The fine structure of operator mice

Farmer Schlutzenberg*
Nam Trang†

April 22, 2026

Abstract

We develop the fine structure theory of *operator-premice*. These are a generalization of standard premice, in which an abstract operator $\mathcal{F}$ is used to form the successor steps in the internal hierarchy of the premouse, instead of Jensen's $\mathcal{J}$-operator (which computes rudimentary closure). Such notions have seen applications in core model induction arguments, but their theory has not previously been developed in detail. We define *fine condensation* for operators $\mathcal{F}$ and show that fine condensation and iterability together ensure that $\mathcal{F}$-mice have the fundamental fine structural properties including universality and solidity of the standard parameter.

Contents

Key words: Inner model, operator, mouse, fine structure
2010 MSC: 03E45, 03E55

[*] Institute of Discrete Mathematics and Geometry, TU Vienna, Email: farmer.schlutzenberg@tuwien.ac.at

[†] Department of Mathematics, University of North Texas, USA, Email: nam.trang@unt.edu

1 Introduction

The core model induction is the most general and successful method for computing lower bounds for the consistency strengths of strong theories, like PFA, MM and many others. It is used to construct models of AD^+ of high complexity – which themselves contain inner models for large cardinals – from such theories; the papers [24], [21], [28], [26], [1], [27], [7], [6], [2] develop the general theory of the core model induction and give various applications. The construction is inductive in structure. Roughly, one proves that the universe V (or some part thereof, such as the hereditarily countable sets HC) is closed under certain kinds of operators (functions) $\mathcal{F}$ which yield inner models for large cardinals. The proof is by induction on the complexity of such $\mathcal{F}$, as measured in terms of the Wadge hierarchy in models of AD^+. Suppose we have constructed an operator $\mathcal{F}$ of the right kind, which captures a pointclass $\underset{\sim}{\Gamma}$ of AD^+.[1] We would like to construct an operator of higher complexity, and a pointclass $\underset{\sim}{\Omega}$ of AD^+ that strictly contains $\underset{\sim}{\Gamma}$. In one case, we can construct such objects by constructing the operator $x \mapsto \mathcal{M}_1^{\mathcal{F},\sharp}(x)$; this is the sharp for the canonical iterable $\mathcal{F}$-closed model for one Woodin cardinal over x, analogous to $\mathcal{M}_1^{\#}(x)$. This is usually achieved by showing that some fully backgrounded construction or partially backgrounded construction (K^c-construction) over x, which is built relative to $\mathcal{F}$, reaches $\mathcal{M}_1^{\mathcal{F},\sharp}(x)$. Here "built relative to $\mathcal{F}$" means that the successor stages of the construction are applications of $\mathcal{F}$, instead of Jensen's operator $\mathcal{J}$. (Given a set X, $\mathcal{J}(X)$ is the closure of $X \cup \{X\}$ under the rudimentary set functions. Standard premice are constructed using $\mathcal{J}$ to extend the model at successor levels, and extenders are added at certain limit levels.) In order for this kind of construction to work, $\mathcal{F}$ should satisfy special properties, generalizing many of those that $\mathcal{J}$ satisfies. This paper defines precisely these concepts and generalizes fine-structural and iterability results from ordinary mice to $\mathcal{F}$-mice.

In an $\mathcal{F}$*-premouse*, $\mathcal{F}$ is used to extend the model at successor levels, instead of $\mathcal{J}$. The operator $\mathcal{F}$ can be used to feed different kinds of information into a model. For example, ordinary mice, or an iteration strategy, or the specification of term relations for a self-justifying system, are some examples of the kind of

---

[1] In the same sense that $\mathcal{M}_1^{\sharp}(x)$ captures $\Sigma^1_2(x)$.

information that might be fed in. We will define *$\mathcal{F}$-premice* for a fairly wide class of operators $\mathcal{F}$ with nice condensation properties, and develop their basic theory. Versions of this theory have been outlined and used by others (see particularly [21, §1.3] and [28, §2.1]), but without supplying a very thorough development of the theory. We give here a more thorough development. Aside from providing more details, some of the basic definitions we use here differ from those in [21] and [28] in important ways. But other than in Remark 3.13 (which can be omitted), this paper has no formal dependence on those two papers, though they do provide significant motivation for what we do here.

If X is a transitive set in the domain of an operator $\mathcal{F}$ of the kind in which we are interested, then $\mathcal{N} = \mathcal{F}(X)$ will be a transitive structure with $X \in \mathcal{N}$, and $\mathcal{F}(X)$ will be a (very) simple instance of an *$\mathcal{F}$-premouse over X*. More generally, if $\mathcal{R}$ is a sound $\mathcal{F}$-premouse over X, then $\mathcal{F}(\mathcal{R})$ will be an $\mathcal{F}$-premouse over X, with $\mathcal{R} \triangleleft \mathcal{F}(\mathcal{R})$ (that is, $\mathcal{R}$ is a proper initial segment of $\mathcal{F}(\mathcal{R})$, in fact the largest such). An essential feature of the operators suitable for our purposes is their behaviour under *condensation*, which should be reasonably analogous to that of the $\mathcal{J}$-operator. For example, if X is a transitive set in the domain of $\mathcal{F}$, $\mathcal{R}, \mathcal{S}$ are $\mathcal{F}$-premice over X, $\mathcal{M}$ is a transitive structure with $\mathcal{R} \in \mathcal{M}$ and $\pi : \mathcal{M} \to \mathcal{F}(\mathcal{S})$ is elementary with $\pi(\mathcal{R}) = \mathcal{S}$ and $\pi \restriction X \cup \{X\} = \mathrm{id}$, then we might want to know that $\mathcal{M} = \mathcal{F}(\mathcal{R})$. In fact, we will also want to consider such condensation with respect to partially elementary maps that show up in fine structural contexts. We will formulate such properties – *coarse condensation* and *fine condensation* – in Definitions 3.10 and 3.25 respectively. But before we can discuss those properties, we need to describe the more basic and general properties of *operator-premice*, which will be the abstract form for $\mathcal{F}$-premice, irrespective of the features of any particular $\mathcal{F}$. This is the subject of §2. After having developed this theory, we will prove in Theorem 3.45 that if $\mathcal{F}$ is an operator with fine condensation, then $\mathcal{F}$-iterable $\mathcal{F}$-premice satisfy the fundamental fine structural properties that are essential to our understanding of standard premice.

As mentioned above, some basic notions we use differ significantly from their analogues in [21] and [28]. Let us now try to convey some idea about this. The fine version of condensation for operators (described roughly in the previous paragraph) which is employed in [21] and [28] is that of *condenses well* (see [21, Definition 1.3.2] and [28, Definition 2.1.10]), and this notion is central to the theory in those papers. As we explain in Remark 3.13, this property does not fully function as one would like, and in particular, in the terminology of [28], the operators F_G derived from mouse operators G typically do not condense well, contrary to [21, Lemma 2.1.12].[2] We will show that the variant we introduce, *condenses finely* (see 3.25), behaves as desired. Because Remark 3.13 motivates some key aspects of the definition of *condenses finely*, we have placed it just prior to the formulation of that definition. But 3.13 does not rely on the material in

---

[2] Actually, there is a minor further issue in [21, Lemma 2.1.12], or more to the point, in [21, Definition 2.1.8], upon which [21, 2.1.12] relies; F_G (as specified in [21, 2.1.8]) is typically not well-defined in the first place. There is a natural correction to this, but employing the correction, [21, Lemma 2.1.12] fails.

the paper prior to where it appears, and the reader who wishes to start with it should have no difficulty in doing so.

A second key definition of [21] and [28] is that of *model operator* (see [21, Definition 1.3.1], and the similar [28, Definition 2.1.4]). The authors were not able to develop the theory at the level of generality of model operators with condensation properties, because we could not see how to define appropriate Σ_1-Skolem functions, nor prove facts such as the preservation of the 1st standard parameter under iteration maps. Thus, we make stronger hypotheses on the kinds of structures we work with, ensuring more properties familiar from $\mathcal{J}$-structues;[3] see especially Definitions 2.21, 2.7 and 2.39.

Our proof of the solidity of the standard parameter, part of Theorem 3.45, is based on that in the union of [4], [25] and [9]. But we provide some details which are not discussed explicitly in those papers, which are also relevant in the case of ordinary premice (as opposed to operator-premice), and which the authors believe are non-trivial. In the introduction to §3.8, we isolate the point in the proof at which the details are relevant.

In the paper, we will mostly focus on material that is new, skipping certain parts which are immediate transcriptions of the theory of standard premice, although for purposes of readability and self-containment, we do include some fairly standard material.

We have tried to develop the theory in a manner that its content is mostly compatible with the literature. This is part of our motivation for developing the theory of $\mathcal{F}$-premice abstractly, dealing with operators $\mathcal{F}$ more general than those given by $\mathcal{J}$-structures; cf. the developments in [21] and [28], which are abstract. Of course the abstract development also makes the work more general, and has the advantage of showing which properties of $\mathcal{J}$-structures are most essential to the theory. But it does incur the cost of increasing complexity somewhat. A reasonable alternative would have been to give a more concrete development by restricting attention to operators given by $\mathcal{J}$-structures, and in the end, all applications known to the authors are of this form. Also, if one deals exclusively with $\mathcal{J}$-structures, one can more naturally formulate fine structural condensation properties regarding *all* $\mathcal{J}$-initial segments of the model. But at least the most straightforward analogues of condensation for abstract $\mathcal{F}$-mice apply only to $\mathcal{F}$-initial segments of the model.[4] This seems to be a significant complication for abstract $\mathcal{F}$-mice.[5] Also, there are important operators, like

---

[3] A *$\mathcal{J}$-structure* is one of form $(\mathcal{J}^A_\alpha, B)$ for some A, where α is an ordinal or $\alpha = \mathrm{Ord}$, and some predicate $B \subseteq \mathcal{J}^A_\alpha$. Here $\mathcal{J}^A_\alpha$ refers to the αth iterate of the $\mathcal{J}$-operator relativized to A.

[4] That is, given a reasonably closed $\mathcal{F}$-mouse $\mathcal{M}$, it is straightforward to formulate condensation properties with respect to embeddings $\mathcal{H} \to \mathcal{M}$, or $\mathcal{H} \to \mathcal{F}(\mathcal{M})$, or $\mathcal{H} \to \mathcal{F}(\mathcal{F}(\mathcal{M}))$, etc, but it is not so clear how this should be done with respect to embeddings $\mathcal{H} \to \mathcal{N}$ when $\mathcal{M} \in \mathcal{N} \in \mathcal{F}(\mathcal{M})$.

[5] For example, strategy mice can either be defined as an instance of the general theory here, or as $\mathcal{J}$-structures. The latter approach is taken in [10], and that approach is more convenient, as it gives us the right notation to prove strong condensation properties like [10, Lemma 4.1(***)]. If one defines strategy mice as an instance of the general theory here, one would then need to define new notation to refer to arbitrary $\mathcal{J}$-initial segments in order to prove the analogue of [10, Lemma 4.1(***)]. But then one might as well have defined strategy

$x \mapsto C_\Gamma(x)$ for pointclasses Γ, which are not known to be given by $\mathcal{J}$-structures. We hope the work here will fuel the developments of a more general theory of operators that can accommodate those like C_Γ in the future. (C_Γ probably need not yield the kind of operator that is appropriate to define operator-mice as we define them. But maybe some appropriate variant can be worked out.)

The paper proceeds as follows. In §2 we define precursors to $\mathcal{F}$-premice, culminating in *operator premice*. We analyse these structures and cover basic fine structure and iteration theory. In §3, we introduce *operators* $\mathcal{F}$, and $\mathcal{F}$*-premice*, which will be instances of operator premice. We define *fine condensation* for operators; this notion is integral to the paper. We describe *mouse operators* in Definition 3.18 (a basic example of abstract operators), and show in Proposition 3.28 that mouse operators condense finely. We then prove, in 3.45, the main result of the paper – that the fundamental fine structural facts (such as solidity of the standard parameter) hold for $\mathcal{F}$-iterable $\mathcal{F}$-premice, given that $\mathcal{F}$ condenses finely.

1.1 Conventions and Notation

We work in `ZF` throughout the paper, indicating choice assumptions where we use them. We write Ord for the class of ordinals. Given a transitive set M, $\mathrm{Ord}^M = \mathrm{Ord}(M)$ denotes $\mathrm{Ord} \cap M$. We write $\mathrm{card}(X)$ for the cardinality of X, $\mathcal{P}(X)$ for the power set of X, and for $\theta \in \mathrm{Ord}$, $\mathcal{P}(<\theta)$ is the set of bounded subsets of θ and $\mathscr{H}_\theta$ the set of sets hereditarily of cardinality $<\theta$. We write $f : X \dashrightarrow Y$ to denote a partial function.

