The fine structure of operator mice

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Abstract

We develop the theory of abstract fine structural *operators* and *operator-premice*. We identify properties, which we require of operator-premice and operators, which ensure that certain basic facts about standard premice generalize. We define *fine condensation* for operators $\mathcal{F}$ and show that fine condensation and iterability together ensure that $\mathcal{F}$-mice have the fundamental fine structural properties including universality and solidity of the standard parameter.

1 Introduction

Given a set X, we write $\mathcal{J}(X)$ for the rud closure of $X \cup \{X\}$. Standard premice are constructed using $\mathcal{J}$ to take steps at successor stages, adding extenders at certain limits. One often wants to generalize this picture, replacing $\mathcal{J}$ with some *operator* $\mathcal{F}$. The resulting structures are $\mathcal{F}$*-premice*, in which $\mathcal{F}$ is used to take steps at successor stages, instead of $\mathcal{J}$.

In this paper, we will define $\mathcal{F}$*-premice* for a fairly wide class of operators $\mathcal{F}$ with nice condensation properties, and develop their basic theory. (We define *operator* precisely in §3.) Versions of this theory have been presented and used by others (see particularly [12] and [10]), but there are some problems with those presentations. Thus, we give here a (hopefully) complete

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development of the theory. We focus on what is new, skipping the parts which are immediate transcriptions of the theory of standard premice. One of the problems just mentioned relates to the preservation of the standard parameter under ultrapower maps; in order to prove the latter it is important that we restrict to *stratified* structures, as one can see in the proof of 2.42. Another problem, discussed in 3.13, relates to the notion *condenses well*; we introduce *condenses finely* as a replacement, and show that works as desired. The complications in the definition of *condenses finely* are motivated by the latter problem and other details mentioned in 3.13, as well as the desire to handle *mouse operators*, as explained in 3.41, and the condensation requirements in the proof of solidity, etc., as seen in 3.36.

This paper was initially written as a component of [6], and the material presented here is used (rather implicitly) in that paper. In the end it seemed better to separate the two papers. However, there is some common ground, and a significant part of the theory in this paper has an analogue in [6, §2] (though things are simpler in the latter). In order to keep both papers reasonably readable, for the most part the common themes are presented in both papers. In some situations proofs are essentially identical, and in these cases we have omitted the proof from one or the other.

We have tried to develop the theory in a manner which is as compatible as possible with the earlier presentations (though in places we have opted for choosing more suggestive notation and terminology over sticking with tradition). Partly because of this, we develop the theory of $\mathcal{F}$-premice abstractly, dealing with operators $\mathcal{F}$ more general than those given by $\mathcal{J}$-structures. This does incur the cost of increasing the complexity somewhat. A reasonable alternative would have been to restrict attention to operators given by $\mathcal{J}$-structures, since all applications known to the authors are of this form. Also, when dealing with $\mathcal{J}$-structures, one can easily formulate – and maybe prove – condensation properties regarding *all* $\mathcal{J}$-initial segments of the model. But the most straightforward analogues for abstract $\mathcal{F}$-mice apply only to $\mathcal{F}$-initial segments of the model.[1] This seems to be a significant deficit for abstract $\mathcal{F}$-mice.[2] On the other hand, aside from making the work

[1] That is, given a reasonably closed $\mathcal{F}$-mouse $\mathcal{M}$, condensation with respect to embeddings $\mathcal{H} \to \mathcal{M}$, or $\mathcal{H} \to \mathcal{F}(\mathcal{M})$, or $\mathcal{H} \to \mathcal{F}(\mathcal{F}(\mathcal{M}))$, etc, but not with respect to $\mathcal{H} \to \mathcal{N}$ when $\mathcal{M} \in \mathcal{N} \in \mathcal{F}(\mathcal{M})$.

[2] For example, strategy mice can either be defined as an instance of the general theory here, or as $\mathcal{J}$-structures. The latter approach is taken in [6], and that approach is more convenient, as it gives us the right notation to prove strong condensation properties like [6,

more general, the abstraction has the advantage of showing what properties of $\mathcal{J}$-structures are most essential to the theory.

The paper proceeds as follows. In §2 we define precursors to $\mathcal{F}$-premice, culminating in *operator premice*. We analyse these structures and cover basic fine structure and iteration theory. In §3, we introduce *operators* $\mathcal{F}$, and *$\mathcal{F}$-premice*, which will be instances of operator premice. We define *fine condensation* for operators; this notion is integral to the paper. (We also discuss in 3.13 the motivation for some of the details of this definition, as this might not be clear.) We then prove, in 3.36, the main result of the paper – that the fundamental fine structural facts (such as solidity of the standard parameter) hold for $\mathcal{F}$-iterable $\mathcal{F}$-premice, given that $\mathcal{F}$ condenses finely. We complete the paper in 3.41 by sketching a proof that mouse operators condense finely.

1.1 Conventions and Notation

We use **boldface** to indicate a term being defined (though when we define symbols, these are in their normal font). Citations such as [6, Lemma 4.1(?)] are used to indicate a referent that may change in time – that is, at the time of writing, [6] is a preprint and its Lemma 4.1 is the intended referent.

We work under ZF throughout the paper, indicating choice assumptions where we use them. Ord denotes the class of ordinals. Given a transitive set M, $\mathrm{o}(\mathcal{M})$ denotes $\mathrm{Ord}\cap M$. We write $\mathrm{card}(X)$ for the cardinality of X, $\mathfrak{P}(X)$ for the power set of X, and for $\theta \in \mathrm{Ord}$, $\mathfrak{P}(<\theta)$ is the set of bounded subsets of θ and $\mathscr{H}_\theta$ the set of sets hereditarily of size $<\theta$. We write $f : X \dashrightarrow Y$ to denote a partial function.

We identify $\in [\mathrm{Ord}]^{<\omega}$ with the strictly decreasing sequences of ordinals, so given $p, q \in [\mathrm{Ord}]^{<\omega}$, $p{\upharpoonright}i$ denotes the upper i elements of p, and $p \trianglelefteq q$ means that $p = q{\upharpoonright}i$ for some i, and $p \triangleleft q$ iff $p \trianglelefteq q$ but $p \neq q$. The default ordering of $[\mathrm{Ord}]^{<\omega}$ is lexicographic (largest element first), with $p < q$ if $p \triangleleft q$.

Given a first-order structure $\mathcal{M} = (X, A_1, \ldots)$ with universe X and predicates, constants, etc, $A_1, \ldots$, we write $\lfloor\mathcal{M}\rfloor$ for X. A **transitive structure** is a first-order structure with with transitive universe. We sometimes blur the distinction between the terms *transitive* and *transitive structure*. For exam-

Lemma 4.1(?)]. If one defines strategy mice as an instance of the general theory here, one would then need to define new notation to refer to arbitrary $\mathcal{J}$-initial segments in order to prove the analogue of [6, Lemma 4.1(?)]. But then one might as well have defined strategy mice as in [6] to begin with. (In fact, this paragraph describes some of the evolution of the present paper and [6].)

ple, when we refer to a transitive structure as being **rud closed**, it means that its universe is rud closed. For $\mathcal{M}$ a transitive structure, $\mathrm{o}(\mathcal{M}) = \mathrm{o}(\lfloor\mathcal{M}\rfloor)$. An arbitrary transitive set X is also considered as the transitive structure (X). We write $\mathrm{trancl}(X)$ for the transitive closure of X.

Given a transitive structure $\mathcal{M}$, we write $\mathcal{J}_\alpha(\mathcal{M})$ for the α^{th} step in Jensen's $\mathcal{J}$-hierarchy over $\mathcal{M}$ (for example, $\mathcal{J}_1(\mathcal{M})$ is the rud closure of $\mathrm{trancl}(\{\mathcal{M}\})$). We similarly use $\mathcal{S}$ to denote the function giving Jensen's more refined $\mathcal{S}$-hierarchy. And $\mathcal{J}(\mathcal{M}) = \mathcal{J}_1(\mathcal{M})$.

We take (standard) **premice** as in [11], and our definition and theory of *strategy premice* is modelled on [11],[1]. Throughout, we define most of the notation we use, but hopefully any unexplained terminology is either standard or as in those papers. For discussion of generalized solidity witnesses, see [13].

Our notation pertaining to iteration trees is fairly standard, but here are some points. Let $\mathcal{T}$ be a putative iteration tree. We write $\leq_\mathcal{T}$ for the tree order of $\mathcal{T}$ and $\mathrm{pred}^\mathcal{T}$ for the $\mathcal{T}$-predecessor function. Let $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ and $\beta = \mathrm{pred}^\mathcal{T}(\alpha+1)$. Then $M^{*\mathcal{T}}_{\alpha+1}$ denotes the $\mathcal{N} \trianglelefteq M^\mathcal{T}_\beta$ such that $M^\mathcal{T}_{\alpha+1} = \mathrm{Ult}_n(\mathcal{N}, E)$, where $n = \deg^\mathcal{T}(\alpha+1)$ and $E = E^\mathcal{T}_\alpha$, and $i^{*\mathcal{T}}_{\alpha+1}$ denotes $i^\mathcal{N}_E$, for this $\mathcal{N}, E$. And for $\alpha + 1 \leq_\mathcal{T} \gamma$, $i^{*\mathcal{T}}_{\alpha+1,\gamma} = i^\mathcal{T}_{\alpha+1,\gamma} \circ i^{*\mathcal{T}}_{\alpha+1}$. Also let $M^{*\mathcal{T}}_0 = M^\mathcal{T}_0$ and $i^{*\mathcal{T}}_0 = \mathrm{id}$. If $\mathrm{lh}(\mathcal{T}) = \gamma + 1$ then $M^\mathcal{T}_\infty = M^\mathcal{T}_\gamma$, etc, and $b^\mathcal{T}$ denotes $[0,\gamma]_\mathcal{T}$. For $\alpha < \mathrm{lh}(\mathcal{T})$, $\mathrm{base}^\mathcal{T}(\alpha)$ denotes the least $\beta \leq_\mathcal{T} \alpha$ such that $(\beta, \alpha]_\mathcal{T}$ does not drop in model or degree. (Therefore either $\beta = 0$ or β is a successor.)

A premouse $\mathcal{P}$ is η-**sound** iff for every $n < \omega$, if $\eta < \rho^\mathcal{P}_n$ then $\mathcal{P}$ is n-sound, and if $\rho^\mathcal{P}_{n+1} \leq \eta$ then letting $p = p^\mathcal{P}_{n+1}$, $p\backslash\eta$ is $(n+1)$-solid for $\mathcal{P}$, and $\mathcal{P} = \mathrm{Hull}^\mathcal{P}_{n+1}(\eta \cup p)$. Here Hull_{n+1} is defined in 2.24.

2 The fine structural framework

In this section, we introduce and analyse an increasingly focused sequence of approximations to $\mathcal{F}$-*premice*. We first define *hierarchical model*, which describes the most basic structure of $\mathcal{F}$-premice. We refine this by defining *adequate model*, adding some semi-fine-structural structural requirements (such as *acceptability*). We then develop some basic facts regarding adequate models and their cardinal structure. From there we can define *potential operator premouse* (*potential opm*) (analogous to a potential premouse); this definition makes new restrictions on the information encoded by the predicates (most significantly that the predicate $\dot{E}$ encodes extenders analogous to those of premice), and adds some pre-fine structural requirements. Using the latter, we can define the central fine structural concepts for potential opms. We then define *Q-operator premouse* (*Q-opm*) by requiring that every proper segment be fully sound, and show that the first-order content of Q-opm-hood is *almost* expressed by a Q-formula.[3] We then define *operator premouse* (analogous to *premouse*). We prove various fine structural facts regarding operator premice, and discuss the basic iterability theory.

In §3, we will introduce *operators* $\mathcal{F}$, and $\mathcal{F}$*-premice*. In an $\mathcal{F}$-premouse $\mathcal{M}$, the predicate $\dot{E}$ is used to encode an extender, $\dot{P}$ to encode auxiliary information given by $\mathcal{F}$ (e.g if $\mathcal{F}$ is an iteration strategy and $\mathcal{T} \in \mathcal{M}$ is a tree according to $\mathcal{F}$, then $\dot{P}$ codes a branch b of $\mathcal{T}$ given by $\mathcal{F}$), $\dot{S}$ to encode the sequence of proper initial segments of $\mathcal{M}$, $\dot{X}$ to encode the extensions of all (not just proper) segments of $\mathcal{M}$, $\dot{cb}$ to refer to the coarse *base* of $\mathcal{M}$ (a coarse, transitive set at the bottom of the structure), and $\dot{cp}$ to refer to a coarse *parameter*.[4] An $\mathcal{F}$-premouse $\mathcal{M}$ is *over* its base $A = \dot{cb}^{\mathcal{M}}$. Here $A \in \mathcal{M}$ and A is in all proper segments of $\mathcal{M}$. When we form fine structural cores, all elements of $A \cup \{A\}$ will be the relevant hulls. But in some contexts we will be interested in hulls which do not include all elements of A.

We now commence with the details.

Definition 2.1. Let Y be transitive. Then $\varrho_Y : Y \to \text{rank}(Y)$ denotes the rank function. And $\hat{Y}$ denotes $\text{trancl}(\{(Y, \omega, \varrho_Y)\})$. For M transitive, we say that M is **rank closed** iff for every $Y \in M$, we have $\hat{Y} \in M$ and $\hat{Y}^{<\omega} \in M$. Note that if M is rud closed and rank closed then $\text{rank}(M) = \text{Ord} \cap M$. $\dashv$

Definition 2.2 (Hulls)**.** Let $\mathcal{L} = \{\dot{B}, \vec{P}, \vec{c}\}$ be a finite first-order language, where $\dot{B}$ is a binary predicate, $\vec{P} = \langle \dot{P}_i \rangle_{i<m}$ is a tuple of unary predicates and $\vec{c} = \langle \dot{c}_i \rangle_{i<n}$ a tuple of constants. Let $\mathcal{N}$ be a first-order $\mathcal{L}$-structure and $B = \dot{B}^{\mathcal{N}}$, etc. Let Γ be a collection of $\mathcal{L}$-formulas with "$x = \dot{c}_i$" in Γ for each $i < n$. Let $X \subseteq \lfloor \mathcal{N} \rfloor$. Then

$$\text{Hull}_\Gamma^{\mathcal{N}}(X) =_{\text{def}} (H, B \cap H^2, P_0 \cap H, \ldots, P_{m-1} \cap H, c_0, \ldots, c_{n-1}),$$

where H is the set of all $y \in \lfloor \mathcal{N} \rfloor$ such that for some $\varphi \in \Gamma$ and $\vec{x} \in X^{<\omega}$, y is the unique $y' \in \mathcal{N}$ such that $\mathcal{N} \models \varphi(\vec{x}, y')$. If $\mathcal{N}$ is transitive, then

[3] As in [1], we consider two cases: type 3, and non-type 3. For example, the property of being a non-type 3 Q-opm is expressed by a Q-formula modulo transitivity and the Pairing Axiom.

[4] E is for *extender*, P for *predicate*, S for *segments*, eX for *extensions*, *cb* for *coarse base*, *cp* for *coarse parameter*.

$\mathcal{C} = \mathrm{cHull}^{\mathcal{N}}_{\Gamma}(X)$ denotes the $\mathcal{L}$ structure which is the transitive collapse of $\mathrm{Hull}^{\mathcal{N}}_{\Gamma}(X)$. (That is, $\lfloor\mathcal{C}\rfloor$ is the transitive collapse of H, and letting $\pi : \lfloor\mathcal{C}\rfloor \to H$ be the uncollapse, $P_i^{\mathcal{C}} = \pi^{-1}(P_i)$, etc.) $\dashv$

Definition 2.3. Let $\mathcal{L}_0$ be the language of set theory expanded by unary predicate symbols $\dot{E}, \dot{P}, \dot{S}$, $\dot{X}$, and constant symbols $\dot{cb}$, $\dot{cp}$. Let $\mathcal{L}_0^+$ be $\mathcal{L}_0$ expanded by constant symbols $\dot{\mu}$, $\dot{e}$. Let $\mathcal{L}_0^- = \mathcal{L}_0 \backslash \{\dot{E}, \dot{P}\}$. $\dashv$

Definition 2.4. A **hierarchical model** is an $\mathcal{L}_0$-structure

$$\mathcal{M} = (\lfloor\mathcal{M}\rfloor ; E, P, S, X, b, p),$$

where $\dot{E}^{\mathcal{M}} = E$, etc, $b = \dot{cb}^{\mathcal{M}}$ and $p = \dot{cp}^{\mathcal{M}}$, such that for some ordinal $\lambda > 0$, the following hold.

1. $\mathcal{M}$ is amenable, $\lfloor\mathcal{M}\rfloor$ is transitive, rud closed and rank closed.
2. (Base, Parameter) $b = \hat{Y}$ for some transitive Y and $p \in \mathcal{J}(b)$; we say that $\mathcal{M}$ is **over** the **(coarse) base** b and has **(coarse) parameter** p.
3. (Segments) $S = \langle S_\xi \rangle_{\xi<\lambda}$ where $S_0 = b$ and for each $\xi \in [1, \lambda)$, S_ξ is a hierarchical model over b with parameter p, with $\dot{S}^{S_\xi} = S{\restriction}\xi$. Let $S_\lambda = \mathcal{M}$.
4. (Continuity) If λ is a limit then $\lfloor\mathcal{M}\rfloor = \bigcup_{\alpha<\lambda} \lfloor S_\alpha \rfloor$.
5. (Extensions) $X\colon \lfloor\mathcal{M}\rfloor \to \lambda$, and $X(x)$ is the least α such that $x \in S_{\alpha+1}$.

Let $l(\mathcal{M})$ denote λ, the **length** of $\mathcal{M}$. For $\alpha \le \lambda$ let $\mathcal{M}|\alpha = S_\alpha$. A hierarchical model $\mathcal{M}$ is a **successor** iff $l(\mathcal{M})$ is a successor $\xi + 1$; in this case let $\mathcal{M}^- = \mathcal{M}|\xi$. If $l(\mathcal{M})$ is a limit, let $\mathcal{M}^- = \mathcal{M}$. We say that $\mathcal{N}$ is an **(initial) segment** of $\mathcal{M}$, and write $\mathcal{N} \trianglelefteq \mathcal{M}$, iff $\mathcal{N} = \mathcal{M}|\alpha$ for some $\alpha \in [1, \lambda]$, and say that $\mathcal{N}$ is a **proper (initial) segment** of $\mathcal{M}$, and write $\mathcal{N} \triangleleft \mathcal{M}$, iff $\mathcal{N} \trianglelefteq \mathcal{M}$ and $\mathcal{N} \neq \mathcal{M}$. (Note that $\mathcal{M}|0 = b \ntrianglelefteq \mathcal{M}$.) We write $E^{\mathcal{M}} = E$, etc.[5] For any transitive Y, let $cb^{\hat{Y}} = \hat{Y}$; so $cb^{\mathcal{M}|\alpha} = \mathcal{M}|0$ for all α. $\dashv$

[5] We opted to use cp instead of p to avoid conflict with notation for standard parameters. We use cb instead of b because to avoid conflict with notation associated to strategy mice. For better readability, we will typically use the variable A to represent $cb^{\mathcal{M}}$.

Definition 2.5. Let $\mathcal{M}$ be a hierarchical model over A.

Let $p \in [\mathrm{o}(\mathcal{M})]^{<\omega}$. If $\mathcal{M}$ is a successor, we say that $\mathcal{M}$ is **$(1,p)$-solid** iff for each $i < \mathrm{lh}(p)$,

$$\mathrm{Th}^{\mathcal{M}}_{\Sigma_1}(cb^{\mathcal{M}} \cup p_i \cup \{p{\restriction}i\}) \in \mathcal{M}.$$

(The language used here is $\mathcal{L}_0$.[6])

We say that $\mathcal{M}$ is **soundly projecting** iff for every successor $\mathcal{N} \trianglelefteq \mathcal{M}$, there is $p \in \mathrm{o}(\mathcal{N})^{<\omega}$ such that $\mathcal{N}$ is $(1,p)$-solid and

$$\mathcal{N} = \mathrm{Hull}^{\mathcal{N}}_{\Sigma_1}(\mathcal{N}^- \cup \{\mathcal{N}^-, p\})$$

We say that $\mathcal{M}$ is **acceptable** iff for every successor $\mathcal{N} \trianglelefteq \mathcal{M}$, for every $\tau \in \mathrm{o}(\mathcal{N}^-)$, if there is some $X \in \mathfrak{P}(A^{<\omega} \times \tau^{<\omega})$ such that $X \in \mathcal{N}\backslash\mathcal{N}^-$ then in $\mathcal{N}$ there is a map $A^{<\omega} \times \tau^{<\omega} \overset{\text{onto}}{\to} \mathcal{N}^-$.

We say that $\mathcal{M}$ is an **adequate model** iff $\mathcal{M}$ an acceptable hierarchical model and every *proper* segment of $\mathcal{M}$ is soundly projecting.

An **adequate model-plus** is an $\mathcal{L}_0^+$-structure $\mathcal{M}$ such that $\mathcal{M}{\restriction}\mathcal{L}_0$ is an adequate model. ⊣

Definition 2.6. Given a language $\mathcal{L}$ extending the language of set theory, an **$\mathcal{L}$-simple-Q-formula** is a formula of the form

$$\varphi(v_0, \ldots, v_{n-1}) \iff \forall x \exists y[x \subseteq y \;\&\; \psi(y, v_0, \ldots, v_{n-1})],$$

for some Σ_1 formula ψ of $\mathcal{L}$. (Here all free variables are displayed; hence, x is not free in ψ.)

Let φ_{pair} be the Pairing Axiom. ⊣

It is easy to see that neither φ_{pair}, nor rud closure, can be expressed, modulo transitivity, by a simple-Q-formula.[7] However:

Lemma 2.7. *There is an $\mathcal{L}_0$-simple-Q-formula φ_{am} such that for all transitive $\mathcal{L}_0$-structures $\mathcal{M}$, $\mathcal{M}$ is an adequate model iff $\mathcal{M} \vDash [\varphi_{\mathrm{pair}} \;\&\; \varphi_{\mathrm{am}}]$.*

[6] For the most part, definability over hierarchical models $\mathcal{M}$ will literally be computed over $\mathfrak{C}_0(\mathcal{M})$ (to be defined later), which will be an $\mathcal{L}_0^+$-structure. But for successors $\mathcal{M}$, we will have $\mathfrak{C}_0(\mathcal{M}) = (\mathcal{M}, \dot{\mu}^{\mathfrak{C}_0(\mathcal{M})}, \dot{e}^{\mathfrak{C}_0(\mathcal{M})})$ and $\dot{\mu}^{\mathfrak{C}_0(\mathcal{M})} = \emptyset = \dot{e}^{\mathfrak{C}_0(\mathcal{M})}$. So in this case, definability over $\mathcal{M}$ (using $\mathcal{L}_0$) will be equivalent to that over $\mathfrak{C}_0(\mathcal{M})$ (using $\mathcal{L}_0^+$).

[7] If $\mathcal{L}$ is a first-order language extending the language of set theory, and X, Y are rud closed transitive $\mathcal{L}$-structures such that $c^X = c^Y$ for each constant symbol $c \in \mathcal{L}$, and $P^X = P^Y$ for each predicate symbol $P \in \mathcal{L}$ with $P \neq \dot{\in}$, then any $\mathcal{L}_0$-Q-formula true in both X, Y is also true in the "union" of X, Y.

Proof Sketch. This is a routine calculation, which we omit. (First find an $\mathcal{L}_0$-Q-formula φ_{rud} such that $[\varphi_{\text{pair}} \ \&\ \varphi_{\text{rud}}]$ expresses rud closure; this uses the the finite basis for rud functions.) $\square$

If $\mathcal{M}$ is an adequate model over A and $\xi < l(\mathcal{M})$ then $\mathcal{M}$ has a map

$$A^{<\omega} \times \xi^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\xi.$$

In fact, by the following lemma, this is true uniformly. Its proof is routine, using the sound-projection of proper segments of $\mathcal{M}$, much like in the proof of the corresponding fact for L.

