On supercompactness of ω_1

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Abstract. This paper studies structural consequences of supercompactness of ω_1 under ZF. We show that the Axiom of Dependent Choice (DC) follows from "ω_1 is supercompact". "ω_1 is supercompact" also implies that AD^+, a strengthening of the Axiom of Determinacy (AD), is equivalent to $\mathsf{AD}_{\mathbb{R}}$. It is shown that "ω_1 is supercompact" does not imply AD. The most one can hope for is Suslin determinacy. We show that this follows from "ω_1 is supercompact" and Hod Pair Capturing (HPC), an inner-model theoretic hypothesis that imposes certain smallness conditions on the universe of sets. "ω_1 is supercompact" on its own implies that every Suslin set is the projection of a determined (in fact, homogenously Suslin) set. "ω_1 is supercompact" also implies all sets in the Chang model have all the usual regularity properties, like Lebesgue measurability and the Baire property.

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1 Introduction

Under ZFC, successor cardinals (like ω_1) are "small". If $\alpha = \beta^+$ is a successor cardinal, then there is an injection from α into $\mathcal{P}(\beta)$.[3] Without the Axiom of Choice, it is possible for successor cardinals like ω_1 to exhibit large cardinal properties. For instance, it has been known since the 1960's that ω_1 can be measurable under ZF; this in particular implies that ω_1 is regular and there is no injection of ω_1 into $\mathcal{P}(\omega)$. We believe this result is independently due to Jech ([4]) and Takeuti ([18]). Furthermore, Takeuti, in the same paper [18], is able to show that "$\mathsf{ZF} + \omega_1$ is supercompact" is consistent relative to "$\mathsf{ZFC}+$ there is a supercompact cardinal". Takeuti's model $\mathcal{T}$ in which ω_1 is supercompact is

[3] This is equivalent to "there is a surjection from $\mathcal{P}(\beta)$ onto α" under ZFC. Without the Axiom of Choice, the equivalence can fail.

the same as Solovay's model while Takeuti used the method of Boolean valued models to describe his model. Suppose ZFC holds and there is a supercompact cardinal. Let κ be a supercompact cardinal. Let $g \subset Coll(\omega, <\kappa)$ be V-generic for the collapse forcing. Let $\mathbb{R}^* = \mathbb{R}^{V[g]}$. Takeuti's model $\mathcal{T}$ is (in modern terms) the symmetric model $V(\mathbb{R}^*)$.

Another major development started in the 1960's in set theory concerns the theory of infinite games with perfect information. The Axiom of Determinacy (AD) asserts that in an infinite game where players take turns to play integers, one of the players has a winning strategy (see the next section for more detailed discussions on AD and its variations). It is well-known that AD contradicts the Axiom of Choice. Solovay has shown that AD implies ω_1 is measurable and $\mathsf{AD}_{\mathbb{R}}$ implies that there is a supercompact (countably complete, normal, fine) measure on $\mathcal{P}_{\omega_1}\mathbb{R}$. Structural consequences of AD have been extensively investigated, most notably by the Cabal seminar members. Through work of Harrington, Kechris, Neeman, Woodin amongst others, we know that ω_1 is α-supercompact for every ordinal $\alpha < \Theta$ under AD^+, a strengthening of AD.[4] By [22], AD and $\mathsf{AD}_{\mathbb{R}}$ cannot imply ω_1 is supercompact. Woodin (see below) shows that AD is consistent with "ω_1 is supercompact."

It can be shown that the theory "ZF + ω_1 is measurable" is equiconsistent with "ZFC+ there is a measurable cardinal". The question of whether "ZF+ω_1 is supercompact" is equiconsistent with "ZFC+ there is a supercompact cardinal" is much more subtle. Woodin, in an unpublished work in the 1990's, is able to show that the former is much weaker than the latter. Woodin's model is a variation of the Chang model.[5] For each λ, let $\mathcal{F}_\lambda$ be the club filter on $\mathcal{P}_{\omega_1}\lambda^\omega$. Woodin's model is defined as

$$\mathcal{C}^+ = \mathrm{L}(\textstyle\bigcup_{\lambda\in Ord}\lambda^\omega)[(\mathcal{F}_\lambda \mid \lambda \in Ord)].$$

The model $\mathcal{C}^+$ is the least inner model M of ZF such that for all λ, $\lambda^\omega \in M$ and $M \cap \mathcal{F}_\lambda \in M$.[6]

Woodin shows that if there is a proper class of Woodin cardinals which are limits of Woodin cardinals, then $\mathcal{C}^+$ satisfies AD and ω_1 is supercompact. We note that in Takeuti's model, AD fails.[7] This is because the model $\mathrm{V}(\mathbb{R}^*)$ satisfies that $\Theta = \omega_2$ (that is κ^+ in V) while AD implies $\Theta > \omega_2$.[8]

[4]Harrington and Kechris [2] proved that under AD, ω_1 is α-supercompact for all α below a Suslin cardinal. Neeman [11] improved the result to that in $\mathrm{L}(\mathbb{R})$, ω_1 is α-supercompact for all $\alpha < \Theta$. Woodin extended Neeman's result by replacing the assumption $\mathrm{V} = \mathrm{L}(\mathbb{R})$ with AD^+. Unfortunately, Woodin's result is unpublished.

[5]The Chang model is defined as $\mathrm{L}(\bigcup_{\lambda\in Ord}\lambda^\omega)$, the least inner model N of ZF such that for all λ, $\lambda^\omega \in N$.

[6]We need a suitable coding of the sequence $(\mathcal{F}_\lambda \mid \lambda \in Ord)$ to let the model satisfy the above properties of M.

[7]This gives a proof that "ω_1 is supercompact" does not imply AD.

[8]This argument is due to the anonymous referee, which is simpler than what the authors presented in the previous manuscript. We thank the referee for showing us this argument.

The theory "ω_1 is supercompact" and variations of Woodin's model $\mathcal{C}^+$ are intimately related to determinacy theory as well as modern developments in descriptive inner model theory, cf. [13, Conjecture 1.8]. The following conjecture captures some of these relationships and is an important test question for the future development of descriptive inner model theory and the core model induction.

Conjecture 1. The following theories are equiconsistent.

(i) $\mathsf{ZF} + \omega_1$ is supercompact.
(ii) $\mathsf{ZF} + \mathsf{AD} + \omega_1$ is supercompact.
(iii) $\mathsf{ZFC}+$ there are proper class many Woodin cardinals which are limits of Woodin cardinals.

[22], [20], [21] made some progress in resolving the conjecture by exploring consistency strength and structural consequences of various fragments of supercompactness of ω_1.

This paper studies structural consequences of (full) supercompactness of ω_1 under ZF. We first show the following basic structural consequences.

Theorem 1. *Assume that ω_1 is supercompact. Then*

1. *the Axiom of Dependent Choices (DC) holds, while*
2. *(Folklore) there is no injection from ω_1 to 2^ω.*

The useful fact that DC holds can be used to derive other determinacy-like consequences such as:

Theorem 2. *Assume ω_1 is supercompact. Then every tree is weakly homogeneous.*

Remark 1. Note that under $\mathsf{ZF}+\mathsf{DC}$, every weakly homogeneously Suslin set is co-Suslin. So if ω_1 is supercompact, then every Suslin set is also co-Suslin.

Theorem 3. *Assume ω_1 is supercompact and Hod Pair Capturing (HPC). Then for any A such that A is Suslin, A is determined.*

See Section 7 for more detailed discussions on the hypothesis HPC. Under "ω_1 is supercompact", we also show that AD^+ and $\mathsf{AD}_{\mathbb{R}}$ are equivalent.

Theorem 4. *Assume ω_1 is supercompact. Then the following theories are equivalent:*

1. *AD^+.*
2. *$\mathsf{AD}_{\mathbb{R}}$.*

"ω_1 is supercompact" also implies a large collection of sets of reals are determined ([22]) and perhaps an even larger collection of sets of reals admit ∞-Borel representations.

Theorem 5. *Assume that ω_1 is supercompact. Then every subset of 2^ω in the Chang model $\mathrm{L}(\bigcup_{\lambda \in Ord} \lambda^\omega)$ is ∞-Borel.*

The paper is organized as follows. Section 2 summarizes basic concepts and definitions used in this paper. In Section 3, we prove Theorem 1. The proof of Theorem 5 is given in Section 4. In Section 5, we prove Theorem 2. Section 6 proves Theorem 4. Finally, Section 7 explains HPC and proves Theorem 3.