We identify $[\mathrm{Ord}]^{<\omega}$ with the strictly decreasing sequences of ordinals, so given $p, q \in [\mathrm{Ord}]^{<\omega}$, $p{\restriction}i$ denotes the upper i elements of p, and $p \trianglelefteq q$ means that $p = q{\restriction}i$ for some i, and $p \triangleleft q$ iff $p \trianglelefteq q$ but $p \neq q$. The default ordering of $[\mathrm{Ord}]^{<\omega}$ is lexicographic, with $p < q$ iff $p \neq q$ and $\max(p \Delta q) \in q$.

Given a first-order structure $\mathcal{M} = (X, A_1, \ldots)$ with universe X and predicates, constants, etc, $A_1, \ldots$, we write $\lfloor \mathcal{M} \rfloor = X$. A **transitive structure** is a first-order structure with transitive universe. We sometimes blur the distinction between the terms *transitive* and *transitive structure*. For example, when we refer to a transitive structure as being **rud closed**, it means that its universe is closed under rudimentary functions. For $\mathcal{M}$ a transitive structure, $\mathrm{Ord}(\mathcal{M}) = \mathrm{Ord}(\lfloor \mathcal{M} \rfloor)$. An arbitrary transitive set X is also considered as the transitive structure (X). We write $\mathrm{trancl}(X)$ for the transitive closure of X. We say that $\mathcal{M}$ is *amenable* if for predicate A of $\mathcal{M}$, we have $X \cap A \in \mathcal{M}$ for all $X \in \lfloor \mathcal{M} \rfloor$.

Given a transitive structure $\mathcal{M}$, we write $\mathcal{J}_\alpha(\mathcal{M})$ for the α^{th} step in Jensen's $\mathcal{J}$-hierarchy over $\mathcal{M}$ (so for example, $\mathcal{J}_1(\mathcal{M})$ is the rud closure of $\mathrm{trancl}(\{\mathcal{M}\})$). We similarly use $\mathcal{S}$ to denote the function giving Jensen's more refined $\mathcal{S}$-hierarchy, so $\mathcal{S}_\omega(\mathcal{M}) = \mathcal{J}_1(\mathcal{M})$. And $\mathcal{J}(\mathcal{M}) = \mathcal{J}_1(\mathcal{M})$.

We take (standard) **premice** as in [25], except that we allow superstrong extenders on their sequence, as discussed in Remark 2.48. Our definition and

---

mice as in [10] to begin with.

theory of *operator premice* is mostly modelled on [25] and [4], and fine structure is mostly in those papers, but adopting the simplifications in [19, §5]. For discussion of generalized solidity witnesses, see [29].

Our notation pertaining to iteration trees is fairly standard, but here are some points. Let $\mathcal{T}$ be a putative iteration tree. We write $<^{\mathcal{T}}$ for the tree order of $\mathcal{T}$ and $\mathrm{pred}^{\mathcal{T}}$ for the $<^{\mathcal{T}}$-predecessor function. Let $\alpha+1 < \mathrm{lh}(\mathcal{T})$ and $\beta = \mathrm{pred}^{\mathcal{T}}(\alpha+1)$. Then $M^{*\mathcal{T}}_{\alpha+1}$ denotes the $\mathcal{N} \trianglelefteq M^{\mathcal{T}}_{\beta}$ such that $M^{\mathcal{T}}_{\alpha+1} = \mathrm{Ult}_n(\mathcal{N}, E)$, where $n = \deg^{\mathcal{T}}(\alpha+1)$ and $E = E^{\mathcal{T}}_{\alpha}$, and $i^{*\mathcal{T}}_{\alpha+1} = i^{\mathcal{N},n}_{E}$ denotes the corresponding ultrapower embedding. And for $\alpha+1 \leq_{\mathcal{T}} \gamma$, $i^{*\mathcal{T}}_{\alpha+1,\gamma} = i^{\mathcal{T}}_{\alpha+1,\gamma} \circ i^{*\mathcal{T}}_{\alpha+1}$. Also let $M^{*\mathcal{T}}_0 = M^{\mathcal{T}}_0$ and $i^{*\mathcal{T}}_0 = \mathrm{id}$. If $\mathrm{lh}(\mathcal{T}) = \gamma+1$ then $M^{\mathcal{T}}_{\infty} = M^{\mathcal{T}}_{\gamma}$, etc, and $b^{\mathcal{T}}$ denotes $[0,\gamma]_{\mathcal{T}}$. For $\alpha < \mathrm{lh}(\mathcal{T})$, $\mathrm{base}^{\mathcal{T}}(\alpha)$ denotes the least $\beta \leq_{\mathcal{T}} \alpha$ such that $(\beta, \alpha]_{\mathcal{T}}$ does not drop in model or degree. (Therefore either $\beta = 0$ or β is a successor.)

A premouse $\mathcal{P}$ is **η-sound** iff for every $n < \omega$, if $\eta < \rho^{\mathcal{P}}_n$ then $\mathcal{P}$ is n-sound, and if $\rho^{\mathcal{P}}_{n+1} \leq \eta < \rho^{\mathcal{P}}_n$ then letting $p = p^{\mathcal{P}}_{n+1}$, $p \backslash \eta$ is $(n+1)$-solid for $\mathcal{P}$, and $\mathcal{P} = \mathrm{Hull}^{\mathcal{P}}_{n+1}(\eta \cup \{p, \vec{p}^{\mathcal{P}}_n\})$, where $p^{\mathcal{P}}_i$ is the i-th standard parameter of $\mathcal{P}$, $\vec{p}^{\mathcal{P}}_n = \{p^{\mathcal{P}}_1, \ldots, p^{\mathcal{P}}_n\}$, and Hull_{n+1} is defined via the union of 2.2 and 2.27.

Let $\mathcal{M}$ be a first order structure and Γ a set of formulas in the signature of $\mathcal{M}$. Let $X \subseteq \mathcal{M}$. Then $\mathrm{Th}^{\mathcal{M}}_{\Gamma}(X)$ denotes the set of pairs $(\varphi, \vec{x})$ such that $\varphi \in \Gamma$, $\vec{x} \in X^{<\omega}$ and $\mathcal{M} \vDash \varphi(\vec{x})$.

2 The fine structural framework

In this section, we introduce and analyse an increasingly focused sequence of approximations to *$\mathcal{F}$-premice* (which were outlined in the introduction, but will be defined formally later). We first define *hierarchical model*, which describes the most basic structure of $\mathcal{F}$-premice. We refine this by defining *adequate model*, adding some semi-fine-structural requirements (such as *acceptability*). We then develop some basic facts regarding adequate models and their cardinal structure. From there we can define *potential operator premouse* (*potential opm*), which are analogous to potential premice; this definition makes new restrictions on the information encoded by the predicates (most significantly that the predicate $\dot{E}$ encodes extenders analogous to those of premice), and adds some pre-fine structural requirements. Using the latter, we can define the central fine structural concepts for potential opms. We then define *Q-operator premouse* (*Q-opm*) by requiring that every proper segment be fully sound, and show that the first-order content of Q-opm-hood is *almost* expressed by a Q-formula.[6] We then define *operator premouse* (analogous to *premouse*). We prove various fine structural facts regarding operator premice, and discuss the basic iterability theory.

Later in §3, we will introduce *operators* $\mathcal{F}$, and *$\mathcal{F}$-premice*. In order to

---

[6] As in [4], we consider two cases: type 3, and non-type 3. For example, the property of being a non-type 3 Q-opm is expressed by a Q-formula modulo transitivity and the Pairing Axiom.

motivate the language $\mathcal{L}_0$ of hierarchical models (see Definition 2.3), we mention now the basic setup for $\mathcal{F}$-premice. In an $\mathcal{F}$-premouse $\mathcal{M}$, the predicate $\dot{E}$ will be used to encode an extender, $\dot{P}$ to encode auxiliary information given by $\mathcal{F}$ (for example, if $\mathcal{F}$ codes an iteration strategy Σ and $\mathcal{T} \in \mathcal{M}$ is a tree according to Σ, then $\dot{P}$ could code a branch b of $\mathcal{T}$ according to Σ), $\dot{S}$ to encode the sequence of proper initial segments of $\mathcal{M}$, $\dot{X}$ to encode the extensions of all (not just proper) segments of $\mathcal{M}$, $\dot{cb}$ to refer to the coarse *base* of $\mathcal{M}$ (a coarse, transitive set at the bottom of the structure), and $\dot{cp}$ to refer to a coarse *parameter*, which will be useful if there is some special element of the coarse base to which we want to be able to refer to directly with the language (continuing the same example of $\mathcal{F}$ coding an iteration strategy, $\dot{cp}$ might specify the structure for which $\mathcal{F}$ is an iteration strategy). The choice of symbols has the following linguistic justification: E stands for *extender*, P for *predicate*, S for *segments*, X for *extensions*, cb for *coarse base*, cp for *coarse parameter*. We use cp instead of p to avoid conflict with notation for standard parameters. We use cb instead of b to avoid conflict with notation associated to strategy mice. For better readability, we will typically use the variable A to represent $cb^{\mathcal{M}}$. An $\mathcal{F}$-premouse $\mathcal{M}$ is *over* its base $A = \dot{cb}^{\mathcal{M}}$. Here $A \in \mathcal{M}$ and A is in all proper segments of $\mathcal{M}$. When we form fine structural cores, all elements of $A \cup \{A\}$ will be in the relevant hulls. But in some contexts we will also be interested in hulls which do not include all elements of A.

2.1 Hierarchical models

Definition 2.1. Let Y be transitive. Then $\varrho_Y : Y \to \mathrm{rank}(Y)$ denotes the rank function. And $\hat{Y}$ denotes $\mathrm{trancl}(\{(Y, \omega, \varrho_Y)\})$. For M transitive, we say that M is **rank closed** iff for every $Y \in M$, we have $\hat{Y} \in M$ and $\hat{Y}^{<\omega} \in M$.[7] Note that if M is rud closed and rank closed then $\mathrm{rank}(M) = \mathrm{Ord} \cap M$. $\dashv$

Definition 2.2 (Hulls)**.** Let $\mathcal{L} = \{\dot{B}, \vec{P}, \vec{c}\}$ be a finite first-order language,[8] where $\dot{B}$ is a binary predicate, $\vec{P} = \left\langle \dot{P}_i \right\rangle_{i<m}$ is a tuple of unary predicates and $\vec{c} = \langle \dot{c}_i \rangle_{i<n}$ a tuple of constants. Let $\mathcal{N}$ be a first-order $\mathcal{L}$-structure and $B = \dot{B}^{\mathcal{N}}$, etc. Let Γ be a collection of $\mathcal{L}$-formulas with "$x = \dot{c}_i$" in Γ for each $i < n$. Let $X \subseteq \lfloor \mathcal{N} \rfloor$. Then

$$\mathrm{Hull}_\Gamma^{\mathcal{N}}(X) =_{\mathrm{def}} (H, B \cap H^2, P_0 \cap H, \ldots, P_{m-1} \cap H, c_0, \ldots, c_{n-1}),$$

where H is the set of all $y \in \lfloor \mathcal{N} \rfloor$ such that for some $\varphi \in \Gamma$ and $\vec{x} \in X^{<\omega}$, y is the unique $y' \in \mathcal{N}$ such that $\mathcal{N} \models \varphi(\vec{x}, y')$. If $\mathcal{N}$ is transitive and H is extensional, then $\mathcal{C} = \mathrm{cHull}_\Gamma^{\mathcal{N}}(X)$ denotes the $\mathcal{L}$ structure which is the transitive collapse of $\mathrm{Hull}_\Gamma^{\mathcal{N}}(X)$. (That is, $\lfloor \mathcal{C} \rfloor$ is the transitive collapse of H, and letting $\pi : \lfloor \mathcal{C} \rfloor \to H$ be the uncollapse, $P_i^{\mathcal{C}} = \pi^{-1}``P_i$, etc.) $\dashv$

---

[7] We take finite sequences over Y as functions $f : n \to Y$ for $n < \omega$, so if Y is infinite then $\mathrm{rank}(\hat{Y}^{<\omega}) < \mathrm{rank}(Y) + \omega$.

[8] We include an equality symbol in all first-order languages by default, interpreted as true equality.