Lemma 2.8. *There is a Σ_1 formula φ in $\mathcal{L}_0^-$, of two free variables, such that for all A and adequate models $\mathcal{M}$ over A, φ defines a map $F : l(\mathcal{M}) \to \mathcal{M}$, and for $\xi < l(\mathcal{M})$, letting $h_\xi = F(\xi)$, we have*

$$h_\xi : A^{<\omega} \times \xi^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\xi$$

and for all $\alpha \leq \xi$, we have $h_\alpha \subseteq h_\xi$.

Definition 2.9. Given an adequate model $\mathcal{M}$ over A and $\xi < l(\mathcal{M})$, let $h_\xi^{\mathcal{M}}$ be the function h_ξ of the preceding lemma. Let $h^{\mathcal{M}} = \bigcup_{\xi < l(\mathcal{M})} h_\xi^{\mathcal{M}}$. $\dashv$

Remark 2.10. So $h^{\mathcal{M}}$ is $\mathcal{L}_0^- {-} \Sigma_1^{\mathcal{M}}$, uniformly in adequate $\mathcal{M}$, and

$$h^{\mathcal{M}} : A^{<\omega} \times l(\mathcal{M}^-)^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}^-$$

(recall that if $\mathcal{M}$ is a limit then $\mathcal{M}^- = \mathcal{M}$), and if $\mathcal{M}$ is a successor then $h^{\mathcal{M}} \in \mathcal{M}$.

Definition 2.11. Let $\mathcal{M}$ be an adequate model over A and $\lambda = l(\mathcal{M})$. Let $\rho < \mathrm{o}(\mathcal{M})$. Then ρ is an **A-cardinal** of $\mathcal{M}$ iff $\mathcal{M}$ has no map $A^{<\omega} \times \gamma^{<\omega} \overset{\text{onto}}{\to} \rho$ where $\gamma < \rho$. We let $\Theta^{\mathcal{M}}$ denote the least A-cardinal of $\mathcal{M}$, if such exists. We say that ρ is **A-regular** in $\mathcal{M}$ iff $\mathcal{M}$ has no map $A^{<\omega} \times \gamma^{<\omega} \overset{\text{cof}}{\to} \rho$ where $\gamma < \rho$. We say that ρ is an **ordinal-cardinal** of $\mathcal{M}$ iff $\mathcal{M}$ has no map $\gamma^{<\omega} \overset{\text{onto}}{\to} \rho$ where $\gamma < \rho$. We say that ρ is **relevant** iff $\rho \leq \mathrm{o}(\mathcal{M}^-)$. $\dashv$

The next four results are proved just like [6, 2.6–2.9(?)]:

Lemma 2.12. *Let $\mathcal{M}$ be an adequate model over A and $\lambda = l(\mathcal{M}) > \xi > 0$. Let κ be an A-cardinal of $\mathcal{M}$ such that $\kappa \leq \mathrm{o}(\mathcal{M}|\xi)$. Then* $\mathrm{rank}(A) < \kappa \leq \xi$ *and $\kappa = \mathrm{o}(\mathcal{M}|\kappa)$.*

Lemma 2.13. *There is a Σ_1 formula φ in $\mathcal{L}_0^-$ such that, for any A and adequate model $\mathcal{M}$ over A, we have the following.*

Suppose $\Theta = \Theta^{\mathcal{M}}$ exists and is relevant. Then:

1. *Θ is the least α such that $\mathfrak{P}(A^{<\omega})^{\mathcal{M}} \subseteq \mathcal{M}|\alpha$.*
2. *$\lfloor \mathcal{M}|\Theta \rfloor$ is the set of all $x \in \mathcal{M}$ such that $\mathrm{trancl}(x)$ is the surjective image of $A^{<\omega}$ in $\mathcal{M}$.*
3. *Over $\mathcal{M}|\Theta$, $\varphi(0, \cdot, \cdot)$ defines a function $G : \Theta \to \mathcal{M}|\Theta$ such that for all $\alpha < \Theta$, we have $G(\alpha) : A^{<\omega} \overset{\mathrm{onto}}{\to} \mathcal{M}|\alpha$.*
4. *Θ is A-regular in $\mathcal{M}$.*

Let $\kappa_0 < \kappa_1$ be consecutive relevant A-cardinals of $\mathcal{M}$. Then:

5. *κ_1 is the least α such that $\mathfrak{P}(A^{<\omega} \times \kappa_0^{<\omega})^{\mathcal{M}} \subseteq \mathcal{M}|\alpha$.*
6. *$\lfloor \mathcal{M}|\kappa_1 \rfloor$ is the set of all $x \in \mathcal{M}$ such that $\mathrm{trancl}(x)$ is the surjective image of $A^{<\omega} \times \kappa_0^{<\omega}$ in $\mathcal{M}$.*
7. *Over $\mathcal{M}|\kappa_1$, $\varphi(\kappa_0, \cdot, \cdot)$ defines a map $G : \kappa_1 \to \mathcal{M}|\kappa_1$ such that for all $\alpha < \kappa_1$, we have $G(\alpha) : A^{<\omega} \times \kappa_0^{<\omega} \overset{\mathrm{onto}}{\to} \mathcal{M}|\alpha$.*
8. *κ_1 is A-regular in $\mathcal{M}$.*

Corollary 2.14. *Let $\mathcal{M}$ be an adequate model over A and let γ be a relevant A-cardinal of $\mathcal{M}$. If γ is a limit of A-cardinals of $\mathcal{M}$ then $\mathcal{M}|\gamma$ satisfies Separation and Power Set. If γ is not a limit of A-cardinals of $\mathcal{M}$ then $\mathcal{M}|\gamma \models \mathsf{ZF}^-$. In particular, $\mathcal{M}|\Theta^{\mathcal{M}} \models \mathsf{ZF}^-$.*

Lemma 2.15. *Let $\mathcal{M}$ be an adequate model over A such that $\Theta^{\mathcal{M}}$ exists and is relevant. Let $\kappa \in [\Theta^{\mathcal{M}}, \mathrm{o}(\mathcal{M}))$ be relevant. Then κ is an A-cardinal of $\mathcal{M}$ iff κ is an ordinal-cardinal of $\mathcal{M}$.*

Definition 2.16. Let $\mathcal{M}$ be an adequate model over A and let $\kappa < \mathrm{o}(\mathcal{M})$. Then $(\kappa^+)^{\mathcal{M}}$ denotes either the least ordinal-cardinal γ of $\mathcal{M}$ such that $\gamma > \kappa$, if there is such, and denotes $\mathrm{o}(\mathcal{M})$ otherwise. By 2.15, if $\mathcal{M}$ is a limit and $\Theta^{\mathcal{M}} \leq \kappa$, then $(\kappa^+)^{\mathcal{M}}$ is the least A-cardinal γ of $\mathcal{M}$ such that $\gamma > \kappa$, if there is such, or is $\mathrm{o}(\mathcal{M})$ otherwise. This applies when $E^{\mathcal{N}} \neq \emptyset$ in 2.19 below. $\dashv$

Definition 2.17. Let $\mathcal{M}$ be an adequate model over A. Then $\rho^{\mathcal{M}}$ denotes the least $\rho \in \mathrm{Ord}$ such that $\rho \geq \omega$ and $\mathfrak{P}(A^{<\omega} \times \rho^{<\omega}) \cap \mathcal{J}(\mathcal{M}) \not\subseteq \mathcal{M}$. $\dashv$

Remark 2.18. We now proceed to the definition of *potential operator-premouse*. We first give some motivation for some of the finer clauses. *Projectum amenability* ensures that we record all essential segments of a potential operator-premouse $\mathcal{N}$ in its history $S^{\mathcal{N}}$. For example, suppose we are forming an n-maximal iteration tree and we wish to apply an extender E to some piece of $\mathcal{N}$, but E is not $\mathcal{N}$-total. Projectum amenability will ensure that there is some $\mathcal{M} \triangleleft \mathcal{N}$ such that E is $\mathcal{M}$-total and $\mathcal{M}$ projects to $\mathrm{crit}(E)$. The property of Σ_1-*ordinal-generation* is used in making sense of fine structure; it ensures for example that the 1st standard parameter p_1 is well-defined. The *stratification* of $\mathcal{N}$ lets us establish facts regarding the preservation of fine structure (including the preservation of p_1, assuming 1-solidity) under degree 0 ultrapower maps. It also ensures that $\mathrm{Hull}^{\mathcal{N}}_{\Sigma_1}(cb^{\mathcal{N}} \cup Y) \preccurlyeq_1 \mathcal{N}$ for any $Y \subseteq \mathcal{N}$. And the existence of $cb^{\mathcal{N}}$-*ordinal-surjections*, together with stratification, will be used in proving that Σ_1-ordinal-generation is propagated under degree 0 ultrapower maps.

Definition 2.19. We say that $\mathcal{N}$ is a **potential operator-premouse (potential opm)** iff $\mathcal{N}$ is an adequate model, over A, such that for every $\mathcal{M} \trianglelefteq \mathcal{N}$,

1. (P-goodness) If $P^{\mathcal{M}} \neq \emptyset$ then $\mathcal{M}$ is a successor and $P^{\mathcal{M}} \subseteq \mathcal{M} \backslash \mathcal{M}^-$.[8]

2. (E-goodness) If $E^{\mathcal{M}} \neq \emptyset$ then $\mathcal{M}$ is a limit and there is an extender F over $\mathcal{M}$ such that, letting $S = S^{\mathcal{M}}$ and $E = E^{\mathcal{M}}$ and $\kappa = \mathrm{crit}(F)$:

   – F is $A^{<\omega} \times \gamma^{<\omega}$-complete for all $\gamma < \kappa$, and

   – the premouse axioms [12, Definition 2.2.1] hold for $(\lfloor \mathcal{M} \rfloor, S, E)$ (so E is the amenable code for F, as in [11]).

   (It follows that $\mathcal{M}$ has a largest cardinal δ, and $\delta \leq i_F(\kappa)$, and $\mathrm{o}(\mathcal{M}) = (\delta^+)^U$ where $U = \mathrm{Ult}(\mathcal{M}, F)$, and $i_F(S{\restriction}(\kappa^+)^{\mathcal{M}}){\restriction}\mathrm{o}(\mathcal{M}) = S$.)

3. If $\mathcal{M}$ is a successor then:

   (a) (Projectum amenability) If $l(\mathcal{M}) > 1$ and $\omega, \alpha < \rho^{\mathcal{M}^-}$ then

$$\mathfrak{P}(A^{<\omega} \times \alpha^{<\omega}) \cap \mathcal{M} \subseteq \mathcal{M}^-.$$

[8] The requirement that $P^{\mathcal{M}} \subseteq \mathcal{M} \backslash \mathcal{M}^-$ does not restrict the information that can be encoded in $P^{\mathcal{M}}$, because given any $X \subseteq \mathcal{M}$, one can always replace it with $\{\mathcal{M}^-\} \times X$.

(b) (A-ordinal-surjections) For every $x \in \mathcal{M}$ there is $\alpha < \mathrm{o}(\mathcal{M})$ a map $A^{<\omega} \times \alpha^{<\omega} \overset{\text{onto}}{\to} x$ in $\mathcal{M}$.

(c) (Σ_1-ordinal-generation) $\mathcal{M} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{M}}(\mathcal{M}^- \cup \{\mathcal{M}^-\} \cup \mathrm{o}(\mathcal{M}))$.

(d) (Stratification) There is a limit $\gamma \in \mathrm{Ord}$ and sequence $\widetilde{\mathcal{M}} = \left\langle \widetilde{\mathcal{M}}_\alpha \right\rangle_{\alpha<\gamma}$ such that:

i. $\widetilde{\mathcal{M}}$ is a continuous, strictly increasing sequence with $\mathcal{M}^- \in \widetilde{\mathcal{M}}_0$ and $\mathcal{M} = \bigcup_{\alpha<\gamma} \widetilde{\mathcal{M}}_\alpha$,

ii. for each $\alpha < \gamma$, $\widetilde{\mathcal{M}}_\alpha$ is an $\mathcal{L}_0$-structure such that $\left\lfloor \widetilde{\mathcal{M}}_\alpha \right\rfloor$ is transitive and $\widetilde{\mathcal{M}}_\alpha = \mathcal{M}{\restriction} \left\lfloor \widetilde{\mathcal{M}}_\alpha \right\rfloor$; that is, $cb^{\widetilde{\mathcal{M}}_\alpha} = A$ and $cp^{\widetilde{\mathcal{M}}_\alpha} = cp^{\mathcal{M}}$ and $E^{\widetilde{\mathcal{M}}_\alpha} = E^{\mathcal{M}} \cap \widetilde{\mathcal{M}}_\alpha$, etc,

iii. $\widetilde{\mathcal{M}}{\restriction}\alpha \in \mathcal{M}$ for every $\alpha < \gamma$, and the function $\alpha \mapsto \widetilde{\mathcal{M}}{\restriction}\alpha$, with domain γ, is $\Sigma_1^{\mathcal{M}}(\{\mathcal{M}^-\})$.

$\dashv$

Remark 2.20. Let $\mathcal{N}$ be a potential opm over A. Suppose $E^{\mathcal{N}}$ codes an extender F. Clearly $\kappa = \mathrm{crit}(F) > \Theta^{\mathcal{M}} > \mathrm{rank}(A)$. By [12, Definition 2.2.1], we have $(\kappa^+)^{\mathcal{M}} < \mathrm{o}(\mathcal{M})$; cf. 2.16. Note that *we allow F to be of superstrong type* (see 2.21) in accordance with [12], not [11, Definition 2.4].[9]

Definition 2.21. Let $\mathcal{M}$ be a potential opm over A. We say that $\mathcal{M}$ is ***E*-active** iff $E^{\mathcal{M}} \neq \emptyset$, and ***P*-active** iff $P^{\mathcal{M}} \neq \emptyset$. **Active** means either E-active or P-active. ***E*-passive** means not E-active. ***P*-passive** means not P-active. **Passive** means not active. **Type 0** means passive. **Type 4** means P-active. **Type 1, 2** or **3** mean E-active, with the usual numerology.

We write $F^{\mathcal{M}}$ for the extender F coded by $E^{\mathcal{M}}$ (where $F = \emptyset$ if $E^{\mathcal{M}} = \emptyset$). We write $\mathbb{E}^{\mathcal{M}}$ for the function with domain $l(\mathcal{M})$, sending $\alpha \mapsto F^{\mathcal{M}|\alpha}$. Likewise for $\mathbb{E}_+^{\mathcal{M}}$, but with domain $l(\mathcal{M}) + 1$.

If $F = F^{\mathcal{M}} \neq \emptyset$, we say $\mathcal{M}$, or F, is **superstrong** iff $i_F(\mathrm{crit}(F)) = \nu(F)$. We say that $\mathcal{M}$ is **super-small** iff $\mathcal{M}$ has no superstrong initial segment.

Suppose $\mathcal{M}$ is a successor. A **stratification** of $\mathcal{M}$ is a sequence $\widetilde{\mathcal{M}}$ witnessing 2.19(3d) for $\mathcal{M}$. For a Σ_1 formula $\varphi \in \mathcal{L}_0$, we say that $\mathcal{M}$ is

[9] The main point of permitting superstrong extenders is that it simplifies certain things. But it complicates others. If the reader prefers, one could instead require that F *not* be superstrong, but various statements throughout the paper regarding condensation would need to be modified, along the lines of [1, Lemma 3.3].

φ**-stratified** iff $\varphi(\mathcal{M}^-, \cdot)^{\mathcal{M}}$ defines the set of all proper restrictions $\widetilde{\mathcal{M}}\restriction\alpha$ of a stratification $\widetilde{\mathcal{M}}$ of $\mathcal{M}$.[10] $\dashv$

Lemma 2.22. *Let $\mathcal{M}$ be a successor potential opm, over A. Let $\widetilde{\mathcal{M}} = \left\langle \widetilde{\mathcal{M}}_\alpha \right\rangle_{\alpha<\gamma}$ be a stratification of $\mathcal{M}$. For $\alpha < \gamma$ let*

$$H_\alpha = \mathrm{Hull}_1^{\widetilde{\mathcal{M}}_\alpha}(A^{<\omega} \cup \mathrm{o}(\widetilde{\mathcal{M}}_\alpha)).$$

Then for every $x \in \mathcal{M}$ there is $\alpha < \gamma$ such that $x \subseteq H_\alpha$.

Proof. Use Σ_1-ordinal-generation and A-ordinal-surjections. $\square$

Definition 2.23. Let $\mathcal{N}$ be a structure for a finite first-order language $\mathcal{L}$. We say that $\mathcal{N}$ is **pre-fine** iff:

- $\mathcal{L}$ is a finite and $\{\dot\in, \dot{cb}\} \subseteq \mathcal{L}$, where $\dot\in$ is a binary relation symbol and $\dot{cb}$ is a constant symbol.
- $\mathcal{N}$ is an amenable $\mathcal{L}$-structure with transitive, rud closed, rank closed universe $\lfloor\mathcal{N}\rfloor$ and $\dot\in^{\mathcal{N}} = \in \cap \lfloor\mathcal{N}\rfloor^2$ and $\dot{cb}^{\mathcal{N}}$ is transitive.
- $\mathcal{N} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{N}}(\dot{cb}^{\mathcal{N}} \cup \mathrm{o}(\mathcal{N}))$ (note the language here is $\mathcal{L}$).

$\dashv$

Definition 2.24 (Fine structure)**.** Let $\mathcal{N}$ be pre-fine for the language $\mathcal{L}$. We sketch a description of the **fine structural notions** for $\mathcal{N}$. For details refer to [1],[11]; we also adopt some simplifications explained in [4].[11] Let $A = cb^{\mathcal{N}}$.

We say that $\mathcal{N}$ is 0**-sound** and let $\rho_0^{\mathcal{N}} = \mathrm{o}(\mathcal{N})$ and $p_0^{\mathcal{N}} = \emptyset$ and $\mathfrak{C}_0(\mathcal{N}) = \mathcal{N}$ and $\mathrm{r}\Sigma_1^{\mathcal{N}} = \Sigma_1^{\mathfrak{C}_0(\mathcal{N})}$ (here and in what follows, definability is with respect to $\mathcal{L}$). Let $T_0^{\mathcal{N}} = \mathcal{N}$.

Now let $n < \omega$ and suppose that $\mathcal{N}$ is n-sound (which will imply that $\mathcal{N} = \mathfrak{C}_n(\mathcal{N})$) and that $\omega < \rho_n^{\mathcal{N}}$. We write $\vec{p}_n^{\mathcal{N}} = (p_1^{\mathcal{N}}, \ldots, p_n^{\mathcal{N}})$. Then $\rho = \rho_{n+1}^{\mathcal{N}}$ is the least ordinal $\rho \geq \omega$ such that for some $X \subseteq A^{<\omega} \times \rho^{<\omega}$, X is $\utilde{\mathrm{r}\Sigma}_{n+1}^{\mathcal{N}}$ but $X \notin \lfloor\mathcal{N}\rfloor$.

[10] The φ-stratification of $\mathcal{M}$ need not imply that every successor $\mathcal{N} \triangleleft \mathcal{M}$ is φ-stratified.

[11] The simplifications involve dropping the parameters u_n, and replacing the use of generalized theories with pure theories. These changes are not important, and if the reader prefers, one could redefine things more analogously to [1],[11].

Define $\mathrm{r}\Sigma_{n+1}^{\mathcal{N}}$ from $T = T_n^{\mathcal{N}}$ as usual[12] (the definition of $T_{n+1}^{\mathcal{N}}$ is given below). And $p_{n+1}^{\mathcal{N}}$ is the least tuple $p \in \mathrm{Ord}^{<\omega}$ such that some such X is

$$\mathrm{r}\Sigma_{n+1}^{\mathcal{N}}(A \cup \rho \cup \{p, \vec{p}_n^{\mathcal{N}}\}).$$

Here $p_{n+1}^{\mathcal{N}}$ is well-defined by Σ_1-ordinal-generation. For any $X \subseteq \mathcal{N}$, let

$$\mathrm{Hull}_{n+1}^{\mathcal{N}}(X) = \mathrm{Hull}_{\mathrm{r}\Sigma_{n+1}}^{\mathcal{N}}(X),$$

and $\mathrm{cHull}_{n+1}^{\mathcal{N}}(X)$ be its transitive collapse. Likewise let

$$\mathrm{Th}_{n+1}^{\mathcal{N}}(X) = \mathrm{Th}_{\mathrm{r}\Sigma_{n+1}}^{\mathcal{N}}(X)$$

(this denotes the *pure* $\mathrm{r}\Sigma_{n+1}$ theory, as opposed to the *generalized* $\mathrm{r}\Sigma_{n+1}$ theory of [1].[13]) Then we let

$$\mathcal{C} = \mathfrak{C}_{n+1}(\mathcal{N}) = \mathrm{cHull}_{n+1}^{\mathcal{N}}(A \cup \rho_{n+1}^{\mathcal{N}} \cup \vec{p}_{n+1}^{\mathcal{N}}),$$

and the uncollapse map $\pi : \mathcal{C} \to \mathcal{N}$ is the associated **core embedding**. Define $(n+1)$-**solidity** and $(n+1)$-**universality** for $\mathcal{N}$ as usual (putting the parameters in A into every relevant hull). We say that $\mathcal{N}$ is $(n+1)$-**sound** iff $\mathcal{N}$ is $(n+1)$-solid and $\mathcal{C} = \mathcal{N}$ and $\pi = \mathrm{id}$.

Now suppose that $\mathcal{N}$ is $(n+1)$-sound and $\rho_{n+1}^{\mathcal{N}} > \omega$ (so $\rho_{n+1}^{\mathcal{N}} > \mathrm{rank}(A)$). Define $T = T_{n+1}^{\mathcal{N}} \subseteq \mathcal{N}$ by letting $t \in T$ iff for some $q \in \mathcal{N}$ and $\alpha < \rho_{n+1}^{\mathcal{N}}$,

$$t = \mathrm{Th}_{n+1}^{\mathcal{N}}(A \cup \alpha \cup \{q\}).$$

$\dashv$

Definition 2.25. Let $\mathcal{L}_0^+$ be $\mathcal{L}_0$ augmented with constant symbols $\dot{\mu}, \dot{e}$.[14]

Let $\mathcal{N}$ be a potential opm.

If $\mathcal{N}$ is E-active then $\mu^{\mathcal{N}} =_{\mathrm{def}} \mathrm{crit}(F^{\mathcal{N}})$, and otherwise $\mu^{\mathcal{N}} =_{\mathrm{def}} \emptyset$.

If $\mathcal{N}$ is E-active type 2 then $e^{\mathcal{N}}$ denotes the trivial completion of the largest non-type Z proper segment of F; otherwise $e^{\mathcal{N}} =_{\mathrm{def}} \emptyset$.[15]

If $\mathcal{N}$ is not type 3 then $\mathfrak{C}_0(\mathcal{N}) = \mathcal{N}^{\mathrm{sq}}$ denotes the $\mathcal{L}_0^+$-structure $(\mathcal{N}, \mu^{\mathcal{N}}, e^{\mathcal{N}})$ (with $\dot{\mu}^{\mathcal{N}} = \mu^{\mathcal{N}}$ etc).

[12] θ is $\mathrm{r}\Sigma_{n+1}^{\mathcal{N}}$ iff there is an $\mathrm{r}\Sigma_1$ formula $\psi(t, v) \in \mathcal{L}$ such that $\theta = \exists t(T(t) \wedge \psi(t, v))$.

[13] As in [1, §2], it does not matter which we use.

[14] μ is for *μeasurable*, and e is for *extender*.

[15] In [1], the (analogue of) e is referred to by its code $\gamma^{\mathcal{M}}$. We use e instead because this does not depend on having (and selecting) a wellorder of $\mathcal{M}$.

If $\mathcal{N}$ is type 3 then define the $\mathcal{L}_0^+$-structure $\mathfrak{C}_0(\mathcal{N}) = \mathcal{N}^{\text{sq}}$ essentially as in [1]; so

$$\mathcal{N}^{\text{sq}} = (R, E', P', S', X'; cb^{\mathcal{N}}, cp^{\mathcal{N}}, \mu^{\mathcal{N}}, e^{\mathcal{N}})$$

where $\nu = \nu(F^{\mathcal{N}})$, $R = \lfloor \mathcal{N}|\nu \rfloor$, E' is the usual squashed predicate coding $F^{\mathcal{N}}$, $P' = \emptyset$, $S' = S^{\mathcal{N}} \cap R$ and $X' = X^{\mathcal{N}} \cap R$.

