sh}}_{\mathcal{N}}$ iff for every limit $\lambda < \mathrm{lh}(\mathcal{T})$, $\mathcal{Q}(\mathcal{T}, b)$ exists where $b = [0, \lambda)_{\mathcal{T}}$.*

Proof. Part (1) follows from the admissibility of $\mathcal{M}|\alpha_0$.

Consider (2). Let $\mathcal{T}$ on $\mathcal{N}$ be normal, of limit length, via both Σ and $\Psi^{\mathrm{sh}}_{\mathcal{N}}$. Let $b = \Sigma(\mathcal{T})$. It suffices to show that (a) if $\mathcal{Q}(\mathcal{T}, b)$ exists then $\mathcal{T}$ is short, and (b) if $\mathcal{T}$ is short then $b = \Psi^{\mathrm{sh}}_{\mathcal{N}}(\mathcal{T})$. (Note that if $\mathcal{Q}(\mathcal{T}, b)$ does not exist then $M^{\mathcal{T}}_b$ is k-suitable so $\mathcal{T}$ is maximal.)

Consider (a); suppose $\mathcal{Q} = \mathcal{Q}(\mathcal{T}, b)$ exists. If b does not drop then $M^{\mathcal{T}}_b$ is suitable and $\delta \neq \delta_i(M^{\mathcal{T}}_b)$ for any $i < k$. So $C_\Gamma(M(\mathcal{T})) \vDash$"$\delta$ is not Woodin", so our mouse capturing hypothesis implies that $\mathcal{T}$ is short. So suppose that b drops. We can't have $C_\Gamma(M(\mathcal{T})) \subseteq \mathcal{Q}$, by suitability strictness. If δ is a cutpoint of $\mathcal{Q}$ (and so a strong cutpoint) we can then compare $\mathcal{Q}$ with $\mathrm{Lp}^{\Gamma}_{+}(M(\mathcal{T}))$; since the comparison is above δ, we get that $\mathcal{Q} \trianglelefteq \mathrm{Lp}^{\Gamma}_{+}(M(\mathcal{T}))$, so $\mathcal{T}$ is short. So suppose δ is not a cutpoint of $\mathcal{Q}$. Let $E \in \mathbb{E}_{+}(\mathcal{Q})$ be least such that $\kappa = \mathrm{crit}(E) < \delta$ and let $\mathcal{T}'$ be the normal tree given by $\mathcal{T} \frown \langle b, E \rangle$. Then $M^{\mathcal{T}'}_\infty \vDash$"$\kappa$ is a limit of Woodins", so $b^{\mathcal{T}'}$ drops and $C_\Gamma(M(\mathcal{T})) \not\subseteq M^{\mathcal{T}'}_\infty$ (by suitability strictness). Also $M^{\mathcal{T}'}_\infty \vDash$"$\delta$ is Woodin" and δ is a cutpoint of $M^{\mathcal{T}'}_\infty$. So $M^{\mathcal{T}'}_\infty = \mathcal{Q}^{\Gamma}_{\mathrm{t}}(M(\mathcal{T}))$ exists, so $\mathcal{T}$ is short.

Consider (b). Since $\mathcal{T}$ is short, $\mathcal{Q} = \mathcal{Q}(\mathcal{T}, b)$ exists. We claim that $\mathcal{T} \frown b$ is Γ-guided, which suffices. For it's easy to reduce to the case that δ is not a cutpoint of $\mathcal{Q}$. Let $\mathcal{T}'$ be as above, let $\lambda = \mathrm{lh}(\mathcal{T})$ and $\alpha = \mathrm{pred}^{\mathcal{T}'}(\lambda + 1)$. Let $M^{*\mathcal{T}'}_{\lambda+1} = M^{\mathcal{T}}_\alpha|\gamma$. Then $M^{\mathcal{T}}_\alpha|\gamma \vDash$"$\kappa$ is a limit of cutpoints". It follows that $\mathcal{T}\upharpoonright[\alpha, \mathrm{lh}(\mathcal{T}))$ can be considered an above-κ, normal tree on $M^{\mathcal{T}}_\alpha|\gamma$, and the iterability of the phalanx $\Phi(\mathcal{T}) \frown (\mathcal{Q}, \delta)$ reduces to the above-κ iterability of $M^{\mathcal{T}}_\alpha|\gamma$, which reduces to the above-δ iterability of $M^{\mathcal{T}'}_\infty$ (because of the existence of $i^{\mathcal{T}'}_{\alpha,\lambda+1}$). But $M^{\mathcal{T}'}_\infty \trianglelefteq \mathrm{Lp}^{\Gamma}_{+}(M(\mathcal{T}))$, so we are done. $\square$

[63] But it seems that we might have $\Psi^{\mathrm{sh}}_{\mathcal{N}} \notin \mathcal{M}|\alpha_0$.

Definition 5.35. Let $A \in \mathfrak{P}(\mathbb{R}) \cap \mathcal{M}$. We define the phrase $\mathcal{T}$ **respects** A as in [19], except that we also require that $\mathcal{T}$ be suitability strict (and making any obvious adaptations to our setting). We define $\mathcal{N}$ **is normally** A**-iterable** as in [19], except that we also require that $\mathcal{N}$ be short tree iterable. Using these definitions, we then define **(almost, locally)** A**-iterable** as in [19]. ⊣

Lemma 5.36. *The analogue of* [19, Lemma 1.9.1] *holds.*

Proof. This is mostly an immediate generalization. The proof in [19] can be run inside $\mathcal{J}(\mathcal{M})$ (in fact, inside $\mathcal{M}$, since $\mathcal{M} \models \mathsf{DC}_{\mathbb{R}}$). Use suitability strictness to see that, for example, in the comparison of $\mathcal{R}|0$ with $\mathcal{N}|0$ (notation as in [19]), no tree drops on its main branch. □

Remark 5.37. We make a further observation on the comparison above. Let $(\mathcal{T},\mathcal{U})$ be the Γ-guided portion of the comparison of, for example, $(\mathcal{R}|0, \mathcal{N}|0)$. Let $\lambda < \mathrm{lh}(\mathcal{T},\mathcal{U})$ be a limit; suppose $\mathcal{T}\upharpoonright\lambda$ is cofinally non-padded. So $\mathcal{Q} = \mathcal{Q}(\mathcal{T}\upharpoonright\lambda, [0,\lambda]_{\mathcal{T}})$ exists. Then in fact, $\delta(\mathcal{T}\upharpoonright\lambda)$ is a strong cutpoint of $\mathcal{Q}$. For otherwise, by the proof of 5.34, $[0,\lambda]_{\mathcal{T}}$ drops in a manner which cannot be undone; i.e., for all $\alpha \geq \lambda$, $[0,\alpha]_{\mathcal{T}}$ drops, a contradiction. Similar remarks pertain to genericity iterations on k-suitable models.

Lemma 5.38. *Let $A \in \mathcal{M} \cap \mathfrak{P}(\mathbb{R})$. Then for a cone of $s \in \mathbb{R}$ there is an ω-suitable, A-iterable premouse over $(\mathfrak{M}, s)$.*

Proof. The following account is based on the sketch given in [19, 1.12.1].[64] We give full detail here, since the proof is rather involved and the possibility of non-tame mice was not covered explicitly in [19], and moreover, comparing our proof with the remarks in [19, Footnote 12], we will not manage to establish the full Dodd-Jensen property for the iteration strategy we construct, but we will verify a version of said property which suffices for our purposes.

[64] We are using g-organized mice as our mice over reals. The authors believe that, had we used a hierarchy Z of mice over reals more closely related to Θ-g-organized mice, then the proof in [2, §7(?)] could be adapted to work in the present context. (One needs to define Z such that Θ-g-organized mice can be realized as derived models of Z-mice, in a reasonably level-by-level manner.) Such a proof would have the advantage of providing some extra information. However, one would need to define and use the relevant Prikry forcing, so it seems to be more work overall, and our approach also has the advantage that it is less dependent on the precise hierarchy of mice over reals that is used. One might alternatively start out like [2, §7(?)], but instead of using Prikry forcing, finish more like in our present proof.

Say that a set of reals constituting a counterexample to the theorem is **Γ-bad**. Suppose there is a Γ-bad set. For other pointclasses $\bar{\Gamma}$ we define **$\bar{\Gamma}$-bad** analogously.