Definition 2.3. Let $\mathcal{L}_0$ be the language of set theory augmented with unary predicate symbols $\dot{E}$, $\dot{P}$, $\dot{S}$, $\dot{X}$, and constant symbols $\dot{cb}$, $\dot{cp}$. Let $\mathcal{L}_0^+$ be $\mathcal{L}_0$ augmented with constant symbols $\dot{\mu}$, $\dot{e}$.[9] Let $\mathcal{L}_0^- = \mathcal{L}_0 \backslash \{\dot{E}, \dot{P}\}$. $\dashv$

Definition 2.4. A **hierarchical model** is an $\mathcal{L}_0$-structure

$$\mathcal{M} = (\lfloor \mathcal{M} \rfloor ; \in\restriction \lfloor \mathcal{M} \rfloor^2, E, P, S, X, b, p),$$

where $\dot{\in}^{\mathcal{M}} = \in\restriction \lfloor \mathcal{M} \rfloor^2$, $\dot{E}^{\mathcal{M}} = E$, etc, $b = \dot{cb}^{\mathcal{M}}$ and $p = \dot{cp}^{\mathcal{M}}$, and such that for some ordinal $\lambda > 0$, the following conditions hold:

1. (Base, Parameter) $b = \hat{Y}$ for some transitive Y, and $p \in \mathcal{J}(b)$; we say that $\mathcal{M}$ is **over** the **(coarse) base** b and has **(coarse) parameter** p.
2. (Segments) $S = \langle S_\xi \rangle_{\xi < \lambda}$ where $S_0 = b$ and for each $\xi \in [1, \lambda)$, S_ξ is a $\mathcal{L}_0$-structure with $\dot{cb}^{S_\xi} = b$, $\dot{cp}^{S_\xi} = p$, and $\dot{S}^{S_\xi} = S\restriction\xi$. Write $S_\lambda = \mathcal{M}$.
3. For each $\xi \in [1, \lambda]$, $\lfloor S_\xi \rfloor$ is transitive, rud closed and rank closed, and S_ξ is amenable (note that this includes in particular $\mathcal{M} = S_\lambda$).[10]
4. (Continuity) $\lfloor S_\xi \rfloor = \bigcup_{\alpha < \xi} \lfloor S_\alpha \rfloor$ for each limit $\xi \leq \lambda$.
5. (Extensions) $X^{S_\xi} \colon \lfloor S_\xi \rfloor \to \xi$, and $X^{S_\xi}(x)$ is the least α such that $x \in \lfloor S_{\alpha+1} \rfloor$.

Let $l(\mathcal{M})$ denote λ, the **length** of $\mathcal{M}$. For $\alpha \leq \lambda$ let $\mathcal{M}|\alpha = S_\alpha$. A hierarchical model $\mathcal{M}$ is a **successor** iff $l(\mathcal{M})$ is a successor ordinal $\xi + 1$; in this case let $\mathcal{M}^- = \mathcal{M}|\xi$. If $l(\mathcal{M})$ is a limit ordinal, let $\mathcal{M}^- = \mathcal{M}$. We say that $\mathcal{N}$ is an **(initial) segment** of $\mathcal{M}$, and write $\mathcal{N} \trianglelefteq \mathcal{M}$, iff $\mathcal{N} = \mathcal{M}|\alpha$ for some $\alpha \in [1, \lambda]$, and say that $\mathcal{N}$ is a **proper (initial) segment** of $\mathcal{M}$, and write $\mathcal{N} \triangleleft \mathcal{M}$, iff $\mathcal{N} \trianglelefteq \mathcal{M}$ and $\mathcal{N} \neq \mathcal{M}$. (Note that $\mathcal{M}|0 = b \ntrianglelefteq \mathcal{M}$.) We write $E^{\mathcal{M}} = E$, etc. For any transitive Y, let $cb^{\hat{Y}} = \hat{Y}$; so $cb^{\mathcal{M}|\alpha} = \mathcal{M}|0$ for all α.[11] $\dashv$

The first observation follows easily from the definition:

Lemma 2.5. *Let $\mathcal{M}$ be a hierarchical model and $\mathcal{N} \trianglelefteq \mathcal{M}$. Then $\mathcal{N}$ is a hierarchical model.*

Remark 2.6. For the most part, definability over hierarchical models $\mathcal{M}$ will literally be computed over $\mathfrak{C}_0(\mathcal{M})$ (to be defined later), which will be an $\mathcal{L}_0^+$-structure. But for successors $\mathcal{M}$, we will have $\mathfrak{C}_0(\mathcal{M}) = (\mathcal{M}, \dot{\mu}^{\mathfrak{C}_0(\mathcal{M})}, \dot{e}^{\mathfrak{C}_0(\mathcal{M})})$ and $\dot{\mu}^{\mathfrak{C}_0(\mathcal{M})} = \emptyset = \dot{e}^{\mathfrak{C}_0(\mathcal{M})}$. So in this case, definability over $\mathcal{M}$ (using $\mathcal{L}_0$) will be equivalent to that over $\mathfrak{C}_0(\mathcal{M})$ (using $\mathcal{L}_0^+$).

---

[9] μ is for *measurable*, and will represent the critical point of an active extender, and e is for *extender*, and will represent the largest witness to the Initial Segment Condition for a type 2 active extender.

[10] Note that it follows that $S_\alpha \in \lfloor S_\beta \rfloor$ and $\lfloor S_\alpha \rfloor \subsetneq \lfloor S_\beta \rfloor$ for all $\alpha < \beta \leq \lambda$.

[11] That is, we have $cb^{\mathcal{M}|\alpha} = b = S_0 = \mathcal{M}|0$ for all $\alpha \in (0, \lambda]$ by definition. But recall that $b = \hat{Y}$ for some transitive Y, so $cb^{\mathcal{M}|0} = cb^b = cb^{\hat{Y}} = \hat{Y} = b = \mathcal{M}|0$.

Definition 2.7. Let $\mathcal{M}$ be a hierarchical model over A.

If $\mathcal{M}$ is a successor, then for $p \in [\mathrm{Ord}^{\mathcal{M}}]^{<\omega}$, we say that $\mathcal{M}$ is $(1,p)$**-solid** iff for every $\alpha \in p$, we have[12]

$$\mathrm{Hull}_1^{\mathcal{M}}\big(A \cup \alpha \cup (p \backslash (\alpha+1))\big) \preccurlyeq_1 \mathcal{M}$$

and

$$\mathrm{cHull}_1^{\mathcal{M}}\big(A \cup \alpha \cup (p \backslash (\alpha+1))\big) \in \mathcal{M}.$$

Note that it follows that $\mathrm{Th}_{\Sigma_1}^{\mathcal{M}}\big(A \cup \alpha \cup (p \backslash (\alpha+1))\big) \in \mathcal{M}$, and note that the Σ_1-elementarity ensures that the uncollapsed hull is extensional (in fact Σ_0 suffices), and hence the transitive collapse cHull is well-defined.

We say that $\mathcal{M}$ is **soundly projecting** iff for every successor $\mathcal{N} \trianglelefteq \mathcal{M}$, there is $p \in [\mathrm{Ord}(\mathcal{N})]^{<\omega}$ such that $\mathcal{N}$ is $(1,p)$-solid and

$$\mathcal{N} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{N}}(\mathcal{N}^- \cup \{\mathcal{N}^-, p\}).$$

We say that $\mathcal{M}$ is **acceptable** iff for every successor $\mathcal{N} \trianglelefteq \mathcal{M}$, for every $\tau \in \mathrm{Ord}(\mathcal{N}^-)$, if there is some $X \in \mathcal{P}(A^{<\omega} \times \tau^{<\omega})$ such that $X \in \mathcal{N} \backslash \mathcal{N}^-$ then in $\mathcal{N}$ there is a map $A^{<\omega} \times \tau^{<\omega} \overset{\text{onto}}{\to} \mathcal{N}^-$.

We say that $\mathcal{M}$ is an **adequate model** iff $\mathcal{M}$ an acceptable hierarchical model and every *proper* segment of $\mathcal{M}$ is soundly projecting.

An **adequate model-plus** is an $\mathcal{L}_0^+$-structure $\mathcal{M}$ such that the $\mathcal{L}_0$-reduct of $\mathcal{M}$ is an adequate model. ⊣

In the end we will be primarily interested in structures for which every initial segment is soundly projecting, not just the proper segments. For certain kinds of operators $\mathcal{F}$, such as the usual operators used to encode an iteration strategy in a hybrid mouse or strategy mouse, the successor structures $\mathcal{N}$ produced (as $\mathcal{F}$-premice) will in fact have the stronger property that $\mathcal{N} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{N}}(\mathcal{N}^- \cup \{\mathcal{N}^-\})$. Of course, this is the case when $\mathcal{F}$ is the usual $\mathcal{J}$-operator. For mouse operators $\mathcal{F}$, $\mathcal{N} = \mathcal{F}(\mathcal{N}^-)$ will be equivalent to a sound mouse over $\mathcal{N}^-$ which projects to $\mathcal{N}^-$. By coding that mouse via its nth reduct for the relevant n (with $\rho_{n+1}^{\mathcal{N}} \leq \mathrm{Ord}^{\mathcal{N}^-} < \rho_n^{\mathcal{N}^-}$), we will get a structure which is soundly projecting, with the p of the definition being $p_{n+1}^{\mathcal{N}}$.

As in [4], etc, it is useful to consider what can be expressed with *Q-formulas* and variants thereof, as they are preserved well downward under Σ_1-elementary maps, and upward under ultrapower embeddings:

Definition 2.8. Given a language $\mathcal{L}$ extending the language of set theory, an $\mathcal{L}$**-simple-Q-formula** is a formula of the form

$$\varphi(v_0, \ldots, v_{n-1}) \iff \forall x \exists y [x \subseteq y \ \& \ \psi(y, v_0, \ldots, v_{n-1})],$$

---

[12] Clearly this implies that $\mathrm{Th}_{\Sigma_1}^{\mathcal{M}}(X) \in \mathcal{M}$ also, where $X = A \cup \alpha \cup (p \backslash (\alpha+1))$. Recall that for standard premice $\mathcal{M}$, when defining the solidity of $p_1^{\mathcal{M}}$, it does not matter whether we demand that the relevant Σ_1-hulls are in $\mathcal{M}$, or their corresponding Σ_1-theories are in $\mathcal{M}$; the two requirements are equivalent. But this does not seem clear for the structures we consider. It is important that we use the stronger condition.

for some Σ_1 formula ψ of $\mathcal{L}$. (Here all free variables are displayed; hence, x is not free in ψ.)

Let φ_{pair} be the Pairing Axiom. $\dashv$

It is easy to see that neither φ_{pair}, nor rud closure, can be expressed, modulo transitivity, by a simple-Q-formula.[13] However:

Lemma 2.9. *There is an $\mathcal{L}_0$-simple-Q-formula φ_{am} such that for all transitive $\mathcal{L}_0$-structures $\mathcal{M}$, $\mathcal{M}$ is an adequate model iff $\mathcal{M} \models [\varphi_{\text{pair}} \ \& \ \varphi_{\text{am}}]$.*

Proof Sketch. This is a routine calculation, which we omit. (First find an $\mathcal{L}_0$-Q-formula φ_{rud} such that $\varphi_{\text{pair}} \wedge \varphi_{\text{rud}}$ expresses rud closure; this uses the the finite basis for rud functions.) $\square$

If $\mathcal{M}$ is an adequate model over A and $\xi < l(\mathcal{M})$ then $\mathcal{M}$ has a map

$$A^{<\omega} \times \xi^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\xi.$$

In fact, by the following lemma, this is true uniformly.

Lemma 2.10. *There is a Σ_1 formula ψ of $\mathcal{L}_0^-$, of two free variables, such that for all A and adequate models $\mathcal{M}$ over A, ψ defines a map $F : l(\mathcal{M}) \to \mathcal{M}$, and for $\xi < l(\mathcal{M})$, letting $h_\xi = F(\xi)$, we have*

$$h_\xi : A^{<\omega} \times \xi^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\xi$$

and for all $\alpha \leq \xi$, we have $h_\alpha \subseteq h_\xi$.

Proof. The proof is quite routine, using the sound-projection of proper segments of $\mathcal{M}$, much like in the proof of the corresponding fact for L. At the referee's request, we provide details. We will define h_ξ for $\xi < l(\mathcal{M})$, by recursion on ξ. We leave it to the reader to see that the definitions are sufficiently uniform and local that one can write down a Σ_1 formula ψ witnessing the lemma.

Recall that $\mathcal{M}|0 = A$. Note $0^{<\omega} = \{\emptyset\}$. We define $h_0 : A^{<\omega} \times 0^{<\omega} \overset{\text{onto}}{\to} A$ by setting $h_0(\vec{x}, \emptyset) = \vec{x}(0)$, in case $\text{lh}(\vec{x}) > 0$, and $h_0(\emptyset, \emptyset) = \emptyset$.

Given a soundly projecting successor $\mathcal{N}$, let $g^{\mathcal{N}}$ be the least $g \in [\text{Ord}(\mathcal{N})]^{<\omega}$ such that $\mathcal{N} = \text{Hull}_1^{\mathcal{N}}(\mathcal{N}^- \cup \{\mathcal{N}^-, g\})$.