We define the **fine structural notions** for $\mathcal{N}$ (n-soundness, $\rho_{n+1}^{\mathcal{N}}$, $\text{Hull}_{n+1}^{\mathcal{N}}$, $\text{Th}_{n+1}^{\mathcal{N}}$, etc) as those for $\mathfrak{C}_0(\mathcal{N})$.[16]

The classes of (non-simple) **$\mathcal{L}_0^+$-Q-formulas** and **$\mathcal{L}_0^+$-P-formulas** are defined analogously to in [1, §§2,3] (but with Σ_1 in place of the rΣ_1 of [1]). $\dashv$

In the proof of the solidity, etc, of iterable opms, one must also deal with structures which are almost active opms, except that they may fail the ISC. The details are immediate modifications of the standard notions, so we leave them to the reader.

Definition 2.26. Let $\mathcal{M}$ be a Q-opm. Let $\mathcal{R}$ be an $\mathcal{L}_0^+$-structure (possibly illfounded). Let $\pi : \mathcal{R} \to \mathfrak{C}_0(\mathcal{M})$.

We say that π is an **weak 0-embedding** iff π is Σ_0-elementary (therefore $\mathcal{R}$ is extensional and wellfounded, so assume $\mathcal{R}$ is transitive) and there is $X \subseteq \mathcal{R}$ such that X is $\in$-cofinal in $\mathcal{R}$ and π is Σ_1-elementary on elements of X, and if $\mathcal{M}$ is type 1 or 2, then letting $\mu = \mu^{\mathcal{R}}$, there is $Y \subseteq \mathcal{R}|(\mu^+)^{\mathcal{R}} \times \mathcal{R}$ such that Y is $\in \times \in$-cofinal in $\mathcal{R}|(\mu^+)^{\mathcal{R}} \times \mathcal{R}$ and π is Σ_1-elementary on elements of Y. $\dashv$

Definition 2.27. For $k \leq \omega$, a **(near) k-embedding** $\pi : \mathcal{M} \to \mathcal{N}$ between k-sound opms is defined analogously to [11], and a **weak k-embedding** is analogous to [8, Definition 2.1(?)].[17] Recall that when $k = \omega$, each of these notions are equivalent with full elementarity. (According to the standard convention, literally $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{N})$ and the elementarity of π is with respect to $\mathfrak{C}_0(\mathcal{M}), \mathfrak{C}_0(\mathcal{N})$.)

We say that $\pi : \mathcal{M} \to \mathcal{N}$ is **(weakly, nearly) k-good** iff π is a (weak, near) k-embedding and $cb^{\mathcal{M}} = cb^{\mathcal{N}}$ and $\pi{\restriction}cb^{\mathcal{M}} = \text{id}$. $\dashv$

[16] Thus, when we write, say, $\mathcal{M} = \text{cHull}_{n+1}^{\mathcal{N}}(X)$, we will have $X \subseteq \mathfrak{C}_0(\mathcal{N})$ and literally mean that $\mathfrak{C}_0(\mathcal{M}) = \mathcal{R}$ where $\mathcal{R} = \text{cHull}_{n+1}^{\mathfrak{C}_0(\mathcal{N})}(X)$. So $\mathcal{M}$ is produced by unsquashing $\mathcal{R}$. However, if $\mathcal{N}$ is type 3 and $n = 0$ it is possible that unsquashing $\mathcal{R}$ produces an illfounded structure $\mathcal{M}$, in which case $\mathfrak{C}_0(\mathcal{M})$ has not literally been defined. In this case, we define $\mathcal{M}$ to be this illfounded structure, and define $\mathfrak{C}_0(\mathcal{M}) = \mathcal{R}$.

[17] Note that this definition of *weak k-embedding* diverges slightly from the definitions given in [1] and [11].

Definition 2.28. Let $\mathcal{N}$ be an ω-sound potential opm. We say that $\mathcal{N}$ is $<\omega$**-condensing** iff for every $k<\omega$, for every soundly projecting, $(k+1)$-sound potential opm $\mathcal{M}$, for every near k-embedding $\pi:\mathcal{M}\to\mathcal{N}$ such that $\rho=\rho_{k+1}^{\mathcal{M}}\leq\mathrm{crit}(\pi)$ and $\rho<\rho_{k+1}^{\mathcal{N}}$, we have the following. If $\mathcal{M}|\rho$ is E-passive let $\mathcal{Q}=\mathcal{M}$, and otherwise let $\mathcal{Q}=\mathrm{Ult}(\mathcal{M}|\rho,F^{\mathcal{M}|\rho})$. Then either:

- $\mathcal{M}\triangleleft\mathcal{Q}$, or
- $\mathcal{M}^-\triangleleft\mathcal{Q}$, and $\mathcal{M}\in\mathcal{R}$ where $\mathcal{R}\triangleleft\mathcal{Q}$ is such that $\mathcal{R}^-=\mathcal{M}^-$.

⊣

Note that if we have $\mathcal{M}\in\mathcal{R}$ as above, then $\rho_\omega^{\mathcal{M}}=\rho_\omega^{\mathcal{M}^-}$.

Definition 2.29. A **Q-operator-premouse (Q-opm)**[18] is a potential operator-premouse $\mathcal{M}$ such that every $\mathcal{N}\triangleleft\mathcal{M}$ is ω-sound and $<\omega$-condensing. ⊣

In [1], there are no condensation requirements made regarding proper segments of premice. We make this demand here so that we can avoid stating it as an explicit axiom at certain points later (and it holds for the structures we care about).

Definition 2.30. An **adequate model-plus** is an $\mathcal{L}_0^+$-structure $\mathcal{N}$ such that $\mathcal{N}\restriction\mathcal{L}_0$ is an adequate model. ⊣

Lemma 2.31. *There are $\mathcal{L}_0^+$-Q-formulas φ_1,φ_2, a $\mathcal{L}_0^+$-P-formula φ_3, an $\mathcal{L}_0^+$-simple-Q-formula $\varphi_{0,\mathrm{limit}}$, and for each Σ_1 formula $\psi\in\mathcal{L}_0$ there are $\mathcal{L}_0^+$-simple-Q-formulas $\varphi_{0,\psi},\varphi_{4,\psi}$ such that for any adequate model-plus $\mathcal{N}'$:*

1. *$\mathcal{N}'\models\varphi_{0,\mathrm{limit}}$ iff $\mathcal{N}'=\mathfrak{C}_0(\mathcal{N})$ for some limit passive Q-opm $\mathcal{N}$.*
2. *$\mathcal{N}'\models\varphi_{4,\psi}$ iff $\mathcal{N}'=\mathfrak{C}_0(\mathcal{N})$ for some ψ-stratified P-active Q-opm $\mathcal{N}$.*
3. *$\mathcal{N}'\models\varphi_{0,\psi}$ iff $\mathcal{N}'=\mathfrak{C}_0(\mathcal{N})$ for some passive Q-opm $\mathcal{N}$ which is either a limit or is ψ-stratified.*
4. *$\mathcal{N}'\models\varphi_1$ (respectively, $\mathcal{N}'\models\varphi_2$) iff $\mathcal{N}'=\mathfrak{C}_0(\mathcal{N})$ for some type 1 (respectively, type 2) Q-opm $\mathcal{N}$.*

[18] Q is for *Q-formula*. We will see that the first-order aspects of Q-opm-hood are expressible with Q-formulas and P-formulas.

5. *If $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ for some type 3 Q-opm $\mathcal{N}$ then $\mathcal{N}' \vDash \varphi_3$. If $\mathcal{N}' \vDash \varphi_3$ then $E^{\mathcal{N}'}$ codes an extender F over $\mathcal{N}'$ such that if $\mathrm{Ult}(\mathcal{N}', F)$ is wellfounded then $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ for some type 3 Q-opm $\mathcal{N}$.*

Proof. Part 1 is routine and parts 4, 5 are straightforward adaptations of their analogues [1, Lemma 2.5], [1, Lemma 3.3] respectively, with the added point that we can drop the clause "or $\mathcal{N}$ is of superstrong type" of [1, Lemma 3.3], because we allow extenders of superstrong type. Part 2 is an easy adaptation of part 3, using the fact that if $\mathcal{N}$ is P-active then $P^{\mathcal{N}} \subseteq \mathcal{N} \backslash \mathcal{N}^-$. So we just sketch the proof of part 3.

Consider an adequate model-plus $\mathcal{N}'$ and $\mathcal{N} = \mathcal{N}' \upharpoonright \mathcal{L}_0$. We leave it to the reader to verify that here is an $\mathcal{L}_0$-simple-Q-formula asserting (when interpreted over $\mathcal{N}'$) that every $\mathcal{M} \triangleleft \mathcal{N}$ is a $< \omega$-condensing ω-sound potential opm, and an $\mathcal{L}_0^+$-simple-Q-formula asserting that $P^{\mathcal{N}} = E^{\mathcal{N}} = \mu^{\mathcal{N}} = e^{\mathcal{N}} = \emptyset$. It remains to see that we can assert that 2.19(3) holds for $\mathcal{M} = \mathcal{N}$ (the assertion will include the possibility that $\mathcal{N}$ is a limit). For 2.19(3a), use the formula "$\forall x \exists y[x \subseteq y \& \varphi(y)]$", where $\varphi(y)$ asserts "either there is $s \in S^{\mathcal{M}}$ such that $y \in s$ or there are S, A such that $S = y \cap S^{\mathcal{M}}$ and $A = cb^{\mathcal{M}}$ and S has a largest element $\mathcal{P}$ and for each $\tau < \mathrm{o}(\mathcal{P})$, if there is $X \in y \backslash \mathcal{P}$ such that $X \subseteq A^{<\omega} \times \tau^{<\omega}$, then there is $n < \omega$ such that $\rho_{n+1}^{\mathcal{P}} \leq \tau$, as witnessed by a satisfaction relation in y" (use the fact that $\mathcal{N}$ is rud closed).

Clause 2.19(3b) is easy, and it is fairly straightforward to assert that either $\mathcal{N}$ is a limit or $\mathcal{N}$ is ψ-stratified, identifying candidates for $\mathcal{N}^-$ as in the previous paragraph. We can therefore assert 2.19(3c) as "$\forall x \exists y[x \subseteq y$ and there is $\alpha < \gamma$ such that $y \subseteq H_\alpha$", where γ, H_α are defined as in 2.22, using the stratification given by ψ. □

Lemma 2.32. *The natural adaptations of* [1, Lemmas 2.4, 3.2] *hold.*

In fact, we can also give a version of those lemmas for weak 0-embeddings.

Lemma 2.33. *Let $\mathcal{M}$ be a Q-opm, let $\mathcal{N}'$ be an $\mathcal{L}_0^+$-structure and let $\pi : \mathcal{N}' \to \mathfrak{C}_0(\mathcal{M})$ be a weak 0-embedding.*

For any $\mathcal{L}_0^+$-Q-formula φ, if $\mathfrak{C}_0(\mathcal{M}) \vDash \varphi$ then $\mathcal{N}' \vDash \varphi$. If $\mathcal{M}$ is a type i Q-opm, $i \neq 3$, then $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ for some type i Q-opm $\mathcal{N}$. [19]

Suppose $\mathcal{M}$ is type 3. For any $\mathcal{L}_0^+$-P-formula φ, if $\mathfrak{C}_0(\mathcal{M}) \vDash \varphi$ then $\mathcal{N}' \vDash \varphi$. If $\mathrm{Ult}(\mathcal{M}, F^{\mathcal{M}})$ is wellfounded then $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ for some type 3 Q-opm $\mathcal{N}$.

[19] Possibly $\mathcal{N}, \mathcal{M}$ are passive and $\mathcal{M}$ is a successor but $\mathcal{N}$ a limit.

The proof is routine, so we omit it.

Lemma 2.34. *Let $\mathcal{M}$ be an n-sound Q-opm over A with $\omega < \rho_n^{\mathcal{M}}$. Let $X \subseteq \mathfrak{C}_0(\mathcal{M})$, let*

$$\mathcal{N} = \mathrm{cHull}_{n+1}^{\mathcal{M}}(A \cup X \cup \vec{p}_n^{\mathcal{M}})$$

and let $\pi : \mathcal{N} \to \mathcal{M}$ be the uncollapse. Then:

1. *If either $n > 1$ or $\mathcal{M}$ is not type 3 or $\mathrm{Ult}(\mathcal{M}, F^{\mathcal{M}})$ is wellfounded then $\mathcal{N}$ is a Q-opm.*

2. *If $\mathcal{N}$ is a Q-opm then π is nearly n-good.*

Proof. Suppose $n = 0$ and $\mathcal{M}$ is a successor. Then it suffices to see that π is $\mathrm{r}\Sigma_1$-elementary. Let $x \in \mathcal{N}$, let φ be $\mathrm{r}\Sigma_0$ and suppose that $\mathcal{M} \vDash \exists y \varphi(y, \pi(x))$. We want to see that there is some $y \in \mathrm{rg}(\pi)$ such that $\mathcal{M} \vDash \varphi(y, \pi(x))$.

Note that $\xi \in \mathrm{rg}(\pi)$, where ξ is least such that $\pi(x) \in \mathcal{M}|(\xi + 1)$ and there is $y \in \mathcal{M}|(\xi + 1)$ such that $\mathcal{M} \vDash \varphi(y, \pi(x))$. Suppose $\xi + 1 < \mathrm{lh}(\mathcal{M})$. Let $\vec{a} \in A^{<\omega}$ be such that there is $\vec{\beta} \in (\xi + 1)^{<\omega}$ such that $\mathcal{M} \vDash \varphi(y, \pi(x))$ where $y = h_{\xi+1}^{\mathcal{M}}(\vec{a}, \vec{\beta})$. Taking $\vec{\beta}$ least such, then $\vec{\beta} \in \mathrm{rg}(\pi)$, so $y \in \mathrm{rg}(\pi)$, as required. Now suppose instead that $\xi + 1 = \mathrm{lh}(\mathcal{M})$. Let $\langle H_\alpha \rangle_{\alpha < \gamma}$ be as in 2.22, with respect to some stratification $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. Then $\alpha \in \mathrm{rg}(\pi)$, where α is least such that $\pi(x) \in H_\alpha$ and there is $y \in H_\alpha$ such that $\mathcal{M} \vDash \varphi(y, \pi(x))$ (use here that for each $\beta < \gamma$, $\widetilde{\mathcal{M}}_\beta \preccurlyeq_0 \mathcal{M}$). So as before, there is some such $y \in \mathrm{rg}(\pi)$.

If $n = 0$ and $\mathcal{M}$ is a limit it is similar, but easier. (However, if $\mathcal{M}$ is type 3, possibly $\mathcal{N}$ is illfounded. This is ruled out by the hypotheses in part 1.)

If $n > 0$, then the proof for standard premice adapts routinely, using the fact that $A \subseteq \mathrm{rg}(\pi)$ as above.[20] (If $\mathcal{M}$ is type 3 and $n > 1$, there is $(a, f) \in \mathrm{rg}(\pi)$ such that $\nu(F^{\mathcal{M}}) = [a, f]^{\mathcal{M}}_{F^{\mathcal{M}}}$, which easily gives that $\mathcal{N}$ is wellfounded.) $\square$

Using stratifications and standard calculations, we also have:

Lemma 2.35. *Let $\pi : \mathcal{N} \to \mathcal{M}$ be nearly n-good, and $A = cb^{\mathcal{N}}$. Suppose that $\mathcal{N} \notin \mathcal{M}$ and $\mathcal{N} = \mathrm{Hull}_{n+1}^{\mathcal{N}}(A \cup \rho \cup \{q\})$, where $\rho \in \mathrm{Ord}$ and $\rho \leq \mathrm{crit}(\pi)$. Then π is n-good.*

If $\mathcal{N} = \mathfrak{C}_{n+1}(\mathcal{M})$ and π is the core embedding, then π is n-good.

[20] The fine structural setup here is a little different from that in [1], as we have dropped the use of $u_i^{\mathcal{M}}$. See [4] for calculations which deal with this difference.

Definition 2.36. An **operator-premouse (opm)** is a soundly projecting Q-opm. For an opm $\mathcal{M}$, let $q^{\mathcal{M}} = p_1^{\mathcal{M}} \cap (\mathrm{o}(\mathcal{M}^-), \mathrm{o}(\mathcal{M}))$ (so if $\mathcal{M}$ is a limit then $q^{\mathcal{M}} = \emptyset$). $\dashv$

Definition 2.37. Let $\mathcal{M}$ be a k-sound opm over A and $q \in (\rho_k^{\mathcal{M}})^{<\omega}$. We say that $\mathcal{M}$ is $(k+1, q)$**-solid** iff for each $\alpha \in q$, letting $q' = q \backslash (\alpha + 1)$ and $X = A \cup \alpha \cup q' \cup \vec{p}_k^{\mathcal{M}}$, we have $\mathrm{Th}_{k+1}^{\mathcal{M}}(X) \in \mathcal{M}$ (recall that this is the $\mathrm{r}\Sigma_{k+1}$ theory, computed over $\mathfrak{C}_0(\mathcal{M})$). $\dashv$

Lemma 2.38. *Let $\mathcal{M}$ be a successor opm and $l(\mathcal{M}) = \xi + 1$. Let $\rho = \rho_\omega^{\mathcal{M}^-}$ and $p = p_1^{\mathcal{M}} \backslash \rho$. Then $\mathcal{M}$ is ρ-sound and $\rho_1^{\mathcal{M}} \leq \rho$ and either $p \subseteq \xi + 1$ or $p = q^{\mathcal{M}}$. Therefore either $\mathcal{M}$ is ω-sound and $\rho_\omega^{\mathcal{M}} = \rho_\omega^{\mathcal{M}^-}$, or there is $k < \omega$ such that $\mathcal{M}$ is k-sound and $\rho_{k+1}^{\mathcal{M}} < \rho_\omega^{\mathcal{M}^-} \leq \rho_k^{\mathcal{M}}$.*

Proof. If $q^{\mathcal{M}} \neq \emptyset$ then $p \cap [\rho, \mathrm{o}(\mathcal{M}^-)] = \emptyset$, as letting $A = cb^{\mathcal{M}}$,

$$\mathcal{M}^- \cup \{\mathcal{M}^-\} \subseteq \mathrm{Hull}_1^{\mathcal{M}}(A \cup \rho \cup p)$$

as $X^{\mathcal{M}}$ is $\Sigma_1^{\mathcal{M}}$, and this suffices since $\mathcal{M}$ is soundly projecting. So suppose $q^{\mathcal{M}} = \emptyset$. Then p is the least $r \in (\xi + 1)^{<\omega}$ such that

$$\mathcal{M}^- \in H = \mathrm{Hull}_1^{\mathcal{M}}(A \cup \rho \cup r).$$

Moreover, $\mathcal{M}$ is $(1, p)$-solid. For $\mathcal{M} = H$ by sound-projection and since $q^{\mathcal{M}} = \emptyset$. Therefore $p \leq r$. But letting $\alpha \in r$ and $r' = r \backslash (\alpha + 1)$ and

$$H' = \mathrm{Hull}_1^{\mathcal{M}}(A \cup \alpha \cup r'),$$

we have $\mathcal{M}^- \notin H'$, so $H' \subseteq \mathcal{M}^-$, because $X^{\mathcal{M}}$ is $\Sigma_1^{\mathcal{M}}$. This suffices. $\square$

Lemma 2.39. *Let $\mathcal{N}$ be a successor operator-premouse and let $\pi : \mathcal{M} \to \mathcal{N}$. Suppose that either (i) π is Σ_1-elementary and $q^{\mathcal{N}} = \emptyset$, or (ii) π is Σ_2-elementary and $q^{\mathcal{N}} \in \mathrm{rg}(\pi)$. Then $\mathcal{M}$ is an operator-premouse of the same type as $\mathcal{N}$, and $\pi(q^{\mathcal{M}}) = q^{\mathcal{N}}$.*

Proof. By 2.31, $\mathcal{M}$ is a Q-opm and we may assume that $\mathcal{N}^- \in \mathrm{rg}(\pi)$, so $\mathcal{M}$ is a successor and $\pi(\mathcal{M}^-) = \mathcal{N}^-$, and $\mathcal{M}$ is ψ-stratified where $\mathcal{N}$ is ψ-stratified. In part (i) the ψ-stratification gives $\mathcal{M} = \mathrm{Hull}_1^{\mathcal{M}}(\mathcal{M}^- \cup \{\mathcal{M}^-\})$. In part (ii) use generalized solidity witnesses. $\square$

However, if π is just Σ_1-elementary and $p_1^{\mathcal{N}} \neq \emptyset$, $\mathcal{M}$ might not be soundly projecting, even if $p_1^{\mathcal{N}} \in \mathrm{rg}(\pi)$. Such embeddings arise when we take Σ_1 hulls, like in the proof of 1-solidity.

Let X be transitive. Then $X^\#$ determines naturally an opm $\mathcal{M}$ over $\hat{X}$ of length 1, so $U = \mathrm{Ult}_0(\mathcal{M}, F^{X^\#})$ is also a Q-opm over $\hat{X}$ of length 1, but U is not an opm.[21] So opm-hood is not expressible with Q-formulas. However, given a successor opm $\mathcal{N}$, we will only form ultrapowers of $\mathcal{N}$ with extenders E such that $\mathrm{crit}(E) < \mathrm{o}(\mathcal{N}^-)$, and under these circumstances, opm-hood is preserved. In fact, we will only form ultrapowers and fine structural hulls under further fine structural assumptions:

Definition 2.40. Let $k \leq \omega$. An opm $\mathcal{M}$ is k-**relevant** iff $\mathcal{M}$ is k-sound, and either $\mathcal{M}$ is a limit or $k = \omega$ or $\rho_{k+1}^{\mathcal{M}} < \rho_\omega^{\mathcal{M}^-}$.

A Q-opm $\mathcal{M}$ which is not an opm (so $\mathcal{M}$ is a successor) is k-**relevant** iff $k = 0$ and $\rho_1^{\mathcal{M}} < \rho_\omega^{\mathcal{M}^-}$. $\dashv$

For the development of the basic fine structure theory of opms, one only need to iterate k-relevant opms (and phalanxes of such structures, and bicephali and pseudo-premice); see 2.43. For instance, the following lemma follows from 2.38:

Lemma 2.41. *Let $k < \omega$ and $\mathcal{M}$ be a k-sound operator-premouse which is not k-relevant. Then $\mathcal{M}$ is $(k+1)$-sound.*

In the following lemma we establish the preservation of fine structure under degree k ultrapowers, for k-relevant opms. The proof involves a key use of stratification.

Lemma 2.42. *Let $\mathcal{M}$ be a k-relevant opm and E an extender over $\mathcal{M}$, weakly amenable to $\mathcal{M}$, with $\mathrm{crit}(E) < \rho_k^{\mathcal{M}}$, and $\mathrm{crit}(E) < \rho_\omega^{\mathcal{M}^-}$ if $\mathcal{M}$ is a successor. Let $\mathcal{N} = \mathrm{Ult}_k(\mathcal{M}, E)$ and $j = i_{E,k}^{\mathcal{M}}$ be the ultrapower embedding. Suppose $\mathcal{N}$ is wellfounded. Then:*

1. *$\mathcal{N}$ is a k-relevant opm of the same type as $\mathcal{M}$.*
2. *$\mathcal{N}$ is a successor iff $\mathcal{M}$ is. If $\mathcal{M}$ is a successor then $j(l(\mathcal{M})) = l(\mathcal{N})$ and if $\mathcal{M}$ is ψ-stratified then $\mathcal{N}$ is ψ-stratified.*
3. *j is k-good.*

[21] U is not soundly projecting.

4. *For any* $q \in (\rho_k^{\mathcal{M}})^{<\omega}$, *if* $\mathcal{M}$ *is* $(k+1, q)$*-solid then* $\mathcal{N}$ *is* $(k+1, j(q))$*-solid.*

5. $\rho_{k+1}^{\mathcal{N}} \leq \sup j``\rho_{k+1}^{\mathcal{M}}$.