Let $\zeta_0 < \alpha_0$ and $\psi_\Omega \in \Sigma_1(\mathcal{L}^-)$ be such that Ω^* is definable over $\mathcal{M}|\zeta_0$ from z_0 and $\mathcal{M}|(\zeta_0+1) \vDash \psi_\Omega(z_0)$ but $\mathcal{M}|\zeta_0 \vDash \neg\psi_\Omega(z_0)$. Recall there is $\xi+1 \in (\theta, l(\mathcal{M}))$ such that $\mathcal{M}|\xi \vDash \mathsf{ZF}$. So by 4.1 there are $\bar{\alpha}, \bar{\xi}, \bar{\beta}, \bar{\Gamma}, A$ such that:

- $\zeta_0 < \bar{\alpha} < \bar{\xi} < \bar{\beta} < \alpha_0$,
- $\mathcal{M}|\bar{\alpha} \preccurlyeq_1^{\mathbb{R}} \mathcal{M}|\bar{\beta}$ but $\mathcal{M}|\alpha' \not\preccurlyeq_1^{\mathbb{R}} \mathcal{M}|\bar{\alpha}$ for all $\alpha' < \bar{\alpha}$,
- $\Theta^{\mathcal{M}|\bar{\beta}} < \bar{\xi}$,
- $\bar{\Gamma} = \mathrm{r}\Sigma_1^{\mathcal{M}|\bar{\alpha}}$ and $A \in \mathfrak{P}(\mathbb{R})^{\mathcal{M}|\bar{\xi}}$ and $\mathcal{M}|\bar{\xi} \vDash \mathsf{ZF}+$"$A$ is $\bar{\Gamma}$-bad".

As $\mathcal{M}|\bar{\xi} \vDash \mathsf{ZF}$, A really is $\bar{\Gamma}$-bad. We may assume that $\bar{\beta}$ is least such that there are $\bar{\alpha}, \bar{\xi}$ as above (relative to the fixed ζ_0). Then $\bar{\beta} = \bar{\xi}+1$, $\rho_1^{\mathcal{M}|\bar{\beta}} = \mathbb{R}$, $p_1^{\mathcal{M}|\bar{\beta}} = \{\bar{\xi}\}$ and $[\bar{\alpha}, \bar{\beta}]$ is a weak gap of $\mathcal{M}$ (the type $\mathrm{r}\Sigma^{\mathcal{M}|\bar{\beta}}_{1,(\{\bar{\xi}\},z_0)}$ does not reflect, using the choice of ζ_0, z_0). We will show that A is *not* $\bar{\Gamma}$-bad, a contradiction.

Let $\langle A_i \rangle_{i<\omega}$ be a self-justifying system at the end of the gap $[\bar{\alpha}, \bar{\beta}]$, with $A_0 = A$. By AD, in $\mathcal{M}|\bar{\xi}$ there is a cone of reals s such that there is no ω-suitable, A-iterable premouse over $(\mathfrak{M}, s)$. Let $z_1 \geq_T z_0$ be a base for this cone such that for every $i < \omega$ there is $\zeta < \Theta^{\mathcal{M}|\bar{\beta}}$ such that A_i is definable over $\mathcal{M}|\zeta$ from z_1, and a scale on $\mathrm{Th}_{\mathrm{r}\Pi_1}^{\mathcal{M}|\bar{\alpha}}(\mathbb{R})$ is definable over $\mathcal{M}|\bar{\beta}$ from z_1. We write $\overline{\mathrm{Lp}}$ for $\mathrm{Lp}^{\bar{\Gamma}}$. Recalling that z_0 codes $\mathfrak{M}$, it follows that

$$C_{\bar{\Gamma}}(\mathfrak{M}, z_1) = C_{\bar{\Gamma}}(z_1) \subsetneq C_\Gamma(z_1) = C_\Gamma(\mathfrak{M}, z_1).$$

So $\overline{\mathrm{Lp}}(\mathfrak{M}, z_1) \triangleleft \mathrm{Lp}^\Gamma(\mathfrak{M}, z_1)$ and both are super-small, by our mouse capturing hypothesis. Let $\mathcal{P} \triangleleft \mathrm{Lp}^\Gamma(\mathfrak{M}, z_1)$ be least such that $\rho_\omega^{\mathcal{P}} = \omega$ and $\mathcal{P} \ntrianglelefteq \overline{\mathrm{Lp}}(\mathfrak{M}, z_1)$. Let $\Sigma_{\mathcal{P}}$ be the (ω, ω_1+1)-strategy for $\mathcal{P}$. So $\Sigma_{\mathcal{P}} \in (\mathcal{M}|\alpha_0) \backslash (\mathcal{M}|\bar{\alpha})$. Let $z_2 \in \mathbb{R}$ code $\mathcal{P}$, with $z_2 \geq_T z_1$.

We say that a pointclass Λ is **lovely** iff $\Lambda = \mathrm{r}\Sigma_1^{\mathcal{N}}(z_2) \cap \mathfrak{P}(\mathbb{R})$ for some passive $\mathcal{N} \triangleleft \mathcal{M}|\alpha_0$. Let $\langle \Gamma_i \rangle_{i \in [0,9]}$ be lovely pointclasses such that $\bar{\Gamma} \subsetneq \Delta_{\Gamma_9}$ and $(\Sigma_{\mathcal{P}} \restriction \mathrm{HC})^{\mathrm{cd}}$ is Δ_{Γ_9} and for each $i \in [1,9]$, $\Gamma_i \subsetneq \Delta_{\Gamma_{i-1}}$. Working in $\mathcal{M}|\xi$, let T_0 be the tree of a scale for a universal Γ_0 set. By Woodin [23] applied in $\mathcal{M}|\xi$ (where $\mathsf{ZF}+\mathsf{AD}$ holds) there is $z_3 \in \mathbb{R}$ such that $z_2 \leq_T z_3$ and

$$H^* =_{\mathrm{def}} \mathrm{HOD}_{T_0, z_2}^{L_\xi[T_0, z_3]} \vDash \text{"}\Delta_0 \text{ is Woodin"},$$

where $\Delta_0 = \omega_2^{L_\xi[T_0,z_3]}$.

Let $T_i, U_i \in H^*$ be trees projecting respectively to a universal Γ_i set and its complement. Let Δ_i be least such that $V_{\Delta_i}^{H^*}$ is Γ_i-Woodin. Let $\lambda < \xi$ be large and such that $(V_\lambda^{H^*}, \Delta_9)$ is a coarse premouse. Let

$$\pi_H : (H, \Delta) \to (V_\lambda^{H^*}, \Delta_9)$$

be elementary, with $H \in \mathrm{HC}^{H^*}$, $\pi_H \in H^*$, and $z_2, T_i, U_i \in \mathrm{rg}(\pi)$ for each $i \leq 9$ (let $U_0 = \emptyset$). Let $\pi_H(T_i^H, U_i^H) = (T_i, U_i)$. Then by arguments in [1] (using $\mathcal{M}|\xi$ as a background $\mathsf{ZF} + \mathsf{AD}$ model):

Fact 5.39. *In $\mathcal{M}|\alpha_0$ there is a unique $(\omega_1, \omega_1 + 1)$-iteration strategy Λ_H for (H, Δ) such that for each countable successor length tree $\mathcal{T}$ via Λ_H, letting $j = i^{\mathcal{T}}$ and $J = M_\infty^{\mathcal{T}}$, then*

$$p[j(T_8^H)] \subseteq p[T_8] \;\&\; p[j(U_8^H)] \subseteq p[U_8].$$

Moreover, the restriction of Λ_H to HC^{H^*} is the unique π_H-realization strategy in H^*. Further, for $i \geq 1$, $J \models$"$j(T_i^H), j(U_i^H)$ are $\mathrm{Col}(\omega, j(\Delta))$-absolutely complementing". Moreover,

$$C^H =_{\mathrm{def}} C_{\bar{\Gamma}} \restriction V_\Delta^H \in H \;\&\; j(C^H) = C_{\bar{\Gamma}} \restriction V_{j(\Delta)}^J;$$
$$\Omega^H =_{\mathrm{def}} \Omega^* \restriction V_\Delta^H \in H \;\&\; j(\Omega^H) = \Omega^* \restriction V_{j(\Delta)}^J.$$

Let $\mathbb{C} = \langle N_\alpha \rangle_{\alpha \leq \Delta}$ be the maximal $L^{\mathrm{g}\Omega^H}[\mathbb{E}, (\mathfrak{M}, z_1)]$-construction as computed in H (see 2.46). For every $\alpha \leq \Delta$ and $n < \omega$, the $(n, \omega_1, \omega_1{+}1)$-strategy for $\mathfrak{C}_n(N_\alpha)$ given by resurrection and lifting to Λ_H, is a ${}^{\mathrm{g}}\Omega^*$-strategy; this is by and 5.39, 3.43 and properties of the resurrection/lifting maps. So by 2.45, this construction does indeed have length $\Delta + 1$.