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2 Definitions and basic concepts

Throughout this paper, we work in ZF without the Axiom of Choice. For a nonempty set A, the axiom DC_A states that for any relation R on A such that for any element x of A there is an element y of A with $(x, y) \in R$, there is a function $f : \omega \to A$ such that for all natural numbers n, $\big(f(n), f(n+1)\big) \in R$. The **Axiom of Dependent Choices** (**DC**) states that for any nonempty set A, DC_A holds.

For a set X, $X^{<\omega}$ denotes the set of all finite sequences of elements of X, and X^ω denotes the set of all functions from ω to X. In particular, 2^ω denotes the set of all function from ω to $2 = \{0, 1\}$, not an ordinal or a cardinal. For a set X, we often consider X^ω as a topological space whose basic open sets are of the form $O_s = \{x \in X^\omega \mid s \subseteq x\}$ for $s \in X^{<\omega}$. For a set X and an infinite cardinal κ, let $\mathcal{P}_\kappa X$ be the set of all subsets σ of X such that σ is well-orderable and its cardinality is less than κ.

Let us review some basic terminology on filters. For a set Z, a **filter on** Z is a collection of subsets of Z closed under supersets and finite intersections. A filter on Z is **σ-complete** if it is closed under countable intersections. A filter on Z is **non-trivial** if the empty set $\emptyset$ does not belong to the filter. A filter on Z is an **ultrafilter** (or a **measure**) if it is non-trivial and for any subset A of Z, either A or $Z \setminus A$ is in the filter. Given a formula ϕ and an ultrafilter μ on Z, if the set $A = \{\sigma \in Z \mid \phi(\sigma)\}$ is in μ, then we say "for μ-measure one many σ, $\phi(\sigma)$ holds".

Let us introduce fineness and normality of ultrafilters on $\mathcal{P}_\kappa X$. An ultrafilter μ on $\mathcal{P}_\kappa X$ is **fine** if for any element x of X, for μ-measure one many σ, x is in σ. An ultrafilter μ on $\mathcal{P}_\kappa X$ is **normal** if for any set A in μ and $f : A \to \mathcal{P}_\kappa X$ with $\emptyset \neq f(\sigma) \subseteq \sigma$ for all $\sigma \in A$, there is an $x_0 \in X$ such that for μ-measure one many σ in A, $x_0 \in f(\sigma)$. Notice that this definition of normality is equivalent to the closure under diagonal intersections in ZF while it may not be equivalent to the standard definition of normality with regressive functions $f\colon A \to X$ without the axiom of choice. An ultrafilter on $\mathcal{P}_\kappa X$ is a **fine measure on** $\mathcal{P}_\kappa X$ if it is σ-complete and fine. A fine measure on $\mathcal{P}_\kappa X$ is a **normal measure on** $\mathcal{P}_\kappa X$ if it is normal.

We now introduce the main definitions of this paper:

Definition 1. *Let κ be an infinte cardinal.*

1. *Let X be a set.*
   (a) *κ is X**-strongly compact** if there is a fine measure on $\mathcal{P}_\kappa X$.*
   (b) *κ is X**-supercompact** if there is a normal measure on $\mathcal{P}_\kappa X$.*
2. *κ is **strongly compact** if for any set X, κ is X-strongly compact.*
3. *κ is **supercompact** if for any set X, κ is X-supercompact.*

We now review basic notions on determinacy axioms. For a nonempty set X, the **Axiom of Determinacy in** X^ω ($\mathsf{AD}_{\mathbf{X}}$) states that for any subset A of X^ω, in the Gale-Stewart game with the payoff set A, one of the players must have a winning strategy. We write AD for AD_ω. The ordinal $\boldsymbol{\Theta}$ is defined as the supremum of ordinals which are surjective images of $\mathbb{R}$. Under $\mathsf{ZF}+\mathsf{AD}$, Θ is very big, e.g., it is a limit of measurable cardinals while under ZFC, Θ is equal to the successor cardinal of the continuum $|\mathbb{R}|$. **Ordinal Determinacy** states that for any $\lambda < \Theta$, any continuous function $\pi\colon \lambda^\omega \to \omega^\omega$, and any $A \subseteq \omega^\omega$, in the Gale-Stewart game with the payoff set $\pi^{-1}(A)$, one of the players must have a winning strategy. In particular, Ordinal Determinacy implies AD while it is still open whether the converse holds under $\mathsf{ZF}+\mathsf{DC}$.

We will introduce the notion of ∞-Borel codes. Before that, we review some terminology on trees. Given a set X, a **tree on** X is a collection of finite sequences of elements of X closed under initial segments. Given an element t of $X^{<\omega}$, $\mathrm{lh}(t)$ denotes its length, i.e., the domain or the cardinality of t. Given a tree T on X and elements s and t of T, s is an **immediate successor of** t **in** T if s is an extension of t and $\mathrm{lh}(s) = \mathrm{lh}(t)+1$. Given a tree T on X and an element t of T, $\mathrm{Succ}_T(t)$ denotes the collection of all immediate successors of t in T. An element t of a tree T on X is **terminal** if $\mathrm{Succ}_T(t) = \emptyset$. For an element t of a tree T on X, $\mathrm{term}(T)$ denotes the collection of all terminal elements of T. Given a tree T on X,[**T**] denotes the collection of all $x \in X^\omega$ such that for all natural numbers n, $x \restriction n$ is in T. A tree T on X is **well-founded** if $[T] = \emptyset$. We often identify a tree T on $X \times Y$ with a subset of the set $\{(s,t) \in X^{<\omega} \times Y^{<\omega} \mid \mathrm{lh}(s) = \mathrm{lh}(t)\}$, and $\mathrm{p}[T]$ denotes the collection of all $x \in X^\omega$ such that there is a $y \in Y^\omega$ with $(x,y) \in [T]$.

Definition 2. *Let λ be a non-zero ordinal.*

1. *An ∞**-Borel code in** λ^ω is a pair (T,ρ) where T is a well-founded tree on some ordinal γ, and ρ is a function from term(T) to $\lambda^{<\omega}$.*
2. *Given an ∞-Borel code $c = (T,\rho)$ in λ^ω, to each element t of T, we assign a subset $B_{c,t}$ of λ^ω by induction on t using the well-foundedness of the tree T as follows:*

(a) *If* t *is a terminal element of* T*, let* $B_{c,t}$ *be the basic open set* $O_{\rho(t)}$ *in* λ^ω*.*
(b) *If* $Succ_T(t)$ *is a singleton of the form* $\{s\}$*, let* $B_{c,t}$ *be the complement of* $B_{c,s}$*.*
(c) *If* $Succ_T(t)$ *has more than one element, then let* $B_{c,t}$ *be the union of all sets of the form* $B_{c,s}$ *where* s *is in* $Succ_T(t)$*.*

We write B_c *for* $B_{c,\emptyset}$*.*

3. *A subset* A *of* λ^ω *is* ∞***-Borel*** *if there is an* ∞*-Borel code* c *in* λ^ω *such that* $A = B_c$*.*

Usually, we use ∞-Borel codes and ∞-Borel sets only in the spaces ω^ω or 2^ω. We use them for general spaces λ^ω in Section 4.

In section 4, we will use the following characterization of ∞-Borelness in the space λ^ω:

Fact 1 *Let* λ *be a non-zero ordinal and* A *be a subset of* λ^ω*. Then the following are equivalent:*

1. *A is* ∞*-Borel, and*
2. *for some formula* ϕ *and some set* S *of ordinals, for all elements* x *of* λ^ω*,* x *is in* A *if and only if* $\mathrm{L}[S,x] \vDash$ "$\phi(S,x)$"*.*

Proof. For the case $\lambda = 2$, one can refer to [8, Theorem 9.0.4]. The general case can be proved in the same way. □

Remark 2. In fact, the second item of Fact 1 is equivalent to the following using Lévy's Reflection Principle:

– for some $\gamma > \lambda$, some formula ϕ, and some set S of ordinals, for all elements x of λ^ω, x is in A if and only if $\mathrm{L}_\gamma[S,x] \vDash$ "$\phi(S,x)$".