6. *If* E *is close to* $\mathcal{M}$ *and* $\mathcal{M}$ *is* $(k+1)$*-solid then* $\rho_{k+1}^{\mathcal{N}} = \sup j``\rho_{k+1}^{\mathcal{M}}$ *and* $p_{k+1}^{\mathcal{N}} = j(p_{k+1}^{\mathcal{M}})$ *and* $\mathcal{N}$ *is* $(k+1)$*-solid.*

Proof. The fact that $\mathcal{N}$ is a Q-opm of the same type as $\mathcal{M}$ is by 2.31. Part 3 is standard and part 2 follows easily. We now verify that $\mathcal{N}$ is soundly projecting; we may assume that $\mathcal{M}, \mathcal{N}$ are successors. If $k > 0$, use elementarity and stratification. Suppose $k = 0$. Let $\rho = \rho_\omega^{\mathcal{M}^-}$ and $q = j(q^{\mathcal{M}})$. The fact that $\mathcal{N}$ is $(1, q)$-solid follows by an easy adaptation of the usual proof of preservation of the standard parameter, using stratification (where in the usual proof, one uses the natural stratification of the $\mathcal{J}$-hierarchy). So it suffices to see that $\mathcal{N} = \mathrm{Hull}_1^{\mathcal{N}}(\mathcal{N}^- \cup \{\mathcal{N}^-, q\})$. But this holds because $\mathcal{M}$ is an opm and

$$\mathcal{N} = \mathrm{Hull}_1^{\mathcal{N}}(\mathrm{rg}(j) \cup \nu_E)$$

and $\nu_E \subseteq \mathcal{N}^-$, the latter because $\mathrm{crit}(E) \leq \mathrm{o}(\mathcal{N}^-)$ (in fact, $\mathrm{crit}(E) < \rho_\omega^{\mathcal{N}^-}$).

Parts 4–6: If $k > 0$ the proof for standard premice works (see, for example, [1, Lemmas 4.5, 4.6], and if $\kappa < \rho_{k+1}^{\mathcal{M}}$, see the calculations in [1, Claim 5 of Theorem 6.2] and [5, §2(?), (p, ρ)-preservation]. If $k = 0$, again use stratification to adapt the usual proof. (In the case that $l(\mathcal{M})$ is a limit, $\mathcal{M}$ is of course "stratified" by its proper segments.)

By part 5, it follows that $\mathcal{N}$ is k-relevant, completing part 1. □

Definition 2.43. Iteration trees $\mathcal{T}$ on opms are as for standard premice, except that for all $\alpha + 1 \leq \mathrm{lh}(\mathcal{T})$, $M_\alpha^{\mathcal{T}}$ is an opm, and if $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ then $E_\alpha^{\mathcal{T}} \in \mathbb{E}_+(\mathcal{M}_\alpha^{\mathcal{T}})$. **Putative iteration trees** $\mathcal{T}$ on opms are likewise, except that if $\mathcal{T}$ has successor length then no demand is made on the nature of $M_\infty^{\mathcal{T}}$; in particular, it might be illfounded (but if $\mathrm{lh}(\mathcal{T}) = \lambda + 1$ for a limit λ then it is still required that $[0, \lambda)_{\mathcal{T}}$ be $\mathcal{T}\upharpoonright\lambda$-cofinal).

Let $k < \omega$ and let $\mathcal{M}$ be a k-sound opm. The iteration game $\mathcal{G}^{\mathcal{M}}(k, \theta)$ is defined completely analogously to the game $\mathcal{G}_k(\mathcal{M}, \theta)$ of [11, §3.1], forming a (putative) iteration tree as above, except for the following difference: Let $\mathcal{T}$ be the putative tree being produced. For $\beta + 1 < \alpha + 1$, we replace the requirement (on player I) that $\mathrm{lh}(E_\beta^{\mathcal{T}}) < \mathrm{lh}(E_\alpha^{\mathcal{T}})$ with the requirement that $\mathrm{lh}(E_\beta^{\mathcal{T}}) \leq \mathrm{lh}(E_\alpha^{\mathcal{T}})$. The rest is like in [11].

A (putative) iteration tree on $\mathcal{M}$ is **k-maximal** iff it is a partial play of $\mathcal{G}^{\mathcal{M}}(k,\infty)$. A **$(k,\theta)$-iteration strategy** for $\mathcal{M}$ is a winning strategy for player II in $\mathcal{G}^{\mathcal{M}}(k,\theta)$.

The iteration game $\mathcal{G}^{\mathcal{M}}(k,\alpha,\theta)$ is defined by analogy with the game $\mathcal{G}_k(\mathcal{M},\alpha,\theta)$ of [11, §4.1], except that each round consists of a run of $\mathcal{G}^{\mathcal{Q}}(q,\theta)$ for some $\mathcal{Q},q$.[22] The iteration game $\mathcal{G}=\mathcal{G}^{\mathcal{M}}_{\max}(k,\alpha,\theta)$ is defined likewise, except that we do not allow player I to drop in model or degree at the beginnings of rounds. That is, (i) round 0 of $\mathcal{G}$ is a run of $\mathcal{G}^{\mathcal{M}}(k,\theta)$, and (ii) letting $0<\gamma<\alpha$ and $\vec{\mathcal{T}}=\langle\mathcal{T}_\beta\rangle_{\beta<\gamma}$ be the sequence of trees played in rounds $<\gamma$ and $\mathcal{N}=M^{\vec{\mathcal{T}}}_\infty$ and $n=\deg^{\vec{\mathcal{T}}}(\infty)$, round γ of $\mathcal{G}$ is a run of $\mathcal{G}^{\mathcal{N}}(n,\theta)$.

A (putative) iteration tree is **k-stack-maximal** iff it is a partial play of $\mathcal{G}^{\mathcal{M}}_{\max}(k,\infty,\infty)$. A **$(k,\alpha,\theta)$-maximal iteration strategy** for $\mathcal{M}$ is a winning strategy for player II in $\mathcal{G}^{\mathcal{M}}_{\max}(k,\alpha,\theta)$, and a **$(k,\alpha,\theta)$-iteration strategy** is likewise for $\mathcal{G}^{\mathcal{M}}(k,\alpha,\theta)$.

Now **(k,θ)-iterability**, **(k,α,θ)-maximal iterability**, etc, are defined by the existence of the appropriate winning strategy. ⊣

Remark 2.44. The requirement, in $\mathcal{G}^{\mathcal{M}}(k,\theta)$, that $\mathrm{lh}(E^{\mathcal{T}}_\beta)\leq\mathrm{lh}(E^{\mathcal{T}}_\alpha)$ for $\beta<\alpha$, is weaker than requiring that $\mathrm{lh}(E^{\mathcal{T}}_\beta)<\mathrm{lh}(E^{\mathcal{T}}_\alpha)$, because opms may have superstrong extenders. For example, we might have that $E^{\mathcal{T}}_0$ is type 2 and $E^{\mathcal{T}}_1$ is superstrong with $\mathrm{crit}(E^{\mathcal{T}}_1)$ the largest cardinal of $\mathcal{M}^{\mathcal{T}}_0|\mathrm{lh}(E^{\mathcal{T}}_0)$, in which case $\mathcal{M}^{\mathcal{T}}_2$ is active but $\mathrm{o}(\mathcal{M}^{\mathcal{T}}_2)=\mathrm{lh}(E^{\mathcal{T}}_1)$, and therefore we might have $\mathrm{lh}(E^{\mathcal{T}}_2)=\mathrm{lh}(E^{\mathcal{T}}_1)$.

The preceding example is essentially general. It is easy to show that if $\mathcal{T}$ is k-maximal and $\alpha<\beta<\mathrm{lh}(\mathcal{T})$ then either $\mathrm{lh}(E^{\mathcal{T}}_\alpha)<\mathrm{o}(M^{\mathcal{T}}_\beta)$ and $\mathrm{lh}(E^{\mathcal{T}}_\alpha)$ is a cardinal of $M^{\mathcal{T}}_\beta$, or $\beta=\alpha+1$ and $\mathrm{lh}(E^{\mathcal{T}}_\alpha)=\mathrm{o}(M^{\mathcal{T}}_{\alpha+1})$ and $E^{\mathcal{T}}_\alpha$ is superstrong and $M^{\mathcal{T}}_{\alpha+1}$ is type 2. Therefore if $\alpha+1<\beta+1<\mathrm{lh}(\mathcal{T})$ then $\nu(E^{\mathcal{T}}_\alpha)<\nu(E^{\mathcal{T}}_\beta)$, and if $\alpha+1\leq\beta<\mathrm{lh}(\mathcal{T})$ then $E^{\mathcal{T}}_\alpha\!\upharpoonright\!\nu(E^{\mathcal{T}}_\alpha)$ is not an initial segment of any extender on $\mathbb{E}_+(M^{\mathcal{T}}_\beta)$.

The comparison algorithm needs to be modified slightly. Suppose we are comparing models $\mathcal{M},\mathcal{N}$, via padded k-maximal trees $\mathcal{T},\mathcal{U}$, respectively,

[22] Recall that for $\gamma<\alpha$, after the first γ rounds have been played, both players having met their commitments so far, we have a γ-sequence $\vec{\mathcal{T}}$ of iteration trees, with wellfounded final model $M^{\vec{\mathcal{T}}}_\infty$ (formed by direct limit if γ is a limit); it follows that this model is an n-sound operator-premouse where $n=\deg^{\vec{\mathcal{T}}}(\infty)$. At the beginning of round γ, player I chooses some $(\mathcal{Q},q)\trianglelefteq(\mathcal{M}^{\vec{\mathcal{T}}}_\infty,n)$, and round γ is a run of $\mathcal{G}^{\mathcal{Q}}(q,\theta)$. If round γ is won by player II and the run produces a tree of length θ, then the run of $\mathcal{G}^{\mathcal{M}}(k,\alpha,\theta)$ is won by player II.

and we have produced $\mathcal{T}\restriction\alpha+1$ and $\mathcal{U}\restriction\alpha+1$. Let γ be least such that $\mathcal{M}^{\mathcal{T}}_\alpha|\gamma \neq \mathcal{M}^{\mathcal{U}}_\alpha|\gamma$. If only one of these models is active, then we use that active extender next. Suppose both are active. If one active extender is type 3 and one is type 2, then we use only the type 3 extender next. Otherwise we use both extenders next. With this modification, and with the remarks in the preceding paragraph, the usual proof that comparison succeeds goes through.

Lemma 2.45. *Let $\mathcal{M}$ be a k-relevant opm and $\mathcal{T}$ a successor length k-stack-maximal tree on $\mathcal{M}$. Then $M^{\mathcal{T}}_\infty$ is a $\deg^{\mathcal{T}}(\infty)$-relevant opm.*

Proof. Given the result for k-maximal trees $\mathcal{T}$, the generalization to k-stack-maximal is routine. But for k-maximal $\mathcal{T}$, the result follows from 2.42, by a straightforward induction on $\mathrm{lh}(\mathcal{T})$. □

In 2.45, it is important that $\mathcal{T}$ is k-stack-maximal; the lemma can fail for trees produced by $\mathcal{G}^{\mathcal{M}}(k,\alpha,\theta)$.

3 $\mathcal{F}$-mice for operators $\mathcal{F}$

We will be interested in opms $\mathcal{M}$ in which the successor steps are taken by some *operator* $\mathcal{F}$; that is, in which $\mathcal{N}=\mathcal{F}(\mathcal{N}^-)$ for each successor $\mathcal{N}\trianglelefteq\mathcal{M}$. We call such an $\mathcal{M}$ an *$\mathcal{F}$-premouse*. A key example that motivates the central definitions is that of *mouse operators*. One can also use the operator framework to define (iteration) *strategy mice*, although a different approach is taken in [6] (to give a more refined hierarchy).

Definition 3.1. We say that X is **swo'd (self-wellordered)** iff $X = x\cup\{x,<\}$ for some transitive set x, and wellorder $<$ of x. In this situation, $<_X$ denotes the wellorder of X extending $<$, and with last two elements $x,<$. Clearly there are uniform methods of passing from an explicitly swo'd X to a wellorder of $A=\hat{X}$. Fix such a method, and for such X, A, let $<_A$ denote the resulting wellorder of A. ⊣

Definition 3.2. We say that a set or class $\mathscr{B}$ is an **operator background** iff (i) $\mathscr{B}$ is transitive, rudimentarily closed and $\omega\in\mathscr{B}$, (ii) for all $x\in\mathscr{B}$ and all y, f, if $f\colon x^{<\omega}\to\mathrm{trancl}(y)$ is a surjection then $y\in\mathscr{B}$, and (iii) $\mathscr{B}\models\mathsf{DC}$. (So $\mathrm{o}(\mathscr{B})=\mathrm{rank}(\mathscr{B})$ is a cardinal; if $\omega<\kappa\leq\mathrm{Ord}$ then $\mathscr{H}_\kappa$ is an operator background, and under ZFC these are the only operator backgrounds.) By (iii), every element of $\mathscr{B}$ has a countable elementary substructure.

Let $\mathscr{B}$ be an operator background. A set C is a **cone of** $\mathscr{B}$ iff there is $a \in \mathscr{B}$ such that C is the set of all $x \in \mathscr{B}$ such that $a \in \mathcal{J}_1(\hat{x})$. With a, C as such, we say C is **the cone above** a. If $b \in \mathcal{J}_1(a)$ we say C is **above** b. A set D is a **swo'd cone of** $\mathscr{B}$ iff $D = C \cap S$, for some cone C of $\mathscr{B}$, and where S is the class of explicitly swo'd sets. Here D is **(the swo'd cone) above** a iff C is (the cone) above a. A **cone** is a cone of $\mathscr{B}$ for some operator background $\mathscr{B}$. Likewise for **swo'd cone**. ⊣

Definition 3.3. An **operatic argument** is a set X such that either $X = \hat{Y}$ for some transitive Y, or X is an ω-sound opm. Given $C \subseteq \mathscr{B}$, let

$$\widehat{C} = \{\hat{Y} \parallel Y \in C \ \& \ Y \text{ is transitive}\}.$$

An **operatic domain over** $\mathscr{B}$ is a set $D = \widehat{C} \cup P \subseteq \mathscr{B}$, where C is a possibly swo'd cone of $\mathscr{B}$, and P is some class of $< \omega$-condensing ω-sound opms, each over some $A \in \widehat{C}$. (We do not make any closure requirements on P.) Write $C^D = C$ and $P^D = P$. Note that $\widehat{C} \cap P = \emptyset$.

An **operatic domain** is an operatic domain over some $\mathscr{B}$. ⊣

Definition 3.4. Let $\mathscr{B}$ be an operator background. An **operator over** $\mathscr{B}$ **with domain** D is a function $\mathcal{F} : D \to \mathscr{B}$ such that (i) D is an operatic domain over $\mathscr{B}$; (ii) for all $X \in D$, $\mathcal{M} = \mathcal{F}(X)$ is a successor opm with $\mathcal{M}^- = X$ (so if $X \in \widehat{C^D}$ then $l(\mathcal{M}) = 1$ and $cb^{\mathcal{M}} = X$). Write $C^{\mathcal{F}} = C^D$ and $P^{\mathcal{F}} = P^D$. ⊣

Remark 3.5. The argument X to an operator should be thought of as having one of two possible types. It is a *coarse object* if $X \in \widehat{C^{\mathcal{F}}}$; it is an opm if $X \in P^{\mathcal{F}}$. Some natural operators $\mathcal{F}$ have the property that, given $\mathcal{N} \in P^{\mathcal{F}}$ (so $\widehat{\mathcal{N}} \in C^{\mathcal{F}}$), $\mathcal{F}(\widehat{\mathcal{N}})$ is inter-computable with $\mathcal{F}(\mathcal{N})$. But operators producing strategy mice do not have this property.

The simplest operator is essentially $\mathcal{J}$:

Definition 3.6. Let $p \in V$. Let C_p be the class of all x such that $p \in \mathcal{J}_1(\hat{x})$. Let P_p be the class of all $< \omega$-condensing ω-sound opms $\mathcal{R}$ over some $Y \in \widehat{C_p}$, with $cp^{\mathcal{R}} = p$. Then $\mathcal{J}_p^{\text{op}}$ denotes the operator over V with domain $D = \widehat{C_p} \cup P_p$, where for $x \in D$, $\mathcal{J}_p^{\text{op}}(x)$ is the passive successor opm $\mathcal{M}$ with universe $\mathcal{J}_1(x)$ and $\mathcal{M}^- = x$ and $cp^{\mathcal{M}} = p$.[23] (So if $x \in \widehat{C_p}$ then $l(\mathcal{M}) = 1$ and $cb^{\mathcal{M}} = x$.) Let $\mathcal{J}^{\mathrm{op}} = \mathcal{J}^{\mathrm{op}}_{\emptyset}$. $\dashv$

Definition 3.7 ($\mathcal{F}$-premouse)**.** For $\mathcal{F}$ an operator, an **$\mathcal{F}$-premouse ($\mathcal{F}$-pm)** is an opm $\mathcal{M}$ such that $\mathcal{N} = \mathcal{F}(\mathcal{N}^-)$ for every successor $\mathcal{N} \trianglelefteq \mathcal{M}$. $\dashv$

Let $\mathcal{M}$ be an $\mathcal{F}$-premouse, where $\mathcal{F}$ is an operator over $\mathscr{B}$. Note that $cb^{\mathcal{M}} \in \widehat{C^{\mathcal{F}}}$, as $\mathcal{M}|1 = \mathcal{F}(\mathcal{M}|0)$ and $\mathcal{M}|0 = cb^{\mathcal{M}} = \hat{x}$ for some x, and $\hat{x} \notin P^{\mathcal{F}}$. Note also that $\mathrm{o}(\mathcal{M}) \leq \mathrm{o}(\mathscr{B})$.

We now define $\mathcal{F}$-iterability for $\mathcal{F}$-premice $\mathcal{M}$. The main point is that the iteration strategy should produce $\mathcal{F}$-premice. One needs to be a little careful, however, because the background $\mathscr{B}$ for $\mathcal{F}$ might only be a set. To simplify things, we restrict our attention to the case that $\mathcal{M} \in \mathscr{B}$.

Definition 3.8. Let $\mathcal{F}$ be an operator over $\mathscr{B}$. Let $\mathcal{M}$ be an opm and let $\mathcal{T}$ be a putative iteration tree on $\mathcal{M}$. We say that $\mathcal{T}$ is a **putative $\mathcal{F}$-iteration tree** iff $M^{\mathcal{T}}_{\alpha}$ is an $\mathcal{F}$-premouse for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$. We say that $\mathcal{T}$ is a **well-putative $\mathcal{F}$-iteration tree** iff $\mathcal{T}$ is an iteration tree and a putative $\mathcal{F}$-iteration tree (i.e. a putative $\mathcal{F}$-iteration tree whose models are all wellfounded). We say that $\mathcal{T}$ is an **$\mathcal{F}$-iteration tree** iff $M^{\mathcal{T}}_{\alpha}$ is an $\mathcal{F}$-premouse for all $\alpha + 1 \leq \mathrm{lh}(\mathcal{T})$. We may drop the "$\mathcal{F}$-" when it is clear from context.

Let $k < \omega$ and let $\mathcal{M} \in \mathscr{B}$ be a k-sound $\mathcal{F}$-premouse. Let $\theta \leq \mathrm{o}(\mathscr{B}) + 1$. The iteration game $\mathcal{G}^{\mathcal{F},\mathcal{M}}(k, \theta)$ has the rules of $\mathcal{G}^{\mathcal{M}}(k, \theta)$, except for the following difference. Let $\mathcal{T}$ be the putative tree being produced. For $\alpha + 1 \leq \theta$, if both players meet their requirements at all stages $< \alpha$, then, in stage α, player II must first ensure that $\mathcal{T}{\restriction}\alpha + 1$ is a well-putative $\mathcal{F}$-iteration tree, and if $\alpha + 1 < \mathrm{o}(\mathscr{B})$, that $\mathcal{T}{\restriction}\alpha + 1$ is an $\mathcal{F}$-iteration tree. (Given this, if $\alpha + 1 < \theta$, player I then selects $E^{\mathcal{T}}_{\alpha}$.)[24]

Let $\lambda, \alpha \leq \mathrm{o}(\mathscr{B})$, and suppose that either $\mathrm{o}(\mathscr{B})$ is regular or $\lambda < \mathrm{o}(\mathscr{B})$. Let $\theta \leq \lambda + 1$. The iteration game $\mathcal{G}^{\mathcal{F},\mathcal{M}}(k, \alpha, \theta)$ is defined just as $\mathcal{G}^{\mathcal{M}}(k, \alpha, \theta)$,

[23] It is easy to see that $\mathcal{M}$ is indeed an opm, so $\mathcal{J}^{\mathrm{op}}_p$ is an operator.

[24] Thus, if we reach stage $\mathrm{o}(\mathscr{B})$, then after selecting a branch, player II wins iff $M^{\mathcal{T}}_{\mathrm{o}(\mathscr{B})}$ is wellfounded. We cannot in general expect $M^{\mathcal{T}}_{\mathrm{o}(\mathscr{B})}$ to be an $\mathcal{F}$-premouse in this situation. For example, suppose that $\mathscr{B} = \mathrm{HC}$ and $\theta = \omega_1 + 1$ and $\mathrm{lh}(\mathcal{T}) = \omega_1 + 1$. Then $M^{\mathcal{T}}_{\omega_1}$ cannot be an $\mathcal{F}$-premouse, since all $\mathcal{F}$-premice have height $\leq \omega_1$. But in applications such as comparison, we only need to know that $M^{\mathcal{T}}_{\omega_1}$ is wellfounded. So we still decide the game in favour of player II in this situation.

with the differences that (i) the rounds are runs of $\mathcal{G}^{\mathcal{F},\mathcal{Q}}(q,\theta)$ for some $\mathcal{Q}, q$,[25] and (ii) if α is a limit and neither player breaks any rule, and $\vec{\mathcal{T}}$ is the sequence of trees played, then player II wins iff $M_\infty^{\vec{\mathcal{T}}}$ is defined (that is, the trees eventually do not drop on their main branches, etc), wellfounded, and if $\alpha < \mathrm{o}(\mathscr{B})$ then $M_\infty^{\vec{\mathcal{T}}}$ is an $\mathcal{F}$-premouse.[26] Likewise, $\mathcal{G}_{\max}^{\mathcal{F},\mathcal{M}}(k,\alpha,\theta)$ is analogous to $\mathcal{G}_{\max}^{\mathcal{M}}(k,\alpha,\theta)$.

An $\mathcal{F}$**-**(k,θ)**-iteration strategy** for $\mathcal{M}$ is a winning strategy for player II in $\mathcal{G}^{\mathcal{F},\mathcal{M}}(k,\theta)$, an $\mathcal{F}$**-**(k,α,θ)**-maximal iteration strategy** for $\mathcal{M}$ is likewise for $\mathcal{G}_{\max}^{\mathcal{F},\mathcal{M}}(k,\alpha,\theta)$, and an $\mathcal{F}$**-**(k,α,θ)**-iteration strategy** is likewise for $\mathcal{G}^{\mathcal{F},\mathcal{M}}(k,\alpha,\theta)$.

Now $\mathcal{F}$**-**(k,θ)**-iterability**, etc, are defined in the obvious manner. $\dashv$

In order to prove that $\mathcal{F}$-premice built by background constructions are $\mathcal{F}$-iterable, we will need to know that $\mathcal{F}$ has good *condensation* properties.