Claim 5.40. *There is $\gamma < \Delta$ and $k < \omega$ such that $\rho_{k+1}^{N_\gamma} = \omega$ and $\mathfrak{C}_\omega(N_\gamma)$ is not $(k, \omega_1 + 1)$-iterable in $\mathcal{M}|\bar{\alpha}$.*

Proof. It suffices to see that $\mathbb{C}$ reaches $\mathcal{P}$. We have $z_2, \mathcal{P} \in \mathrm{HC}^H$, and by the definability of $\Sigma_{\mathcal{P}} \restriction \mathrm{HC}$, letting $\Sigma_{\mathcal{P}}^H = \Sigma_{\mathcal{P}} \restriction V_\Delta^H$, we have $\Sigma_{\mathcal{P}}^H \in H$, and $\Sigma_{\mathcal{P}}^H$ is moved correctly by $\Lambda_H \restriction \mathrm{HC}$. It follows that the background extenders used in $\mathbb{C}$ all cohere $\Sigma_{\mathcal{P}}^H$, and so we can apply 3.23 (the stationarity of $\mathbb{C}$ with respect to $\mathcal{P}$). So we just need to rule out the possibility that for some normal tree $\mathcal{T}$ on $\mathcal{P}$ via $\Sigma_{\mathcal{P}}$, with last model $\mathcal{P}'$, $N_\Delta \trianglelefteq \mathcal{P}'$. But because $(\Sigma_{\mathcal{P}} \restriction \mathrm{HC})^{\mathrm{cd}}$ and $(\Omega^*)^{\mathrm{cd}}$ are Δ_{Γ_9} and N_Δ is definable over (V_Δ^H, Ω^H), we have $\mathcal{T} \in C_{\Gamma_9}(V_\Delta^H)$. But $C_{\Gamma_9}(V_\Delta^H) \models$"$\Delta$ is Woodin", so by the universality of N_Δ (see [17, Lemma 11.1]), $\mathcal{T} \notin C_{\Gamma_9}(V_\delta^H)$, contradiction. $\square$

We will now look at the least stage where the construction produces a fine structurally nice mouse which is not iterable in $\mathcal{M}|\bar{\alpha}$. This move, and its relation to producing a mouse with ω Woodins and a suitability strict iteration strategy, is related to, and motivated by, an argument shown to the first author by Steel, in a similar situation, though a different context.

Given a k-sound premouse $\mathcal{N} \in \mathrm{HC}$ and $\zeta \in \mathrm{o}(\mathcal{N})$, we say that $\mathcal{N}$ is **$(\bar{\Gamma}, k, \zeta)$-iterable** iff there is an above-ζ, $(k, \omega_1 + 1)$-iteration strategy for $\mathcal{N}$ in $\mathcal{M}|\bar{\alpha}$. We say that $\mathcal{N}$ is $(\bar{\Gamma}, \zeta)$-iterable iff $\mathcal{N}$ is $(\bar{\Gamma}, m, \zeta)$-iterable, where m is defined in the next paragraph.

By the previous claim, we may let $(\gamma, m, \eta') \in \mathrm{Ord}^3$ be lex-least such that, letting $\mathcal{S} = \mathfrak{C}_m(N_\gamma)$, $\mathcal{S}|\eta'$ is a g-whole cutpoint of $\mathcal{S}$ and

$$\mathcal{R}' =_{\mathrm{def}} \mathrm{cHull}^{\mathcal{S}}_{m+1}(\eta' \cup p^{\mathcal{S}}_{m+1})$$

is η'-sound and not $(\bar{\Gamma}, m, \eta')$-iterable. Let $\pi' : \mathcal{R}' \to \mathcal{S}$ be the uncollapse. (It follows that $\pi'(p^{\mathcal{R}'}_{m+1} \backslash \eta') = p^{\mathcal{S}}_{m+1} \backslash \eta'$. We allow $\eta' < \rho^{\mathcal{S}}_{m+1}$, so we do need to assume η'-soundness explicitly.) It seems that η' could be measurable in $\mathcal{R}'$, which is slightly inconvenient. So we first replace $\mathcal{R}'$ with a slightly larger hull $\mathcal{R}$, and replace η' with a strong cutpoint η of $\mathcal{R}$.

Given a premouse $\mathcal{N}$ and $\eta < \mathrm{o}(\mathcal{N})$, we say that η is **$\mathcal{N}$-finely measurable** iff $\eta = \mathrm{crit}(E)$ for some $\mathcal{N}$-total measure E such that either $E \in \mathbb{E}^{\mathcal{N}}_+$, or $E \in \mathbb{E}^{\mathrm{Ult}(\mathcal{N},F)}_+$ for some $F \in \mathbb{E}^{\mathcal{N}}_+$.

We claim that $\eta' < \min(\rho^{\mathcal{R}'}_m, \rho^{\mathcal{S}}_m)$ and η' is not measurable in H, nor $\mathcal{S}$-finely measurable. For $\rho^{\mathcal{R}'}_m$ is the least ρ such that either $\rho \notin \mathrm{dom}(\pi')$ or $\pi'(\rho) \geq \rho^{\mathcal{S}}_m$, by elementarity. We have $\eta' < \rho^{\mathcal{R}'}_m$ (as otherwise $\mathcal{R}'$ is not $(\bar{\Gamma}, m-1, \eta')$-iterable, which implies that $\mathfrak{C}_{m-1}(N_\gamma)$ is not $(\bar{\Gamma}, m-1, \rho^{N_\gamma}_m)$-iterable, contradicting the minimality of m), so also $\eta' < \rho^{\mathcal{S}}_m$. Since $\eta' < \rho^{\mathcal{S}}_m$, if η' is $\mathcal{S}$-finely measurable then η' is measurable in H. But if $H \vDash$"μ is a normal measure on η'" and $j : H \to \mathrm{Ult}(H, \mu)$ is the ultrapower map, then

$$\mathcal{R}' = \mathrm{cHull}^{j(\mathcal{S})}_{m+1}(\eta' \cup p^{j(\mathcal{S})}_{m+1}),$$

which contradicts the minimality of $j(\eta')$ in $\mathrm{Ult}(H, \mu)$. (The minimality can be computed correctly in H and its Λ_H-iterates by 5.39.)

Now let $\eta = ((\eta')^+)^{\mathcal{S}}$. We claim that $\eta < \rho^{\mathcal{S}}_m$ and $\mathcal{S}|\eta$ is a g-whole strong cutpoint of $\mathcal{S}$ and

$$\mathcal{R} =_{\mathrm{def}} \mathrm{cHull}^{\mathcal{S}}_{m+1}(\eta \cup p^{\mathcal{S}}_{m+1})$$

is η-sound and not $(\bar{\Gamma}, m, \eta)$-iterable. (Therefore $\eta = ((\eta')^+)^{\mathcal{R}}$ is also a strong cutpoint of $\mathcal{R}$.) For suppose $\eta = \rho^{\mathcal{S}}_m$. Then $\pi'(\eta') = \eta'$ because otherwise we contradict the minimality of m, as above. So $\rho_m^{\mathcal{R}'} = ((\eta')^+)^{\mathcal{R}'}$ and η' is not $\mathcal{R}'$-finely measurable. But then any above-η' tree on $\mathcal{R}'$ immediately drops either in model or to degree $\leq m-1$, which contradicts the minimality of (γ, m). In particular, $\eta = ((\eta')^+)^{\mathcal{S}} < \mathrm{o}(\mathcal{S})$, so $\mathcal{S}|\eta$ is g-whole, and since η' is not $\mathcal{S}$-finely measurable, η is a strong cutpoint of $\mathcal{S}$. Clearly $\mathcal{R}$ is η-sound. If $\pi'(\eta') > \eta'$ then $\eta \leq \pi'(\eta') < \rho_m^{\mathcal{S}}$, which easily gives that $\mathcal{R}$ is not $(\bar{\Gamma}, m, \eta)$-iterable. If $\pi'(\eta') = \eta'$ then η' is not $\mathcal{R}'$-finely measurable, which implies that $\mathcal{R}'$ is not $(\bar{\Gamma}, m, ((\eta')^+)^{\mathcal{R}'})$-iterable, so $\mathcal{R}$ is not $(\bar{\Gamma}, m, \eta)$-iterable.

Let $\pi_0 : \mathcal{R} \to \mathcal{S}$ be the uncollapse embedding. Let $\Sigma_{\mathcal{R}}$ be the above-η, $(m, \omega_1, \omega_1 + 1)$-strategy for $\mathcal{R}$ given by resurrection and lifting to Λ_H, taking π_0 as the base lifting map. Let $\mathcal{T}$ be on $\mathcal{R}$ via $\Sigma_{\mathcal{R}}$ and $\lambda < \mathrm{lh}(\mathcal{T})$, and let $\mathcal{U}$ be the lifted tree on H. Write $\mathbb{C}_\lambda = i^{\mathcal{U}}_{0,\lambda}(\mathbb{C})$. Let $n = \deg^{\mathcal{T}}(\lambda)$. Let $(\gamma^{\mathcal{T}}_\lambda, \mathcal{S}^{\mathcal{T}}_\lambda, \pi^{\mathcal{T}}_\lambda)$ be the $(\gamma', \mathcal{S}', \pi')$ produced by lifting/resurrection such that $\gamma' \leq i^{\mathcal{U}}_{0,\lambda}(\gamma)$ and $\mathcal{S}' = \mathfrak{C}_n(N^{\mathbb{C}_\lambda}_{\gamma'})$ and $\pi' : M^{\mathcal{T}}_\lambda \to \mathcal{S}'$ is the lifting map. (In particular, $\pi^{\mathcal{T}}_\lambda$ is a weak n-embedding, and $\gamma^{\mathcal{T}}_\lambda = i^{\mathcal{U}}_{0,\lambda}(\gamma)$ iff $[0,\lambda]_{\mathcal{T}}$ does not drop in model. Here if $[0,\lambda]_{\mathcal{T}}$ does not drop in model, the codomain of π_λ is $i^{\mathcal{U}}_{0,\lambda}(\mathcal{S})$, not $i^{\mathcal{U}}_{0,\lambda}(\mathcal{R})$.)