Now suppose $1 < l(\mathcal{M})$. We define $h_1 : A^{<\omega} \times 1^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|1$. Set $h_1(\vec{x}, \emptyset) = h_0(\vec{x}, \emptyset)$, so $h_0 \subseteq h_1$. For $k < \omega$, let $\vec{0}_k$ denote the sequence $(0, \ldots, 0)$ of 0's of length k. So $1^{<\omega} = \{\vec{0}_k \mid k < \omega\}$. Let us now define $h_1(\vec{x}, \vec{0}_{n+1})$ for $n < \omega$. Let φ be a Σ_1 formula with free variables exactly $v_0, \ldots, v_{k+2}$, where $k < \omega$. Let n be the Gödel number of φ, and $\vec{x} \in A^k$. If there is a unique $y \in \mathcal{M}|1$ such that $\mathcal{M} \models \varphi(\vec{x}, A, g^{\mathcal{M}|1}, y)$, then we define $h_1(\vec{x}, \vec{0}_{n+1})$ = that unique y; otherwise

---

[13] If $\mathcal{L}$ is a first-order language extending the language of set theory, and X, Y are rud closed transitive $\mathcal{L}$-structures such that $c^X = c^Y$ for each constant symbol $c \in \mathcal{L}$, and $P^X = P^Y$ for each predicate symbol $P \in \mathcal{L}$ with $P \neq \dot{\in}$, then any $\mathcal{L}_0$-Q-formula true in both X, Y is also true in the "union" of X, Y.

define $h_1(\vec{x},\vec{0}_{n+1}) = \emptyset$. By the definition of $g^{\mathcal{M}|1}$ and since $A = (\mathcal{M}|1)^-$, $h_1 : A^{<\omega} \times 1^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|1$, and since h_1 is definable (in fact without parameters) over $\mathcal{M}|1$, we have $h_1 \in \mathcal{M}$.

Now suppose $0 < \gamma < \gamma+1 < l(\mathcal{M})$ and we have defined $h_\gamma : A^{<\omega} \times \gamma^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\gamma$. We define $h_{\gamma+1} : A^{<\omega} \times (\gamma+1)^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|(\gamma+1)$. We start by setting $h_\gamma \subseteq h_{\gamma+1}$. It remains to define $h(\vec{x},\vec{\alpha})$ in case $\vec{\alpha} \in (\gamma+1)^{<\omega} \backslash \gamma^{<\omega}$. Here we will use $(\vec{x},\vec{\alpha})$ to determine some formula φ (with Gödel code n) and some elements $y_0, \ldots, y_{m-1} \in \mathcal{M}|\gamma$, and use these data, along with the parameters $\mathcal{M}|\gamma$ and $g^{\mathcal{M}|(\gamma+1)}$, to attempt to define some $y \in \mathcal{M}|(\gamma+1)$; if this attempt is successful, we will set $h_{\gamma+1}(\vec{x},\vec{\alpha}) = y$. Here we use $y_i = h_\gamma(\vec{x}_i,\vec{\ell}_i)$, where $\vec{x}_i$ is a certain substring of $\vec{x}$, and $\vec{\ell}_i$ a certain substring of $\vec{\alpha}$, determined as follows. Suppose $\vec{\alpha}$ has form

$$(\gamma,\vec{0}_n,1,\vec{0}_{j_0},1,\vec{0}_{k_0},1,\vec{\ell}_0,\vec{0}_{j_1},1,\vec{0}_{k_1},1,\vec{\ell}_1,\ldots,\vec{0}_{j_{m-1}},1,\vec{0}_{k_{m-1}},1,\vec{\ell}_{m-1}) \qquad (2.1)$$

where $\vec{\ell}_i$ has length $\mathrm{lh}(\vec{\ell}_i) = k_i$ for each $i < m$.[14] Note that any such sequence is uniquely readable, in that the form above is uniquely determined by the sequence. Suppose that $\vec{\ell}_i \in \gamma^{<\omega}$ for each $i < m$. Let φ be the Σ_1 formula with Gödel code n. Suppose that the free variables of φ are exactly $v_0, \ldots, v_{m+2}$. Let $\vec{x} \in A^{<\omega}$ have length $j_0 + j_1 + \ldots + j_{m-1}$, and write

$$\vec{x} = \vec{x}_0 \,\hat{}\, \ldots \,\hat{}\, \vec{x}_{m-1}$$

where $\mathrm{lh}(\vec{x}_i) = j_i$. Let $y_i = h_\gamma(\vec{x}_i,\vec{\ell}_i)$, so $y_i \in \mathcal{M}|\gamma$. If there is a unique $y \in \mathcal{M}|(\gamma+1)$ such that

$$\mathcal{M}|(\gamma+1) \models \varphi(y_0,\ldots,y_{m-1},\mathcal{M}|\gamma,g^{\mathcal{M}|(\gamma+1)},y),$$

then define $h_{\gamma+1}(\vec{x},\vec{\alpha}) = y$, and otherwise define $h_{\gamma+1}(\vec{x},\vec{\alpha}) = \emptyset$. For all other $(\vec{x},\vec{\alpha})$, define $h_{\gamma+1}(\vec{x},\vec{\alpha}) = \emptyset$. Then $h_{\gamma+1}$ is surjective and definable over $\mathcal{M}|(\gamma+1)$, so $h_{\gamma+1} \in \mathcal{M}$.

Given h_α for all $\alpha < \xi$ where $\xi < l(\mathcal{M})$ is a limit, (we must) set $h_\xi = \bigcup_{\alpha<\xi} h_\alpha$. By the uniformity of the definitions, h_ξ is $\Sigma_1^{\mathcal{M}|\xi}$, so $h_\xi \in \mathcal{M}$. □

Definition 2.11. Given an adequate model $\mathcal{M}$ over A and $\xi < l(\mathcal{M})$, let $h_\xi^{\mathcal{M}}$ be the function h_ξ of the preceding lemma. Let $h^{\mathcal{M}} = \bigcup_{\xi<l(\mathcal{M})} h_\xi^{\mathcal{M}}$. ⊣

Remark 2.12. So $h^{\mathcal{M}}$ is $\Sigma_1^{\mathcal{M}}$ via a formula in $\mathcal{L}_0^-$, uniformly in adequate $\mathcal{M}$, and

$$h^{\mathcal{M}} : A^{<\omega} \times l(\mathcal{M}^-)^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}^-$$

(recall that if $\mathcal{M}$ is a limit then $\mathcal{M}^- = \mathcal{M}$), and if $\mathcal{M}$ is a successor then $h^{\mathcal{M}} \in \mathcal{M}$.

---

[14] The sequence in line (2.1) has length $1+n+1+(j_0+1+k_0+1+\mathrm{lh}(\vec{\ell}_0))+\ldots+(j_{m-1}+1+k_{m-1}+1+\mathrm{lh}(\vec{\ell}_{m-1}))$. The first entry is γ, followed by n-many 0s, one 1, j_0-many 0s, one 1, k_0-many 0s, one 1, then $\mathrm{lh}(\vec{\ell}_0)$-many ordinals $\leq \gamma$, etc.

We now want to analyse somewhat the cardinal structure of adequate models. This will be useful when we come to defining *potential operator-premice*, in particular regarding the properties of extenders on their sequence.

Definition 2.13. Let $\mathcal{M}$ be an adequate model over A and $\lambda = l(\mathcal{M})$. Let $\rho < \mathrm{Ord}(\mathcal{M})$. Then ρ is an **A-cardinal** of $\mathcal{M}$ iff $\mathcal{M}$ has no map $A^{<\omega} \times \gamma^{<\omega} \overset{\text{onto}}{\to} \rho$ where $\gamma < \rho$. We let $\Theta^{\mathcal{M}}$ denote the least A-cardinal of $\mathcal{M}$, if such exists. We say that ρ is **A-regular** in $\mathcal{M}$ iff $\mathcal{M}$ has no map $A^{<\omega} \times \gamma^{<\omega} \overset{\text{cof}}{\to} \rho$ where $\gamma < \rho$. We say that ρ is an **ordinal-cardinal** of $\mathcal{M}$ iff $\mathcal{M}$ has no map $\gamma^{<\omega} \overset{\text{onto}}{\to} \rho$ where $\gamma < \rho$. We say that ρ is **relevant** iff $\rho \leq \mathrm{Ord}(\mathcal{M}^-)$. $\dashv$

Lemma 2.14. *Let $\mathcal{M}$ be an adequate model over A and $\lambda = l(\mathcal{M}) > \xi > 0$. Let κ be an A-cardinal of $\mathcal{M}$ such that $\kappa \leq \mathrm{Ord}(\mathcal{M}|\xi)$. Then $\mathrm{rank}(A) < \kappa \leq \xi$ and $\kappa = \mathrm{Ord}(\mathcal{M}|\kappa)$.*

Lemma 2.15. *There is a Σ_1 formula φ in $\mathcal{L}_0^-$ such that, for any A and adequate model $\mathcal{M}$ over A, if $\Theta = \Theta^{\mathcal{M}}$ exists and is relevant then:*

1. *Θ is the least α such that $\mathcal{P}(A^{<\omega})^{\mathcal{M}} \subseteq \mathcal{M}|\alpha$.*
2. *$\lfloor \mathcal{M}|\Theta \rfloor$ is the set of all $x \in \mathcal{M}$ such that $\mathrm{trancl}(x)$ is the surjective image of $A^{<\omega}$ in $\mathcal{M}$.*
3. *Over $\mathcal{M}|\Theta$, $\varphi(0, \cdot, \cdot)$ defines a function $G : \Theta \to \mathcal{M}|\Theta$ such that for all $\alpha < \Theta$, we have $G(\alpha) : A^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\alpha$.*
4. *Θ is A-regular in $\mathcal{M}$.*

Let $\kappa_0 < \kappa_1$ be consecutive relevant A-cardinals of $\mathcal{M}$. Then:

5. *κ_1 is the least α such that $\mathcal{P}(A^{<\omega} \times \kappa_0^{<\omega})^{\mathcal{M}} \subseteq \mathcal{M}|\alpha$.*
6. *$\lfloor \mathcal{M}|\kappa_1 \rfloor$ is the set of all $x \in \mathcal{M}$ such that $\mathrm{trancl}(x)$ is the surjective image of $A^{<\omega} \times \kappa_0^{<\omega}$ in $\mathcal{M}$.*
7. *Over $\mathcal{M}|\kappa_1$, $\varphi(\kappa_0, \cdot, \cdot)$ defines a map $G : \kappa_1 \to \mathcal{M}|\kappa_1$ such that for all $\alpha < \kappa_1$, we have $G(\alpha) : A^{<\omega} \times \kappa_0^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\alpha$.*
8. *κ_1 is A-regular in $\mathcal{M}$.*

Proof. We just prove parts 1–4; the others are similar. Let $\gamma \in [1, l(\mathcal{M})]$ be least such that $\mathcal{P}(A^{<\omega}) \cap \mathcal{M} \subseteq \mathcal{M}|\gamma$. For part 1, we must see that $\gamma = \Theta$.

Let us first observe that γ is a limit ordinal. Suppose $\gamma = \xi + 1$ for some ξ. By acceptability, there is a surjection $\pi : A^{<\omega} \to \mathcal{M}|\xi$ with $\pi \in \mathcal{M}|\gamma$. So $\mathrm{Ord}(\mathcal{M}|\xi) < \Theta$. So if $\gamma = l(\mathcal{M})$ then $\mathrm{Ord}(\mathcal{M}^-) = \mathrm{Ord}(\mathcal{M}|\xi) < \Theta$, contradicting the assumption that Θ is relevant. So $\gamma < l(\mathcal{M})$. But then because $\mathcal{M}$ is adequate, $\mathcal{M}|\gamma$ is soundly projecting, so there is a surjection $\pi' : (\mathcal{M}|\xi)^{<\omega} \to \mathcal{M}|\gamma$ with $\pi' \in \mathcal{M}$, so $\mathrm{Ord}(\mathcal{M}|\gamma) < \Theta$. But then the usual diagonalization gives a contradiction.

Now by acceptability, for every $\alpha < \gamma$, $\mathcal{M}|\gamma$ has a map $A^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\alpha$.

We now claim that $\gamma = \Theta$. For $\gamma \leq \Theta$ by acceptability. So suppose $\gamma < \Theta$, and let $g : A^{<\omega} \overset{\text{onto}}{\to} \gamma^{<\omega}$ be in $\mathcal{M}$. Let $h = h^{\mathcal{M}|\gamma}$. Then because $g, h \in \mathcal{M}$, clearly $\mathcal{M}$ has a map $f : A^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\gamma$, so $\mathcal{M}$ has a map $A^{<\omega} \overset{\text{onto}}{\to} \mathcal{P}(A^{<\omega})^{\mathcal{M}}$, again a contradiction.