Definition 3.9. Let $\pi : \mathcal{M} \to \mathcal{N}$ be an embedding and b be transitive. We say that π is **above** b iff $b \cup \{b\} \subseteq \mathrm{dom}(\pi)$ and $\pi \restriction b \cup \{b\} = \mathrm{id}$. $\dashv$

Definition 3.10. Let $\mathcal{F}$ be an operator over $\mathscr{B}$ and $p \in \mathscr{B}$ be transitive. We say that $\mathcal{F}$ **condenses coarsely above** p (or $\mathcal{F}$ **has almost coarse condensation above** p) iff for every successor $\mathcal{F}$-pm $\mathcal{N}$, every set-generic extension $V[G]$ of V and all $\mathcal{M}, \pi \in V[G]$, if $\mathcal{M}^- \in V$ and $\pi : \mathcal{M} \to \mathcal{N}$ is fully elementary and above p, then $\mathcal{M}$ is an $\mathcal{F}$-pm (so in particular, $\mathcal{M}^- \in \mathrm{dom}(\mathcal{F})$ and $\mathcal{M} = \mathcal{F}(\mathcal{M}^-) \in V$).

We say that $\mathcal{F}$ **almost condenses coarsely above** b iff the preceding holds for $G = \emptyset$. $\dashv$

Definition 3.11. An operator $\mathcal{F}$ over $\mathscr{B}$ is **total** iff $P^{\mathcal{F}}$ includes all $<\omega$-condensing ω-sound $\mathcal{F}$-pms in $\mathscr{B}$. $\dashv$

Lemma 3.12. *Let $\mathcal{F}$ be a total operator which almost condenses coarsely above some $p \in \mathrm{HC}$. Then $\mathcal{F}$ condenses coarsely above p.*

[25] By some straightforward calculations using the restrictions on α, θ, one can see that for any $\gamma < \alpha$, if neither player has lost the game after the first γ rounds, and $\vec{\mathcal{T}} \restriction \gamma$ is the sequence of trees played thus far, then $M_\infty^{\vec{\mathcal{T}}\restriction\gamma} \in \mathscr{B}$ and $M_\infty^{\vec{\mathcal{T}}\restriction\gamma}$ is an $\mathcal{F}$-premouse, so $\mathcal{G}^{\mathcal{F},\mathcal{Q}}(q,\theta)$ is defined for the relevant $(\mathcal{Q}, q)$. This uses the rule that if one of the rounds produces a tree of length θ, then the game terminates.

[26] It follows that if $\lambda = \mathrm{o}(\mathscr{B})$ then $M_\infty^{\vec{\mathcal{T}}} | \mathrm{o}(\mathscr{B})$ is an $\mathcal{F}$-premouse.

Proof Sketch. Suppose the lemma fails and let $\mathbb{P}$ be a poset, and $G \subseteq \mathbb{P}$ be V-generic, such that in $V[G]$ there is a counterexample $\pi : \mathcal{M} \to \mathcal{N}$. We may easily assume that $\mathcal{M}^-$ is an $\mathcal{F}$-pm, and therefore that $\mathcal{M}^- \in \mathrm{dom}(\mathcal{F})$. So $\mathcal{M} \neq \mathcal{F}(\mathcal{M}^-)$. By Σ^1_1-absoluteness, we may assume that $\mathbb{P} = \mathrm{Col}(\omega, \mathcal{F}(\mathcal{M}^-) \cup \mathcal{N})$. Therefore there is a transitive, rud closed set $X \in \mathscr{B}$, where $\mathcal{F}$ is over $\mathscr{B}$, such that $\mathbb{P} \in X$ and $X \models$"It is forced by $\mathbb{P}$ that there is an $\mathcal{M}$ and a fully elementary $\pi : \mathcal{M} \to \mathcal{N}$, with $\mathcal{M} \neq \mathcal{F}(\mathcal{M}^-)$." Because $\mathscr{B} \models \mathsf{DC}$, we can take a countable elementary hull of X, such that letting $\sigma : \bar{X} \to X$ be the uncollapse, $\mathrm{rg}(\sigma)$ includes all relevant objects and all points in $p \cup \{p\} \subseteq \mathrm{rg}(\sigma)$. But we can find generics for $\bar{X}$, and because $\mathcal{F}$ almost condenses coarsely above p, this easily leads to contradiction. $\square$

Remark 3.13. We soon proceed toward the central notion of *condenses finely*, a refinement of *condenses coarsely*. This notion is based on that of *condenses well*, [12, 2.1.10] (*condenses well* also appeared in the original version of [10], in the same form). We have modified the latter notion in several respects, for multiple reasons. Before beginning we motivate two of the main changes.

Regarding the first, we can demonstrate a concrete problem with *condenses well*, at least when it is used in concert with other definitions in [12]. The following discussion uses the definitions and notation of [12, §2], without further explanation here; the terminology differs from this paper. (The remainder of this remark is for motivation only; nothing in it is needed later.)

Let K be the function $x \mapsto \mathcal{J}_2(x)$. Clearly K is a mouse operator (see [12, 2.1.7]). Let $F = F_K$ (see [12, 2.1.8]). Then we claim that F does not condense well (contrary to [12, 2.1.12]). We verify this.

Clearly regular premice $\mathcal{M}$ whose ordinals are closed under "$+\omega$" can be arranged as models $\tilde{\mathcal{M}}$ with parameter $\emptyset$ (see [12, 2.1.1]), such that for each $\alpha < l(\tilde{\mathcal{M}})$, $\tilde{\mathcal{M}}|\alpha + 1 = F(\tilde{\mathcal{M}}|\alpha)$.

Now let $\mathcal{M}$ be a premouse such that for some $\kappa < \mathrm{o}(\mathcal{M})$, κ is measurable in $\mathcal{M}$, via some measure on $\mathbb{E} = \mathbb{E}^{\mathcal{M}}$, and $\mathcal{M} \models$"$\lambda = \kappa^{+\kappa}$ exists", $\rho^{\mathcal{M}}_\omega = \lambda$, and $\mathcal{M} = \mathcal{J}_1(\mathcal{M}_0)$ where $\mathcal{M}_0 = \mathcal{J}^{\mathbb{E}}_\lambda$. Let $\mathcal{M}^* = \mathcal{J}(\tilde{\mathcal{M}}_0)$, arranged as a model with parameter $\emptyset$ extending $\tilde{\mathcal{M}}_0$. We have $\rho^{\mathcal{M}}_\omega = \lambda = \rho(\mathcal{M}_0)$ and $\tilde{\mathcal{M}}_0 \in \mathcal{M}^* \in F(\tilde{\mathcal{M}}_0)$ and $l(\mathcal{M}^*) = \lambda + 1$ and $(\mathcal{M}^*)^- = \tilde{\mathcal{M}}_0$ (see [12, 2.1.3]). (We can't say $\mathcal{M}^* = \tilde{\mathcal{M}}$, because $\tilde{\mathcal{M}}$ is not defined.)

Let $E \in \mathbb{E}$ be $\mathcal{M}$-total with $\mathrm{crit}(E) = \kappa$. Let $\mathcal{N} = \mathrm{Ult}_0(\mathcal{M}, E)$ and $\pi = i_E$. Then $\rho^{\mathcal{N}}_1 = \sup \pi``\lambda < \pi(\lambda)$. Let $\mathcal{N}_0 = \pi(\mathcal{M}_0)$ and $\mathcal{N}^* = \mathcal{J}_1(\tilde{\mathcal{N}}_0)$, arranged as a model with parameter $\emptyset$ extending $\tilde{\mathcal{N}}_0$. Then $\rho_1(\mathcal{N}^*) < \pi(\lambda) =$

$\rho(\tilde{\mathcal{N}_0})$, and therefore $\mathcal{N}^* = F(\tilde{\mathcal{N}_0})$. But $\pi : \mathcal{M}^* \to \mathcal{N}^*$ is a 0-embedding (and $\pi(\tilde{\mathcal{M}_0}) = \tilde{\mathcal{N}_0}$). Since $\mathcal{M}^* \neq F(\tilde{\mathcal{M}_0})$, F does not condense well (see [12, 2.1.10(1)]). (Note also that by using $\mathrm{Ult}_1(\mathcal{M}, E)$ in place of $\mathrm{Ult}_0(\mathcal{M}, E)$, we would get that π is *both* a 0-embedding and Σ_2-elementary, so even this hypothesis is consistent with having $\mathcal{M}^* \neq F(\tilde{\mathcal{M}_0})$.)

However, as pointed out by Steel, the preceding example is somewhat unnatural, because we could have taken a degree ω ultrapower. (Note that $\mathcal{M}$ is not 0-relevant. The example motivates our focus on forming k-ultrapowers of k-relevant opms.) So here is a second example, and one in which the embedding is the kind that can arise in the proof of solidity of the standard parameter – certainly in this context we would want to make use of *condenses well*. We claim there are (consistently) mice $\mathcal{M}$, containing large cardinals, and $\rho, \alpha \in \mathrm{Ord}^{\mathcal{M}}$ such that:

– $\mathcal{M} = \mathcal{J}(\mathcal{N})$ where $\mathcal{N} = \mathcal{M}|(\rho^+)^{\mathcal{M}}$,

– $\mathcal{M}$ is 1-sound,

– $\rho_1^{\mathcal{M}} = \rho < \alpha < (\rho^+)^{\mathcal{M}}$,

– $p_1^{\mathcal{M}} = \{(\rho^+)^{\mathcal{M}}, \alpha\}$, and

– letting $\mathcal{H} = \mathrm{cHull}_1^{\mathcal{M}}(\alpha \cup \{(\rho^+)^{\mathcal{M}}\})$, we have $\rho_\omega^{\mathcal{H}} = \alpha$.

(In fact, this happens in L, excluding the large cardinal assumption.) Given such $\mathcal{M}$, note that $\alpha = (\rho^+)^{\mathcal{H}}$ and $\mathcal{H} = \mathcal{J}(\mathcal{M}||\alpha)$. Then $\mathcal{H}$ is a 1-solidity witness for $\mathcal{M}$, and the 0-embedding $\pi : \mathcal{H} \to \mathcal{M}$ is the one that would be used in the proof of the 1-solidity of $\mathcal{M}$. Moreover, with F as before, "$\mathcal{M} = \mathcal{J}(\mathcal{N}) = F(\mathcal{N})$" (since $\mathcal{M}$ projects below $\mathrm{Ord}^{\mathcal{N}}$) but "$\mathcal{H} \neq F(\mathcal{M}||\alpha) = \mathcal{J}(\mathcal{J}(\mathcal{M}||\alpha))$". So we again have a failure of *condenses well*, and one which is arising in the context of the proof of solidity. (Of course, in the example we are already assuming 1-solidity, but the example seems to indicate that we cannot really expect to use *condenses well* in the proof of solidity for F-mice.)

Now let us verify that such an $\mathcal{M}$ exists. Let $\mathcal{P}$ be any mouse (with large cardinals) and ρ a cardinal of $\mathcal{P}$ such that $(\rho^{++})^{\mathcal{P}} < \mathrm{Ord}^{\mathcal{P}}$. Let $\gamma = (\rho^+)^{\mathcal{P}}+1$. For $\alpha < (\rho^+)^{\mathcal{P}}$ let

$$\mathcal{H}_\alpha = \mathrm{cHull}_1^{\mathcal{P}|\gamma}(\alpha \cup \{(\rho^+)^{\mathcal{P}}\}).$$

Because $\rho_\omega^{\mathcal{P}|\gamma} = (\rho^+)^{\mathcal{P}}$, it is easy to find α with $\rho < \alpha < (\rho^+)^{\mathcal{P}}$ and such that the uncollapse map $\mathcal{H}_\alpha \to \mathcal{P}|\gamma$ is fully elementary, and so $\rho_\omega(\mathcal{H}_\alpha) = \alpha = (\rho^+)^{\mathcal{H}_\alpha}$. Fix such an α. Let $\mathcal{H} = \mathcal{H}_\alpha$ and

$$\mathcal{M} = \mathrm{cHull}_1^{\mathcal{P}|\gamma}(\rho \cup \{(\rho^+)^{\mathcal{P}}, \alpha\}).$$

We claim that $\mathcal{M}, \rho, \alpha$ are as required. For $\mathcal{M} \in \mathcal{P}$, which easily gives that $\rho_1^{\mathcal{M}} = \rho$. Clearly $\mathcal{M} = \mathcal{J}(\mathcal{N})$ where $\mathcal{N} = \mathcal{M}|(\rho^+)^{\mathcal{M}}$. The 1-solidity witness associated to $(\rho^+)^{\mathcal{M}}$ is

$$\mathrm{cHull}_1^{\mathcal{M}}((\rho^+)^{\mathcal{M}}),$$

which is just $\mathcal{M}|(\rho^+)^{\mathcal{M}}$, as $\mathcal{M}|(\rho^+)^{\mathcal{M}} \preccurlyeq_1 \mathcal{M}$, as $\mathcal{M}|(\rho^+)^{\mathcal{M}} \models \mathsf{ZF}^-$. And the 1-solidity witness associated to α is

$$\mathrm{cHull}_1^{\mathcal{M}}(\alpha \cup \{(\rho^+)^{\mathcal{M}}\}),$$

which is just $\mathcal{H} = \mathcal{J}(\mathcal{P}||\alpha) \in \mathcal{M}$. All of the required properties follow.

The preceding examples seem to extend to any (first-order) mouse operator K such that $\mathcal{J}(x) \in K(x)$ for all x.

To get around the problem just described, we will need to weaken the conclusion of *condenses well*, as will be seen.

The second change is not based on a definite problem, but on a suspicion. It relates to, in the notation used in clause (2) of [12, 2.1.10], the embedding $\sigma : F(\mathcal{P}_0) \to \mathcal{M}$. In at least the basic situations in which one would want to use this clause (or its analogue in *condenses finely*), σ actually arises from something like an iteration map. But in *condenses well*, no hypothesis along these lines regarding σ is made. It seems that this could be a deficit, as it might be that $F(\mathcal{P}_0)$ is lower than $\mathcal{M}$ in the mouse order (if one can make sense of this); we might have $F(\mathcal{P}_0) \triangleleft \mathcal{M}$. Thus, it seems that in proving an operator condenses well, one might struggle to make use of the existence of σ. So, in *condenses finely*, we make stronger demands on σ.

A third change is that we do not require that $\pi \circ \sigma \in V$ (with π, σ as in [12, 2.1.10]). This is explained toward the end of 3.32.

Motivation for the remaining details will be provided by how they arise later, in our proof of the fundamental fine structural properties for $\mathcal{F}$-mice for operators $\mathcal{F}$ which condense finely, and in our proof that mouse operators condense finely. We now return to our terminology and notation. Before we can define *condenses finely*, we need to set up some terminology in order to describe the demands on σ.

The notion of $(z^{\mathcal{M}}_{k+1}, \zeta^{\mathcal{M}}_{k+1})$ below is a direct adaptation from [7, Definition 2.16(?)]. The facts proved there about this notion generalize readily to the present setting.

Definition 3.14. Let $\mathcal{M}$ be a k-sound opm. Let $\mathcal{D}$ be the class of pairs $(z,\zeta) \in [\mathrm{Ord}]^{<\omega} \times \mathrm{Ord}$ such that $\zeta \leq \min(z)$. For $x \in [\mathrm{Ord}]^{<\omega}$ let f_x be the decreasing enumeration of x. For $x = (z,\zeta) \in \mathcal{D}$ let $f_x = f_z \frown \langle\zeta\rangle$. Order $\mathcal{D}$ by $x <^* y$ iff $f_x <_{\mathrm{lex}} f_y$. Then $(z^{\mathcal{M}}_{k+1}, \zeta^{\mathcal{M}}_{k+1})$ denotes the $<^*$-least $(z,\zeta) \in \mathcal{D}$ such that

$$\mathrm{Th}^{\mathcal{M}}_{k+1}(cb^{\mathcal{M}} \cup z \cup \zeta) \notin \mathcal{M}.$$

The $(k+1)$**-solid-core** of $\mathcal{M}$ is

$$\mathfrak{S}_{k+1}(\mathcal{M}) = \mathrm{cHull}^{\mathcal{M}}_{k+1}(cb^{\mathcal{M}} \cup z^{\mathcal{M}}_{k+1} \cup \zeta^{\mathcal{M}}_{k+1}),$$

and the $(k+1)$**-solid-core map** $\sigma^{\mathcal{M}}_{k+1}$ is the uncollapse map. ⊣

If $\mathcal{M}$ is $(k+1)$-solid then $\mathfrak{S}_{k+1}(\mathcal{M}) = \mathfrak{C}_{k+1}(\mathcal{M})$ and $\sigma^{\mathcal{M}}_{k+1}$ is the core map. But we will need to consider the $(k+1)$-solid-core more generally, in the proof of $(k+1)$-solidity.

Definition 3.15. Let $k \leq \omega$, let $\mathcal{L}, \mathcal{M}$ be k-sound opms and $\sigma : \mathcal{L} \to \mathcal{M}$. We say that σ is k**-tight** iff there is $\lambda \in \mathrm{Ord}$ and a sequence $\langle \mathcal{L}_\alpha \rangle_{\alpha \leq \lambda}$ of opms such that $\mathcal{L} = \mathcal{L}_0$ and $\mathcal{M} = \mathcal{L}_\lambda$ and there is a sequence $\langle E_\alpha \rangle_{\alpha < \lambda}$ of extenders such that each E_α is weakly amenable to $\mathcal{L}_\alpha$, with $\mathrm{crit}(E_\alpha) > cb^{\mathcal{L}}$,

$$\mathcal{L}_{\alpha+1} = \mathrm{Ult}_k(\mathcal{L}_\alpha, E_\alpha),$$

and for limit η,

$$\mathcal{L}_\eta = \mathrm{dirlim}_{\alpha<\beta<\eta}(\mathcal{L}_\alpha, \mathcal{L}_\beta; j_{\alpha\beta})$$

where $j_{\alpha\beta} : \mathcal{L}_\alpha \to \mathcal{L}_\beta$ is the resulting ultrapower map, and $\sigma = j_{0\lambda}$. ⊣

Definition 3.16. Let $k \leq \omega$ and $\mathcal{M}, \mathcal{N}$ be k-sound opms and p be transitive.

We say that $\pi : \mathcal{M} \to \mathcal{N}$ is a k**-factor above** p iff π is a weak k-embedding above p, and if $k < \omega$ then there is a k-tight $\sigma : \mathcal{L} \to \mathcal{M}$ such that

$$\pi \circ \sigma \circ \sigma^{\mathcal{L}}_{k+1} : \mathfrak{S}_{k+1}(\mathcal{L}) \to \mathcal{N}$$

is a near k-embedding, σ is above p, and $\mathcal{L}$ is k-relevant.

For an operator $\mathcal{F}$, a k-factor is $\mathcal{F}$**-rooted** iff either $k = \omega$ or we can take $\mathcal{L}$ to be an $\mathcal{F}$-premouse.

A k-factor is **good** iff $A =_{\mathrm{def}} cb^{\mathcal{M}} = cb^{\mathcal{N}}$ and π is above A. ⊣

An ω-factor above p is just an ω-embedding (i.e. fully elementary between ω-sound opms) above p. If $k < \omega$, then both σ and $\sigma^{\mathcal{L}}_{k+1}$, and therefore also $\sigma \circ \sigma^{\mathcal{L}}_{k+1}$, are k-good. Any near k-embedding $\pi : \mathcal{M} \to \mathcal{N}$ between opms is a k-factor, and if $\mathcal{M}$ is an $\mathcal{F}$-pm, then π is $\mathcal{F}$-rooted (if $k < \omega$, use $\mathcal{L} = \mathcal{M}$ and $\sigma = \mathrm{id}$).

Definition 3.17. Let $\mathcal{C}$ be a successor opm and $\mathcal{M}$ a successor Q-opm with $\mathcal{C}^- = \mathcal{M}^-$. We say that $\mathcal{C}$ is a **universal hull** of $\mathcal{M}$ iff there is an above $\mathcal{C}^-$, 0-good embedding $\pi : \mathcal{C} \to \mathcal{M}$ and for every $x \in \mathcal{M}$, $\mathrm{Th}_1^{\mathcal{M}}(\mathcal{M}^- \cup \{x\})$ is $\underset{\sim}{\mathrm{r}\Sigma}{}_1^{\mathcal{C}}$ (after replacing x with a constant symbol). $\dashv$

Definition 3.18. Let $\mathcal{F}$ be an operator over $\mathscr{B}$ and $b \in \mathscr{B}$ be transitive. We say that $\mathcal{F}$ **condenses finely above** b (or $\mathcal{F}$ **has fine condensation above** b) iff (i) $\mathcal{F}$ condenses coarsely above b; and (ii) Let $A, \bar{A}, \mathcal{N}, \mathcal{L} \in V$ and let $\mathcal{M}, \varphi, \sigma \in V[G]$ where G is set-generic over V. Suppose that:

- $b \in \mathcal{J}_1(\bar{A}) \cap \mathcal{J}_1(A)$,
- $\mathcal{M}$ is a Q-opm over $\bar{A}$, $\mathcal{L}$ is an opm over $\bar{A}$, and $\mathcal{N}$ is an opm over A, each of successor length,
- $\mathcal{L}, \mathcal{M}^-, \mathcal{N}$ are $\mathcal{F}$-premice,
- $\varphi : \mathcal{M} \to \mathcal{N}$.

Then:

- If $\mathcal{M}$ is an opm and $k < \omega$ and either
  - φ is k-good, or
  - $V[G] \models$"φ is a k-factor above b, as witnessed by $(\mathcal{L}, \sigma)$" and $\mathcal{M}$ is k-relevant,

  then either $\mathcal{M} \in \mathcal{F}(\mathcal{M}^-)$ or $\mathcal{M} = \mathcal{F}(\mathcal{M}^-)$.
- If $\rho_1^{\mathcal{M}} \leq \mathrm{o}(\mathcal{M}^-)$ and φ is 0-good, then there is a universal hull $\mathcal{H}$ of $\mathcal{M}$ such that either $\mathcal{H} \in \mathcal{F}(\mathcal{M}^-)$ or $\mathcal{H} = \mathcal{F}(\mathcal{M}^-)$.

We say $\mathcal{F}$ **almost condenses finely above** b iff $\mathcal{F}$ almost condenses coarsely above b and condition (ii) above holds for $G = \emptyset$. $\dashv$

As we will see later, there are natural examples of operators which condense finely, but do not condense well. We next observe that in certain key circumstances, we can actually conclude that $\mathcal{M} = \mathcal{F}(\mathcal{M}^-)$.

Lemma 3.19. *Let $k, \mathcal{M}, G, \mathcal{N}$, etc, be as in 3.18. Suppose that either $\mathcal{M} = \mathfrak{C}_{k+1}(\mathcal{N})$ or $\mathcal{M}$ is k-relevant. Then $\mathcal{M} \notin \mathcal{F}(\mathcal{M}^-)$, and if $k = 0$ then there is no universal hull of $\mathcal{M}$ in $\mathcal{F}(\mathcal{M}^-)$.*

Proof. Suppose otherwise. Then by projectum amenability for $\mathcal{F}(\mathcal{M}^-)$, $\mathcal{M}$ is not k-relevant. So $\mathcal{M} = \mathfrak{C}_{k+1}(\mathcal{N}) \notin \mathcal{N}$; let $\varphi : \mathcal{M} \to \mathcal{N}$ be the core map. By 2.35, φ is k-good, so $\varphi(\mathcal{M}^-) = \mathcal{N}^-$. Clearly $\mathcal{M} \neq \mathcal{N}$, so letting $\rho = \rho_{k+1}^{\mathcal{N}}$, we have $\rho < \rho_k^{\mathcal{N}}$, and by 2.41, $\mathcal{N}$ is k-relevant. So $\rho < \rho_\omega^{\mathcal{N}^-}$ and $\rho \leq \text{crit}(\varphi)$. We have $\varphi(\rho_\omega^{\mathcal{M}^-}) = \rho_\omega^{\mathcal{N}^-}$, so $\rho \leq \rho_\omega^{\mathcal{M}^-}$. Since φ is k-good, $\rho < \rho_k^{\mathcal{M}}$. Since $\mathcal{M}$ is not k-relevant, therefore $\rho = \rho_\omega^{\mathcal{M}^-} = \text{crit}(\varphi)$. So because $\mathcal{N}^-$ is $<\omega$-condensing and ρ is a cardinal of $\mathcal{N}^-$, we have $\mathcal{M}^- \triangleleft \mathcal{N}^-$, so $\mathcal{F}(\mathcal{M}^-) \triangleleft \mathcal{N}^-$, so either $\mathcal{M} \in \mathcal{N}$, or $k = 0$ and there is a universal hull $\mathcal{H}$ of $\mathcal{M}$ in $\mathcal{N}$, both of which contradict the fact that $\mathcal{M} = \mathfrak{C}_{k+1}(\mathcal{N})$. □

So under the circumstances of the lemma above, if $\mathcal{M}$ is an opm, fine condensation gives the stronger conclusion that $\mathcal{M} = \mathcal{F}(\mathcal{M}^-)$. But we will need to apply fine condensation more generally, such as in the proof of solidity.