Let $\mathcal{T}$ be an above-η normal tree on $\mathcal{R}$, of countable limit length. Let b be a $\mathcal{T}$-cofinal branch. Let $\mathcal{Q}_b = \mathcal{Q}(\mathcal{T}, b)$. Then $k(\mathcal{T}, b)$ denotes ω if $\mathcal{Q}_b \triangleleft M^{\mathcal{T}}_b$, and denotes $\deg^{\mathcal{T}}(\lambda)$ otherwise. And $\Phi_{\mathcal{Q}}(\mathcal{T}, b)$ denotes the phalanx $\Phi(\mathcal{T}) \frown (\mathcal{Q}_b, k)$, where $k = k(\mathcal{T}, b)$. (In the phalanx notation, k denotes the base degree corresponding to $\mathcal{Q}_b$.) We say that b is **$\bar{\Gamma}$-verified for $\mathcal{T}$** iff $\Phi_{\mathcal{Q}}(\mathcal{T}, b)$ is normally $(\omega_1 + 1)$-iterable in $\mathcal{M}|\bar{\alpha}$.

Claim 5.41. *Let $\mathcal{T}$ be normal on $\mathcal{R}$ via $\Sigma_{\mathcal{R}}$, of length $\lambda + 1$ for some limit $\lambda < \omega_1$. Suppose that $\mathfrak{P} =_{\mathrm{def}} \Phi_{\mathcal{Q}}(\mathcal{T}\upharpoonright\lambda, b)$ is not normally (ω_1+1)-iterable in $\mathcal{M}|\bar{\alpha}$. Let $M_\lambda = M^{\mathcal{T}}_\lambda$, $b = b^{\mathcal{T}}$, $\mathcal{Q} = \mathcal{Q}(\mathcal{T}\upharpoonright\lambda, b)$, $k = k(\mathcal{T}\upharpoonright\lambda, b)$, $\delta = \delta(\mathcal{T}\upharpoonright\lambda)$ and $M_{\mathcal{T}} = M(\mathcal{T}\upharpoonright\lambda)$. Then either:*

(*i*) *δ is a strong cutpoint of $\mathcal{Q} = M_\lambda$, b does not drop in model or degree and $\mathcal{Q}||(\delta^+)^{\mathcal{Q}} = \overline{\mathrm{Lp}}_+(M_{\mathcal{T}})$; or*

(*ii*) *δ is not a cutpoint of $\mathcal{Q}$, and letting $E \in \mathbb{E}^{\mathcal{Q}}_+$ be such that* $\mathrm{crit}(E) < \delta <$ $\mathrm{lh}(E)$*, with* $\mathrm{lh}(E)$ *minimal, and letting $\mathcal{T}^+$ be the normal tree $\mathcal{T} \frown \langle E \rangle$, then $b^{\mathcal{T}^+}$ does not drop in model or degree, and $\mathcal{Q}||\mathrm{lh}(E) = \overline{\mathrm{Lp}}_+(M_{\mathcal{T}})$.*

Proof. Let $(\gamma_\lambda, \mathcal{S}_\lambda, \pi_\lambda) = (\gamma^{\mathcal{T}}_\lambda, \mathcal{S}^{\mathcal{T}}_\lambda, \pi^{\mathcal{T}}_\lambda)$. Suppose δ is a cutpoint (hence strong cutpoint) of $\mathcal{Q}$. Because δ is a cutpoint, the difficulty in iterating $\mathfrak{P}$ gives that $\mathcal{Q}$ is not $(\bar{\Gamma}, k, \delta)$-iterable. Because δ is a strong cutpoint and by standard fine structure, $\mathcal{Q} \trianglelefteq \mathrm{Lp}^{\Gamma}_+(M_{\mathcal{T}})$.

We leave the proof that $\mathcal{Q} = M_\lambda$ to the reader; assume this. We show that b does not drop in model or degree; suppose otherwise. We have

$$\mathcal{Q} = \mathrm{Hull}^{\mathcal{Q}}_{k+1}(\delta \cup p^{\mathcal{Q}}_{k+1}).$$

and $(\gamma_\lambda, k) <_{\mathrm{lex}} (i^{\mathcal{U}}_{0,\lambda}(\gamma), m)$ and $p^{\mathcal{S}_\lambda}_{k+1} = \pi_\lambda(p^{\mathcal{Q}}_{k+1})$ and (by the commutativity between the copy and iteration maps after the last drop) and

$$\mathrm{rg}(\pi_\lambda) \subseteq \mathcal{R}^* =_{\mathrm{def}} \mathrm{Hull}^{\mathcal{S}_\lambda}_{k+1}(\pi_\lambda(\delta) \cup p^{\mathcal{S}_\lambda}_{k+1}).$$

Let $\mathcal{R}'$ be the transitive collapse of $\mathcal{R}^*$ and let $\sigma : \mathcal{Q} \to \mathcal{R}'$ be the obvious map, a weak k-embedding with $\sigma(\delta) = \pi_\lambda(\delta)$. So σ lifts above-δ trees on $\mathcal{Q}$ to above-$\pi_\lambda(\delta)$ trees on $\mathcal{R}'$. Therefore $\mathcal{R}'$ is not $(\bar{\Gamma}, k, \pi_\lambda(\delta))$-iterable. But $\mathcal{R}'$ is $\pi_\lambda(\delta)$-sound, as there are generalized $(k+1)$-solidity witnesses for $(\mathcal{S}_\lambda, p^{\mathcal{S}_\lambda}_{k+1})$ in $\mathrm{rg}(\pi_\lambda)$ (by commutativity as before). This contradicts the minimality of $(i^{\mathcal{U}}_{0,\lambda}(\gamma), m)$ in $M^{\mathcal{U}}_\lambda$.

So $b^{\mathcal{T}}$ does not drop. One can show $\mathcal{Q}||(\delta^+)^{\mathcal{Q}} \trianglelefteq \overline{\mathrm{Lp}}_+(M_{\mathcal{T}})$ much as above. But $\mathcal{Q} \ntrianglelefteq \overline{\mathrm{Lp}}_+(M_{\mathcal{T}})$, as $\mathcal{Q}$ is not $(\bar{\Gamma}, k, \delta)$-iterable. So $\mathcal{Q}||(\delta^+)^{\mathcal{Q}} = \overline{\mathrm{Lp}}_+(M_{\mathcal{T}})$, as required.

Now suppose δ is not a cutpoint of $\mathcal{Q}$. Suppose that $b^{\mathcal{T}^+}$ drops in model or degree. Since δ is a strong cutpoint of $M^{\mathcal{T}^+}_\infty$, then as before, by choice of (γ, m), $M^{\mathcal{T}^+}_\infty$ is $(\bar{\Gamma}, j, \delta)$-iterable, where $j = \deg^{\mathcal{T}^+}(M^{\mathcal{T}^+}_\infty)$. Therefore, letting $\kappa = \mathrm{crit}(E)$ and $\xi = \lambda + 1$, $M^{*\mathcal{T}^+}_\xi$ is $(\bar{\Gamma}, j, \kappa)$-iterable (we can copy trees using i_E). But κ is a cutpoint of $M^{*\mathcal{T}^+}_\xi$. So $\mathcal{T}^+ = (\mathcal{T}\upharpoonright\chi + 1) \frown \mathcal{T}'$, where $\chi = \mathrm{pred}^{\mathcal{T}}(\xi)$ and $\mathcal{T}'$ is an above-κ, j-maximal tree on $M^{*\mathcal{T}^+}_\xi$. Thus, the iterability of $\mathfrak{P}$ can be reduced to the above-κ iterability of $M^{*\mathcal{T}^+}_\xi$. Therefore $\mathfrak{P}$ is iterable in $\mathcal{M}|\bar{\alpha}$, a contradiction. So $b^{\mathcal{T}^+}$ does not drop. We then get $\mathcal{Q}||\mathrm{lh}(E) = \overline{\mathrm{Lp}}_+(M_{\mathcal{T}})$ by the arguments just given. $\square$

Claim 5.42. *Let $\mathcal{T}$ be a normal tree on $\mathcal{R}$, via $\Sigma_{\mathcal{R}}$, of countable limit length. Then there is at most one branch $\bar{\Gamma}$-verified for $\mathcal{T}$. However, the following partial strategy Ψ is not an above-η, (m, ω_1)-strategy for $\mathcal{R}$: Given $\mathcal{T}$, let $\Psi(\mathcal{T})$ be the unique branch which is $\bar{\Gamma}$-verified for $\mathcal{T}$.*

Proof. Uniqueness follows from the usual comparison and fine structural arguments, using the η-soundness of $\mathcal{R}$. If existence holds then by uniqueness and because $\mathcal{M}|\bar{\alpha}$ is admissible, $\mathcal{R}$ is $(\bar{\Gamma}, \eta)$-iterable, contradiction. $\square$