So $\gamma = \Theta$, giving part 1. Part 2: As mentioned above, for every $\alpha < \Theta$, $\mathcal{M}|\Theta$ has a surjection $A^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\alpha$. So letting $Y \in \mathcal{M}$ be transitive and $\pi : A^{<\omega} \overset{\text{onto}}{\to} Y$ with $\pi \in \mathcal{M}$, it suffices to see that $Y \in \mathcal{M}|\Theta$. Let $X \subseteq A^{<\omega}$ be the code for Y determined by π. Then $X \in \mathcal{M}|\gamma = \mathcal{M}|\Theta$. But then $Y \in L_\kappa(X)$ where κ is least such that $L_\kappa(X)$ is admissible, and note that $\kappa < \Theta$, so $Y \in \mathcal{M}|\Theta$.

Part 3: Let $\alpha < \Theta$. We will define $g : A^{<\omega} \times A^{<\omega} \overset{\text{onto}}{\to} \mathcal{M}|\alpha$, and the uniformity in the definition will yield the result. Let $\beta \in [\alpha, \Theta)$ be least such that

$$\mathcal{P}(A^{<\omega}) \cap \mathcal{M}|\beta \not\subseteq \mathcal{M}|\alpha.$$

Let $h = h^{\mathcal{M}|\beta}$. Let $x \in A^{<\omega}$ be such that for some y, $f = h(x, y)$ is such that $f : A^{<\omega} \to \mathcal{M}|\alpha$ is a surjection (such x exists by acceptability). Let y_x be the least such y, and $f_x = h(x, y_x)$. For all such x and for $z \in A^{<\omega}$, define $g(x, z) = f_x(z)$. For all other (x, z), set $g(x, z) = \emptyset$. This completes the definition of g, and the uniformity is clear.

Part 4 now follows. □

Corollary 2.16. *Let $\mathcal{M}$ be an adequate model over A and let γ be a relevant A-cardinal of $\mathcal{M}$. If γ is a limit of A-cardinals of $\mathcal{M}$ then $\mathcal{M}|\gamma$ satisfies Separation and Power Set. If γ is not a limit of A-cardinals of $\mathcal{M}$ then $\mathcal{M}|\gamma \vDash \mathsf{ZF}^-$. In particular, $\mathcal{M}|\Theta^{\mathcal{M}} \vDash \mathsf{ZF}^-$.*

Lemma 2.17. *Let $\mathcal{M}$ be an adequate model over A such that $\Theta^{\mathcal{M}}$ exists and is relevant. Let $\kappa \in [\Theta^{\mathcal{M}}, \mathrm{Ord}(\mathcal{M}))$ be relevant. Then κ is an A-cardinal of $\mathcal{M}$ iff κ is an ordinal-cardinal of $\mathcal{M}$.*

Proof. Suppose $\kappa > \Theta = \Theta^{\mathcal{M}}$ and κ is an ordinal-cardinal, but $\mathcal{M}$ has a map

$$f : A^{<\omega} \times \gamma^{<\omega} \overset{\text{onto}}{\to} \kappa$$

where $\gamma < \kappa$. For each $y \in \gamma^{<\omega}$, let $f_y : A^{<\omega} \to \kappa$ be $f_y(x) = f(x, y)$, and let g_y be the norm on $A^{<\omega}$ associated to f_y (that is, $g_y : A^{<\omega} \to \mathrm{Ord}$, $\mathrm{rg}(g_y)$ is an ordinal, and $g_y(x) < g_y(x')$ iff $f_y(x) < f_y(x')$). Then $g_y \in \mathcal{M}$ and $\mathrm{rg}(g_y) < \Theta$, because the associated prewellorder on $A^{<\omega}$ is in $\mathcal{M}|\Theta$ and $\mathcal{M}|\Theta \vDash \mathsf{ZF}^-$. Similarly, the function $y \mapsto (f_y, g_y)$ is in $\mathcal{M}$. Let

$$h : \Theta \times \gamma^{<\omega} \overset{\text{onto}}{\to} \kappa$$

be as follows. Let $(\alpha, y) \in \Theta \times \gamma^{<\omega}$. If $\alpha \notin \mathrm{rg}(g_y)$ then $h(\alpha, y) = 0$; otherwise $h(\alpha, y) = f(x, y)$ where $g_y(x) = \alpha$. Then $h \in \mathcal{M}$, a contradiction. □

Definition 2.18. Let $\mathcal{M}$ be an adequate model over A and let $\kappa < \mathrm{Ord}(\mathcal{M})$. Then $\kappa^{+\mathcal{M}}$ denotes either the least ordinal-cardinal γ of $\mathcal{M}$ such that $\gamma > \kappa$, if there is such, and denotes $\mathrm{Ord}(\mathcal{M})$ otherwise. By 2.17, if $\mathcal{M}$ is a limit and $\Theta^{\mathcal{M}} \leq \kappa$, then $\kappa^{+\mathcal{M}}$ is the least A-cardinal γ of $\mathcal{M}$ such that $\gamma > \kappa$, if there is such, or is $\mathrm{Ord}(\mathcal{M})$ otherwise. This applies when $E^{\mathcal{N}} \neq \emptyset$ in 2.21 below. $\dashv$

Definition 2.19. Let $\mathcal{M}$ be an adequate model over A. Then $\rho^{\mathcal{M}}$ denotes the least $\rho \in \mathrm{Ord}$ such that $\rho \geq \omega$ and $\mathcal{P}(A^{<\omega} \times \rho^{<\omega}) \cap \mathcal{J}(\mathcal{M}) \not\subseteq \mathcal{M}$. $\dashv$

2.2 Potential operator-premice

Remark 2.20. We now proceed to the definition of *potential operator-premouse*. This will lay out the main first order properties we demand of $\mathcal{F}$-premice. The properties for segments with an active extender are very close to those for standard premice as in [25], generalized to allow superstrong extenders. The properties for successor levels are new, and they consist of four clauses. Let us first give some motivation for these. *Projectum amenability* generalizes the fact that in an ordinary premouse $\mathcal{N}$, if $\mathcal{M} \triangleleft \mathcal{N}$ then there are no new bounded subsets of $\rho_\omega^{\mathcal{M}}$ which are in $\mathcal{J}(\mathcal{M})$. It ensures that we record all essential segments of a potential operator-premouse $\mathcal{N}$ in its history $S^{\mathcal{N}}$. For example, suppose we are forming an n-maximal iteration tree and we wish to apply an extender E to some piece of $\mathcal{N}$, but E is not $\mathcal{N}$-total. Projectum amenability will ensure that there is some $\mathcal{M} \triangleleft \mathcal{N}$ such that E is $\mathcal{M}$-total and $\mathcal{M}$ projects to $\mathrm{crit}(E)$. The property of Σ_1*-ordinal-generation* is used in making sense of fine structure; it ensures for example that the 1st standard parameter p_1 is well-defined. The *stratification* of $\mathcal{N}$ lets us define Σ_1-Skolem functions in the manner usual for $\mathcal{J}$-structures, thereby ensuring that $\mathrm{Hull}_{\Sigma_1}^{\mathcal{N}}(cb^{\mathcal{N}} \cup Y) \preccurlyeq_1 \mathcal{N}$ for any $Y \subseteq \mathcal{N}$, and also allows us to establish facts regarding the preservation of fine structure (including the preservation of p_1, assuming 1-solidity) under degree 0 ultrapower maps. And the existence of $cb^{\mathcal{N}}$*-ordinal-surjections*, together with stratification, will be used in proving that Σ_1-ordinal-generation is propagated under degree 0 ultrapower maps.

Definition 2.21. We say that $\mathcal{N}$ is a **potential operator-premouse (potential opm)** iff $\mathcal{N}$ is an adequate model, over A, such that for every $\mathcal{M} \trianglelefteq \mathcal{N}$,

1. (P-goodness) If $P^{\mathcal{M}} \neq \emptyset$ then $\mathcal{M}$ is a successor and $P^{\mathcal{M}} \subseteq \mathcal{M} \backslash \mathcal{M}^-$.[15]
2. (E-goodness) If $E^{\mathcal{M}} \neq \emptyset$ then $\mathcal{M}$ is a limit and letting $F = \bigcup E^{\mathcal{M}}$, then F is an extender over $\mathcal{M}$ such that, letting $S = S^{\mathcal{M}}$, $E = E^{\mathcal{M}}$ and $\kappa = \mathrm{crit}(F)$:
   (a) F is $A^{<\omega} \times \gamma^{<\omega}$-complete for all $\gamma < \kappa$.
   (b) The (following) premouse axioms hold for $(\lfloor \mathcal{M} \rfloor, S, E)$: letting $U = \mathrm{Ult}_0(\mathcal{M}|\kappa^{+\mathcal{M}}, F)$ and $\nu = \nu(F)$, we have:

---

[15] The requirement that $P^{\mathcal{M}} \subseteq \mathcal{M}\backslash\mathcal{M}^-$ does not restrict the information that can be encoded in $P^{\mathcal{M}}$, because given any $X \subseteq \mathcal{M}$, one can always replace it with $\{\mathcal{M}^-\} \times X$.

    i. (Mitchell-Steel indexing) $\mathrm{Ord}^{\mathcal{M}} = \nu^{+U}$.
    ii. (Coherence) $S = S^{U|\mathrm{Ord}(\mathcal{M})}$.
    iii. (Initial Segment Condition) For every η such that $\kappa^{+\mathcal{M}} < \eta < \nu$ and $\eta = \nu(F{\upharpoonright}\eta)$ (that is, either $\eta = \gamma + 1$ for some generator of F, or η is a limit of generators of F) and $F{\upharpoonright}\eta$ is not of type Z, either:
        A. there is $\mathcal{P} \triangleleft \mathcal{M}$ such that $\bigcup E^{\mathcal{P}}$ is the trivial completion of $F{\upharpoonright}\eta$, or
        B. $E^{\mathcal{M}|\eta} \neq \emptyset$ and there is $\mathcal{P} \triangleleft \mathrm{Ult}_0(\mathcal{M}|\eta, F')$, where $F' = \bigcup E^{\mathcal{M}|\eta}$, such that $\bigcup E^{\mathcal{P}}$ is the trivial completion of $F{\upharpoonright}\eta$.
    iv. (Amenable encoding) E is the amenable code for F; that is, E is the set of all $e \in \mathcal{M}$ such that for some $\xi \in (\kappa, \kappa^{+\mathcal{M}})$, we have $e = F \cap ((\mathcal{M}|\xi) \times [\nu]^{<\omega})$.

(It follows that $\mathcal{M}$ has a largest cardinal δ, and $\delta \leq i_F(\kappa)$, and $\delta \leq \nu < \mathrm{Ord}(\mathcal{M}) = \delta^{+U}$.)

3. If $\mathcal{M}$ is a successor then:
    (a) (Projectum amenability) If $l(\mathcal{M}) > 1$ and $\omega, \alpha < \rho^{\mathcal{M}^-}$ then
    $$\mathcal{P}(A^{<\omega} \times \alpha^{<\omega}) \cap \mathcal{M} \subseteq \mathcal{M}^-.$$
    (b) (A-ordinal-surjections) For every $x \in \mathcal{M}$ there is $\alpha < \mathrm{Ord}^{\mathcal{M}}$ and a map $A^{<\omega} \times \alpha^{<\omega} \overset{\text{onto}}{\to} x$ in $\mathcal{M}$.
    (c) (Σ_1-ordinal-generation) $\mathcal{M} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{M}}(\mathcal{M}^- \cup \{\mathcal{M}^-\} \cup \mathrm{Ord}^{\mathcal{M}})$.
    (d) (Stratification) There is a limit $\gamma \in \mathrm{Ord}$ and sequence $\widetilde{\mathcal{M}} = \left\langle \widetilde{\mathcal{M}}_\alpha \right\rangle_{\alpha<\gamma}$ such that:
        i. for each $\alpha < \gamma$, $\widetilde{\mathcal{M}}_\alpha$ is an $\mathcal{L}_0$-structure such that $\left\lfloor \widetilde{\mathcal{M}}_\alpha \right\rfloor$ is transitive and $\widetilde{\mathcal{M}}_\alpha = \mathcal{M}{\upharpoonright} \left\lfloor \widetilde{\mathcal{M}}_\alpha \right\rfloor$; that is, $cb^{\widetilde{\mathcal{M}}_\alpha} = A$ and $cp^{\widetilde{\mathcal{M}}_\alpha} = cp^{\mathcal{M}}$ and $P^{\widetilde{\mathcal{M}}_\alpha} = P^{\mathcal{M}} \cap \left\lfloor \widetilde{\mathcal{M}}_\alpha \right\rfloor$, etc,
        ii. $\widetilde{\mathcal{M}}$ is a continuous, strictly increasing sequence with $\mathcal{M}^- \in \widetilde{\mathcal{M}}_0$ and $\mathcal{M} = \bigcup_{\alpha<\gamma} \widetilde{\mathcal{M}}_\alpha$,
        iii. $\widetilde{\mathcal{M}}{\upharpoonright}\alpha \in \mathcal{M}$ for every $\alpha < \gamma$, and the function $\alpha \mapsto \widetilde{\mathcal{M}}{\upharpoonright}\alpha$, with domain γ, is $\Sigma_1^{\mathcal{M}}(\{\mathcal{M}^-\})$. $\dashv$

Remark 2.22. Let $\mathcal{N}$ be a potential opm over A. Suppose $E^{\mathcal{N}}$ codes an extender F. Clearly $\kappa = \mathrm{crit}(F) > \Theta^{\mathcal{M}} > \mathrm{rank}(A)$. By [28, Definition 2.2.1], we have $\kappa^{+\mathcal{M}} < \mathrm{Ord}(\mathcal{M})$; cf. 2.18. Note that *we allow F to be of superstrong type* (see 2.23) in accordance with [28], not [25, Definition 2.4].[16] Mitchell-Steel

---

[16] The main point of permitting superstrong extenders is that it simplifies certain things. But it complicates others. If the reader prefers, one could instead require that F *not* be superstrong, but various statements throughout the paper regarding condensation would need to be modified, along the lines of [4, Lemma 3.3].

fine structure for premice with superstrongs has been developed in [16] and [11], and of course, [29] developed it for Jensen fine structure.