Definition 3.20. We say that $(\mathcal{F}, b, A)$ (or $(\mathcal{F}, b, A, B)$) is an **(almost) fine ground** iff $\mathcal{F}$ an operator which (almost) condenses finely above b and $A \in \widehat{C_\mathcal{F}}$ and $b \in \mathcal{J}_1(A)$ (and $B \in \widehat{C_\mathcal{F}}$ and $b \in \mathcal{J}_1(B)$). ⊣

Analogously to 3.12:

Lemma 3.21. *Let $\mathcal{F}$ be a total operator which almost condenses finely above some $p \in \text{HC}$. Then $\mathcal{F}$ condenses finely above p.*

We now show how fine condensation for $\mathcal{F}$ ensures that the copying construction proceeds smoothly for relevant $\mathcal{F}$-premice.

Definition 3.22. Let $\mathcal{M}$ be an opm. If $\mathcal{M}$ is not type 3 then $\mathcal{M}^\uparrow =_{\text{def}} \mathcal{M}$. If $\mathcal{M}$ is type 3 and $\kappa = \mu^\mathcal{M}$ then

$$\mathcal{M}^\uparrow =_{\text{def}} \text{Ult}(\mathcal{M}|(\kappa^+)^\mathcal{M}, F^\mathcal{M}).$$

For $\pi : \mathcal{M} \to \mathcal{N}$, a Σ_0-elementary embedding between opms of the same type, we define $\pi^\uparrow : \mathcal{M}^\uparrow \to \mathcal{N}^\uparrow$ as follows. If $\mathcal{M}$ is not type 3 then $\pi^\uparrow = \pi$. If $\mathcal{M}$ is type 3 then $\pi^\uparrow$ is the embedding induced by π.

Let $\mathcal{M},\mathcal{N}$ be opms. We write $\mathcal{N} \trianglelefteq^{\uparrow} \mathcal{M}$ iff either $\mathcal{N} \trianglelefteq \mathcal{M}$ or $\mathcal{N} \triangleleft \mathcal{M}^{\uparrow}$. We write $\mathcal{N} \triangleleft^{\uparrow} \mathcal{M}$ iff either $\mathcal{N} \triangleleft \mathcal{M}$ or $\mathcal{N} \triangleleft \mathcal{M}^{\uparrow}$. Let $j,k \leq \omega$ be such that $\mathcal{M}$ is j-sound and $\mathcal{N}$ is k-sound. We write

$$(\mathcal{N},k) \trianglelefteq (\mathcal{M},j)$$

iff either $[\mathcal{N} = \mathcal{M}$ and $k \leq j]$ or $\mathcal{N} \triangleleft \mathcal{M}$. We write

$$(\mathcal{N},k) \trianglelefteq^{\uparrow} (\mathcal{M},j)$$

iff either $(\mathcal{N},k) \trianglelefteq (\mathcal{M},j)$ or $\mathcal{N} \triangleleft \mathcal{M}^{\uparrow}$. ⊣

The copying process is complicated by squashing of type 3 structures, as explained in [11] and [8]. In order to reduce these complications, we will consider a trivial *reordering* of the tree order of lifted trees.

Definition 3.23. Let $\mathcal{T}$ be a k-maximal iteration tree. An **insert set** for $\mathcal{T}$ is a set $I \subseteq \mathrm{lh}(\mathcal{T})$ be such that for all $\alpha \in I$, we have $\alpha+1 < \mathrm{lh}(\mathcal{T})$ and $M^{\mathcal{T}}_{\alpha}$ is type 3 and $E^{\mathcal{T}}_{\alpha} = F(M^{\mathcal{T}}_{\alpha})$. Given such an I, the I**-reordering** $<_{\mathcal{T},I}$ of $<_{\mathcal{T}}$ is the iteration tree order defined as follows. Let $\beta+1 < \mathrm{lh}(\mathcal{T})$ and $\gamma = \mathrm{pred}^{\mathcal{T}}(\beta+1)$. Then $\mathrm{pred}^{\mathcal{T},I}(\beta+1) = \gamma$ unless $\beta+1 \in D^{\mathcal{T}}$ and $\gamma = \alpha+1$ for some $\alpha \in I$ and $\mathrm{crit}(E^{\mathcal{T}}_{\beta}) < j(\kappa)$, where $j = i_{E^{\mathcal{T}}_{\alpha}}$ and $\kappa = \mathrm{crit}(E^{\mathcal{T}}_{\alpha})$, in which case $\mathrm{pred}^{\mathcal{T},I}(\beta+1) = \alpha$. For limits $\beta < \mathrm{lh}(\mathcal{T})$, we set $[\gamma,\beta)_{\mathcal{T},I} = [\gamma,\beta)_{\mathcal{T}}$ for all sufficiently large $\gamma <_{\mathcal{T}} \beta$. ⊣

So if $\alpha = \mathrm{pred}^{\mathcal{T},I}(\beta+1) \neq \mathrm{pred}^{\mathcal{T}}(\beta+1)$, then $M^{*\mathcal{T}}_{\beta+1} \triangleleft M^{\mathcal{T}}_{\alpha+1}|j(\kappa)$ (for j,κ as above) so $M^{*\mathcal{T}}_{\beta+1} \triangleleft^{\uparrow} M^{\mathcal{T}}_{\alpha}$, but possibly $M^{*\mathcal{T}}_{\beta+1} \not\trianglelefteq M^{\mathcal{T}}_{\alpha}$.

Definition 3.24. Let $\mathcal{T}$ be a k-maximal tree on an opm $\mathcal{M}$, let I be an insert set for $\mathcal{T}$, let $\mathcal{N} \trianglelefteq \mathcal{M}$ and $\alpha < \mathrm{lh}(\mathcal{T})$. Let $\langle \beta_1,\ldots,\beta_n\rangle$ enumerate $D^{\mathcal{T}} \cap (0,\alpha]_{\mathcal{T},I}$. Let $\beta_0 = 0$, let $\gamma_i = \mathrm{pred}^{\mathcal{T},I}(\beta_{i+1})$ for $i < n$, and let $\gamma_n = \alpha$. Let $\pi_i = i^{*\mathcal{T}}_{\beta_i,\gamma_i}$, where $i^{*\mathcal{T}}_{0,\gamma_0} = i^{\mathcal{T}}_{0,\gamma_0}$. Let $\mathcal{N}_0 = \mathcal{N}$ and $\mathcal{N}_{i+1} = \pi_i^{\uparrow}(\mathcal{N}_i)$ if $\mathcal{N}_i \in \mathrm{dom}(\pi_i^{\uparrow})$, let $\mathcal{N}_{i+1} = M^{\mathcal{T}}_{\gamma_i}$ if $M^{*\mathcal{T}}_{\beta_i} = \mathcal{N}_i$, and $\mathcal{N}_{i+1}$ is undefined otherwise (in the latter case, $\mathcal{N}_j$ is undefined for all $j > i$).

We say that $[0,\alpha]_{\mathcal{T},I}$ **drops below the image of** $\mathcal{N}$ iff $\mathcal{N}_{n+1}$ is undefined. If $[0,\alpha]_{\mathcal{T},I}$ does not drop below the image of $\mathcal{N}$, we define $M^{\mathcal{T},I}_{\mathcal{N},\alpha} = \mathcal{N}' = \mathcal{N}_{n+1}$; and

$$i^{\mathcal{T},I}_{\mathcal{N},0,\alpha} : \mathcal{N} \to \mathcal{N}'$$

as follows. If $\mathcal{N}' = M^{\mathcal{T}}_{\alpha}$ then

$$i^{\mathcal{T},I}_{\mathcal{N},0,\alpha} =_{\mathrm{def}} i^{*\mathcal{T}}_{\beta_n,\alpha} \circ \pi^{\uparrow}_{n-1} \circ \pi^{\uparrow}_{n-2} \circ \ldots \circ \pi^{\uparrow}_0 \restriction \mathfrak{C}_0(\mathcal{N}),$$

and if $\mathcal{N}' \triangleleft^{\uparrow} M^{\mathcal{T}}_{\alpha}$ then

$$i^{\mathcal{T},\mathcal{N}}_{0,\alpha} =_{\text{def}} \pi^{\uparrow}_{n} \circ \pi^{\uparrow}_{n-1} \circ \ldots \circ \pi^{\uparrow}_{0} \restriction \mathfrak{C}_0(\mathcal{N}).$$

Also for $\xi <_{\mathcal{T},I} \alpha$, define $i^{\mathcal{T},I}_{\mathcal{N},\xi,\alpha} : M^{\mathcal{T},I}_{\mathcal{N},\xi} \to M^{\mathcal{T},I}_{\mathcal{N},\alpha}$ to be the natural map j such that $j \circ i^{\mathcal{T},I}_{\mathcal{N},0,\xi} = i^{\mathcal{T},I}_{\mathcal{N},0,\alpha}$ (so j is given by composing restrictions of $\sigma^{\uparrow}$ for iteration maps σ of $\mathcal{T}$ along segments of $[\xi,\alpha]_{\mathcal{T},I}$). ⊣

We now state the basic facts about the copying construction for $\mathcal{F}$-premice. We begin with a simple lemma regarding type 3 $\mathcal{F}$-premice.

Lemma 3.25. *Let $(\mathcal{F}, b, \bar{A}, A)$ be an almost fine ground. Let $\mathcal{N}$ be a type 3 $\mathcal{F}$-pm over A, such that $\mathcal{N}^{\uparrow}$ is an $\mathcal{F}$-pm. Let $\pi : \mathcal{R} \to \mathfrak{C}_0(\mathcal{N})$ be a weak 0-embedding. Then $\mathcal{R} = \mathfrak{C}_0(\mathcal{M})$ for some $\mathcal{F}$-pm $\mathcal{M}$.*

Proof. Because π is a weak 0-embedding, $E = E^{\mathcal{R}}$ is an extender over $\mathcal{R}$. So we can define $\mathcal{R}^{\uparrow}$ and $\pi^{\uparrow} : \mathcal{R}^{\uparrow} \to \mathcal{N}^{\uparrow}$ as in 3.22. By almost coarse condensation, $\mathcal{R}^{\uparrow}$ is an $\mathcal{F}$-pm, which yields the desired conclusion. □

Of course, in the preceding lemma we only actually needed almost *coarse* condensation. Below, the indexing function ι need not be the identity, because of the possibility of ν-high copy embeddings; see [8].

Lemma 3.26. *Let $(\mathcal{F}, b, \bar{A}, A)$ be an almost fine ground. Let $j \leq \omega$ and let $\mathcal{Q}$ be a j-sound $\mathcal{F}$-premouse over A. Let $(\mathcal{N}, k) \trianglelefteq (\mathcal{Q}, j)$. Let $\mathcal{M}$ be a k-relevant $\mathcal{F}$-pm over $\bar{A}$ and $\pi : \mathcal{M} \to \mathcal{N}$ an $\mathcal{F}$-rooted k-factor above b.*

Let $\Sigma_{\mathcal{Q}}$ be an $\mathcal{F}$-(j, ω_1+1)-strategy for $\mathcal{Q}$. Then there is an $\mathcal{F}$-(k, ω_1+1)-strategy $\Sigma_{\mathcal{M}}$ for $\mathcal{M}$ such that trees $\mathcal{T}$ via $\Sigma_{\mathcal{M}}$ lift to trees $\mathcal{U}$ via $\Sigma_{\mathcal{Q}}$. In fact, there is an insert set I for $\mathcal{U}$ and $\iota : \mathrm{lh}(\mathcal{T}) \to \mathrm{lh}(\mathcal{U})$ such that for each $\alpha < \mathrm{lh}(\mathcal{T})$, letting $\alpha' = \iota(\alpha)$, there is $N^{\mathcal{U}}_{\alpha} \trianglelefteq^{\uparrow} M^{\mathcal{U}}_{\alpha'}$ such that

$$(N^{\mathcal{U}}_{\alpha}, \deg^{\mathcal{T}}(\alpha)) \trianglelefteq^{\uparrow} (M^{\mathcal{U}}_{\alpha'}, \deg^{\mathcal{U}}(\alpha')),$$

and there is an $\mathcal{F}$-rooted $\deg^{\mathcal{T}}(\alpha)$-factor above b

$$\pi_{\alpha} : M^{\mathcal{T}}_{\alpha} \to N^{\mathcal{U}}_{\alpha},$$

and if π is good then π_{α} is good. Moreover, $[0,\alpha]_{\mathcal{T}} \cap D^{\mathcal{T}}$ model-drops iff $[0,\alpha']_{\mathcal{U},I}$ drops below the image of $\mathcal{N}$. If $[0,\alpha]_{\mathcal{T}} \cap D^{\mathcal{T}}$ does not model-drop then $N^{\mathcal{U}}_{\alpha} = M^{\mathcal{U},I}_{\mathcal{N},\alpha'}$ and

$$\pi_{\alpha} \circ i^{\mathcal{T}}_{0,\alpha} = i^{\mathcal{U},I}_{\mathcal{N},0,\alpha'} \circ \pi. \tag{3.1}$$

If either $[0,\alpha]_{\mathcal{T}}$ model-drops or $[(\mathcal{N},k)=(\mathcal{Q},j)$ and π is a near j-embedding$]$ then $N^{\mathcal{U}}_\alpha = M^{\mathcal{U}}_{\alpha'}$ and $\deg^{\mathcal{T}}(\alpha)=\deg^{\mathcal{U}}(\alpha')$ and π_α is a near $\deg^{\mathcal{T}}(\alpha)$-embedding.

The previous paragraph also holds with "(j,ω_1,ω_1+1)-maximal" replacing "(j,ω_1+1)" and "(k,ω_1,ω_1+1)-maximal" replacing "(k,ω_1+1)".

Proof. We just sketch the proof, for the k-maximal case. It is mostly the standard copying construction, augmented with propagation of near embeddings (see [3]), and the standard extra details dealing with type 3 premice (see [11] and [8]). We put $\alpha'\in I$ iff either (i) $E^{\mathcal{T}}_\alpha = F(M^{\mathcal{T}}_\alpha)$ and $N^{\mathcal{U}}_\alpha \ntrianglelefteq M^{\mathcal{U},I}_{\alpha'}$ (so $N^{\mathcal{U}}_\alpha \triangleleft^{\uparrow} M^{\mathcal{U},I}_{\alpha'}$) or (ii) $E^{\mathcal{T}}_\alpha \neq F(M^{\mathcal{T}}_\alpha)$ and $\pi^{\uparrow}_\alpha(\mathrm{lh}(E^{\mathcal{T}}_\alpha)) > \mathrm{o}(M^{\mathcal{U},I}_{\alpha'})$. It follows that if $\alpha'\in I$ then $M^{\mathcal{U}}_{\alpha'}$ is type 3 and $[0,\alpha]_{\mathcal{T}}$ does not drop in model; the latter is by arguments in [8]. When $\alpha'\in I$, we set $E^{\mathcal{U}}_{\alpha'} = F(M^{\mathcal{U}}_{\alpha'})$, and then define $E^{\mathcal{U}}_{\alpha'+1}$ by copying $E^{\mathcal{T}}_\alpha$ with π_α (and then $(\alpha+1)'=\alpha'+2$). We omit the remaining, standard, details regarding the correspondence of tree structures and definition of $\iota, N^{\mathcal{U}}_\alpha, \pi_\alpha$.

Now the main thing is to observe that for each α, π_α is an $\mathcal{F}$-rooted $\deg^{\mathcal{T}}(\alpha)$-factor (above b; for the rest of the proof we omit that phrase). For given this, fine condensation, together with 3.25, gives that $M^{\mathcal{T}}_\alpha$ is an $\mathcal{F}$-pm. (If $M^{\mathcal{T}}_\alpha$ might be type 3 (i.e. $N^{\mathcal{U}}_\alpha$ is type 3), then 3.25 applies, because $(N^{\mathcal{U}}_\alpha)^{\uparrow}$ is an $\mathcal{F}$-pm, because we can extend $\mathcal{U}\restriction(\alpha'+1)$ to a tree $\mathcal{U}'$, setting $E^{\mathcal{U}'}_{\alpha'} = F(N^{\mathcal{U}}_\alpha)$.) Fix $(\mathcal{L}_0,\sigma_0)$ witnessing the fact that π is a (good) $\mathcal{F}$-rooted k-factor above b.

Suppose that $[0,\alpha]_{\mathcal{T}}$ does not drop in model. Then it is routine that $[0,\alpha']_{\mathcal{U},I}$ does not drop below the image of $\mathcal{N}$, π_α is a weak $\deg^{\mathcal{T}}(\alpha)$-embedding and line (3.1) holds. If $\deg^{\mathcal{T}}(\alpha)=k$ then it follows that $(\mathcal{L}_0,\sigma)$ witnesses the fact that π_α is a (good) $\mathcal{F}$-rooted k-factor above b, where $\sigma = i^{\mathcal{T}}_{0,\alpha}\circ\sigma_0$, because $i^{\mathcal{U},I}_{\mathcal{N},0,\alpha'}$ and $\pi\circ\sigma_0$ are both near k-embeddings, and $\pi_\alpha\circ i^{\mathcal{T}}_{0,\alpha} = i^{\mathcal{U},I}_{\mathcal{N},0,\alpha'}\circ\pi$.

Suppose further that $[0,\alpha]_{\mathcal{T}}$ drops in degree and let $n=\deg^{\mathcal{T}}(\alpha)$. Then letting $\mathcal{L}=\mathfrak{C}_{n+1}(M^{\mathcal{T}}_\alpha)$ and $\sigma:\mathcal{L}\to M^{\mathcal{T}}_\alpha$ be the core embedding, $(\mathcal{L},\sigma)$ witnesses the fact that π_α is a (good) $\mathcal{F}$-rooted n-factor above b (we have $\mathfrak{S}_{k+1}(\mathcal{L})=\mathcal{L}$ and $\sigma^{\mathcal{L}}_{k+1}=\mathrm{id}$). The fact that $\mathcal{L}$ is n-relevant is verified as follows. There is $\beta+1\leq_{\mathcal{T}}\alpha$ such that $\mathcal{L}=M^{*\mathcal{T}}_{\beta+1}$ and $\sigma = i^{*\mathcal{T}}_{\beta+1,\alpha}$. Suppose that $\mathcal{L}$ is a successor. Then letting $\xi=\mathrm{pred}^{\mathcal{T}}(\beta+1)$, we have $\mathrm{lh}(E^{\mathcal{T}}_\xi)\leq\mathrm{o}(\mathcal{L}^-)$. So letting $\kappa=\mathrm{crit}(\sigma)$, $E^{\mathcal{T}}_\beta$ measures only $\mathfrak{P}(\kappa)\cap\mathcal{L}^-$. But since $\mathcal{L}^-\triangleleft M^{*\mathcal{T}}_{\beta+1}$, therefore $\kappa<\rho^{\mathcal{L}^-}_\omega$. But $\rho^{\mathcal{L}}_{n+1}\leq\kappa$, which suffices. The fact that $\pi_\alpha\circ\sigma$ is a near n-embedding is because $\pi_\alpha\circ\sigma = i^{\mathcal{U},I}_{\mathcal{N},\xi',\alpha'}\circ\pi_\xi$ and π_ξ is a weak $(n+1)$-embedding, and $i^{\mathcal{U},I}_{\mathcal{N},\xi',\alpha'}$ a near n-embedding.

Now suppose that $[0,\alpha]_{\mathcal{T}}$ drops in model. It is straightforward to see that $[0,\alpha']_{\mathcal{U},I}$ drops below the image of $\mathcal{N}$ and that $N^{\mathcal{U}}_\alpha = M^{\mathcal{U}}_{\alpha'}$. The fact that π_α is an $\mathcal{F}$-rooted $\deg^{\mathcal{T}}(\alpha)$-factor is almost the same as in the dropping degree case above. The fact that π_α is in fact a near $\deg^{\mathcal{T}}(\alpha)$-embedding and $\deg^{\mathcal{T}}(\alpha) = \deg^{\mathcal{U}}(\alpha')$ follows from an examination of the proof that near embeddings are propagated by the copying construction in [3]; similar arguments are given in [8]. □

We next consider constructions building $\mathcal{F}$-mice.

Definition 3.27. Let $\mathcal{N}$ be an $\mathcal{F}$-pm and $k \leq \omega$. Then $\mathcal{N}$ is **$\mathcal{F}$-k-fine** iff for each $j \leq k$:

- $\mathfrak{C}_j(\mathcal{N})$ is a j-solid $\mathcal{F}$-pm,
- if $j < k$ then $\mathfrak{C}_j(\mathcal{N})$ is $(j+1)$-universal,
- if $k = \omega$ then $\mathfrak{C}_\omega(\mathcal{N})$ is $< \omega$-condensing.

⊣

Definition 3.28. Let $\mathcal{F}$ be an operator over $\mathscr{B}$. Let $A \in \widehat{C_{\mathcal{F}}}$ and $\chi \leq \mathrm{o}(\mathscr{B})+1$. An **$L^{\mathcal{F}}[\mathbb{E}, A]$-construction (of length χ)** is a sequence $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha<\chi}$ such that for all $\alpha < \chi$:

- $\mathcal{N}_0 = \mathcal{F}(A)$ and $\mathcal{N}_\alpha$ is an $\mathcal{F}$-pm over A.
- If α is a limit then $\mathcal{N}_\alpha = \liminf_{\beta<\alpha} \mathcal{N}_\beta$.
- If $\alpha + 1 < \chi$ then either (i) $\mathcal{N}_{\alpha+1}$ is E-active and $\mathcal{N}_{\alpha+1}||\mathrm{o}(\mathcal{N}_{\alpha+1}) = \mathcal{N}_\alpha$, or (ii) $\mathcal{N}_\alpha$ is $\mathcal{F}$-ω-fine and $\mathcal{N}_{\alpha+1} = \mathcal{F}(\mathfrak{C}_\omega(\mathcal{N}_\alpha))$.

We say that $\mathbb{C}$ is **$\mathcal{F}$-tenable** iff $\mathcal{N}^{\uparrow}$ is an $\mathcal{F}$-pm for each $\alpha < \chi$. ⊣

We will now explain how condensation for $\mathcal{F}$ leads to the $\mathcal{F}$-iterability of substructures $\mathcal{R}$ of $\mathcal{F}$-pms built by background construction. The basic engine behind this is the realizability of iterates of $\mathcal{R}$ back into models of the construction.

Definition 3.29. Let $(\mathcal{F}, b, \bar{A}, A)$ be an almost fine ground $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha\leq\lambda}$ be an $L^{\mathcal{F}}[\mathbb{E}, A]$-construction. Let $k \leq \omega$ and suppose that $\mathcal{N}_\lambda$ is $\mathcal{F}$-k-fine. Let $\mathcal{R}$ be a k-sound $\mathcal{F}$-pm over $\bar{A}$ and $\pi : \mathcal{R} \to \mathfrak{C}_k(\mathcal{N}_\lambda)$ be a weak k-embedding. Let $\mathcal{T}$ be a putative $\mathcal{F}$-iteration tree on $\mathcal{R}$, with $\deg^{\mathcal{T}}(0) = k$. We say that $\mathcal{T}$ is **$(\pi, \mathbb{C})$-realizable above** b iff for every $\alpha < \mathrm{lh}(\mathcal{T})$, letting $\beta = \mathrm{base}^{\mathcal{T}}(\alpha)$ and $m = \deg^{\mathcal{T}}(\alpha)$, there are ζ, τ such that:

– $(\zeta, m) \leq_{\text{lex}} (\lambda, k)$,

– if $[0,\alpha]_{\mathcal{T}}$ does not drop in model or degree then $\zeta = \lambda$ and $\tau = \pi$,

– if $[0,\alpha]_{\mathcal{T}}$ drops in model or degree then $\tau\colon M^{*\mathcal{T}}_\beta \to \mathfrak{C}_m(\mathcal{N}_\zeta)$ is a near m-embedding above b,

– if $M^{*\mathcal{T}}_\beta$ is not type 3 then there is a weak m-embedding $\varphi : M^{\mathcal{T}}_\alpha \to \mathfrak{C}_m(\mathcal{N}_\zeta)$ such that $\varphi \circ i^{*\mathcal{T}}_{\beta,\alpha} = \tau$.