Definition 5.43. We define the term $\bar{\Gamma}$**-k-suitable** analogously to *k-suitable* (cf. 5.28), but with $\bar{\Gamma}$ replacing Γ. We likewise define $\bar{\Gamma}$**-A-iterable** and $\bar{\Gamma}$**-suitability strict**. Let R be $\bar{\Gamma}$-ω-suitable with $z_1 \in R$. Then σ_i^R denotes the $\mathrm{Col}(\omega, \delta_i^R)$-term capturing A_i over R (see [1]). Let Q be a structure and $\pi : Q \to P$. We say that π is an $\vec{A}$**-embedding** iff π is Σ_1-elementary and $\sigma_i^R \in \mathrm{rg}(\pi)$ for all $i < \omega$. ⊣

Claim 5.44. *(i) $\mathcal{S}$ has infinitely many Woodins δ such that $\eta < \delta < \rho_m^{\mathcal{S}}$. Let δ_ω be the supremum of the first ω-many and let $\mathcal{N}$ be the translation of $\mathcal{S}|\delta_\omega$ to a* g*-organized spm over $\widehat{\mathcal{S}|\eta}$ (translated as in 5.27). Then (ii) $\mathcal{N}$ is $\bar{\Gamma}$-ω-suitable.*

Proof. We will construct a $\bar{\Gamma}$-ω-suitable premouse which is an initial segment of a $\Sigma_{\mathcal{R}}$-iterate of $\mathcal{R}$. This is by applying Claim 5.42 and an obvious generalization thereof, in tandem with Claim 5.41, up to ω many times. So let $\mathcal{T}_0$ on $\mathcal{R}_0 = \mathcal{R}$ be via $\Sigma_{\mathcal{R}}$ (so above $\delta_{-1} =_{\mathrm{def}} \eta$), witnessing the failure of "existence" in 5.42, with $\mathcal{T}_0$ of minimal length. Let $\delta_0 = \delta(\mathcal{T}_0)$. Let $b = \Sigma_{\mathcal{R}}(\mathcal{T}_0)$. So 5.41 applies to $\Phi_{\mathcal{Q}}(\mathcal{T}_0, b)$. Use notation as there, so $\mathcal{T} = \mathcal{T}_0 \frown b$ and $\delta = \delta_0$.

Suppose first that 5.41(ii) holds. Let $\kappa = \mathrm{crit}(E)$. Since E overlaps δ and $b^{\mathcal{T}^+}$ does not drop in model or degree, κ is a limit of Woodins of $M_\infty^{\mathcal{T}^+}$, and $\eta < \kappa < \delta < \rho_m(M_\infty^{\mathcal{T}^+})$ (recall we arranged that η is a strong cutpoint of $\mathcal{R}$). And $M_\infty^{\mathcal{T}^+}$ is not $(\bar{\Gamma}, \delta)$-iterable. Now let δ_ω^* be the supremum of the first ω-many Woodins of $M_\infty^{\mathcal{T}^+}$ above η. Let ζ be least such that $\delta_\omega^* < \mathrm{lh}(E_\zeta^{\mathcal{T}})$. So $M_\infty^{\mathcal{T}^+}|\delta_\omega^* = M_\zeta^{\mathcal{T}}|\delta_\omega^*$. Note δ_ω^* is a strong cutpoint of $M_\zeta^{\mathcal{T}}$ and $\zeta \in b^{\mathcal{T}^+}$, so $[0,\zeta]_{\mathcal{T}}$ does not drop in model or degree. Therefore $M_\zeta^{\mathcal{T}}$ is not $(\bar{\Gamma}, \delta_\omega^*)$-iterable. Now let $\mathcal{U}$ be the lifted tree, via Λ_H, on H. Then $\eta < \pi_\zeta^{\mathcal{T}}(\delta_\omega^*) < \rho_m(\mathcal{S}_\zeta^{\mathcal{T}})$ and $\pi_\zeta^{\mathcal{T}}(\delta_\omega^*)$ is the sup of the first ω Woodins of $\mathcal{S}_\zeta$ above η, and $\mathcal{S}_\zeta$ is not $(\bar{\Gamma}, \pi_\zeta^{\mathcal{T}}(\delta_\omega^*))$-iterable. By the elementarity of $i_{0,\zeta}^{\mathcal{U}}$, this gives (i).

We now verify condition (c) of $\bar{\Gamma}$-ω-suitability. Let $\kappa \geq \eta$ be a cutpoint of $\mathcal{S}|\delta_\omega$ with $\eta \leq \kappa$. Let $\mathcal{C}_\kappa$ be the κ-core of $\mathcal{S}$. We claim that $(*)$ $\mathcal{C}_\kappa$ is not $(\bar{\Gamma}, \kappa)$-iterable. For let $\xi \in b^{\mathcal{T}}$ be least such that $\pi_\xi(\mathrm{lh}(E_\xi^{\mathcal{T}})) > i_{0,\xi}^{\mathcal{U}}(\kappa)$. Let $\bar{\kappa}$ be the least such that $\pi_\xi(\bar{\kappa}) \geq i_{0,\xi}^{\mathcal{U}}(\kappa)$. Then $\nu(E_\alpha^{\mathcal{T}}) \leq \bar{\kappa}$ for all $\alpha + 1 \leq_{\mathcal{T}} \xi$, and $\bar{\kappa}$ is a cutpoint of $M_\xi^{\mathcal{T}}$ (as κ is a cutpoint of $\mathcal{S}$). Therefore $M_\xi^{\mathcal{T}}$ is not $(\bar{\Gamma}, \bar{\kappa})$-iterable, and

$$\mathrm{rg}(\pi_\xi) \subseteq \mathrm{Hull}_{m+1}^{\mathcal{S}_\xi}(i_{0,\xi}^{\mathcal{U}}(\kappa) \cup p_{m+1}^{\mathcal{S}_\xi}),$$

so $i_{0,\xi}^{\mathcal{U}}(\mathcal{C}_\kappa)$ is not $(\bar{\Gamma}, i_{0,\xi}^{\mathcal{U}}(\kappa))$-iterable, giving $(*)$.

Now let $\mathcal{S}|\kappa$ be a g-whole strong cutpoint of $\mathcal{S}|\delta_\omega$. By the choice of γ, we have $\mathcal{S}|(\kappa^+)^{\mathcal{S}} \trianglelefteq \overline{\mathrm{Lp}}_+(\mathcal{S}|\kappa)$. But letting $\mathcal{C}_{\kappa+1}$ be the $(\kappa+1)$-core of $\mathcal{S}$, by $(*)$, we have $\overline{\mathrm{Lp}}_+(\mathcal{S}|\kappa) \triangleleft \mathcal{C}_{\kappa+1}$. Condition (c) follows.

We now verify condition (d). Let $\eta \leq \xi < \delta_\omega$ with $\mathcal{S} \models$"ξ is not Woodin"; we must show that $C_{\bar{\Gamma}}(\mathcal{S}|\xi) \models$"$\xi$ is not Woodin". We may assume that $\mathcal{S}|\xi$ is g-whole, and by (c), that ξ is not a strong cutpoint of $\mathcal{S}$. Let $F \in \mathbb{E}^{\mathcal{S}}$ be least such that $\mu = \mathrm{crit}(F) \leq \xi < \mathrm{lh}(F)$. Note that μ is a limit of strong cutpoints of $\mathcal{S}|\xi$. So if $\mu = \xi$ then $\mathcal{S}|\xi$ is the Q-structure for ξ, so we are done. So suppose $\mu < \xi$. We may assume that $\mathcal{S}||\mathrm{lh}(F) \models$"$\xi$ is Woodin", because otherwise there is $\mathcal{Q} \triangleleft \mathcal{S}||\mathrm{lh}(F)$ such that $\mathcal{Q}$ is a Q-structure for ξ and ξ is a strong cutpoint of $\mathcal{Q}$, and so $\mathcal{Q} \triangleleft \overline{\mathrm{Lp}}_+(\mathcal{S}|\xi)$ (by choice of γ). Therefore μ is not a cardinal of $\mathcal{S}$. Let $\mathcal{Q} \triangleleft \mathcal{S}$ be least such that $\mathrm{lh}(F) \leq \mathrm{o}(\mathcal{Q})$ and $\rho_\omega^{\mathcal{Q}} < \mu$. Then $\mathcal{Q}$ collapses ξ. Let $\zeta \in [\rho_\omega^{\mathcal{Q}}, \mu)$ be a g-whole strong cutpoint of $\mathcal{Q}$. Then $\mathcal{Q} \triangleleft \overline{\mathrm{Lp}}_+(\mathcal{S}|\zeta)$, so $\mathcal{Q} \in C_{\bar{\Gamma}}(\mathcal{S}|\zeta)$, which suffices. This completes the proof that $\mathcal{S}|\delta_\omega$ is $\bar{\Gamma}$-ω-suitable in this case.