Throughout this paper, we will freely use either of the equivalent conditions of ∞-Borelness.

We now introduce the axiom AD^+, and reivew some notions on Suslin sets. The axiom AD^+ states that (a) $\mathsf{DC}_\mathbb{R}$ holds, (b) Ordinal Determinacy holds, and (c) every subset of 2^ω is ∞-Borel. Since AD^+ demands Ordinal Determinacy, AD^+ implies AD while it is open whether the converse holds in ZF+DC. A subset A of 2^ω (ot ω^ω) is **Suslin** if there are some ordinal λ and a tree T on $2\times\lambda$ ($\omega\times\lambda$ respectively) such that $A = \mathrm{p}[T]$. A is **co-Suslin** if the complement of A is Suslin. An infinite cardinal λ is a **Suslin cardinal** if there is a subset A of 2^ω (ω^ω) such that there is a tree on $2\times\lambda$ ($\omega\times\lambda$) such that $A = \mathrm{p}[T]$ while there are no $\gamma < \lambda$ and a tree S on $2\times\gamma$ ($\omega\times\lambda$) such that $A = \mathrm{p}[S]$. Under ZF+$\mathsf{DC}_\mathbb{R}$, AD^+ is equivalent to the assertion that Suslin cardinals are closed below Θ in the order topology of $(\Theta, <)$.

3 Choice principles and supercompactness of ω_1

In this section, we prove Theorem 1.

Proof (Theorem 1). 1. Let A be any nonempty set and R be any relation on A such that for any $x \in A$ there is a $y \in A$ such that $(x, y) \in R$. We will show that there is a function $f\colon \omega \to A$ such that for all natural numbers n, $\big(f(n), f(n+1)\big) \in R$.

Since ω_1 is supercompact, there is a fine normal measure on $\mathcal{P}_{\omega_1} A$. We fix such a measure μ.

Claim 1 *For μ-measure one many elements σ of $\mathcal{P}_{\omega_1} A$, the following holds:*

$$(\forall x \in \sigma)\ (\exists y \in \sigma)\ (x, y) \in R$$

Proof (Claim 1). Suppose not. We will derive a contradiction using μ. Since μ is an ultrafilter on $\mathcal{P}_{\omega_1} A$, for μ-measure one many elements σ of $\mathcal{P}_{\omega_1} A$, the following holds:

$$(\exists x \in \sigma)\ (\forall y \in \sigma)\ (x, y) \notin R$$

By normality of μ, there is an $x_0 \in A$ such that for μ-measure one many elements σ of $\mathcal{P}_{\omega_1} A$ with $x_0 \in \sigma$, for all $y \in \sigma$, $(x_0, y) \notin R$.

On the other hand, by the assumption on R, there is a $y_0 \in A$ such that $(x_0, y_0) \in R$. By fineness of μ, for μ-measure one many elements σ of $\mathcal{P}_{\omega_1} A$, both x_0 and y_0 are elements of σ.

Since μ is a filter, for μ-measure one many elements σ of $\mathcal{P}_{\omega_1} A$ with $x_0 \in \sigma$, for all $y \in \sigma$, $(x_0, y) \notin R$ while both x_0 and y_0 are elements of σ and $(x_0, y_0) \in R$. This gives us both $(x_0, y_0) \notin R$ and $(x_0, y_0) \in R$, a contradiction. This finishes the proof of the claim. □

We now know that for μ-measure one many elements σ of $\mathcal{P}_{\omega_1} A$, the following holds:

$$(\forall x \in \sigma)\ (\exists y \in \sigma)\ (x, y) \in R$$

Let us pick such a σ. Then for any $x \in \sigma$, there is a $y \in \sigma$ such that $(x, y) \in R$. Since σ is an element of $\mathcal{P}_{\omega_1} A$, it is countable, so we can fix a surjection $\pi\colon \omega \to \sigma$. Using this π, the above property of σ, and the well-orderedness of $(\omega, <)$, one can easily construct a desired $f\colon \omega \to A$. This finishes the proof of 1..

2. This is a well-known fact to the experts. Nevertheless, we will give a proof for the sake of completeness. Suppose that there was an injection $i\colon \omega_1 \to 2^\omega$. We will derive a contradiction using supercompactness of ω_1. For each $\alpha < \omega_1$, we write x_α for $i(\alpha)$.

We first note that there is a non-principal σ-complete ultrafilter on ω_1, i.e., ω_1 is measurable. Since ω_1 is supercompact, we can fix a fine normal measure μ on $\mathcal{P}_{\omega_1}\omega_1$. Let ν be as follows:

$$\nu = \{A \subseteq \omega_1 \mid \text{ for } \mu\text{-measure one many elements } \sigma \text{ of } \mathcal{P}_{\omega_1}\omega_1,\ \sup\sigma \in A\}$$

Then it is easy to see that ν is a non-principal σ-complete ultrafilter on ω_1.

Using this ν, we will derive a contradiction as follows. Since ν is an ultrafilter on ω_1, for any natural number n, there is an $k_n \in \{0,1\}$ such that the set $A_n = \{\alpha < \omega_1 \mid x_\alpha(n) = k_n\}$ is of ν-measure one. Since ν is σ-complete, the set $A = \bigcap_{n\in\omega} A_n$ is of ν-measure one. By the property of each A_n, for any α in A, for all natural numbers n, $x_\alpha(n) = k_n$. But since i is injective, A has at most one element. This contradicts that A is of ν-measure one and ν is non-principal. This finishes the proof of 2.. This completes the proof of Theorem 1. □

Remark 3. (2) of Theorem 1 is the best one can hope for. "ω_1 is supercompact" does not imply "there is no injection $f : \omega_2 \to \mathcal{P}(\omega_1)$". To see this, assume ZFC and there is a supercompact cardinal κ. Let $f : \kappa^+ \to \mathcal{P}(\kappa)$ be an injection in V. Let $\mathcal{T}$ be the Takeuti model defined at κ. Then clearly $f \in \mathcal{T}$ and in $\mathcal{T}$, $\kappa = \omega_1$ and $(\kappa^+)^V = \omega_2$.

4 Chang model and supercompactness of ω_1

In this section, we prove Theorem 5. As a corollary, one can obtain usual regularity properties for sets of reals in the Chang model:

Corollary 1. *Assume that ω_1 is supercompact. Then every subset of 2^ω in the Chang model is Lebesgue measurable and has the Baire property.*

Corollary 1 directly follows from Theorem 1, Theorem 5, and the following fact:

Fact 2 (Essentially Solovay) *Assume that there is no injection from ω_1 to 2^ω. Let A be a subset of 2^ω which is ∞-Borel. Then A is Lebesgue measurable and has the Baire property.*

For the proof of Fact 2, one can refer to e.g., [3, Theorem 2.4.2 & Proposition 3.2.13].

To prove Theorem 5, we use the following lemma:

Lemma 1. $\mathrm{L}(\bigcup_{\lambda\in Ord}\lambda^\omega) = \bigcup_{\lambda\in Ord}\mathrm{L}(\lambda^\omega)$.

Proof. Given a set X, let $J(X)$ be the rudimental closure of $X \cup \{X\}$. Let $(C_\alpha \mid \alpha \in \mathrm{Ord})$ be the following sequence: $C_0 = \mathrm{L}_\omega$, $C_{\alpha+1} = J(C_\alpha \cup \alpha^\omega)$, and $C_\beta = \bigcup_{\alpha<\beta} C_\alpha$ when β is a limit ordinal. Set $C = \bigcup_{\alpha\in\mathrm{Ord}} C_\alpha$.

We first argue that C is equal to the Chang model $\mathrm{L}(\bigcup_{\lambda\in\mathrm{Ord}}\lambda^\omega)$. It is easy to see that C is contained in the Chang model because the construction of the sequence $(C_\alpha \mid \alpha \in \mathrm{Ord})$ is absolute between the Chang model and V. So it is enough to prove that C contains the Chang model. For that it is enough to show that C is an inner model of ZF containing all sets in Ord^ω. By the construction of $(C_\alpha \mid \alpha \in \mathrm{Ord})$, it is easy to see that C contains all the sets in Ord^ω, is rudimentarily closed, satisfies Comprehension Scheme, and for any subset X of

C in V, there is a set Y in C such that $X \subseteq Y$ (namely C_α for some big α). Therefore, C is an inner model of ZF containing all the sets in Ord^ω, as desired.