– if $M^{*\mathcal{T}}_\beta$ is type 3 then there is a weak m-embedding $\varphi : \mathcal{S} \to \mathfrak{C}_m(N_\zeta)$ such that $\varphi \circ i^{*\mathcal{T}}_{\beta,\alpha} = \tau$, where $\mathcal{S}$ is "$(M^{\mathcal{T}}_\alpha)^{\text{sq}}$".[27]

We say that $\mathcal{T}$ is **weakly $(\pi, \mathbb{C})$-realizable** iff in some set-generic extension $V[G]$, either $\mathcal{T}$ is $(\pi,\mathbb{C})$-realizable, or there is a limit $\lambda \leq \text{lh}(\mathcal{T})$ and a $(\mathcal{T}\restriction\lambda)$-cofinal branch b such that $(\mathcal{T}\restriction\lambda) \frown b$ is $(\pi,\mathbb{C})$-realizable. ⊣

Definition 3.30. A **putative $\mathcal{F}$-(k,θ)-iteration strategy** for a k-sound $\mathcal{F}$-pm $\mathcal{N}$ is a function Σ such that for every k-maximal $\mathcal{F}$-tree $\mathcal{T}$ on $\mathcal{N}$, with $\mathcal{T}$ via Σ and $\text{lh}(\mathcal{T}) < \theta$ a limit, $\Sigma(\mathcal{T})$ is a $\mathcal{T}$-cofinal branch. ⊣

Lemma 3.31. *Let $(\mathcal{F}, b, \bar{A}, A)$ be an almost fine ground. Let $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha<\chi}$ be a tenable $L^{\mathcal{F}}[\mathbb{E}, A]$-construction. Let $\lambda < \chi$ and $k \leq \omega$ be such that $\mathcal{N}_\lambda$ is $\mathcal{F}$-k-fine, and let $\mathcal{S} = \mathfrak{C}_k(\mathcal{N}_\lambda)$. Let $\mathcal{R}$ be a k-relevant $\mathcal{F}$-pm over $\bar{A}$. Let $\pi : \mathcal{R} \to \mathcal{S}$ be an $\mathcal{F}$-rooted k-factor above b. Let Σ be either:*

– *a putative $\mathcal{F}$-$(k, \omega_1 + 1)$-iteration strategy for $\mathcal{R}$, or*

– *a putative $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)$-maximal iteration strategy for $\mathcal{R}$.*

Suppose that every putative $\mathcal{F}$-tree via Σ is $(\pi, \mathbb{C})$-realizable above b. Then Σ is an $\mathcal{F}$-$(k, \omega_1 + 1)$, or $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)$-maximal, iteration strategy.

Proof. The argument is almost that used for 3.26, using the maps provided by $(\pi, \mathbb{C})$-realizability in place of copy maps. The tenability of $\mathbb{C}$ is used to see that 3.25 applies where needed. □

[27] $(M^{\mathcal{T}}_\alpha)^{\text{sq}}$ might not make literal sense, if say $M^{\mathcal{T}}_\alpha$ is not wellfounded. By "$(M^{\mathcal{T}}_\alpha)^{\text{sq}}$" we mean that either $\alpha = \xi + 1$ and $\mathcal{S} = \text{Ult}_m((M^{*\mathcal{T}}_\alpha)^{\text{sq}}, E^{\mathcal{T}}_\xi)$ (formed without unsquashing), or α is a limit and $\mathcal{S}$ is the direct limit of the structures $(M^{\mathcal{T}}_\xi)^{\text{sq}}$ for $\xi \in [\beta, \alpha)_{\mathcal{T}}$, under the iteration maps.

In practice, we will take $\mathcal{R}$ and $\pi : \mathcal{R} \to \mathcal{S}$ to be fully elementary, which will give that π is an $\mathcal{F}$-rooted k-factor. The above proof does not work with $(k, \omega_1, \omega_1 + 1)$-*maximal* replaced by $(k, \omega_1, \omega_1 + 1)$.

Remark 3.32. We digress to mention a key application of the extra strength that *condenses finely* has compared to *almost condenses finely*; this essentially comes from [9]. Adopt the assumptions and notation of the first paragraph of 3.31. Assume further that $(\mathcal{F}, b, \bar{A}, A)$ is a fine ground (not just almost), $\mathscr{B} = V$ and $\mathcal{F}$ is total. For an $\mathcal{F}$-premouse $\mathcal{M}$, say that $\mathcal{M}$ is **$\mathcal{F}$-full** iff there is no $\alpha \in \mathrm{Ord}$ such that $\mathcal{F}^\alpha(\mathcal{M})$ projects $< \mathrm{o}(\mathcal{M})$.[28] Assume also that there is no $\mathcal{F}$-full $\mathcal{M}$ such that $\mathrm{o}(\mathcal{M})$ is Woodin in $\mathcal{F}^{\mathrm{Ord}}(\mathcal{M})$. Let κ be a cardinal. Suppose that every k-maximal putative $\mathcal{F}$-tree $\mathcal{T}$ on $\mathcal{R}$ of length $\leq \kappa$ is weakly $(\pi, \mathbb{C})$-realizable. Then $\mathcal{R}$ is $\mathcal{F}$-$(k, \kappa + 1)$-iterable, via the strategy guided by Q-structures of the form $\mathcal{F}^\alpha(M(\mathcal{T}))$ for some $\alpha \in \mathrm{Ord}$.[29] This follows by a straightforward adaptation of the proof for standard premice (cf. [9]). In the argument one needs to apply *condenses finely* to embeddings φ, σ when $\varphi \circ \sigma \notin V$. We can only expect $\varphi \circ \sigma \in V$ if the realized branch does not drop in model or degree (indeed, in the latter case, $\varphi \circ \sigma = \pi$), or if all relevant objects are countable.

From now on we will only deal with *almost condenses finely*.

We use the following variant of the weak Dodd Jensen property of [2], extended to deal partially with good k-factors, analogously to how weak k-embeddings are dealt with in [8, §4.2].

Definition 3.33. Let $k \leq \omega$ and $\mathcal{M}$ be a countable k-relevant opm.

A k-factor $\pi : \mathcal{M} \to \mathcal{N}$ is **simple** iff it is witnessed by $(\mathcal{L}, \sigma) = (\mathcal{M}, \mathrm{id})$.

An iteration tree is **relevant** iff it has countable, successor length. We say that $(\mathcal{T}, \mathcal{Q}, \pi)$ is **$(\mathcal{M}, k)$-simple** iff $\mathcal{T}$ is a relevant (k, ∞, ∞)-maximal tree, $\mathcal{Q} \trianglelefteq M^{\mathcal{T}}_\infty$ and $\pi : \mathcal{M} \to \mathcal{Q}$ is a good simple k-factor.[30]

Let Σ be an iteration strategy for $\mathcal{M}$. Let $\vec{\alpha} = \langle \alpha_n \rangle_{n<\omega}$ enumerate $\mathrm{o}(\mathcal{M})$. We say that Σ has the **k-simple Dodd-Jensen (DJ) property for $\vec{\alpha}$** iff

[28] Here $\mathcal{F}^\alpha(\mathcal{M})$ is the unique $\mathcal{F}$-pm $\mathcal{N}$ such that $\mathcal{M} \trianglelefteq \mathcal{N}$ and $l(\mathcal{N}) = l(\mathcal{M}) + \alpha$ and $\mathcal{N}|\beta$ is E-passive for every $\beta \in (l(\mathcal{M}), l(\mathcal{N})]$.

[29] It might be that the Q-structure satisfies "$\delta(\mathcal{T})$ is not Woodin", but in this case, $\alpha = \beta + 1$ for some β and $\mathcal{F}^\beta(M(\mathcal{T}))$ satisfies "$\delta(\mathcal{T})$ is Woodin".

[30] So $\mathcal{Q}$ is k-sound; the (k, ∞, ∞)-maximality of $\mathcal{T}$ then implies that if $\mathcal{Q} = M^{\mathcal{T}}_\infty$ then $\deg^{\mathcal{T}}(\infty) \geq k$. So we do not need to explicitly stipulate that $\deg^{\mathcal{T}}(\infty) \geq k$, unlike in [8].

for all $(\mathcal{M},k)$-simple $(\mathcal{T},\mathcal{Q},\pi)$ with $\mathcal{T}$ via Σ, we have $\mathcal{Q}=M^{\mathcal{T}}_\infty$ and $b^{\mathcal{T}}$ does not drop in model (or degree), and if π is also nearly k-good, then

$$i^{\mathcal{T}}\restriction\mathrm{o}(\mathcal{M})\leq^{\vec{\alpha}}_{\mathrm{lex}}\pi\restriction\mathrm{o}(\mathcal{M})$$

(that is, either $i^{\mathcal{T}}\restriction\mathrm{o}(\mathcal{M})=\pi\restriction\mathrm{o}(\mathcal{M})$, or $i^{\mathcal{T}}(\alpha_n)<\pi(\alpha_n)$ where $n<\omega$ is least such that $i^{\mathcal{T}}(\alpha_n)\neq\pi(\alpha_n)$). ⊣

Note that in the context above, if $i^{\mathcal{T}}\restriction\mathrm{o}(\mathcal{M})=\pi\restriction\mathrm{o}(\mathcal{M})$, then $i^{\mathcal{T}}=\pi$, because $i^{\mathcal{T}},\pi$ are both nearly 0-good, and $\mathcal{M}=\mathrm{Hull}^{\mathcal{M}}_1(cb^{\mathcal{M}}\cup\mathrm{o}(\mathcal{M}))$.

Lemma 3.34. *Assume* $\mathsf{DC}_{\mathbb{R}}$. *Let* $(\mathcal{F},b,A)$ *be an almost fine ground with* $A\in\mathrm{HC}$. *Let* $\mathcal{M}$ *be a countable,* $\mathcal{F}$-(k,ω_1,ω_1+1)-*maximally iterable* k-*relevant* $\mathcal{F}$-*pm. Let* $\vec{\alpha}=\langle\alpha_n\rangle_{n<\omega}$ *enumerate* $\mathrm{o}(\mathcal{M})$. *Then there is an* $\mathcal{F}$-(k,ω_1,ω_1+1)-*maximal strategy for* $\mathcal{M}$ *with the* k-*simple DJ property for* $\vec{\alpha}$.

Proof Sketch. The proof is mostly like the usual one (see [2]), with adaptations much as in [8, Lemma 4.6(?)]. Let Σ be an $\mathcal{F}$-(k,ω_1,ω_1+1)-maximal strategy for $\mathcal{M}$. Given a relevant tree $\mathcal{T}$ via Σ, $\mathcal{P}=M^{\mathcal{T}}_\infty$ and $m=\deg^{\mathcal{T}}(\infty)$, let $\Sigma^{\mathcal{T}}_{\mathcal{P}}$ be the (m,ω_1,ω_1+1)-maximal tail of Σ for $\mathcal{P}$. If $(\mathcal{T},\mathcal{Q},\pi)$ is also $(\mathcal{M},k)$-simple, let $\Sigma^{\mathcal{T},\mathcal{Q},\pi}_{\mathcal{M}}$ be the (k,ω_1,ω_1+1)-maximal strategy for $\mathcal{M}$ given by π-pullback (as in 3.26).

Note that $(\mathcal{T},\mathcal{M},\mathrm{id})$ is $(\mathcal{M},k)$-simple where $\mathcal{T}$ is trivial on $\mathcal{M}$. Let $(\mathcal{T}_0,\mathcal{Q}_0,\pi_0)$ be $(\mathcal{M},k)$-simple, with $\mathcal{T}_0$ via Σ, and $\mathcal{P}_0=M^{\mathcal{T}_0}_\infty$, such that for any $(\mathcal{M},k)$-simple $(\mathcal{T},\mathcal{Q},\pi)$ via $\Sigma^{\mathcal{T}_0}_{\mathcal{P}_0}$, we have that $b^{\mathcal{T}}$ does not drop in model or degree, if $\mathcal{Q}_0=\mathcal{P}_0$ then $\mathcal{Q}=M^{\mathcal{T}}_\infty$, and if $\mathcal{Q}_0\triangleleft\mathcal{P}_0$ then $(i^{\mathcal{T}})^{\uparrow}(\mathcal{Q}_0)\trianglelefteq\mathcal{Q}$ (see 3.22). (The existence of $\mathcal{T}_0$, etc, follows from $\mathsf{DC}_{\mathbb{R}}$.)

Let $\Sigma_1=\Sigma^{\mathcal{T}_0,\mathcal{Q}_0,\pi_0}_{\mathcal{M}}$. Working as in the standard proof (see [2]), let $\mathcal{T}_1$ be a relevant tree via Σ_1, with $b^{\mathcal{T}_1}$ not dropping in model or degree, and let $\pi_1:\mathcal{M}\to\mathcal{P}_1=M^{\mathcal{T}_1}_\infty$ be nearly k-good, such that for all relevant trees $\mathcal{T}$ via $\Sigma^{\mathcal{T}_1}_{\mathcal{P}_1}$, if $b^{\mathcal{T}}$ does not drop in model or degree, then for any near k-embedding $\pi:\mathcal{M}\to\mathcal{M}^{\mathcal{T}}_\infty$, we have $i^{\mathcal{T}}\circ\pi_1\leq^{\vec{\alpha}}_{\mathrm{lex}}\pi$.

Let $\Sigma_2=(\Sigma_1)^{\mathcal{T}_1,\mathcal{P}_1,\pi_1}_{\mathcal{M}}$. Then Σ_2 is as desired; cf. [8]. (Use the propagation of near embeddings after drops in model given by 3.26, as in [8].) □

Definition 3.35. Let $\mathcal{M}$ be a k-sound opm and let $q=p^{\mathcal{M}}_{k+1}$. For $i<\mathrm{lh}(p^{\mathcal{M}}_{k+1})$, $\mathcal{H}=\mathfrak{W}_{k+1,i}(\mathcal{M})$ denotes the corresponding solidity witness

$$\mathcal{H}=\mathrm{cHull}^{\mathcal{M}}_{k+1}(q_i\cup\{q\restriction i\}\cup\vec{p}^{\,\mathcal{M}}_k),$$

and $\varsigma_{k+1,i}(\mathcal{M})$ denotes the uncollapse map $\mathcal{H}\to\mathcal{M}$. ⊣

We can now state the central result of the paper – the fundamental fine structural facts for $\mathcal{F}$-premice. The definitions **$\mathcal{F}$-pseudo-premouse** and **$\mathcal{F}$-bicephalus**, and the **$\mathcal{F}$-iterability** of such structures, are the obvious ones. Likewise the definition of **$\mathcal{F}$-iterability** for phalanxes of $\mathcal{F}$-pms.

Theorem 3.36. *Let $(\mathcal{F}, b, A)$ be an almost fine ground with $b \in \mathrm{HC}$. Then:*

1. *For $k < \omega$, every k-sound, $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)$-maximally iterable $\mathcal{F}$-premouse over A is $\mathcal{F}$-$(k+1)$-fine.*
2. *Every ω-sound, $\mathcal{F}$-$(\omega, \omega_1, \omega_1 + 1)$-maximally iterable $\mathcal{F}$-premouse over A is $< \omega$-condensing.*
3. *Every $\mathcal{F}$-$(0, \omega_1, \omega_1 + 1)$-maximally iterable $\mathcal{F}$-pseudo-premouse over A is an $\mathcal{F}$-premouse.*
4. *There is no $\mathcal{F}$-$(0, \omega_1, \omega_1 + 1)$-maximally iterable $\mathcal{F}$-bicephalus over A.*

Proof Sketch. We sketch enough of the proof of parts 1 and 2, focusing on the new aspects, that by combining these sketches with the full proofs of these facts for standard premice, one obtains a complete proof. So one should have those proofs in mind (see [1], [11], [8]). Part 3 involves similar modifications to the standard proof, and part 4 is an immediate transcription. We begin with part 1.

Let $\mathcal{M}$ be a k-sound, $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)$-maximally iterable $\mathcal{F}$-premouse. We may assume that $\rho_{k+1}^{\mathcal{M}} < \rho_k^{\mathcal{M}}$, and by 2.41, that $\mathcal{M}$ is k-relevant. We may assume that $\mathcal{M}$ is countable (otherwise we can replace $\mathcal{M}$ with a countable elementary substructure, because $\mathcal{F}$ almost condenses coarsely above $b \in \mathrm{HC}$ and $\mathscr{B} \models \mathsf{DC}$).

Let Σ_0 be an $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)$-maximal iteration strategy for $\mathcal{M}$. We would like to use 3.34, but that lemma assumes $\mathsf{DC}_{\mathbb{R}}$. But we may assume $\mathsf{DC}_{\mathbb{R}}$. For we can pass to $W = L^{\mathcal{F}, \Sigma_0}[x]$, where $x \in \mathbb{R}$ codes $\mathcal{M}$.[31] (The hypotheses of the theorem hold in W regarding $b, A, \mathcal{M}, \mathcal{F}^W, \Sigma_0^W$, (and $\mathscr{B}^W$), where $\mathscr{B}^W, \mathcal{F}^W, \Sigma_0^W$ are the natural restrictions of $\mathscr{B}, \mathcal{F}, \Sigma_0$.)

Now using 3.34, let Σ be an $\mathcal{F}$-$(k, \omega_1 + 1)$ iteration strategy for $\mathcal{M}$ with the k-simple DJ property for some enumeration of $\mathrm{o}(\mathcal{M})$. We assume that $\mathcal{M}$ is a successor, since the contrary case is simpler and closer to the standard proof.

[31] We don't care about the fine structure of W, so it doesn't matter exactly how we feed in $\mathcal{F}, \Sigma_0$.

We first establish $(k+1)$-universality and that $\mathcal{C} = \mathfrak{C}_{k+1}(\mathcal{M})$ is an $\mathcal{F}$-pm. Let $\pi : \mathcal{C} \to \mathcal{M}$ be the core map. We may assume that $\mathcal{M}$ is k-relevant, because otherwise $\mathcal{C} = \mathcal{M}$ and $\pi = \mathrm{id}$.

First suppose $k = 0$, and consider 1-universality. Because π is 0-good and by 2.33, $\mathcal{C}$ is a Q-opm, $\mathcal{C}$ is a successor and $\pi(\mathcal{C}^-) = \mathcal{M}^-$. By fine condensation and 3.19, $\mathcal{H} = \mathcal{F}(\mathcal{C}^-)$ is a universal hull of $\mathcal{C}$, as witnessed by $\sigma : \mathcal{H} \to \mathcal{C}$. Also, $\mathcal{C}$ is 0-relevant. For otherwise, by the proof of 3.19, $\mathcal{H} \in \mathcal{M}$, but then $\mathcal{C} \in \mathcal{M}$, a contradiction. So

$$\rho =_{\mathrm{def}} \rho_1^{\mathcal{M}} = \rho_1^{\mathcal{C}} < \rho_\omega^{\mathcal{C}^-},$$

and since $\mathcal{H}^- = \mathcal{C}^-$, therefore $\mathcal{C}||(\rho^+)^{\mathcal{C}} = \mathcal{H}||(\rho^+)^{\mathcal{H}}$. So it suffices to see that $\mathcal{M}||(\rho^+)^{\mathcal{M}} = \mathcal{H}||(\rho^+)^{\mathcal{H}}$.

Let $\rho = \rho_1^{\mathcal{M}}$. The phalanx $\mathfrak{P} = ((\mathcal{M}, <\rho), \mathcal{H})$ is $\mathcal{F}$-$((0,0), \omega_1 + 1)$-maximally iterable.[32] Moreover, we get an $\mathcal{F}$-$((0,0), \omega_1+1)$-iteration strategy for $\mathfrak{P}$ given by lifting to k-maximal trees on $\mathcal{M}$ via Σ. This is proved as usual, using $\pi \circ \sigma$ to lift $\mathcal{H}$ to $\mathcal{M}$, and using calculations as in 3.26 to see that the strategy is indeed an $\mathcal{F}$-strategy. Since our strategies are $\mathcal{F}$-strategies, we can therefore compare $\mathfrak{P}$ with $\mathcal{M}$. The analysis of the comparison is mostly routine, using the k-simple DJ property. (Here all copy embeddings are near embeddings, so we only actually need the weak DJ property.) The only, small, difference is when $b^{\mathcal{T}}$ is above $\mathcal{H}$ without drop and $M_\infty^{\mathcal{T}} \trianglelefteq M_\infty^{\mathcal{U}}$. Because $\mathcal{H}$ is a universal hull of $\mathcal{C} = \mathfrak{C}_1(\mathcal{M})$, this implies that $b^{\mathcal{U}}$ does not drop and $M_\infty^{\mathcal{T}} = M_\infty^{\mathcal{U}}$; now deduce that $\mathcal{M}||(\rho^+)^{\mathcal{M}} = \mathcal{H}||(\rho^+)^{\mathcal{H}}$ as usual, completing the proof.

We now show that $\mathcal{C} = \mathcal{H}$, and therefore that $\mathcal{C}$ is an $\mathcal{F}$-pm. Because $\mathcal{H}$ is a universal hull of $\mathcal{C}$ and $\mathcal{C}$ is 0-relevant, we have $\rho_1^{\mathcal{H}} = \rho < \rho_\omega^{\mathcal{H}^-}$ (as $\mathcal{H}^- = \mathcal{C}^-$) and $p_1^{\mathcal{C}} \leq \sigma(p_1^{\mathcal{H}})$. But $\mathcal{H}$ is $(1, q^{\mathcal{H}})$-solid, so $\mathcal{C}$ is $(1, \sigma(q^{\mathcal{H}}))$-solid (using stratification), so $\sigma(q^{\mathcal{H}}) \trianglelefteq p_1^{\mathcal{C}}$. And since σ is above $\mathcal{C}^-$, it follows that $\sigma(p_1^{\mathcal{H}}) = p_1^{\mathcal{C}}$. But by 1-universality, $\pi(p_1^{\mathcal{C}}) = p_1^{\mathcal{M}}$, so $\mathcal{C} = \mathrm{Hull}_1^{\mathcal{C}}(A \cup \rho \cup p_1^{\mathcal{C}})$, so $\mathcal{H} = \mathcal{C}$ and $\sigma = \mathrm{id}$, completing the proof.

Now suppose $k > 0$. Then $\mathcal{C} = \mathfrak{C}_{k+1}(\mathcal{M})$ is an opm by 2.39, and is k-relevant as $\rho_{k+1}^{\mathcal{C}} < \rho_k^{\mathcal{C}} \leq \rho_\omega^{\mathcal{C}^-}$. So by fine condensation and 3.19, $\mathcal{C} = \mathcal{F}(\mathcal{C}^-)$ is an $\mathcal{F}$-pm. The rest is a simplification of the argument for $k = 0$.

[32] A $(k_0, k_1, \ldots, k)$-maximal tree on a phalanx $((M_0, \rho_0), (M_1, \rho_1), \ldots, H)$, is one formed according to the usual rules for k-maximal trees, except that an extender E with $\rho_{i-1} \leq \mathrm{crit}(E) < \rho_i$ (where $\rho_{-1} = 0$) is applied to M_i, at degree k_i.