Now suppose that conclusion (a) of Claim 5.41 holds. Let $\mathcal{T}_0^+ = \mathcal{T}_0 \frown b$ and let $\mathcal{R}_1 = M_\infty^{\mathcal{T}_0^+}$. Then $b^{\mathcal{T}_0^+}$ does not drop in model or degree. And δ_0 is a strong cutpoint of $\mathcal{R}_1$, $\mathcal{R}_1$ is δ_0-sound, projects $< \delta_0$, and is not $(\bar{\Gamma}, \delta_0)$-iterable. So the obvious modification of Claim 5.42 applies to $\mathcal{R}_1$ above δ_0. Pick $\mathcal{T}_1$ on $\mathcal{R}_1$, above δ_0, like before. Again apply Claim 5.41. If its conclusion (b) holds proceed as before, and otherwise let $\mathcal{R}_1 = M_\infty^{\mathcal{T}_1^+}$ and pick $\mathcal{T}_2$ on $\mathcal{R}_1$, etc.

If the above process produces $\mathcal{R}_n$ and $\mathcal{T}_n$ for all $n < \omega$, then we get (i) much as before, and note that, letting δ_n be the n^{th} Woodin of $\mathcal{S}$ above η, then $\mathcal{S}$ is not $(\bar{\Gamma}, \delta_n)$-iterable. Part (ii) follows much like before. □

Claim 5.45. *Let $\mathcal{P}$ be $\bar{\Gamma}$-ω-suitable and let $\pi : \mathcal{Q} \to \mathcal{P}$ be an $\vec{A}$-embedding. Then (i) $\mathcal{Q}$ is $\bar{\Gamma}$-ω-suitable and for each $i < \omega$, (ii) $\pi(\sigma_i^{\mathcal{Q}}) = \sigma_i^{\mathcal{P}}$, and (iii) $\mathrm{rg}(\pi)$ is cofinal in $\delta_i^{\mathcal{P}}$.*

Proof. Parts (i) and (ii) are by condensation of term relations for self-justifying-systems; see [1]. Consider (iii). If $\mathrm{rg}(\pi) \cap \delta_i^{\mathcal{P}}$ is bounded in $\delta_i^{\mathcal{P}}$, then we may assume that $\mathrm{crit}(\pi) = \delta_i^{\mathcal{Q}}$, by taking the appropriate hull (cf. the first part of the proof of [19, Lemma 1.16.2]). But then $\mathcal{Q}|\delta_i^{\mathcal{Q}} = \mathcal{P}|\delta_i^{\mathcal{Q}}$, and $\mathcal{P}|\delta_i^{\mathcal{Q}}$ is not $\bar{\Gamma}$-Woodin, but $\mathcal{Q} \models$"$\delta_i^{\mathcal{Q}}$ is Woodin", so $\mathcal{Q}$ is not $\bar{\Gamma}$-ω-suitable, contradiction. □

Definition 5.46. Let $\mathcal{T} = \langle \mathcal{T}_\alpha \rangle_{\alpha \le \gamma}$ be a stack of normal iteration trees. We say that $\mathcal{T}$ is **relevant** iff for every $\alpha < \gamma$, $b^{\mathcal{T}_\alpha}$ does not drop. (Here we allow $\mathcal{T}_\gamma$ to be trivial, and it might drop.) (Recall from 2.39 that a *hod* iteration strategy acts on relevant trees.) $\dashv$

From now on we fix $\mathcal{N}$ as defined in Claim 5.44. Let $\Sigma_{\mathcal{N}}$ be the hod-$(\omega, \omega_1, \omega_1 + 1)$ strategy for $\mathcal{N}$ given by resurrection and lifting to Λ_H. The next claim follows from 5.39.

Claim 5.47. *For any successor length tree $\mathcal{U}$ on H via Λ_H, $i^{\mathcal{U}}(N)$ is $\bar{\Gamma}$-ω-suitable and $i^{\mathcal{U}} \restriction \mathcal{N} : \mathcal{N} \to i^{\mathcal{U}}(\mathcal{N})$ is an $\vec{A}$-embedding.*

Claim 5.48. *$\Sigma_{\mathcal{N}}$ is $\bar{\Gamma}$-suitability strict. Moreover, let $\mathcal{T}$ be via $\Sigma_{\mathcal{N}}$, of successor length, such that $b^{\mathcal{T}}$ does not drop. Then $i^{\mathcal{T}}$ is an $\vec{A}$-embedding.*

Proof. Let $\mathcal{T}$ be via $\Sigma_{\mathcal{N}}$, of successor length. If $b^{\mathcal{T}}$ does not drop, then the lemma's conclusions regarding $M^{\mathcal{T}}_\infty$ and $i^{\mathcal{T}}$ follow from 5.45 and 5.47.

Suppose $b^{\mathcal{T}}$ drops and that $i < \omega$ is as in 5.32(2), but some $\mathcal{R} \trianglelefteq M^{\mathcal{T}}_\infty$ is $\bar{\Gamma}$-$(i+1)$-suitable. For simplicity assume that $\mathcal{T}$ consists of just one normal tree and that $\mathcal{T}$ has minimal possible length. It follows that for every extender E used in $\mathcal{T}$, $\nu(E) < \delta = \delta^{\mathcal{R}}_i$. Let $n = \deg^{\mathcal{T}}(b^{\mathcal{T}})$. Then $\rho_{n+1}(M^{\mathcal{T}}_\infty) < \mathrm{o}(\mathcal{R})$ and $M^{\mathcal{T}}_\infty$ is δ-sound. So let $\mathcal{Q} \trianglelefteq M^{\mathcal{T}}_\infty$ be least such that $\mathcal{R} \trianglelefteq \mathcal{Q}$ and $\rho^{\mathcal{Q}}_\omega \le \delta$. So

$$\mathcal{Q}|(\delta^+)^{\mathcal{Q}} = \mathcal{R}|(\delta^+)^{\mathcal{R}} = \mathrm{Lp}^{\bar{\Gamma}}_+(\mathcal{R}|\delta),$$

$\mathcal{Q} \vDash$"δ is Woodin", $\mathcal{Q}$ is δ-sound and δ is a strong cutpoint of $\mathcal{Q}$. So letting $j < \omega$ be such that $\rho^{\mathcal{Q}}_{j+1} \le \delta < \rho^{\mathcal{Q}}_j$, $\mathcal{Q}$ is not $(\bar{\Gamma}, j, \delta)$-iterable. Let $\mathcal{U}$ be the Λ_H-tree on H given by lifting $\mathcal{T}$. Suppose for simplicity that $\mathcal{Q} = M^{\mathcal{T}}_\infty$. Because of the drop, $\mathcal{S}^{\mathcal{T}}_\infty$ is $(\bar{\Gamma}, j, \pi^{\mathcal{T}}_\infty(\delta))$-iterable, so $\mathcal{Q} = M^{\mathcal{T}}_\infty$ is $(\bar{\Gamma}, j, \delta)$-iterable, contradiction. If $\mathcal{Q} \triangleleft M^{\mathcal{T}}_\infty$ it is similar.[65] $\square$

Definition 5.49. Let $\mathcal{Q}$ be $\bar{\Gamma}$-ω-suitable. Let Σ be a hod-$(\omega, \omega_1, \omega_1 + 1)$-strategy for $\mathcal{Q}$. We say that $(\mathcal{T}, \mathcal{P})$ is a **Σ-pair** iff $\mathcal{T}$ is a countable tree on $\mathcal{Q}$ via Σ, with last model $\mathcal{P}$. We say that a Σ-pair $(\mathcal{T}, \mathcal{P})$ is **non-dropping** iff $b^{\mathcal{T}}$ does not drop. We say that Σ is **$\vec{A}$-good** iff for every non-dropping Σ-pair $(\mathcal{T}, \mathcal{P})$, $\mathcal{P}$ is $\bar{\Gamma}$-ω-suitable and $i^{\mathcal{T}}$ is an $\vec{A}$-embedding. If $(\mathcal{T}, \mathcal{P})$ is a non-dropping Σ-pair, we write $\Sigma^{\mathcal{T}}_{\mathcal{P}}$ for the $(\mathcal{T}, \mathcal{P})$-tail of Σ (that is, $\Sigma^{\mathcal{T}}_{\mathcal{P}}$ is the hod-$(\omega, \omega_1, \omega_1 + 1)$ iteration strategy Λ for $\mathcal{P}$ where $\Lambda(\mathcal{U}) = \Sigma(\mathcal{T}, \mathcal{U})$). $\dashv$

[65] Suppose $M^{\mathcal{T}}_\infty$ is active type 3 and $\nu(E(M^{\mathcal{T}}_\infty)) < \mathrm{o}(\mathcal{Q}) < \mathrm{o}(M^{\mathcal{T}}_\infty)$. Let $E^* \in M^{\mathcal{U}}_\infty$ be a background extender for $\mathcal{S}^{\mathcal{T}}_\infty$ and lift $\mathcal{Q}$ to a model in $\mathrm{Ult}(M^{\mathcal{U}}_\infty, E^*)$.