Next, we claim that for all ordinals λ, $C_\lambda \in \mathrm{L}(\lambda^\omega)$. For this, it is enough to see that the construction of the sequence $(C_\alpha \mid \alpha \leq \lambda)$ is absolute between $\mathrm{L}(\lambda^\omega)$ and V, which follows by observing that λ^ω is in $\mathrm{L}(\lambda^\omega)$.

We now argue that the Chang model $\mathrm{L}(\bigcup_{\lambda\in\mathrm{Ord}} \lambda^\omega)$ is equal to $\bigcup_{\lambda\in\mathrm{Ord}} \mathrm{L}(\lambda^\omega)$. The inclusion $\bigcup_{\lambda\in\mathrm{Ord}} \mathrm{L}(\lambda^\omega) \subseteq \mathrm{L}(\bigcup_{\lambda\in\mathrm{Ord}} \lambda^\omega)$ is clear. We will see the other inclusion. Let A be any set in the Chang model. By the second to last paragraph, A is in C and hence there is an ordinal λ such that A is in C_λ. By the last paragraph, C_λ is in $\mathrm{L}(\lambda^\omega)$. Therefore, A is in $\mathrm{L}(\lambda^\omega)$, as desired. This completes the proof of Lemma 1. □

We now come to the proof of Theorem 5.

Proof (Theorem 5).

We assume that ω_1 is supercompact and will show that every subset of 2^ω in the Chang model is ∞-Borel.

By Lemma 1, to obtain Theorem 5, it is enough to prove that for all λ, every subset of 2^ω in $\mathrm{L}(\lambda^\omega)$ is ∞-Borel.

Throughout the rest of this section, we fix an infinite ordinal λ and a fine measure μ on $\mathcal{P}_{\omega_1}\lambda^\omega$ whose existence is ensured by the supercompactness of ω_1. We will show that every subset of λ^ω in $\mathrm{L}(\lambda^\omega)$ is ∞-Borel using μ, which will give us that every subset of 2^ω in $\mathrm{L}(\lambda^\omega)$ is ∞-Borel. The arguments are a generalization of the proof of Woodin's theorem in [1, Theorem 1.9].

By Fact 1, it is enough to show that for any subset A of λ^ω in $\mathrm{L}(\lambda^\omega)$, there are some formula ϕ and a set S of ordinals such that for all elements x of λ^ω,

$$x \in A \iff \mathrm{L}[S, x] \vDash \phi[S, x]. \quad (\dagger)$$

If (†) holds for all elements x of λ^ω, then we say that A is **defined from the pair** (ϕ, S) and we write $B_{(\phi,S)}$ for A.

The following is the key claim in this section:

Claim 2 *There is a function F which is OD from μ such that*

1. *F is defined for all pairs (ϕ, S) where ϕ is a formula and S is a set of ordinals,*
2. *$F(\phi, S)$ is of the form (ψ, T) such that if A is a subset of $(\lambda^\omega)^{n+1}$ defined from (ϕ, S), then $pA = \{\mathbf{x} \in (\lambda^\omega)^n \mid (\exists y)\ (\mathbf{x}, y) \in A\}$ is defined from (ψ, T), i.e., if $A = B_{(\phi,S)}$, then $pA = B_{F(\phi,S)}$.*

To prove Claim 2, we use a variant of Vopěnka algebra: Let S be a set of ordinals and σ be an element of $\mathcal{P}_{\omega_1}\lambda^\omega$. We fix an injection $\iota \colon \mathrm{OD}_{S,\sigma} \cap \mathcal{P}(\sigma) \to \mathrm{HOD}_{S,\sigma}$ which is OD from S and σ such that for all $t \in \lambda^{<\omega}$, $\iota(O_t) = t$ where $O_t = \{x \in \sigma \mid t \subseteq x\}$.[9] Set $B_\sigma = \{\iota(A) \mid A \in \mathrm{OD}_{S,\sigma} \cap \mathcal{P}(\sigma)\}$. For $p, q \in B_\sigma$,

[9] We are demanding that for all $t \in \lambda^{<\omega}$, $\iota(O_t) = t$ to have that "x is in $\mathrm{L}[S, B_\sigma][G_x]$" in Fact 3. Namely, x should be easily computable from B_σ and G_x without referring to ι because ι may not be definable in $\mathrm{L}[S, B_\sigma][G_x]$.

$p \le q$ if $\iota^{-1}(p) \subseteq \iota^{-1}(q)$. Note that the structure $(B_\sigma, \le)$ is in $\mathrm{HOD}_{S,\sigma}$. For an element x of σ, set $G_x = \{p \in B_\sigma \mid x \in \iota^{-1}(p)\}$.

Fact 3 (Vopěnka) *1. In $HOD_{S,\sigma}$, B_σ is a complete Boolean algebra, and*
2. for any element x of σ, G_x is B_σ-generic over $HOD_{S,\sigma}$, and x is in $\mathrm{L}[S, B_\sigma][G_x]$, which is a subclass of $HOD_{S,\sigma}[G_x]$.

Recall that we have fixed the fine measure μ on $\mathcal{P}_{\omega_1}\lambda^\omega$. For each $\sigma \in \lambda^\omega$, let $Q_\sigma = (B_\sigma)^{\mathrm{L}(S,\sigma)}$. We will consider the ultraproducts $\prod_\sigma \mathrm{L}[S, Q_\sigma][x]/\mu$ for $x \in \lambda^\omega$. Using the fineness of μ, one can prove Łos' theorem for these ultraproducts (the proof is essentially the same as the one given in [19, Lemma 2.3]).[10] By DC from Theorem 1, the above ultraproducts are all well-founded and we identify them with their transitive collapses. For each $y \in \bigcap_\sigma \mathrm{L}[S, Q_\sigma]$, let $y_\infty = \prod_\sigma y/\mu$. In particular, $S_\infty = \prod_\sigma S/\mu$. Let $h\colon \lambda \to \mathrm{Ord}$ be such that $h(\alpha) = \alpha_\infty$ for all $\alpha < \lambda$. We also set $Q_\infty = \prod_\sigma Q_\sigma/\mu$.

We are now ready to prove Claim 2.

Proof (Claim 2).

For simplicity, we will assume $n = 1$ (the general case is treated in the same way). Let $A \subseteq (\lambda^\omega)^2$ be defined from (ϕ, S), i.e., $A = B_{(\phi,S)}$. Then for all $x \in \lambda^\omega$,

$$\begin{aligned} & x \in \mathrm{p}A \\ \iff & (\exists y \in \lambda^\omega)\ (x, y) \in B_{(\phi,S)} \\ \iff & \text{for } \mu\text{-measure one many } \sigma,\ \mathrm{L}(S,\sigma) \models \text{“}(\exists y)\ (x, y) \in B_{(\phi,S)}\text{”} \\ \iff & \text{for } \mu\text{-measure one many } \sigma, \\ & \mathrm{L}[S, Q_\sigma, x] \models \text{“}\Big(\exists p \in \mathrm{Coll}\big(\omega, |\mathcal{P}(Q_\sigma)|\big)\Big) p \Vdash (\exists y)\ (\check{x}, y) \in B_{(\phi,S)}\text{”} \\ \iff & \prod_\sigma \mathrm{L}[S, Q_\sigma, x]/\mu \models \text{“}\Big(\exists p \in \mathrm{Coll}\big(\omega, |\mathcal{P}(Q_\infty)|\big)\Big)\ p \Vdash (\exists y)\ (\check{x_\infty}, y) \in B_{(\phi,S_\infty)}\text{”} \\ \iff & \mathrm{L}[S_\infty, Q_\infty, x_\infty] \models \text{“}\Big(\exists p \in \mathrm{Coll}\big(\omega, |\mathcal{P}(Q_\infty)|\big)\Big)\ p \Vdash (\exists y)\ (\check{x_\infty}, y) \in B_{(\phi,S_\infty)}\text{”} \end{aligned}$$