Now consider $(k+1)$-solidity. Let $q = p_{k+1}^{\mathcal{M}}$ and $i < \mathrm{lh}(q)$ and $\mathcal{W} = \mathfrak{W}_{k+1,i}(\mathcal{M})$ and $\pi = \varsigma_{k+1,i}$. We have

$$\rho_{k+1}^{\mathcal{W}} \leq \mu =_{\mathrm{def}} \mathrm{crit}(\pi) = q_i.$$

By 2.35 we may assume that π is k-good, so $\mathcal{W}$ is a k-sound successor Q-opm and $\pi(\mathcal{W}^-) = \mathcal{M}^-$. By 2.38 we may assume that $\mu < \rho_\omega^{\mathcal{M}^-}$, so $\mu \leq \rho_\omega^{\mathcal{W}^-}$. Suppose $\mu = \rho_\omega^{\mathcal{W}^-}$. Then since $\mathcal{M}^-$ is $<\omega$-condensing, $\mathcal{F}(\mathcal{W}^-) \in \mathcal{M}^-$. But by the fine condensation of $\mathcal{F}$, $\mathcal{W}$ is computable from $\mathcal{F}(\mathcal{W}^-)$, so $\mathcal{W} \in \mathcal{M}$, as required. So we may assume that $\mu < \rho_\omega^{\mathcal{W}^-}$, so $\mathcal{W}$ is k-relevant, so $\mathcal{W} \notin \mathcal{F}(\mathcal{W}^-)$ and if $k = 0$ then $\mathcal{W}$ has no universal hull in $\mathcal{F}(\mathcal{W}^-)$.

If $k = 0$, let $\mathcal{H} = \mathcal{F}(\mathcal{W}^-)$; by fine condensation, $\mathcal{H}$ is an $\mathcal{F}$-pm, and is a universal hull of $\mathcal{W}$. If $k > 0$ then $\mathcal{W}$ is an opm, so by fine condensation, $\mathcal{W} = \mathcal{F}(\mathcal{W}^-)$ is an $\mathcal{F}$-pm. If $k > 0$, let $\mathcal{H} = \mathcal{W}$.

Let us assume that μ is not a cardinal of $\mathcal{M}$, since the contrary case is easier. So $\mu = (\kappa^+)^{\mathcal{H}} = (\kappa^+)^{\mathcal{W}}$ for some $\mathcal{M}$-cardinal κ. Let $\mathcal{R} \triangleleft \mathcal{M}$ be least such that $\mu \leq \mathrm{o}(\mathcal{R})$ and $\rho_\omega^{\mathcal{R}} = \kappa$. Let $\mathfrak{P} = ((\mathcal{M}, <\kappa), (\mathcal{R}, <\mu), \mathcal{H})$. Then $\mathfrak{P}$ is (k, r, k)-maximally iterable, where r is least such that $\rho_{r+1}^{\mathcal{R}} = \kappa$, by lifting to k-maximal trees $\mathcal{V}$ on $\mathcal{M}$ (possibly $r = -1$, i.e. $\mathcal{R}$ is active type 3 with $\mu = \mathrm{o}(\mathcal{R})$). Let $I \subseteq \mathrm{lh}(\mathcal{V})$ be the resulting insert set. Let $(\mathcal{T}, \mathcal{U})$ be the successful comparison of $(\mathfrak{P}, \mathcal{M})$. The analysis of the comparison is now routine except in the case that either (i) $k = 0$ and $b^{\mathcal{T}}$ is above $\mathcal{H}$ without drop and $M_\infty^{\mathcal{T}} \trianglelefteq M_\infty^{\mathcal{U}}$, or (ii) $b^{\mathcal{T}}$ is above $\mathcal{R}$ and does not model-drop, $b^{\mathcal{U}}$ does not drop in model or degree and $M_\infty^{\mathcal{T}} = \mathcal{Q} = M_\infty^{\mathcal{U}}$. (As in [8], when we are not in case (ii), the final copy map π_∞ is a near $\deg^{\mathcal{T}}(\infty)$-embedding.)

We deal with case (i) much as in the proof of 1-universality. Let $\mathcal{H}' = M_\infty^{\mathcal{T}}$. Suppose that $b^{\mathcal{U}}$ does not drop and $\mathcal{H}' = M_\infty^{\mathcal{U}}$. As usual, we have that $\rho \leq \mathrm{crit}(i^{\mathcal{U}})$. So letting $t = \mathrm{Th}_1^{\mathcal{M}}(A \cup \rho \cup p_1^{\mathcal{M}})$, t is $\utilde{\Sigma}_1^{\mathcal{H}'}$, so is $\utilde{\Sigma}_1^{\mathcal{H}}$, so is $\utilde{\Sigma}_1^{\mathcal{W}}$, a contradiction as usual. So either $b^{\mathcal{U}}$ drops or $\mathcal{H}' \triangleleft M_\infty^{\mathcal{U}}$. But then as usual, $\mathcal{H} \in \mathcal{M}$, so $\mathcal{W} \in \mathcal{M}$, so we are done.

Now consider case (ii), under which $r \geq 0$. So $k \leq l =_{\mathrm{def}} \deg^{\mathcal{T}}(\infty)$, and the final copy map $\pi_\infty : M_\infty^{\mathcal{T}} \to M_{\mathcal{R},\infty}^{\mathcal{V},I}$ is a weak l-embedding. If $k < l$ then π_∞ is near k, which contradicts k-simple DJ (in fact weak DJ). So suppose $k = l$. If $k = r$ then fairly standard arguments (such as in [8]) give a contradiction, so suppose $k < r$. Then

$$\pi_\infty \circ i^{\mathcal{U}} : \mathcal{M} \to M_{\mathcal{R},\infty}^{\mathcal{V},I}$$

is a good simple k-factor, as witnessed by $\mathcal{L} = \mathcal{M}$ and $\sigma = \mathrm{id}$; indeed,

$$\pi_\infty \circ i^{\mathcal{U}} \circ \sigma_{k+1}^{\mathcal{M}} : \mathfrak{S}_{k+1}(\mathcal{M}) \to M_{\mathcal{R},\infty}^{\mathcal{V},I}$$

is nearly k-good, which is proved just as in [8], which also implies that $\pi_\infty \circ i^{\mathcal{U}}$ is weakly k-good, because $\sigma^{\mathcal{M}}_{k+1}$ is k-good. Since $\mathcal{R} \triangleleft \mathcal{M}$, this contradicts k-simple DJ. (This is the only place we need k-simple DJ beyond weak DJ.)

Now consider part 2. Let $k < \omega$ and let $\mathcal{H}$ be a $(k+1)$-sound potential opm which is soundly projecting. Let $\pi : \mathcal{H} \to \mathcal{M}$ be nearly k-good, with $\rho = \rho^{\mathcal{H}}_{k+1} < \rho^{\mathcal{M}}_{k+1}$. Then $\mathcal{H}$ is in fact an opm. Let us assume that $\mathcal{H}, \mathcal{M}$ are both successors, so $\pi(\mathcal{H}^-) = \mathcal{M}^-$. By fine condensation of $\mathcal{F}$, $\mathcal{H}^-$ is an $\mathcal{F}$-pm, and either $\mathcal{H} \in \mathcal{F}(\mathcal{H}^-)$ or $\mathcal{H} = \mathcal{F}(\mathcal{H}^-)$. If $\mathcal{H}$ is not k-relevant then the result follows from the fact that $\mathcal{M}^-$ is $< \omega$-condensing and $\mathcal{H}^-$ is an $\mathcal{F}$-pm. So assume $\mathcal{H}$ is k-relevant, so $\mathcal{H} = \mathcal{F}(\mathcal{H}^-)$.

Now use weak DJ (at degree ω) and the usual phalanx comparison argument to reach the desired conclusion. Say $\mathfrak{P} = ((\mathcal{M}, < \rho), \mathcal{H})$ is the phalanx. Then $\mathfrak{P}$ is $\mathcal{F}$-$((\omega, k), \omega_1 + 1)$-iterable, lifting to $\mathcal{F}$-(ω, ω)-maximal trees $\mathcal{V}$ on $\mathcal{M}$. (It could be that $\mathcal{M}$ is not k-relevant. So we want to keep the degrees of nodes of $\mathcal{V}$ at ω where possible, to ensure that each $M^{\mathcal{V}}_\alpha$ is an $\mathcal{F}$-pm.) Suppose $\mathcal{T}$ is non-trivial. Because $k < \omega$, if $M^{\mathcal{T}}_\infty$ is above $\mathcal{H}$ without drop in model or degree, π_∞ need only be a weak k-embedding. But in this case, $M^{\mathcal{T}}_\infty$ is not ω-sound, which implies $M^{\mathcal{U}}_\infty \triangleleft M^{\mathcal{T}}_\infty$, which contradicts weak DJ. The rest is routine. □

We next describe mouse operators, using *op-$\mathcal{J}$-structures*:

Definition 3.37 (op-$\mathcal{J}$-structure)**.** Let $\alpha \in \mathrm{Ord}\backslash\{0\}$, let Y be an operatic argument, let

$$D = \mathrm{Lim} \cap [\mathrm{o}(Y) + \omega, \mathrm{o}(Y) + \omega\alpha)$$

and let $\vec{P} = \langle P_\beta \rangle_{\beta \in D}$ be given.

We define $\mathcal{J}^{\vec{P}}_\beta(Y)$ for $\beta \in [1, \alpha]$, if possible, by recursion on β, as follows. We set $\mathcal{J}^{\vec{P}}_1(Y) = \mathcal{J}(Y)$ and take unions at limit β. For $\beta + 1 \in [2, \alpha]$, let $R = \mathcal{J}^{\vec{P}}_\beta(Y)$ and suppose that $P =_{\mathrm{def}} P_{\mathrm{o}(R)} \subseteq R$ and is amenable to R. In this case we define

$$\mathcal{J}^{\vec{P}}_{\beta+1}(Y) = \mathcal{J}(R, \vec{P} {\restriction} R, P).$$

Note then that by induction, $\vec{P} {\restriction} R \subseteq R$ and $\vec{P} {\restriction} R$ is amenable to R.

Let $\mathcal{L}_{\mathcal{J}}$ be the language with binary relation symbol $\dot{\in}$, predicate symbols $\dot{\vec{P}}$ and $\dot{P}$, and constant symbol $\dot{cb}$.

Let Y be an operatic argument. An **op-$\mathcal{J}$-structure over** Y is an amenable $\mathcal{L}_{\mathcal{J}}$-structure

$$\mathcal{M} = (\mathcal{J}^{\vec{P}}_\alpha(Y), \in^{\mathcal{M}}, \vec{P}, P, Y),$$

where $\alpha \in \mathrm{Ord}\backslash\{0\}$ and $\vec{P} = \left\langle \vec{P}_\gamma \right\rangle_{\gamma\in D}$ with domain D defined as above, $\lfloor\mathcal{M}\rfloor = \mathcal{J}_\alpha^{\vec{P}}(Y)$ is defined, $\dot{\vec{P}}^{\mathcal{M}} = \vec{P}$, $\dot{P}^{\mathcal{M}} = P$, $\dot{cb}^{\mathcal{M}} = Y$.

Let $\mathcal{M}$ be an op-$\mathcal{J}$-structure, and adopt the notation above. Let $l(\mathcal{M})$ denote α. For $\beta \in [1, \alpha]$ and $R = \mathcal{J}_\beta^{\vec{P}}(Y)$ and $\gamma = \mathrm{o}(R)$, let

$$\mathcal{M}|^{\mathcal{J}}\beta = (R, \in^R, \vec{P}\restriction R, P_\gamma, Y).$$

We write $\mathcal{N} \trianglelefteq^{\mathcal{J}} \mathcal{M}$, and say that $\mathcal{N}$ is a **$\mathcal{J}$-initial segment** of $\mathcal{M}$, iff $\mathcal{N} = \mathcal{M}|^{\mathcal{J}}\beta$ for some β. Clearly if $\mathcal{N} \trianglelefteq^{\mathcal{J}} \mathcal{M}$ then $\mathcal{N}$ is an op-$\mathcal{J}$-structure over Y. We write $\mathcal{N} \triangleleft^{\mathcal{J}} \mathcal{M}$, and say that $\mathcal{N}$ is a **$\mathcal{J}$-proper segment** of $\mathcal{M}$, iff $\mathcal{N} \trianglelefteq^{\mathcal{J}} \mathcal{M}$ but $\mathcal{N} \neq \mathcal{M}$.

Let $\mathcal{M}$ be an op-$\mathcal{J}$-structure. Note that $\mathcal{M}$ is pre-fine. We define the **fine-structural notions** for $\mathcal{M}$ using 2.24. ⊣

From now on we omit "$\in$" from our notation for op-$\mathcal{J}$-structures.

Definition 3.38 (Pre-operator)**.** Let $\mathscr{B}$ be an operator background. A **pre-operator over** $\mathscr{B}$ is a function $G : D \to \mathscr{B}$, with D an operatic domain over $\mathscr{B}$, such that for each $Y \in D$, $G(Y)$ is an op-$\mathcal{J}$-structure $\mathcal{M}$ over Y such that (i) every $\mathcal{N} \trianglelefteq \mathcal{M}$ is ω-sound, and (ii) for some $n < \omega$, $\rho_{n+1}^{\mathcal{M}} = \omega$. Let $C^G = C^D$ and $P^G = P^D$. ⊣

Definition 3.39 (Operator $\mathcal{F}_G$)**.** Let G be a pre-operator over $\mathscr{B}$, with domain D. We define a corresponding operator $\mathcal{F} = \mathcal{F}_G$, also with domain D, as follows.

Let $X \in \widehat{C^D}$ and $\mathcal{N} = G(X) = (\lfloor\mathcal{N}\rfloor, \vec{P}^{\mathcal{N}}, P^{\mathcal{N}}, X)$. Let $n < \omega$ be such that $\rho_{n+1}^{\mathcal{N}} = \omega$ and $\mathrm{o}(X) < \sigma =_{\mathrm{def}} \rho_n^{\mathcal{N}}$. If $n = 0$ then let $\mathcal{M} = \mathcal{N}$. If $n > 0$ then let $\mathcal{Q} = \mathcal{N}|^{\mathcal{J}}\sigma$ and let $\mathcal{M}$ be the op-$\mathcal{J}$-structure

$$\mathcal{M} = (\lfloor\mathcal{Q}\rfloor, \vec{P}^{\mathcal{N}}\restriction\sigma, T, X),$$

where $T \subseteq \lfloor\mathcal{Q}\rfloor$ codes

$$\mathrm{Th}_n^{\mathcal{N}}(\lfloor\mathcal{Q}\rfloor \cup \vec{p}_n^{\mathcal{N}})$$

in some uniform fashion, amenably to $\lfloor\mathcal{Q}\rfloor$, such as with mastercodes.[33] Note that in either case, $\mathcal{M} = (\lfloor\mathcal{M}\rfloor, \vec{P}^{\mathcal{M}}, P^{\mathcal{M}}, X)$ is an ω-sound op-$\mathcal{J}$-structure over X and $\rho_1^{\mathcal{M}} = \omega$.

[33] For concreteness, we take T to be the set of pairs (α, t') such that for some t, $(\vec{p}_n^{\mathcal{M}}, \alpha, t) \in T_n^{\mathcal{M}}$, and t' results from t by replacing $\vec{p}_n^{\mathcal{M}}$ with $\mathcal{R}$ (the latter is not a parameter of the theory t, so we can unambiguously use it as a constant symbol).

Define $\mathcal{F}(X)$ as the hierarchical model $\mathcal{K}$ over X, of length 1 (so $S^{\mathcal{K}} = \emptyset$), with $\lfloor\mathcal{K}\rfloor = \lfloor\mathcal{M}\rfloor$, $E^{\mathcal{K}} = \emptyset = cp^{\mathcal{K}}$,[34] and

$$P^{\mathcal{K}} = \{X\} \times \left(\vec{P}^{\mathcal{M}} \oplus P^{\mathcal{M}}\right).$$

(We use $\{X\} \times \cdots$ to ensure that $P^{\mathcal{K}} \subseteq \mathcal{K} \backslash \mathcal{K}^{-}$.)

Now let $\mathcal{R} \in P^D$; we define $\mathcal{F}(\mathcal{R})$. Let $A = cb^{\mathcal{R}}$ and $\rho = \rho_\omega^{\mathcal{R}}$. Let $\mathcal{P} = G(\mathcal{R})$. Let $\mathcal{N} \trianglelefteq \mathcal{P}$ be largest such that for all $\alpha < \rho$, we have

$$\mathfrak{P}(A^{<\omega} \times \alpha^{<\omega})^{\mathcal{N}} = \mathfrak{P}(A^{<\omega} \times \alpha^{<\omega})^{\mathcal{R}}.$$

Let $n < \omega$ be such that $\rho_{n+1}^{\mathcal{N}} = \omega$ and $\mathrm{o}(\mathcal{R}) < \rho_n^{\mathcal{N}}$. Now define $\mathcal{M}$ from $(\mathcal{N}, n)$ as in the definition of $\mathcal{F}(X)$ for $X \in \widehat{C^D}$, but with $cb^{\mathcal{M}} = \mathcal{R}$. Much as there, $\mathcal{M} = (\lfloor\mathcal{M}\rfloor, \vec{P}^{\mathcal{M}}, P^{\mathcal{M}}, \mathcal{R})$ is an ω-sound op-$\mathcal{J}$-structure over $\mathcal{R}$ and $\rho_1^{\mathcal{M}} = \omega$.

Now set $\mathcal{F}(\mathcal{R})$ to be the unique hierarchical model $\mathcal{K}$ of length $l(\mathcal{R}) + 1$ with $\lfloor\mathcal{K}\rfloor = \lfloor\mathcal{M}\rfloor$, $\mathcal{R} \triangleleft \mathcal{K}$ (so $S^{\mathcal{K}} = S^{\mathcal{R}} \frown \langle\mathcal{R}\rangle$), $E^{\mathcal{K}} = \emptyset$, and

$$P^{\mathcal{K}} = \{\mathcal{R}\} \times \left(\vec{P}^{\mathcal{M}} \oplus P^{\mathcal{M}}\right).$$

This completes the definition. ⊣

With notation as above, let $\mathcal{R} \in D$. Note that $\mathcal{F}(\mathcal{R})$ easily codes $G(\mathcal{R})$, *unless* $\mathcal{R} \in P^D$ and $\mathcal{N} \triangleleft \mathcal{P}$ where $\mathcal{N}, \mathcal{P}$ are as in the definition of $\mathcal{F}(\mathcal{R})$.

$\mathcal{F}_G$ is indeed an operator:

Lemma 3.40. *Let G be a pre-operator over $\mathscr{B}$ with domain D. Then $\mathcal{F}_G$ is an operator over $\mathscr{B}$. Moreover, for any $\mathcal{F}_G$-premouse $\mathcal{M}$ of length $\alpha+\omega$, for all sufficiently large $n < \omega$, $\mathcal{F}_G(\mathcal{M}|(\alpha+n))$ does not project early.*

Proof Sketch. We first show that $\mathcal{F}_G$ is an operator. Let $\mathcal{F} = \mathcal{F}_G$ and $X \in D = \mathrm{dom}(\mathcal{F})$. We must verify that $\mathcal{M} = \mathcal{F}(X)$ is an opm. This follows from (i) the choice of $\lfloor\mathcal{F}(X)\rfloor$ (i.e. the choice of $\mathcal{N} \trianglelefteq G(X)$ in the definition of $\mathcal{F}(X)$, which gives, for example, projectum amenability for $\mathcal{F}(X)$), (ii) if $X \in P^D$ then X is an ω-sound opm (acceptability follows from this and projectum amenability), (iii) standard properties of $\mathcal{J}$-structures (e.g. for

[34] A natural generalization of this definition would set $cp^{\mathcal{K}}$ to be some fixed non-empty object. For example, if one uses operators to define strategy mice, one might set $cp^{\mathcal{K}}$ to be the structure that the iteration strategy is for.

stratification), and (iv) with $\mathcal{P}$ as in the definition $\mathcal{F}(X)$, the fact that $\mathcal{P}$ is ω-sound and $\rho_1^{\mathcal{P}} = \omega$ (for sound projection).

Now let $\mathcal{M}$ be an $\mathcal{F}$-premouse of limit length $\alpha + \omega$. Then for all m,

$$\rho_\omega^{\mathcal{M}|(\alpha+m+1)} \leq \rho_\omega^{\mathcal{M}|(\alpha+m)},$$

because $\mathcal{M}|(\alpha+m+1)$ is soundly projecting and $\mathcal{M}|(\alpha+m)$ is ω-sound. So if $n < \omega$ is such that $\rho_\omega^{\mathcal{M}|(\alpha+n)}$ is as small as possible, n works. □

So any limit length $\mathcal{F}_G$-premouse $\mathcal{M}$ is "closed under G" in the sense that for $\in$-cofinally many $X \in \mathcal{M}$, we have $G(X) \in \mathcal{M}$.

We finish by illustrating how things work for *mouse operators*. The details involved provide some further motivation for the definition of fine condensation.

Example 3.41. Let $\varphi \in \mathcal{L}_0$. Let $\mathscr{B}$ be an operator background. Suppose that for every transitive structure $x \in \mathscr{B}$ there is $\mathcal{M} \triangleleft \mathrm{Lp}(x)$ such that $\mathcal{M} \models \varphi$, and let $\mathcal{M}_x$ be the least such. Let $G : \mathscr{B} \dashrightarrow \mathscr{B}$ be the pre-operator where for $x \in \mathscr{B}$ a transitive structure, $G(\hat{x})$ is the op-$\mathcal{J}$-structure over $\hat{x}$ naturally coding $\mathcal{M}_x$, and for $x \in \mathscr{B}$ a $< \omega$-condensing ω-sound opm, $G(x)$ is the op-$\mathcal{J}$-structure over x naturally coding $\mathcal{M}_x$.

The **mouse operator** $\mathcal{F}_\varphi$ determined by φ is $\mathcal{F}_{G_\varphi}$. A straightforward argument shows that $\mathcal{F}_\varphi$ almost condenses finely. We describe some of it, to illustrate how it relates to fine condensation. Let $\mathcal{F} = \mathcal{F}_\varphi$ and let $\mathcal{N}$ be a successor $\mathcal{F}$-pm. Let $\mathcal{M}$ be a successor Q-opm with $\rho_1^{\mathcal{M}} \leq \mathrm{o}(\mathcal{M}^-)$ and let $\pi : \mathcal{M} \to \mathcal{N}$ be a 0-embedding, so $\pi(\mathcal{M}^-) = \mathcal{N}^-$. Here $\mathcal{M}$ might not be an opm. Let $\mathcal{N}^* \triangleleft \mathrm{Lp}(\mathcal{N}^-)$ be the premouse over $\mathcal{N}^-$ coded by $\mathcal{N}$. (So $\mathcal{N}^*$ has no proper segment satisfying φ, and either $\mathcal{N}^* \models \varphi$ or $\mathcal{N}^*$ projects $< \rho_\omega^{\mathcal{N}^-}$.) Let $n < \omega$ be such that $\rho_{n+1}^{\mathcal{N}^*} \leq \mathrm{o}(\mathcal{N}^-) < \rho_n^{\mathcal{N}^*}$. Then there is an n-sound premouse $\mathcal{M}^*$ over $\mathcal{M}^-$ and an n-embedding $\pi^* : \mathcal{M}^* \to \mathcal{N}^*$ with $\pi \subseteq \pi^*$. Because $\rho_1^{\mathcal{M}} \leq \mathrm{o}(\mathcal{M}^-)$, $\rho_{n+1}^{\mathcal{M}^*} \leq \mathrm{o}(\mathcal{M}^-)$. So *if* $\mathcal{M}^*$ is sound, then $\mathcal{M}^* \triangleleft \mathrm{Lp}(\mathcal{M}^-)$, and it is easy to see that $\mathcal{M}^* \trianglelefteq \mathcal{M}'$, where $\mathcal{M}'$ is the premouse coded by $\mathcal{F}(\mathcal{M}^-)$. Suppose soundness fails, and let $\mathcal{H}^* = \mathfrak{C}_{n+1}(\mathcal{M}^*)$. Then $\mathcal{H}^* \trianglelefteq \mathcal{M}'$, and the n^{th} master code $\mathcal{H}$ of $\mathcal{H}^*$ is a universal hull of $\mathcal{M}$, and either $\mathcal{H} \in \mathcal{F}(\mathcal{M}^-)$ or $\mathcal{H} = \mathcal{F}(\mathcal{M}^-)$, as required. Note that we made significant use of the fact that $\rho_1^{\mathcal{M}} \leq \mathrm{o}(\mathcal{M}^-)$.

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