The following claim is immediate:

Claim 5.50. *Let Σ be a hod-$(\omega, \omega_1, \omega_1 + 1)$-iteration strategy for $\mathcal{Q}$. Let $(\mathcal{T}, P)$ be a non-dropping Σ-pair. If Σ is suitability strict then $\Sigma_P^{\mathcal{T}}$ is suitability strict. If Σ is $\vec{A}$-good then $\Sigma_P^{\mathcal{T}}$ is $\vec{A}$-good.*

Claim 5.51. *Let $\mathcal{Q}$ be $\bar{\Gamma}$-ω-suitable. Then there is at most one suitability strict $\vec{A}$-good hod-$(\omega, \omega_1, \omega_1 + 1)$ iteration strategy for $\mathcal{Q}$.*

Proof. Let Σ, Λ be two such strategies, and let $\mathcal{T}$ be of limit length, via Σ, Λ, such that $b = \Sigma(\mathcal{T}) \neq \Lambda(\mathcal{T}) = c$. We may assume that $\mathcal{T}$ is normal. We can compare the phalanx $\Phi(\mathcal{T}) \frown b$ with the phalanx $\Phi(\mathcal{T}) \frown c$, forming trees $\mathcal{U}, \mathcal{V}$, using Σ, Λ, respectively. The comparison is successful. By suitability strictness, we have $M_\infty^{\mathcal{U}} = \mathcal{P} = M_\infty^{\mathcal{V}}$. By standard fine structure, $b^{\mathcal{U}}$ and $b^{\mathcal{V}}$ do not drop and $M_\infty^{\mathcal{U}} \vDash$"$\delta(\mathcal{T})$ is Woodin". In particular, $\delta(\mathcal{T}) = \delta_k^{\mathcal{P}}$ for some $k < \omega$. Because Σ, Λ are $\vec{A}$-strategies and by 5.45, therefore $\mathrm{rg}(i^{\mathcal{U}}) \cap \mathrm{rg}(i^{\mathcal{V}})$ is unbounded in $\delta_k^{\mathcal{P}}$. But then $\mathrm{rg}(i_b^{\mathcal{T}}) \cap \mathrm{rg}(i_c^{\mathcal{T}})$ is unbounded in $\delta_k^{\mathcal{P}}$, so $b = c$. □

We are now in a position to establish a version of Dodd-Jensen.

Claim 5.52. *Let Σ be an $\vec{A}$-good, suitability strict strategy for $\mathcal{Q}$. Let $(\mathcal{T}, \mathcal{P})$ be a non-dropping Σ-pair.*

(1) Let $\pi : \mathcal{R} \to \mathcal{P}$ be an $\vec{A}$-embedding. Then the π-pullback Λ of $\Sigma_{\mathcal{P}}^{\mathcal{T}}$ is $\vec{A}$-good and suitability strict. Therefore if $\mathcal{R} = \mathcal{Q}$ then $\Lambda = \Sigma$.

(2) Let $\pi : \mathcal{Q} \to \mathcal{P}$ be an $\vec{A}$-embedding. Then for all $\alpha < \mathrm{o}(\mathcal{Q})$, $i^{\mathcal{T}}(\alpha) \leq \pi(\alpha)$.

Proof. The first clause of (1) is proven like 5.48. This together with 5.51 yields the second clause. For (2), the standard Dodd-Jensen proof works; the copying does not break down by (1). □

One can now deduce that $\mathcal{N}$ is $\bar{\Gamma}$-A-iterable, because 5.50 and 5.52 apply to $\mathcal{N}$ and $\Sigma_{\mathcal{N}}$, which is enough Dodd-Jensen for $\Sigma_{\mathcal{N}}$ to apply the proof of [14, Theorem 4.6]. Recall that $\mathcal{N}$ is over $\widehat{\mathcal{S}|\eta}$. Let $g \subseteq \mathrm{Col}(\omega, \mathcal{S}|\eta)$ be $\mathcal{N}$-generic. Let $x \in \mathbb{R} \cap (\mathcal{N}|1)[g]$ code $(\mathcal{N}|\eta, g)$. Then we can reorganize $\mathcal{N}[x]$ as a premouse $\mathcal{N}^*$ over $(\mathfrak{M}, x)$, and $\mathcal{N}^*$ is $\bar{\Gamma}$-ω-suitable and $\bar{\Gamma}$-A-iterable; these facts all follow by S-construction (for g-organized spms; cf. 4.11). But $x \geq_T z_1$, contradicting the choice of z_1. This completes the proof of 5.38. □

Now for simplicity assume $n = 0$ and $\beta = l(\mathcal{M})$ is a limit ordinal; we allow that $\Upsilon^{\mathcal{M}} \neq \emptyset$. Let $p, w_1, W, \Sigma, \langle \beta_i, Y_i, \psi_i \rangle_{i<\omega}$ be as in the proof of 5.17. Claim 5.18 holds. Let $z = w_1$, $G = p$, $\Upsilon = \Upsilon^{\mathcal{M}}$, $U = U^{\mathcal{M}}$ and $U' = U'^{\mathcal{M}}$. Define the language

$$\mathcal{L}^* = \mathcal{L} \cup \{\dot{\beta}_i, \dot{\mathcal{M}}_i\}_{i<\omega} \cup \{\dot{G}, \dot{p}, \dot{W}, \dot{z}, \dot{\Upsilon}, \dot{U}, \dot{U}'\};$$

each symbol in $\mathcal{L}^* \backslash \mathcal{L}$ is a constant symbol. Relative to these definitions, let B_0, $\langle B_0^i \rangle_{i<\omega}$ and $\vec{S} = \langle S_i \rangle_{i<\omega}$ be as in [19].[66] The analogue of [19, Corollary 1.14] holds (the proof should be executed in $\mathcal{J}(\mathcal{M})$, where we have $\langle S_i \rangle_{i<\omega}$, and where $\mathsf{DC}_{\mathbb{R}}$ holds – this allows us to "intersect all the cones" without introducing new reals, and also the resulting iterate $\mathcal{N}$ is in $\mathcal{J}(\mathcal{M})$, hence in $\mathcal{M}$). Regarding [19, Lemma 1.15.1], the overall proof is executed in V, where $\mathcal{M}$ is countable, and so we may take $\bar{\mathcal{M}} = \mathcal{M}$, and we need not take any countable substructure of V. The proper segments of the iteration are all in $\mathcal{M}$. Also see [9] for details on the process of interleaving comparison with genericity iteration.[67] Consider the analogue of [19, Lemma 1.16.2]:

Lemma 5.53. *Let $\mathcal{N}$ be ω-suitable and $\vec{S}$-iterable. Let $\pi \colon \mathcal{Q} \to \mathcal{N}$ be Σ_1-elementary with $\tau_{i,j}^{\mathcal{N}} \in \mathrm{rg}(\pi)$ for all $i, j < \omega$. Then there is some $m < \omega$ such that for all $n \geq m$, $\mathrm{rg}(\pi)$ is cofinal in $\delta_n^{\mathcal{N}}$.*

Proof. The proof mostly follows that of [19, 1.16.2]. But consider the proof of its Claim; we adopt the same notation. Within that proof, consider the proof that $\mathcal{M}^* = \bar{\mathcal{M}}$. We prove this, as things are different. As $\mathcal{M}$ is countable we have $\bar{\mathcal{M}} = \mathcal{M}$ and $\bar{\mathbb{R}} = \mathbb{R}^{\mathcal{M}}$. Let Υ^*, U^*, etc, be $\dot{\Upsilon}^{\mathcal{M}^*}$, $\dot{U}^{\mathcal{M}^*}$, etc. Let $\Upsilon = \Upsilon^{\mathcal{M}}$ and $\Upsilon^- = \Upsilon^{\mathcal{M}^-}$, etc. We have $\rho : \mathcal{M}^- \to \mathcal{M}$ and $\psi^* : \mathcal{H}^* \to \mathcal{H}^-$.

First note that $\Upsilon^* = \Upsilon$, for $\rho \circ \psi^*$ yields order-preserving maps $U^* \to U$ and $U'^* \to U'$. Therefore $cb^{\mathcal{M}^*} = cb^{\mathcal{M}}$. So essentially as in the proof of 5.17, $\mathcal{M}^*$ is a 1-sound hpm over $cb^{\mathcal{M}}$ with $\rho_1^{\mathcal{M}^*} = \omega$ and $p_1^{\mathcal{M}^*} = p$.

By 3.43, as $\rho^* \circ \psi^* \colon \mathcal{H}^* \to \mathcal{H}$ is Σ_1-elementary, we have that $\mathcal{H}^*$ is a $(0, \omega_1 + 1)$-iterable g-organized Ω-pm over $T^{\mathcal{M}^*}$; likewise for $\mathcal{H}^{\mathcal{M}^*|\eta}$ for every η such that $\mathcal{M}^*|\eta$ is relevant. So $\mathcal{M}^*$ is a $(0, \omega_1 + 1)$-iterable Θ-g-organized Ω-pm over $\Upsilon^{\mathcal{M}}$. So we can compare $\mathcal{M}^*$ with $\mathcal{M}$. Because they are both 1-sound and minimal for realizing Σ, $\mathcal{M}^* = \mathcal{M}$. □

[66] As before, we use the symbol $\mathcal{L}^*$ where [19] uses $\mathcal{L}$, and vice versa.