The first equivalence follows from the assumption that A is defined from (ϕ, S). The second equivalence follows from the fineness of μ. The forward direction of the third equivalence follows from the property of the Vopěnka algebra Q_σ given in Fact 3: Given a y in σ with $(x, y) \in B_{\phi,S}$, letting z code x and y in a simple way, $z \in \mathrm{L}[S, Q_\sigma][G_z]$ by Fact 3. Hence y is in a set generic extension of $\mathrm{L}[S, Q_\sigma, x]$ whose poset is of size at most $|Q_\sigma|$ in $\mathrm{L}[S, Q_\sigma, x]$. In particular, in $\mathrm{L}[S, Q_\sigma, x]$, one can force to add such a y over $\mathrm{Coll}\big(\omega, |\mathcal{P}(Q_\sigma)|\big)$. The backward direction of the third equivalence follows from the fact that $\mathcal{P}\big(\mathcal{P}(Q_\sigma)\big)^{\mathrm{L}[S,Q_\sigma,x]}$ is countable in V because Q_σ is countable by the fact that $\mathrm{OD}^{\mathrm{L}(S,\sigma)} \cap \mathcal{P}(\sigma)$ is well-orderable and σ is countable in V, and because $\mathrm{L}[S, Q_\sigma, x]$ is a transitive model of ZFC. The fourth & fifth equivalences follow from Łos' theorem for the ultraproduct $\prod_\sigma \mathrm{L}[S, Q_\sigma][x]/\mu$ and the definitons of S_∞, Q_∞, and x_∞.

[10] For the proof of Łos' theorem, we do not need the normality of μ because for each σ, one can define a well-order on $\mathrm{L}[S, Q_\sigma][x]$ uniformly in σ.

Now let T be the set of ordinals simply coding S_∞, Q_∞, and h. Then for each $x \in \lambda^\omega$, $x_\infty \in \mathrm{L}[T,x]$ because $x_\infty = h[x]$. Let ψ be the formula stating "$\mathrm{L}[S_\infty, Q_\infty, x_\infty] \models \Big(\exists p \in \mathrm{Coll}\big(\omega, |\mathcal{P}(Q_\infty)|\big)\Big)\ p \Vdash (\exists y)\ (\check{x}, y) \in B_{(\phi, S_\infty)}$". Then for each $x \in \lambda^\omega$,

$$\begin{aligned} & x \in \mathrm{p}A \\ \iff & \mathrm{L}[S_\infty, Q_\infty, x_\infty] \models \text{“}\Big(\exists p \in \mathrm{Coll}\big(\omega, |\mathcal{P}(Q_\infty)|\big)\Big)\ p \Vdash (\exists y)\ (\check{x_\infty}, y) \in B_{(\phi, S_\infty)}\text{”} \\ \iff & \mathrm{L}[T,x] \models \text{“}\psi[T,x]\text{”} \end{aligned}$$

Therefore, $F(\phi, S) = (\psi, T)$ satisfies the desired equivalence. This completes the proof of Claim 2. □

As is mentioned before Claim 2, we shall prove that for all subsets A of λ^ω in $\mathrm{L}(\lambda^\omega)$, A is defined from some pair (ϕ, S) as in (†), which gives us Theorem 5. The idea is to look at the hierarchy $\big(\mathrm{L}_\alpha(\lambda^\omega) \mid \alpha \in \mathrm{Ord}, \alpha \geq \omega\big)$, and by induction on α, to each definition of an element A of $\mathrm{L}_\alpha(\lambda^\omega)$, we assign certain ϕ and S such that A is defined from (ϕ, S). We fix an F from Claim 2.

Definition 3. *The hierarchy* $\big(\mathrm{L}_\alpha(\lambda^\omega) \mid \alpha \in \mathit{Ord}, \alpha \geq \omega\big)$ *is defined as follows:*

1. $\mathrm{L}_\omega(\lambda^\omega) = \lambda \cup \lambda^\omega$,
2. $\mathrm{L}_{\alpha+1}(\lambda^\omega) = \mathit{Def}\,\big(\mathrm{L}_\alpha(\lambda^\omega), \in\big)$, *and*
3. $\mathrm{L}_\beta(\lambda^\omega) = \bigcup_{\alpha<\beta} \mathrm{L}_\alpha(\lambda^\omega)$ *when* β *is a limit ordinal bigger than* ω.

Note that the above definition is not standard in the following two senses: One is that the indexing of the hierarchy starts with $\alpha = \omega$, not $\alpha = 0$. The other is that the first stage $\mathrm{L}_\omega(\lambda^\omega)$ is not transitive. However, the above definition satisfies that for all $\alpha \geq \omega + \omega$, $\mathrm{L}_\alpha(\lambda^\omega)$ is transitive and that the union of all $\mathrm{L}_\alpha(\lambda^\omega)$ $(\alpha \geq \omega)$ coincides with $\mathrm{L}(\lambda^\omega)$, the least inner model of ZF containing λ^ω as an element. These two properties are enough for us to prove Theorem 5. We start the indexing with $\alpha = \omega$ to ensure that one can code any formula with a natural number smaller than α in the arguments below. Also we use the above definition of $\mathrm{L}_\omega(\lambda^\omega)$ for convenience of proving Claim 3 below.

Remark 4. There is a sequence of partial surjections $\big(\pi_\alpha \colon \alpha^{<\omega} \times \lambda^\omega \to \mathrm{L}_\alpha(\lambda^\omega) \mid \alpha \in \mathrm{Ord}, \alpha \geq \omega\big)$ which is OD such that

1. for all $\alpha \geq \omega$, $\pi_\alpha(\emptyset, x) = x$,
2. $\pi_0(\boldsymbol{\beta}, x) = x(0)$ when $\boldsymbol{\beta} \neq \emptyset$,
3. if $\omega \leq \beta < \alpha$, then $\pi_\beta = \pi_\alpha \restriction \beta^{<\omega} \times \lambda^\omega$, and
4. if $\alpha > \omega$, $(\boldsymbol{\beta}, x) \in \alpha^{<\omega} \times \lambda^\omega$, $\pi_\alpha(\boldsymbol{\beta}, x)$ is defined, and $\boldsymbol{\beta} = (\beta_0, \beta_1, \cdots, \beta_k)$, then $\pi_\alpha(\boldsymbol{\beta}, x)$ is an element of $\mathrm{L}_{\beta_0+1}(\lambda^\omega)$ which is defined in the structure $\big(\mathrm{L}_{\beta_0}(\lambda^\omega), \in\big)$ via a formula coded by β_1 with some parameters of the form $\pi_{\beta_0}(\boldsymbol{\gamma}, y)$ where $\boldsymbol{\gamma}$ here depends only on $\boldsymbol{\beta}$, not on x.

Definition 4. *For an ordinal $\alpha \geq \omega$, a formula ϕ, and $\boldsymbol{\beta}^0, \cdots, \boldsymbol{\beta}^{n-1} \in \alpha^{<\omega}$, let*

$$T^{\alpha}_{(\phi,\boldsymbol{\beta}^0,\cdots,\boldsymbol{\beta}^{n-1})} = \{(x_0,\cdots,x_{n-1}) \in (\lambda^\omega)^n \mid \mathrm{L}_\alpha(\lambda^\omega) \vDash \phi\big[\pi_\alpha(\boldsymbol{\beta}^0, x_0), \cdots, \pi_\alpha(\boldsymbol{\beta}^{n-1}, x_{n-1})\big]\}$$

Claim 3 *There is a function which is OD from μ sending (α, p) to an ∞-Borel code $q^\alpha_p = (\psi, S)$, where α is an ordinal with $\alpha \geq \omega$ and $p = (\phi, \boldsymbol{\beta}^0, \cdots, \boldsymbol{\beta}^{n-1})$ is as in Definition 4, such that T^α_p is defined from q^α_p.*

Proof (Claim 3).

We prove the claim by induction on α. Let us fix α. Then we prove the statement by induction on the complexity of ϕ.

Case 1: When ϕ is of the form $v \in w$ or $v = w$.

Suppose that $\alpha = \omega$. This is the base case of the double induction. Let $\boldsymbol{\beta}^0, \boldsymbol{\beta}^1 \in \alpha^{<\omega}$. Then the set $T^0_{\phi,\boldsymbol{\beta}^0,\boldsymbol{\beta}^1}$ is of the form $\emptyset$, $\{(x_0,x_1) \mid x_0(0) \in x_1(0)\}$, $\{(x_0,x_1) \mid x_1(0) \in x_0(0)\}$, $\{(x_0,x_1) \mid x_0(0) = x_1(0)\}$, or $\{(x_0,x_1) \mid x_0 = x_1\}$. In each case, one can assign a suitable code q^α_p in a simple way.