Definition 2.23. Let $\mathcal{M}$ be a potential opm over A. We say that $\mathcal{M}$ is **E-active** iff $E^{\mathcal{M}} \neq \emptyset$, and **$P$-active** iff $P^{\mathcal{M}} \neq \emptyset$. **Active** means either E-active or P-active. **E-passive** means not E-active. **P-passive** means not P-active. **Passive** means not active. **Type 0** means passive. **Type 4** means P-active. **Type 1, 2** or **3** mean E-active, with numerology as in [4].

We write $F^{\mathcal{M}}$ for the extender F coded by $E^{\mathcal{M}}$ (where $F = \emptyset$ if $E^{\mathcal{M}} = \emptyset$). We write $\mathbb{E}^{\mathcal{M}}$ for the function with domain $l(\mathcal{M})$, sending $\alpha \mapsto F^{\mathcal{M}|\alpha}$. Likewise for $\mathbb{E}_+^{\mathcal{M}}$, but with domain $l(\mathcal{M}) + 1$.

For $\alpha \leq \mathrm{lh}(\mathcal{M})$, we define $\mathcal{M}||\alpha$ as follows. If $\mathcal{M}|\alpha$ is E-passive then $\mathcal{M}||\alpha = \mathcal{M}|\alpha$. If $\mathcal{M}$ is E-active then $\mathcal{M}||\alpha$ denotes the (unique) E-passive potential opm $\mathcal{N}$ with $S^{\mathcal{N}} = S^{\mathcal{M}|\alpha}$ (so $\mathrm{lh}(\mathcal{M}) = \alpha$ and $\mathcal{N}$ has the same universe as has $\mathcal{M}|\alpha$).

Suppose $F = F^{\mathcal{M}} \neq \emptyset$ and $\kappa = \mathrm{crit}(F)$. As usual, $\nu(F) = \max(\kappa^{+\mathcal{M}}, \nu'(F))$, where $\nu'(F)$ is the strict sup of generators of F. We say $\mathcal{M}$, or F, is **superstrong** iff $i_F(\mathrm{crit}(F)) = \nu(F)$.

Suppose $\mathcal{M}$ is a successor. A **stratification** of $\mathcal{M}$ is a sequence $\widetilde{\mathcal{M}}$ witnessing 2.21(3d) for $\mathcal{M}$. For a Σ_1 formula $\varphi \in \mathcal{L}_0$, we say that $\mathcal{M}$ is **φ-stratified** iff $\varphi(\mathcal{M}^-, \cdot)^{\mathcal{M}}$ defines the set of all proper restrictions $\widetilde{\mathcal{M}}\restriction\alpha$ of a stratification $\widetilde{\mathcal{M}}$ of $\mathcal{M}$.[17] $\dashv$

Lemma 2.24. *Let $\mathcal{M}$ be a successor potential opm, over A. Let $\widetilde{\mathcal{M}} = \left\langle \widetilde{\mathcal{M}_\alpha} \right\rangle_{\alpha<\gamma}$ be a stratification of $\mathcal{M}$. For $\alpha < \gamma$ let*

$$H_\alpha = \mathrm{Hull}_1^{\widetilde{\mathcal{M}}_\alpha}(A \cup \mathrm{Ord}(\widetilde{\mathcal{M}}_\alpha)).$$

Then for every $x \in \mathcal{M}$ there is $\alpha < \gamma$ such that $x \subseteq H_\alpha$.

Proof. Use Σ_1-ordinal-generation and A-ordinal-surjections. $\square$

Lemma 2.25. *Let $\mathcal{M}$ be a potential opm $\mathcal{M}$ over coarse base A. Then $\mathcal{M} = \mathrm{Hull}_1^{\mathcal{M}}(A \cup \mathrm{Ord}^{\mathcal{M}})$, and $\mathrm{Hull}_1^{\mathcal{M}}(A \cup X) \preccurlyeq_1 \mathcal{M}$ for all $X \subseteq \mathcal{M}$.*

Proof. Use Σ_1-ordinal-generation and Lemma 2.10. $\square$

2.3 Fine structure

We now want to define the various fine structural notions for the structures under consideration, such as soundness, projecta, etc. We will want these definitions to apply not just to potential opms, but also the squash of a type 3 potential opm, once we have defined these. The following definition abstracts out what properties we want for this.

Definition 2.26. Let $\mathcal{N}$ be a structure for a finite first-order language $\mathcal{L}$ consisting of constants and relation symbols. We say that $\mathcal{N}$ is **pre-fine** (for $\mathcal{L}$) iff:

---

[17] The φ-stratification of $\mathcal{M}$ need not imply that every successor $\mathcal{N} \triangleleft \mathcal{M}$ is φ-stratified.

(i) $\mathcal{L}$ is a finite and $\{\dot{\in}, \dot{cb}\} \subseteq \mathcal{L}$, where $\dot{\in}$ is a binary relation symbol and $\dot{cb}$ is a constant symbol.

(ii) $\mathcal{N}$ is an amenable $\mathcal{L}$-structure with transitive, rud closed, rank closed universe $\lfloor\mathcal{N}\rfloor$ and $\dot{\in}^{\mathcal{N}} = {\in} \cap \lfloor\mathcal{N}\rfloor^2$ and $\dot{cb}^{\mathcal{N}}$ is transitive.

(iii) $\mathcal{N} = \mathrm{Hull}^{\mathcal{N}}_{\Sigma_1}(\dot{cb}^{\mathcal{N}} \cup \mathrm{Ord}(\mathcal{N}))$ (note the language here is $\mathcal{L}$),

(iv) there is a Σ_1 formula φ of $\mathcal{L}$, a limit ordinal η and a sequence $\langle\mathcal{N}_\alpha\rangle_{\alpha<\eta}$ such that each $\mathcal{N}_\alpha$ is an $\mathcal{L}$-structure, $\lfloor\mathcal{N}_\alpha\rfloor$ is transitive, $\mathcal{N}_\alpha \in \mathcal{N}$,

$$\mathcal{N}_\alpha = \mathcal{N}{\restriction}\lfloor\mathcal{N}_\alpha\rfloor\,,$$

$\lfloor\mathcal{N}_\alpha\rfloor \subseteq \lfloor\mathcal{N}_\beta\rfloor$ for $\alpha < \beta$, $\lfloor\mathcal{N}\rfloor = \bigcup_{\alpha<\eta}\lfloor\mathcal{N}_\alpha\rfloor$, and for all $X \in \mathcal{N}$,

$$X \in \{\mathcal{N}_\alpha\}_{\alpha<\eta} \iff \mathcal{N} \vDash \varphi(X).$$

(Since $\dot{cb}^{\mathcal{N}_\alpha} = \dot{cb}^{\mathcal{N}}$ and $\mathcal{N}_\alpha$ is transitive, it follows that $\dot{cb}^{\mathcal{N}} \subseteq \mathcal{N}_\alpha$.) ⊣

Definition 2.27 (Fine structure)**.** Let $\mathcal{N}$ be pre-fine for the language $\mathcal{L}$. We sketch a description of the **fine structural notions** for $\mathcal{N}$. For details refer to [4],[25]; we also adopt the simplifications explained in [19, §5].[18] Let $A = cb^{\mathcal{N}}$.

We say that $\mathcal{N}$ is 0**-sound** and let $\rho_0^{\mathcal{N}} = \mathrm{Ord}(\mathcal{N})$, $p_0^{\mathcal{N}} = \emptyset$, $\mathfrak{C}_0(\mathcal{N}) = \mathcal{N}$ and $\mathrm{r}\Sigma_1^{\mathcal{N}} = \Sigma_1^{\mathfrak{C}_0(\mathcal{N})}$ (here and in what follows, definability is with respect to $\mathcal{L}$). Let $T_0^{\mathcal{N}} = \mathcal{N}$.

Now let $n < \omega$ and suppose we have already defined $\rho_n^{\mathcal{N}}$, $\vec{p}_n^{\,\mathcal{N}} = (p_1^{\mathcal{N}}, \ldots, p_n^{\mathcal{N}})$, $\mathfrak{C}_n(\mathcal{N})$, and n-soundness. Suppose that $\mathcal{N}$ is n-sound, which will imply that $\mathfrak{C}_n(\mathcal{N}) = \mathcal{N}$. Suppose we have also defined the class of $\mathrm{r}\Sigma_{n+1}^{\mathcal{N}}$ relations, and that every $\Sigma_1^{\mathcal{N}}$ relation is $\mathrm{r}\Sigma_{n+1}^{\mathcal{N}}$.

Define $\rho_{n+1}^{\mathcal{N}}$ as the least ordinal $\rho \geq \omega$ such that for some $X \subseteq A^{<\omega} \times \rho^{<\omega}$, X is $\mathrm{r}\underset{\sim}{\Sigma}_{n+1}^{\mathcal{N}}$ but $X \notin \lfloor\mathcal{N}\rfloor$. Define $p_{n+1}^{\mathcal{N}}$ as the least $p \in [\mathrm{Ord}]^{<\omega}$ such that some such X is

$$\mathrm{r}\Sigma_{n+1}^{\mathcal{N}}(A \cup \rho_{n+1}^{\mathcal{N}} \cup \{p, \vec{p}_n^{\,\mathcal{N}}\}).$$

Here $p_{n+1}^{\mathcal{N}}$ is well-defined by condition (iii) of pre-fineness. For $X \subseteq \mathcal{N}$, let

$$\mathrm{Hull}_{n+1}^{\mathcal{N}}(X) = \mathrm{Hull}_{\mathrm{r}\Sigma_{n+1}}^{\mathcal{N}}(X),$$

and $\mathrm{cHull}_{n+1}^{\mathcal{N}}(X)$ be its transitive collapse, if this hull is extensional. If $A \subseteq X$ then the hull is indeed extensional, as then $\mathrm{Hull}_1^{\mathcal{N}}(X) \preccurlyeq_1 \mathcal{N}$, since $\mathrm{r}\Sigma_1^{\mathcal{N}} \subseteq \mathrm{r}\Sigma_{n+1}^{\mathcal{M}}$, and using conditions (iii) and (iv) of pre-fineness, we can define Σ_1 Skolem functions in a standard manner.[19] Let

$$\mathrm{Th}_{n+1}^{\mathcal{N}}(X) = \mathrm{Th}_{\mathrm{r}\Sigma_{n+1}}^{\mathcal{N}}(X).$$

---

[18] The simplifications involve dropping the parameters u_n, and replacing the use of generalized theories with pure theories. These changes are not important, and if the reader prefers, one could redefine things more analogously to [4],[25].