[67] The issue is as follows. Let $\mathcal{T}$ be one of the trees involved in the comparison. Let $\alpha < \mathrm{lh}(\mathcal{T})$; it might be that $[0, \alpha]_{\mathcal{T}}$ drops. But then the usual procedure for choosing the least extender on $\mathbb{E}_+(\mathcal{M}_\alpha^{\mathcal{T}})$ producing a bad extender algebra axiom need not make sense, because in fact, the relevant extender algebra is not even in $M_\alpha^{\mathcal{T}}$.

We modify the statement of [19, Lemma 1.20.1] as follows: Let $\mathcal{Q}$ be ω-suitable, j-sound and j-realizable. We claim that with respect to trees above $\delta^{\mathcal{Q}}_{j-1}$, $\mathcal{Q}$ is short tree iterable, and the conclusions of [19, Lemma 1.20.1] hold, except with (a)(ii) replaced by "$\mathcal{Q}$-to-$\mathcal{P}$ drops", and (b)(ii) replaced by "b drops and $\mathcal{T}\,\hat{}\,b$ is Γ-guided". Let us argue that $\mathcal{Q}$ is short tree iterable above $\delta^{\mathcal{Q}}_{j-1}$. Assume $j = 0$ for simplicity. First note that whenever $\pi : \mathcal{Q} \to \mathcal{N}$ is a 0-realization, the π-pullback $(\Psi^{\mathrm{sh}}_{\mathcal{N}})^\pi$ of $\Psi^{\mathrm{sh}}_{\mathcal{N}}$ is suitability strict. To see this argue like in the proof of 5.48. Then, as in the proof of 5.34, it follows that $(\Psi^{\mathrm{sh}}_{\mathcal{N}})^\pi$ is precisely the short tree strategy for $\mathcal{Q}$. This suffices. Now consider the uniqueness of the branch b described in [19, Lemma 1.20.1(b)], as modified above. Given two such branches b, c, we compare the phalanxes $\Phi(\mathcal{T}\,\hat{}\,b), \Phi(\mathcal{T}\,\hat{}\,c)$, producing trees $\mathcal{U}, \mathcal{V}$. If $\mathcal{T}$ is short then note that both $\mathcal{T}\,\hat{}\,b$ and $\mathcal{T}\,\hat{}\,c$ are Γ-guided, so $b = c$. If $\mathcal{T}$ is maximal then b, c cannot drop; rule out the possibility that, for example, $M^{\mathcal{U}}_\infty \triangleleft M^{\mathcal{V}}_\infty$ and $b^{\mathcal{V}}$ drops, by using suitability strictness.

Let $\Sigma, \mathcal{Q}, (\mathcal{F}, \prec^*), \mathcal{Q}_\infty$ be defined as in [19, §2]. Then $\Sigma, (\mathcal{F}, \prec^*) \in \mathcal{J}(\mathcal{M})$ and the analogue of [19, Lemma 2.1.2] holds, but we mention some points. It seems possible that $\mathcal{Q}_\infty$ be illfounded because $\mathrm{o}(\mathcal{J}(\mathcal{M})) = \mathrm{o}(\mathcal{M}) + \omega$. But $\mathcal{J}(\mathcal{M}) \models$"$\mathcal{Q}_\infty$ is wellfounded in the codes". Standard arguments therefore show that $\mathcal{Q}_\infty | \delta_0^{\mathcal{Q}_\infty}$ is wellfounded (in fact that $\delta_0^{\mathcal{Q}_\infty} \leq \Theta^{\mathcal{M}}$).[68] The latter is enough for the scale construction to go through. The rest of the argument is essentially as in [19]. This completes the proof. □

5.5 Scales analysis within core model induction

We finish by explaining how we use the scale existence theorems in application to the core model induction. Assume $\mathsf{DC}_{\mathbb{R}}$.

Suppose that $\Upsilon =_{\mathrm{def}} (\Omega{\restriction}\mathrm{HC}) \times \{z\}$ is self-scaled for some $z \in \mathbb{R}$, with $z \geq_T a_0$. Then using the scales existence theorems 5.1, 5.22, 5.26 together with 5.16, we get the scales analysis for $\mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R}, \Upsilon)$ from optimal determinacy and super-small mouse capturing hypotheses (that is, through any initial segment of $\mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R}, \Upsilon)$ for which these hypotheses hold).

We have dealt with $\mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R}, \Omega{\restriction}\mathrm{HC}, z)$ instead of $\mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R})$ because we seem to need extra assumptions to obtain the scales analysis from optimal

[68]Recall that at the start of the proof we reduced to the case that $\mathcal{M} \models$"Θ exists". This reduction relied on $\mathcal{M}$ being Θ-g-organized. This seems to be a key point at which there is a problem with the scales analysis for g-organized mice.

assumptions in the latter. We now discuss what we need for this. In application, *if* there are no divergent AD pointclasses, Ω will in fact be *very* nice:

Definition 5.54. Let Γ be a boldface pointclass and $X \subseteq \mathbb{R}$. We say that Γ is an **AD-pointclass** iff AD holds with respect to all sets in Γ. We say that Γ, X are **Wadge compatible** iff A, X are Wadge compatible for every $A \in \Gamma$.

We say that Ω is **very nice** iff $\Upsilon =_{\text{def}} (\Omega{\restriction}\mathrm{HC}) \times \{z\}$ is self-scaled for some $z \in \mathbb{R}$, $\mathcal{J}(\mathrm{HC}, \Upsilon) \models \mathsf{AD}$, and Υ^{cd} is Wadge compatible with every boldface AD-pointclass. ⊣

Remark 5.55. Suppose Ω is very nice and let Υ be as above. We want to see that the scales analysis in $\mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R})$ proceeds from optimal determinacy assumptions. Let $\mathcal{N} \triangleleft \mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R})$ be such that $\mathcal{N} \models \mathsf{AD}$ and $\mathcal{N}$ ends a gap $[\alpha, \beta]$ of $\mathrm{Lp}^{^{\mathrm{G}}\Omega}(\mathbb{R})$, such that $[\alpha, \beta]$ is not strong. Suppose that if $[\alpha, \beta]$ is weak and $\Omega{\restriction}\mathrm{HC} \in \mathcal{N}|\alpha$ then super-small mouse capturing for $\Gamma = \Sigma_1^{\mathcal{N}|\alpha}$ holds on a cone. We claim that one of the scale existence theorems 5.1, 5.17, or 5.26 applies.

For by 5.16 and the mouse capturing hypothesis, we may assume that the gap is admissible, and so weak, and that $\Omega{\restriction}\mathrm{HC} \notin \mathcal{N}|\alpha$, so $\Upsilon^{\mathrm{cd}} \notin \mathcal{N}|\alpha$. We claim that then $\mathcal{J}(\mathcal{N}) \models \mathsf{AD}$, so 5.17 applies. If every set of reals in $\mathcal{J}(\mathcal{N})$ is Wadge below Υ^{cd}, this is because $\mathcal{J}(\mathrm{HC}, \Upsilon) \models \mathsf{AD}$. So suppose otherwise. Let $\mathcal{P} \trianglelefteq \mathcal{N}$ be least such that there is $Z \in \mathcal{J}(\mathcal{P})$ such that $Z \not\leq_W \Upsilon^{\mathrm{cd}}$. If $\mathcal{P} \triangleleft \mathcal{N}$ then $\mathcal{J}_1(\mathcal{P}) \models \mathsf{AD}$, so by the Wadge compatibility given by 5.54, we have $\Omega{\restriction}\mathrm{HC} \in \mathcal{J}(\mathcal{P})$, so $\alpha \leq l(\mathcal{P})$. We claim that $\Omega{\restriction}\mathrm{HC} \notin \mathcal{N}|\beta$. Because Ω is very nice and by 5.14, this is clear if $\mathrm{Th}^{\mathcal{N}|\alpha}_{\mathrm{r}\Pi_1} \leq_W \Upsilon^{\mathrm{cd}}$ or $\mathrm{Th}^{\mathcal{N}|\alpha}_{\mathrm{r}\Sigma_1} \leq_W \Upsilon^{\mathrm{cd}}$ (as very niceness would otherwise yield scales on these sets). Otherwise, by Wadge compatibility, $\Upsilon^{\mathrm{cd}} <_W \mathrm{Th}^{\mathcal{N}|\alpha}_{\mathrm{r}\Sigma_1}$. But then because $\mathcal{N}|\alpha$ is admissible, $\Upsilon^{\mathrm{cd}} \in \mathcal{N}|\alpha$, contradiction. So $\mathcal{P} = \mathcal{N}$. Since $\mathcal{N}$ ends a weak gap, there are sets $X_i \in \mathfrak{P}(\mathbb{R}) \cap \mathcal{N}$ such that $\mathfrak{P}(\mathbb{R}) \cap \mathcal{J}(\mathcal{N})$ is exactly the sets which are projective in $\oplus_{i<\omega} X_i$. It follows that $\mathfrak{P}(\mathbb{R}) \cap \mathcal{J}(\mathcal{N}) \subseteq \mathfrak{P}(\mathbb{R}) \cap \mathcal{J}(\mathrm{HC}, \Upsilon)$, so $\mathcal{J}(\mathcal{N}) \models \mathsf{AD}$ (and so $\Upsilon \in \mathcal{J}(\mathcal{N})$).

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