Suppose that $\alpha > \omega$. Let $\beta_* = \max\{\beta^0_0, \beta^1_0\}$. Then $\beta^* < \alpha$ and by Remark 4, both $\pi_\alpha(\boldsymbol{\beta}^0, x_0)$ and $\pi_\alpha(\boldsymbol{\beta}^1, x_1)$ are definable in the structure $\big(\mathrm{L}_{\beta_*}(\lambda^\omega), \in\big)$ with some parameters of the form $\pi_{\beta_*}(\boldsymbol{\gamma}, y)$ where $\boldsymbol{\gamma}$ here depends only on $\boldsymbol{\beta}^0$ and $\boldsymbol{\beta}^1$. Then one can find a formula ϕ' and some $\boldsymbol{\gamma}^0, \boldsymbol{\gamma}^1$ such that $T^\alpha_{\phi,\boldsymbol{\beta}^0,\boldsymbol{\beta}^1} = T^{\beta_*}_{\phi',\boldsymbol{\gamma}^0,\boldsymbol{\gamma}^1}$. By induction hypothesis, one can find a desired code q^α_p.

Case 2: When ϕ is of the form $\neg\,\phi'$.

In this case, by induction hypothesis, letting $p' = (\phi', \boldsymbol{\beta}^0, \cdots, \boldsymbol{\beta}^{n-1})$, we have $q^\alpha_{p'} = (\psi, S)$. Then $q^\alpha_p = (\neg\,\psi, S)$ is the desired code.

Case 3: When ϕ is of the form $\phi_1 \wedge \phi_2$.

In this case, by induction hypothesis, letting $p_1 = (\phi_1, \boldsymbol{\beta}^0, \cdots, \boldsymbol{\beta}^{n-1})$ and $p_2 = (\phi_2, \boldsymbol{\beta}^0, \cdots, \boldsymbol{\beta}^{n-1})$, we have $q^\alpha_{p_1} = (\psi_1, S_1)$ and $q^\alpha_{p_2} = (\psi_2, S_2)$. Then let $q^\alpha_p = (\psi, S)$ where S is a set of ordinals simply coding S_1 and S_2, and $\psi(S, x)$ states that both "$\mathrm{L}[S_1, x] \vDash \psi_1[S_1, x]$" and "$\mathrm{L}[S_2, x] \vDash \psi_2[S_2, x]$" hold. Then q^α_p is the desired code.

Case 4: When ϕ is of the form $\exists v\, \phi'$.

In this case, by induction hypothesis, for each $\boldsymbol{\beta} \in \alpha^{<\omega}$, setting $p_{\boldsymbol{\beta}} = (\phi', \boldsymbol{\beta}, \boldsymbol{\beta}^0, \cdots, \boldsymbol{\beta}^{n-1})$, we have the code $q^\alpha_{p_{\boldsymbol{\beta}}}$. We write $q_{\boldsymbol{\beta}}$ for $q^\alpha_{p_{\boldsymbol{\beta}}}$. Note that

$$\begin{aligned}
&T^\alpha_{\phi,\boldsymbol{\beta}^0,\cdots,\boldsymbol{\beta}^{n-1}}\\
=&\{(x_0,\cdots x_{n-1}) \mid \big(\exists y \in \mathrm{L}_\alpha(\lambda^\omega)\big)\\
&\qquad\qquad \mathrm{L}_\alpha(\lambda^\omega) \vDash \text{“}\phi'[y, \pi_\alpha(\boldsymbol{\beta}^0, x_0), \cdots, \pi_\alpha(\boldsymbol{\beta}^{n-1}, x_{n-1})]\text{”}\}\\
=&\{(x_0,\cdots x_{n-1}) \mid (\exists \boldsymbol{\beta} \in \alpha^{<\omega})\ (\exists x \in \lambda^\omega)\\
&\qquad\qquad \mathrm{L}_\alpha(\lambda^\omega) \vDash \text{“}\phi'[\pi_\alpha(\boldsymbol{\beta}, x), \pi_\alpha(\boldsymbol{\beta}^0, x_0), \cdots, \pi_\alpha(\boldsymbol{\beta}^{n-1}, x_{n-1})]\text{”}\}\\
=&\bigcup_{x\in\lambda^\omega}\ \bigcup_{\boldsymbol{\beta}\in\alpha^{<\omega}} T^\alpha_{\phi',\boldsymbol{\beta},\boldsymbol{\beta}^0,\cdots,\boldsymbol{\beta}^{n-1}}\\
=&\bigcup_{x\in\lambda^\omega}\ \bigcup_{\boldsymbol{\beta}\in\alpha^{<\omega}} B_{q_{\boldsymbol{\beta}}}\\
=&\bigcup_{x\in\lambda^\omega} B_{\bigvee_{\boldsymbol{\beta}\in\alpha^{<\omega}} q_{\boldsymbol{\beta}}}\\
=&B_{F\left(\bigvee_{\boldsymbol{\beta}\in\alpha^{<\omega}} q_{\boldsymbol{\beta}}\right)},
\end{aligned}$$

where $B_{q_{\boldsymbol{\beta}}}$ is the subset of λ^ω defined from the code $q_{\boldsymbol{\beta}}$ as in (†), $\bigvee_{\boldsymbol{\beta}\in\alpha^{<\omega}} q_{\boldsymbol{\beta}}$ is the pair (ψ, S) defining the union $\bigcup_{\boldsymbol{\beta}\in\alpha^{<\omega}} B_{q_{\boldsymbol{\beta}}}$ in a simliar way as Case 3, and F is from Claim 2. Therefore, $q^\alpha_p = F\big(\bigvee_{\boldsymbol{\beta}\in\alpha^{<\omega}} q_{\boldsymbol{\beta}}\big)$ is the desired code. This completes the proof of the claim. □

We are now ready to finish the proof of Theorem 5.

Let A be a subset of λ^ω in $\mathrm{L}(\lambda^\omega)$. By Fact 1, it is enough to find a pair (ϕ, S) which defines A as in (†). Since A is in $\mathrm{L}(\lambda^\omega)$, there is an ordinal α such that $A \in \mathrm{L}_{\alpha+1}(\lambda^\omega) \setminus \mathrm{L}_\alpha(\lambda^\omega)$. Let ψ be a formula defining A in the structure $\big(\mathrm{L}_\alpha(\lambda^\omega), \in\big)$ with some parameters $\pi_\alpha(\boldsymbol{\beta}^0, x_0), \cdots, \pi_\alpha(\boldsymbol{\beta}^{n-1}, x_{n-1})$. By Remark 4, $\pi_\alpha(\emptyset, x) = x$ for all $x \in \lambda^\omega$. Hence

$$\begin{aligned}
A =&\{x \mid \mathrm{L}_\alpha(\lambda^\omega) \vDash \text{“}\psi[\pi_\alpha(\emptyset, x), \pi_\alpha(\boldsymbol{\beta}^0, x_0), \cdots, \pi_\alpha(\boldsymbol{\beta}^{n-1}, x_{n-1})]\text{”}\}\\
=&\{x \mid (x, x_0, \cdots, x_{n-1}) \in T^\alpha_{\psi,\emptyset,\boldsymbol{\beta}^0,\cdots,\boldsymbol{\beta}^{n-1}}\}\\
=&\{x \mid (x, x_0, \cdots, x_{n-1}) \in B_{q^\alpha_p}\},
\end{aligned}$$

where $p = (\psi, \emptyset, \boldsymbol{\beta}^0, \cdots, \boldsymbol{\beta}^{n-1})$ and q^α_p is from Claim 3. This shows that A is defined from q^α_p with parameters $x_0, \cdots, x_{n-1}$, which easily gives us that A is defined from (ϕ, S) for some ϕ and S.