[19] That is, fix $\langle\mathcal{N}_\alpha\rangle_{\alpha<\eta}$ as in (iv). Let ψ be Σ_0 and $\vec{x} \in X^{<\omega}$ with $\mathcal{N} \vDash \exists y\ \psi(y, \vec{x})$. We want to see that there is $y \in \mathrm{Hull}_1^{\mathcal{N}}(X)$ such that $\mathcal{N} \vDash \psi(y, \vec{x})$. Using (iii), let $\vec{a} \in A^{<\omega}$ and

(Recall that $\mathrm{Th}^{\mathcal{N}}_{\mathrm{r}\Sigma_{n+1}}(X)$ was specified at the end of §1.1. Note that this theory is analogous to the *pure* $\mathrm{r}\Sigma_{n+1}$ theory, as opposed to the *generalized* $\mathrm{r}\Sigma_{n+1}$ theory, in the sense of [4].[20].) Then we let

$$\mathfrak{C}_{n+1}(\mathcal{N}) = \mathrm{cHull}^{\mathcal{N}}_{n+1}(A \cup \rho^{\mathcal{N}}_{n+1} \cup \vec{p}^{\mathcal{N}}_{n+1}),$$

and the uncollapse map $\pi : \mathfrak{C}_{n+1}(\mathcal{N}) \to \mathcal{N}$ is the associated **core embedding**. We say that $\mathcal{N}$ is $(n+1)$**-universal** iff

$$\mathcal{N} \cap \mathcal{P}\big(A^{<\omega} \times (\rho^{\mathcal{N}}_{n+1})^{<\omega}\big) \subseteq \mathfrak{C}_{n+1}(\mathcal{N}).$$

For $\alpha \in p^{\mathcal{N}}_{n+1}$, define the $(n+1)$**-solidity witness** $\mathcal{W}^{\mathcal{N}}_{n+1}(\alpha)$ **at** α by setting

$$\mathcal{W}^{\mathcal{N}}_{n+1}(\alpha) = \mathrm{cHull}_{n+1}\big(A \cup \alpha \cup (p^{\mathcal{N}}_{n+1} \backslash (\alpha+1)) \cup \vec{p}^{\mathcal{N}}_{n}\big).$$

We say that $\mathcal{N}$ is $(n+1)$**-solid** iff $\mathcal{W}^{\mathcal{N}}_{n+1}(\alpha) \in \mathcal{N}$ for each $\alpha \in p^{\mathcal{N}}_{n+1}$. (This follows the form of the definition of solidity in [29]. In [4] and [25], $\mathrm{r}\Sigma_{n+1}$-theories are used as solidity witnesses, instead of transitive structures. Note that also following Zeman [29] but not Steel [25], we do not incorporate $(n+1)$-universality into $(n+1)$-solidity.) We say that $\mathcal{N}$ is $(n+1)$-**sound** iff $\mathcal{N}$ is $(n+1)$-solid and $\mathfrak{C}_{n+1}(\mathcal{N}) = \mathcal{N}$ and the core embedding $\pi = \mathrm{id}$.

Now suppose that $\mathcal{N}$ is $(n+1)$-sound and $\rho^{\mathcal{N}}_{n+1} > \omega$ (so $\rho^{\mathcal{N}}_{n+1} > \mathrm{rank}(A)$).[21] Define $T = T^{\mathcal{N}}_{n+1} \subseteq \mathcal{N}$ by letting $t \in T$ iff for some $q \in \mathcal{N}$ and $\alpha < \rho^{\mathcal{N}}_{n+1}$,

$$t = \mathrm{Th}^{\mathcal{N}}_{n+1}(A \cup \alpha \cup \{q\}).$$

Define $\mathrm{r}\Sigma_{n+2}$ from T_{n+1} as usual: an $\mathrm{r}\Sigma_{n+2}$ formula $\varphi(\vec{v})$ (in free variables $\vec{v}$) is one of form

$$\exists t\ \big(T_{n+1}(t) \wedge \psi(t, \vec{v})\big)$$

where ψ is $\mathrm{r}\Sigma_1$. The $\mathrm{r}\Sigma^{\mathcal{N}}_{n+2}$ relations are then given by interepreting these formulas over $\mathcal{N}$, with T_{n+1} interpreted as $T^{\mathcal{N}}_{n+1}$. This completes the definitions. $\dashv$

τ be a Σ_1 formula such that there is $\vec{\beta} \in [\mathrm{Ord}^{\mathcal{N}}]^{<\omega}$ and $y \in \mathcal{N}$ such that

$$\big[\mathcal{N} \models \psi(y, \vec{x})\big] \text{ and } \big[y \text{ is the unique } y' \in \mathcal{N} \text{ such that } \mathcal{N} \models \tau(\vec{a}, \vec{\beta}, y')\big].$$

Then there is $\alpha < \eta$ such that $\vec{x} \in (\mathcal{N}_\alpha)^{<\omega}$ and there is $\vec{\beta} \in [\mathrm{Ord}^{\mathcal{N}_\alpha}]^{<\omega}$ and $y \in \mathcal{N}_\alpha$ such that

$$\big[\mathcal{N}_\alpha \models \psi(y, \vec{x})\big] \text{ and } \big[y \text{ is the unique } y' \in \mathcal{N}_\alpha \text{ such that } \mathcal{N}_\alpha \models \tau(\vec{a}, \vec{\beta}, y')\big].$$

Let α_0 be least such α. Let $\vec{\beta}_0 \in [\mathrm{Ord}(\mathcal{N}_{\alpha_0})]^{<\omega}$ be the least witness to the choice of α_0. Let y_0 be the unique $y \in \mathcal{N}_{\alpha_0}$ such that $\mathcal{N}_{\alpha_0} \models \tau(\vec{a}, \vec{\beta}_0, y)$. Then $\mathcal{N} \models \psi(y_0, \vec{x})$ and $\alpha_0, \vec{\beta}_0, y_0 \in \mathrm{Hull}^{\mathcal{N}}_1(X)$, which suffices.

[20] As in [4, §2], it does not matter which we use.

[21] If $A \cap \mathrm{Ord} < \mathrm{rank}(A)$, we can still definably refer to each $\alpha < \mathrm{rank}(A)$ via elements $x \in A$ of rank α. This is because $\mathcal{N}$ is rud closed and rank closed, so the rank function $r_A : A \to \mathrm{Ord}$ is in $\mathcal{N}$, and since $A = \dot{cb}^{\mathcal{N}}$, note that $\{r_A\}$ is $\mathrm{r}\Sigma^{\mathcal{N}}_1$. It easily follows that we can't have $\omega < \rho^{\mathcal{N}}_{n+1} \leq \mathrm{rank}(A)$.

Definition 2.28. Let $\mathcal{N}$ be a potential opm.

If $\mathcal{N}$ is E-active then $\mu^{\mathcal{N}} =_{\mathrm{def}} \mathrm{crit}(F^{\mathcal{N}})$, and otherwise $\mu^{\mathcal{N}} =_{\mathrm{def}} \emptyset$.[22]

If $\mathcal{N}$ is E-active type 2 then $e^{\mathcal{N}}$ denotes the trivial completion of the largest non-type Z proper segment of F; otherwise $e^{\mathcal{N}} =_{\mathrm{def}} \emptyset$.[23]

If $\mathcal{N}$ is non-type 3 then $\mathfrak{C}_0(\mathcal{N}) = \mathcal{N}^{\mathrm{sq}}$ denotes the $\mathcal{L}_0^+$-structure $(\mathcal{N}, \mu^{\mathcal{N}}, e^{\mathcal{N}})$ (with $\dot{\mu}^{\mathcal{N}} = \mu^{\mathcal{N}}$ etc).

If $\mathcal{N}$ is type 3 then define the $\mathcal{L}_0^+$-structure $\mathfrak{C}_0(\mathcal{N}) = \mathcal{N}^{\mathrm{sq}}$, called the **squash** of $\mathcal{N}$, essentially as in [4]; so

$$\mathcal{N}^{\mathrm{sq}} = (R, E', P', S', X'; cb^{\mathcal{N}}, cp^{\mathcal{N}}, \mu^{\mathcal{N}}, e^{\mathcal{N}})$$

where $\nu = \nu(F^{\mathcal{N}})$, $R = \lfloor \mathcal{N}|\nu \rfloor$, $E' = F^{\mathcal{N}} \restriction \nu$ (which is the usual coding of $F^{\mathcal{N}}$ over the squash), $P' = \emptyset$, $S' = S^{\mathcal{N}} \cap R$ and $X' = X^{\mathcal{N}} \cap R$. Note $e^{\mathcal{N}^{\mathrm{sq}}} = e^{\mathcal{N}} = \emptyset$ here.

We define the **fine structural notions** for $\mathcal{N}$ (n-soundness, $\rho_{n+1}^{\mathcal{N}}$, $\mathrm{Hull}_{n+1}^{\mathcal{N}}$, $\mathrm{Th}_{n+1}^{\mathcal{N}}$, etc) as those for $\mathfrak{C}_0(\mathcal{N})$.[24] $\dashv$

In the proof of the solidity, etc, of iterable opms, one must also deal with structures which are almost active opms, except that they may fail the ISC. The details are immediate modifications of the standard notions, so we leave them to the reader.

The following definition is just the direct adaptation of the usual one:

Definition 2.29. Let $\mathcal{M}$ be a potential opm. Let $\mathcal{R}$ be an $\mathcal{L}_0^+$-structure (possibly illfounded). Let $\pi : \mathcal{R} \to \mathfrak{C}_0(\mathcal{M})$.

We say that π is a **weak 0-embedding** iff π is Σ_0-elementary (therefore $\mathcal{R}$ is extensional and wellfounded, so assume $\mathcal{R}$ is transitive) and there is $X \subseteq \mathcal{R}$ such that X is $\in$-cofinal in $\mathcal{R}$ and π is Σ_1-elementary on elements of X, and if $\mathcal{M}$ is type 1 or 2, then letting $\mu = \mu^{\mathcal{R}}$, there is $Y \subseteq \mathcal{R}|\mu^{+\mathcal{R}} \times \mathcal{R}$ such that Y is $\in \times \in$-cofinal in $(\mathcal{R}|\mu^{+\mathcal{R}}) \times \mathcal{R}$ and π is Σ_1-elementary on elements of Y. $\dashv$

The following definition of *(near) k-embedding* is analogous to that in [25, Definition 2.20, Remark 4.3], and *weak k-embedding* analogous to that introduced in [14] (the change in the definition in *weak k-embedding* between [4] and the one we use here is due to Steve Jackson).[25]

---

[22] Recall the language $\mathcal{L}_0^+ = \mathcal{L}_0 \cup \{\dot{\mu}, \dot{e}\}$ was specified in Definition 2.3.

[23] In [4], the (analogue of) e is referred to by its code $\gamma^{\mathcal{M}}$. We use e instead because this does not depend on having (and selecting) a wellorder of $\mathcal{M}$.

[24] Thus, when we write, say, $\mathcal{M} = \mathrm{cHull}_{n+1}^{\mathcal{N}}(X)$, we will have $X \subseteq \mathfrak{C}_0(\mathcal{N})$ and literally mean that $\mathfrak{C}_0(\mathcal{M}) = \mathcal{R}$ where $\mathcal{R} = \mathrm{cHull}_{n+1}^{\mathfrak{C}_0(\mathcal{N})}(X)$. So if $\mathcal{N}$ is type 3, then $\mathcal{M}$ is produced by first squashing $\mathcal{N}$, forming the transitive collapse $\mathcal{R}$ of the hull of X in $\mathcal{N}^{\mathrm{sq}}$, and then unsquashing $\mathcal{R}$ to reach $\mathcal{M}$. However, if $\mathcal{N}$ is type 3 and $n = 0$ it is possible that unsquashing $\mathcal{R}$ produces an illfounded structure $\mathcal{M}$, in which case $\mathfrak{C}_0(\mathcal{M})$ has not literally been defined. In this case, we define $\mathcal{M}$ to be this illfounded structure, and define $\mathfrak{C}_0(\mathcal{M}) = \mathcal{R}$.

[25] Jackson noticed a difficulty in the proof of the Shift Lemma for weak k-embeddings as defined in [4] and [25], when $0 < k < \omega$. Jackson suggested the definition we use here as a replacement. (The problem is that it does not seem obvious that a weak k-embedding π in the sense of [4] is always such that $\pi``T_k^M \subseteq T_k^N$. But we do not actually know of an example of a weak k-embedding as defined in [4] for which this or the Shift Lemma fails.

Definition 2.30. Let $k \leq \omega$ and let $\mathcal{M}, \mathcal{N}$ be k-sound opms. Let

$$\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{N}).$$

We say that π is a *near k-embedding* iff either $k = \omega$ and π is fully elementary, or $k < \omega$ and:

1. π is $\mathrm{r}\Sigma_{k+1}$-elementary,
2. $\pi(\vec{p}_k^{\mathcal{M}}) = \vec{p}_k^{\mathcal{N}}$,
3. for $i < k$, we have:
   - if $\rho_i^{\mathcal{M}} = \rho_0^{\mathcal{M}}$ then $\rho_i^{\mathcal{N}} = \rho_0^{\mathcal{N}}$, and
   - if $\rho_i^{\mathcal{M}} < \rho_0^{\mathcal{M}}$ then $\pi(\rho_i^{\mathcal{M}}) = \rho_i^{\mathcal{N}}$.

We say π is a *k-embedding* iff π is a near k-embedding and if $k < \omega$ then $\pi``\rho_k^{\mathcal{M}}$ is cofinal in $\rho_k^{\mathcal{N}}$.