This completes the proof of Theorem 5. □

5 Weak homogeneity and supercompactness of $\boldsymbol{\omega_1}$

In this section, we prove Theorem 2. A tree T is said to be on $\omega \times \kappa$ if $T \subset (\omega \times \kappa)^{<\omega}$. For a tree T on $\omega \times \kappa$, for $s \in \omega^{<\omega}$, let $T_s = \{t \in \kappa^{lh(s)} \mid (s, t) \in T\}$. Let also $p_0[T] = \{x \in \omega^\omega \mid \exists f \forall n (x \restriction n, f \restriction n) \in T\}$, $p_1[T] = \{f \in \kappa^\omega \mid \exists x \forall n (x \restriction$

$n, f \restriction n) \in T\}$. Every tree T considered in the following will be on $\omega \times \kappa$ for some κ.

Following [9], we define what it means for a tree T on $\omega \times \kappa$ to be **weakly homogeneous**. First, for $n < \omega$, κ an infinite cardinal, λ a nonzero ordinal, let $\mathrm{MEAS}_n^{\kappa,\lambda}$ be the set of all κ-complete measures on λ^n. For $m < n < \omega$, for $X \subseteq \lambda^m$, let $ext_n(X) = \{t \in \lambda^n \mid t \restriction m \in X\}$. A **$\lambda$-tower of measures** is a sequence $(\mu_n \mid n < \omega)$ such that

(i) for each n, $\mu_n \in \mathrm{MEAS}_n^{\omega_1,\lambda}$, and
(ii) for $m < n$, $\mu_m = proj_m(\mu_n)$, where $proj_m(\mu_n) = \{X \subseteq \lambda^m \mid ext_n(X) \in \mu_n\}$.

A tower $(\mu_n \mid n < \omega)$ is **countably complete** if for every sequence $(X_n \mid n < \omega)$ such that $X_n \in \mu_n$ for all $n < \omega$, there is a function $f : \omega \to \lambda$ such that $f \restriction n \in X_n$ for all n.

Definition 5. *Let T be a tree on $\omega \times \lambda$. T is* ***weakly homogeneous*** *if there is a sequence $(M_s \mid s \in \omega^{<\omega})$ such that*

(i) for each s, M_s is a countable subset of $MEAS_{lh(s)}^{\omega_1,\lambda}$ and for each $\mu \in M_s$, $T_s \in \mu_s$.
(ii) for all $x \in p_0[T]$, there is a countably complete λ-tower of measures $(\mu_n \mid n < \omega)$ such that for each n, $\mu_n \in M_{x \restriction n}$.

Remark 5. We will not work directly with weakly homogenous trees in the proof of Theorem 2. Rather, the conclusion that T is weakly homogeneous is reached by verifying that the hypotheses needed to run the proof in [9] follow from our hypothesis that ω_1 is supercompact. Theorem 2 is similar to one of the main results of [9], which states that "every tree is weakly homogeneous" follows from $\mathsf{AD}_{\mathbb{R}}$.

Proof (Theorem 2). Let T be a tree on $\omega \times \lambda$. [9] shows that T is weakly homogeneous provided the following conditions hold:

(A) There is a countably complete, normal fine measure on $\mathcal{P}_{\omega_1}(\bigcup_n(\mathcal{P}(\lambda^n) \cup \mathrm{MEAS}_n^{\omega_1,\lambda}))$.
(B) The Axiom of Dependent Choice holds for relations on $\mathcal{P}(\lambda)$.
(C) There is a wellorder on $\bigcup_n \mathrm{MEAS}_n^{\omega_1,\lambda}$.

We need to verify (A), (B), (C) follow from the supercompactness of ω_1. (A) is obvious. (B) follows from Theorem 1. Now we verify (C). Let $X = \bigcup_n \mathrm{MEAS}_n^{\omega_1,\lambda}$. We need to show that X is wellorderable.[11]

It is enough to prove that $\mathrm{MEAS}_1^{\omega_1,\lambda}$ is well-orderable. This is because for each $n > 1$, there is a bijection from $\mathrm{MEAS}_1^{\omega_1,\lambda}$ onto $\mathrm{MEAS}_n^{\omega_1,\lambda}$. Such a bijection is induced by a bijection between λ and λ^n. Hence, $\mathrm{MEAS}_n^{\omega_1,\lambda}$ is well-orderable. Using a definable bijection from λ onto $\lambda^{<\omega}$, we conclude that X is well-orderable.

[11] This is similar to Kunen's proof that under AD, every countably complete measure on an ordinal is OD. See [5, Corollary 28.21].

Let $Z = \mathcal{P}(\lambda)$. Let U be a countably complete, normal fine measure on $\mathcal{P}_{\omega_1} Z$. Given $\mu \in$MEAS$_1^{\omega_1,\lambda}$ and $\sigma \in \mathcal{P}_{\omega_1} Z$, let

$$f_\mu(\sigma) = min \bigcap(\sigma \cap \mu).$$

So f_μ is a function from $\mathcal{P}_{\omega_1} Z$ into the ordinals.

Claim 4 *Suppose $\mu \neq \nu$ are in MEAS$_1^{\omega_1,\lambda}$. Then $\forall^*_U \sigma \; f_\mu(\sigma) \neq f_\nu(\sigma)$; here "$\forall^*_U \sigma \varphi(\sigma)$" abbreviates the statement "the set of σ such that $\varphi(\sigma)$ is in U".*

Proof. Let A witness $\mu \neq \nu$. Without loss of generality, assume $A \in \mu$ and $\neg A \in \nu$. By fineness of U, $\forall^*_U \sigma$, $\{A, \neg A\} \subset \sigma$. Fix such a σ. Then $f_\mu(\sigma) \in A$ and $f_\nu(\sigma) \in \neg A$. Since $A, \neg A$ are disjoint, $f_\mu(\sigma) \neq f_\nu(\sigma)$. □

Let $\pi : X \to \prod_{\sigma \in \mathcal{P}_{\omega_1} Z} Ord/U$ be defined as: $\pi(\mu) = [f_\mu]_U$. The claim gives us that π is an injection. By DC, $\prod_{\sigma \in \mathcal{P}_{\omega_1} Z} Ord/U$ is well-founded and furthermore is well-ordered. Therefore, X is well-ordered as desired. □

6 AD$^+$, AD$_\mathbb{R}$, and supercompactness of ω_1

In this section, we prove Theorem 4. The following fundamental fact about AD$^+$ is due to W.H. Woodin (cf. [7]).

Theorem 6 (Woodin). *The following are equivalent.*

1. *AD$^+$.*
2. *AD + the class of Suslin cardinals is closed below Θ.*

We will also need the following results due to D.A. Martin and Woodin.

Theorem 7. *Assume ZF + DC. The following are equivalent.*

1. *AD$_\mathbb{R}$.*
2. *AD$^+$+ every set is Suslin.*

We now use Theorem 6 and Theorem 7 to prove Theorem 4.

Proof (Theorem 4). First, note that by Theorem 1, DC follows from supercompactness of ω_1. The ($\Leftarrow$) direction follows immediately from Theorem 7. For the ($\Rightarrow$) direction, suppose AD$_\mathbb{R}$ fails. Let $\kappa < \Theta$ be the largest Suslin cardinal and $\Gamma = S(\kappa)$. The existence of κ follows from Theorems 6 and 7. By [6, Theorem 1.3], there is a universal Γ-set. Let A be such a universal set. Then by results of Section 5, A is weakly homogeneously Suslin. By the Martin-Solovay construction, $\neg A$ is Suslin. But $\neg A \in \check{\Gamma} \backslash \Gamma$. This contradicts the fact that Γ is the largest Suslin pointclass.

Remark 6. Wilson's methods, cf. [23], using the theory of envelopes of pointclasses can be used to show that ZF + DC+ω_1 is strongly compact implies every Suslin set of reals is co-Suslin directly, without using weak homogeneity.

The following may be a more approachable version of a well-known conjecture that AD is equivalent to AD$^+$.

Conjecture 2. Assume ω_1 is supercompact. AD is equivalent to AD$^+$.

7 HPC and supercompactness of ω_1

In this section, we will prove Theorem 3. First, we note that we do not need the full "ω_1 is supercompact" hypothesis in the proof of the theorem; one just needs:

- ω_1 is $\mathbb{R}$-supercompact, and
- $\neg\square_{\omega_1}$.

Both of these are consequences of ω_1 is supercompact, cf. [22, Section 1].

Now we explain *Hod Pair Capturing* (HPC). This hypothesis and the notion of *least branch hod pair* (lbr hod pair) are formulated by John Steel. The reader can see [16] for a detailed discussion regarding topics concerning least-branch hod premice, lbr hod pairs, and HPC. The main thing one needs from HPC are the facts given by Theorem 8. For basic facts about inner model theoretic notions such as iteration strategies, see [15]. In particular, a complete strategy for $\mathcal{P}$ is an iteration strategy Σ that acts on all finite stacks (of normal trees) on $\mathcal{P}$ that are according to Σ.

Definition 6 (lbr hod pair, [16]). *$(\mathcal{P},\Sigma)$ is an lbr hod pair if $\mathcal{P}$ is an lpm (least-branch hod premouse) and Σ is a complete strategy for $\mathcal{P}$ that normalizes well and has strong hull condensation.*

Definition 7 (HPC, [16]). *Suppose A is Suslin co-Suslin. Then there is an lbr hod pair $(\mathcal{P},\Sigma)$ such that A is Wadge reducible to $Code(\Sigma)$.*

Remark 7. We caution the reader that the formulation of HPC here in Definition 7 is slightly different from Steel's formulation of HPC. The difference is that we do not work under AD^+ in this section. In applications using the core model induction, we are proving that HPC or its variations holds in a universe where AD typically fails (assuming certain smallness hypotheses). From our hypotheses and Steel's results in [16] and [17], we get that such a Σ as in Definition 7 is Suslin co-Suslin.

It is conjectured that AD^+ implies HPC. HPC and its variations have been shown to hold in very strong models of determinacy, cf. [12].

In the above, a complete strategy acts on all countable stacks of countable normal trees. The reader can consult [16] for more details on lbr hod pairs. The basic theory of lbr hod pairs has been worked out in [16]. What we need are a couple of facts about them. In the following, we fix a canonical coding *Code* of subsets of HC by subsets of $\mathbb{R}$.[12] Given an lbr hod pair $(\mathcal{P},\Sigma)$, for $n<\omega$, $\mathcal{M}_n^{\Sigma,\sharp}$ is the minimal, active Σ-mouse that has n Woodin cardinals. See for instance [14] for a precise definition. The following facts are relevant for us.

Lemma 2. *Let $(\mathcal{P},\Sigma)$ be an lbr hod pair. Let $M=\mathcal{M}_n^{\Sigma,\sharp}$ and Λ be M's canonical strategy. Let λ be the largest Woodin cardinal of M. There is a term $\tau_\Sigma\in M^{Coll(\omega,\lambda)}$ such that whenever $i:M\to N$ is an iteration embedding via an iteration according to λ, and $g\subseteq Coll(\omega,i(\lambda))$ is N-generic, then*

[12] One way of defining *Code* is as follows. Let $\pi:\mathbb{R}\to HC$ be a surjection defined as: for any x that codes a well-founded relation E_x on ω, let $\pi(x)$ be the transitive collapse of the structure (ω,E_x). Then for any $A\subseteq HC$, $Code(A)$ is defined to be $\pi^{-1}[A]$.

$$[i(\tau_\Sigma)]_g = Code(\Sigma) \cap N[g].$$

Theorem 8. *Suppose ω_1 is $\mathbb{R}$-supercompact, $\neg\Box_{\omega_1}$. Suppose $(\mathcal{P}, \Sigma)$ is an lbr hod pair. Then*

1. *([17, Section 2]) $Code(\Sigma)$ is Suslin co-Suslin.*
2. *([22, Section 3]) $\mathcal{M}_2^{\Sigma,\sharp}$ exists.*

Remark 8. We note that in the above theorem, the hypothesis ω_1 is $\mathbb{R}$-supercompact is used in (1) to extend Σ to act on all stacks $\boldsymbol{\mathcal{W}}$ such that there is a surjection of $\mathbb{R}$ onto $\boldsymbol{\mathcal{W}}$. The proof of (2) just needs ω_1 is $\mathbb{R}$-strongly compact and $\neg\Box_{\omega_1}$.

The following theorem, due to Neeman, is our main tool for proving determinacy.

Theorem 9 (Neeman, [10]). *Suppose $A \subseteq \mathbb{R}$. Suppose (M, Λ, δ) is such that*

1. *M is a countable, transitive model of ZFC;*
2. *$M \vDash \delta$ is Woodin;*
3. *Λ is an $\omega_1 + 1$-iteration strategy for M;*[13]
4. *there is a term $\tau \in M^{Coll(\omega,\delta)}$ such that whenever $i : M \to N$ is an iteration map according to Λ, $g \subseteq Coll(\omega, i(\delta))$ is N-generic, then $i(\tau)_g = A \cap N[g]$.*

Then A is determined.

Proof (Theorem 3). Let A be Suslin. Then A is also co-Suslin by Remark 1. By HPC, let $(\mathcal{P}, \Sigma)$ be an lbr hod pair such that A is Wadge reducible to $Code(\Sigma)$. Let $x \in \mathbb{R}$ witness this; we let τ_x be the continuous function given by x such that $\tau_x^{-1}[Code(\Sigma)] = A$.[14] By Theorem 8, $Code(\Sigma)$ is Suslin co-Suslin and $\mathcal{M}_2^{\Sigma,\sharp}$ exists. Let Λ be the canonical iteration strategy for $\mathcal{M}_2^{\Sigma,\sharp}$ and $\delta_0 < \delta_1$ be the Woodin cardinals of $\mathcal{M}_2^{\Sigma,\sharp}$. Let $\mathcal{T}$ be an iteration tree with the following properties:

- $\mathcal{T}$ is according to Λ.
- Letting $i : \mathcal{M}_2^{\Sigma,\sharp} \to \mathcal{N}$ be the corresponding iteration embedding, then x is $\mathcal{N}$-generic for the extender algebra at $i(\delta_0)$.[15]

Now we can construe $\mathcal{N}[x]$ as a Σ-mouse over x, which we will call $\mathcal{M}$. Note that $i(\delta_1)$ is a Woodin cardinal of $\mathcal{M}$ and Λ induces a strategy Ψ on $\mathcal{M}$.

We note that $(\mathcal{M}, \Psi, i(\delta_1))$ satisfies the hypothesis of Theorem 9 for A. Let τ_Σ be given as in Lemma 2 for $(\mathcal{M}_2^{\Sigma,\sharp}, \Lambda, \delta_1)$, then $i(\tau_\Sigma)$ induces a term $\sigma_\Sigma \in M^{Coll(\omega,i(\delta_1))}$ satisfying (3) of Theorem 9. Let $\mathbb{P} = Coll(\omega, i(\delta_1))$. The term τ consists of $(1_\mathbb{P}, \sigma)$ where $1_\mathbb{P} \Vdash_\mathbb{P}$ "σ is a real and $\sigma \in \tau_x^{-1}[\sigma_\Sigma]$".

By Theorem 9, A is determined. This completes the proof of Theorem 3. □

[13] Neeman [10] needs less iterability than $\omega_1 + 1$-iterability. $\omega_1 + 1$-iterability is available to us in this context.

[14] We can construe x as a function $\omega^{<\omega} \to \omega^{<\omega}$ and this naturally gives rise to a continuous (in fact Lipschitz) function $\sigma_x : \omega^\omega \to \omega^\omega$.

[15] The reader can see [15] for more detailed discussions about the extender algebra and genericity iteration trees. All we need here is the fact there is some forcing $\mathbb{Q} \in \mathcal{N}$ such that $\text{card}^\mathcal{N}(\mathbb{Q}) < i(\delta_1)$ and that there is some $\mathcal{N}$-generic filter $h \subset \mathbb{Q}$ with $x \in \mathcal{N}[h]$.

We conjecture that HPC is not needed in Theorem 3. [22] has shown that ω_1 is supercompact implies that all sets in $L(\mathbb{R})$ are Suslin and co-Suslin and are determined and much more.[16] One may hope to prove Conjecture 3 by showing that every Suslin set is homogeneously Suslin. Theorem 2 shows that every Suslin set is a projection of a homogenously Suslin, hence determined, set.

Conjecture 3. Assume ω_1 is supercompact. For any Suslin set A, A is determined.

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[16][22] shows that there is a transitive class model M containing all reals such that $M \models \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

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