If $0 < k \leq \omega$, we say π is a *weak k-embedding* iff $k = \omega$ and π is fully elementary, or $k < \omega$ and has the properties of a near k-embedding, except that instead of $\mathrm{r}\Sigma_{k+1}$-elementarity, we only demand that π is $\mathrm{r}\Sigma_k$-elementary and there is a set $X \subseteq \rho_k^{\mathcal{M}}$ such that X is cofinal in $\rho_k^{\mathcal{M}}$ and π is $\mathrm{r}\Sigma_{k+1}$-elementary on parameters in $\mathrm{Hull}_{k+1}^{\mathcal{M}}(X \cup \{\vec{p}_k^{\mathcal{M}}\})$. Therefore $\pi``T_k^{\mathcal{M}} \subseteq T_k^{\mathcal{N}}$. (Note that this definition of *weak k-embedding* diverges slightly from the definitions given in [4, p. 52] and [25, Definition 4.1]; see Footnote 25.)

We say that π is **(weakly, nearly) k-good** iff π is a (weak, near) k-embedding and $cb^{\mathcal{M}} = cb^{\mathcal{N}}$ and $\pi \restriction cb^{\mathcal{M}} = \mathrm{id}$. $\dashv$

The following definition is not intended to be a comprehensive statement of fine structural condensation. It simply encompasses some basic instances of condensation, which for example arise in the basic proofs of fine structure, such as solidity, etc. The definition is also of low enough complexity that it is preserved by the relevant hulls and ultrapower maps.

Definition 2.31. Let $\mathcal{N}$ be an ω-sound potential opm. We say that $\mathcal{N}$ is **$<\omega$-condensing** iff for every $k < \omega$, for every soundly projecting, $(k+1)$-sound potential opm $\mathcal{M}$, for every near k-embedding $\pi : \mathcal{M} \to \mathcal{N}$ such that $\rho = \rho_{k+1}^{\mathcal{M}} \leq \mathrm{crit}(\pi)$ and $\rho < \rho_{k+1}^{\mathcal{N}}$, we have the following. If $\mathcal{N}|\rho$ is E-passive let $\mathcal{Q} = \mathcal{N}$, and otherwise let $\mathcal{Q} = \mathrm{Ult}(\mathcal{N}|\rho, F^{\mathcal{N}|\rho})$. Then either:

- $\mathcal{M} \triangleleft \mathcal{Q}$, or
- $\mathcal{M}^- \triangleleft \mathcal{Q}$, and $\mathcal{M} \in \mathcal{R}$ where $\mathcal{R} \triangleleft \mathcal{Q}$ is such that $\mathcal{R}^- = \mathcal{M}^-$. $\dashv$

The inclusion of the second option above (where $\mathcal{M}^- \triangleleft \mathcal{Q}$ and $\mathcal{M} \in \mathcal{R}$) might appear to diverge from the usual kind of conclusion for condensation, as we do not have $\mathcal{M} \trianglelefteq \mathcal{Q}$ here; instead, $\mathcal{M}$ is strictly "between" $\mathcal{R}^-$ and $\mathcal{R}$. Note that if this clause attains then by projection amenability for $\mathcal{R}$, $\mathcal{M}$ is sound and $\rho_\omega^{\mathcal{M}} = \rho_1^{\mathcal{M}} = \rho_\omega^{\mathcal{M}^-}$, the soundness following from the fact that $\rho_\omega^{\mathcal{M}} = \rho_\omega^{\mathcal{M}^-}$.

2.4 Q-operator-premice

Definition 2.32. A **Q-operator-premouse (Q-opm)**[26] is a potential operator-premouse $\mathcal{M}$ such that every $\mathcal{N} \triangleleft \mathcal{M}$ is ω-sound and $< \omega$-condensing. $\dashv$

Q-operator-premice are basically analogous to *premice* in [4]. However, we will soon refine things one step further, defining *operator-premice*, which will be the primary objects of interest; these are just Q-operator-premice, all of whose segments are soundly projecting (including the top one), and they are also analogous to premice. Of course for premice, there is no distinction between these two notions. The analogy with the premice of [4] does fail, however, in a minor regard: [4] makes no condensation demands of proper segments of premice. We make this requirement here so that we can avoid stating it as an explicit axiom at certain points later (and it holds for the structures we care about).

Much as in [4], modulo wellfoundedness (of the relevant objects), we can capture the property of being a Q-opm with formulas of the following forms:

Definition 2.33. The class of (non-simple) **$\mathcal{L}_0^+$-Q-formulas** is defined as in [4, Definition 2.3.9]; that is, these are the formulas $\varphi(\vec{u})$ of form

$$\forall x\ \forall \xi < \dot{\mu}^+\ \exists y\ \exists \eta < \dot{\mu}^+\ \left[x \subseteq y \wedge \xi \leq \eta \wedge \psi(y, \eta, \vec{u})\right],$$

where $\psi(v_0, v_1, \vec{u})$ is an $\mathrm{r}\Sigma_1$ formula of $\mathcal{L}_0^+$, which has only $v_0, v_1, \vec{u}$ free. We define the class of **$\mathcal{L}_0^+$-P-formulas** just like the conventional notion of Q-formulas (instead of following [4, Definition 3.1.4]); these are the formulas $\varphi(\vec{u})$ of form

$$\forall x\ \exists y\ \left[x \subseteq y \wedge \psi(y, \vec{u})\right]$$

where $\psi(v, \vec{u})$ is an $\mathrm{r}\Sigma_1$ of $\mathcal{L}_0^+$ and has only $v, \vec{u}$ free. $\dashv$

Recall from Definition 2.7 that an adequate model-plus is an $\mathcal{L}_0^+$-structure $\mathcal{N}$ such that $\mathcal{N}{\restriction}\mathcal{L}_0$ is an adequate model.

Lemma 2.34. *There are $\mathcal{L}_0^+$-Q-formulas φ_1, φ_2, an $\mathcal{L}_0^+$-P-formula φ_3, an $\mathcal{L}_0^+$-simple-Q-formula $\varphi_{0,\mathrm{limit}}$, and for each Σ_1 formula $\psi \in \mathcal{L}_0$ there are $\mathcal{L}_0^+$-simple-Q-formulas $\varphi_{0,\psi}, \varphi_{4,\psi}$, obtained recursively from ψ, such that for any adequate model-plus $\mathcal{N}'$:*

1. *$\mathcal{N}' \vDash \varphi_{0,\mathrm{limit}}$ iff $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ for some limit passive Q-opm $\mathcal{N}$.*
2. *$\mathcal{N}' \vDash \varphi_{4,\psi}$ iff $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ for some ψ-stratified P-active Q-opm $\mathcal{N}$.*
3. *$\mathcal{N}' \vDash \varphi_{0,\psi}$ iff $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ for some passive Q-opm $\mathcal{N}$ which is either a limit or is ψ-stratified.*
4. *$\mathcal{N}' \vDash \varphi_1$ (respectively, $\mathcal{N}' \vDash \varphi_2$) iff $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ for some type 1 (respectively, type 2) Q-opm $\mathcal{N}$.*

---

[26] Q is for *Q-formula*. We will see that the first-order aspects of Q-opm-hood are expressible with Q-formulas and P-formulas.

5. *If* $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ *for some type 3 Q-opm* $\mathcal{N}$ *then* $\mathcal{N}' \vDash \varphi_3$. *If* $\mathcal{N}' \vDash \varphi_3$ *then* $E^{\mathcal{N}'}$ *codes an extender* F *over* $\mathcal{N}'$ *such that if* $\mathrm{Ult}(\mathcal{N}', F)$ *is wellfounded then* $\mathcal{N}' = \mathfrak{C}_0(\mathcal{N})$ *for some type 3 Q-opm* $\mathcal{N}$.

Proof. Part 1 is routine. Part 4 is a straightforward adaptation of its analogue [4, Lemma 2.5]. Part 5 is likewise adaptation of [4, Lemma 3.3], with the following two remarks. Firstly, the P-formulas of [4, Definition 3.1.4] are more liberal than $\mathcal{L}_0^+$-P-formulas. But note that each of the sentences $\theta_1, \ldots, \theta_5$ of the proof of [4, Lemma 3.3], adapted to our context, are expressible with an $\mathcal{L}_0^+$-P-formula (for θ_1, this is as in part 1 of the current lemma). Now the rest of the proof of [4, Lemma 3.3] goes through, noting that we have been able to drop the clause "or $\mathcal{N}$ is of superstrong type" from the statement of [4, Lemma 3.3], because we allow extenders of superstrong type as the active extenders of Q-opms.

Part 2 is an easy adaptation of part 3, using the fact that if $\mathcal{N}$ is P-active then $P^{\mathcal{N}} \subsetneq \mathcal{N} \backslash \mathcal{N}^-$. So it just remains to consider part 3; we just sketch the proof of this.

Consider an adequate model-plus $\mathcal{N}'$ and $\mathcal{N} = \mathcal{N}' {\upharpoonright} \mathcal{L}_0$. We leave it to the reader to verify that here is an $\mathcal{L}_0$-simple-Q-formula asserting (when interpreted over $\mathcal{N}'$) that every $\mathcal{M} \triangleleft \mathcal{N}$ is a $<\omega$-condensing ω-sound potential opm, and an $\mathcal{L}_0^+$-simple-Q-formula asserting that $P^{\mathcal{N}} = E^{\mathcal{N}} = \mu^{\mathcal{N}} = e^{\mathcal{N}} = \emptyset$. It remains to see that we can assert that 2.21(3) holds for $\mathcal{M} = \mathcal{N}$ (the assertion will include the possibility that $\mathcal{N}$ is a limit). For 2.21(3a), use the formula "$\forall x \exists y \big[x \subseteq y \wedge \varphi(y)\big]$", where $\varphi(y)$ asserts "either there is $s \in S^{\mathcal{M}}$ such that $y \in s$ or there are S, A such that $S = y \cap S^{\mathcal{M}}$ and $A = cb^{\mathcal{M}}$ and S has a largest element $\mathcal{P}$ and for each $\tau < \mathrm{Ord}(\mathcal{P})$, if there is $X \in y \backslash \mathcal{P}$ such that $X \subseteq A^{<\omega} \times \tau^{<\omega}$, then there is $n < \omega$ such that $\rho_{n+1}^{\mathcal{P}} \leq \tau$, as witnessed by a satisfaction relation in y" (use the fact that $\mathcal{N}$ is rud closed).

Clause 2.21(3b) is easy, and it is fairly straightforward to assert that either $\mathcal{N}$ is a limit or $\mathcal{N}$ is ψ-stratified, identifying candidates for $\mathcal{N}^-$ as in the previous paragraph. We can therefore assert 2.21(3c) as "$\forall x \exists y \big[x \subseteq y$ and there is $\alpha < \gamma$ such that $y \subseteq H_\alpha\big]$", where γ, H_α are defined as in 2.24, using the stratification given by ψ. □

The natural adaptations of [4, Lemmas 2.4, 3.2] hold, and the proofs are straightforward:

Lemma 2.35. *Let* $\mathcal{R}, \mathcal{S}$ *be transitive* $\mathcal{L}_0^+$*-structures. Let* $\pi : \mathcal{R} \to \mathcal{S}$. *Let* $\vec{x} \in \mathcal{R}^{<\omega}$. *Then:*

1. *Let* φ *be an* $\mathcal{L}_0^+$*-P-formula. Then:*
   - (a) *If* π *is* $\mathrm{r}\Sigma_1$*-elementary and* $\mathcal{S} \vDash \varphi(\pi(\vec{x}))$ *then* $\mathcal{R} \vDash \varphi(\vec{x})$.
   - (b) *If* π *is* Σ_0*-elementary and* $\subseteq$*-cofinal and* $\mathcal{R} \vDash \varphi(\vec{x})$ *then* $\mathcal{S} \vDash \varphi(\pi(\vec{x}))$.
2. *Let* φ *be an* $\mathcal{L}_0^+$*-Q-formula. Suppose* $\dot{\mu}^{\mathcal{R}} \in \mathrm{Ord}^{\mathcal{R}}$. *Then:*
   - (a) *If* π *is* $\mathrm{r}\Sigma_1$*-elementary and* $\mathcal{S} \vDash \varphi(\pi(\vec{x}))$ *then* $\mathcal{R} \vDash \varphi(\vec{x})$.

  (b) *If* π *is* Σ_0*-elementary and* $\mathrm{rg}(\pi)$ *is* $\subseteq\times\subseteq$*-cofinal in* $(\mathcal{S}|\dot{\mu}^{+\mathcal{S}})\times\mathcal{S}$ *and* $\mathcal{R}\models\varphi(\vec{x})$ *then* $\mathcal{S}\models\varphi(\pi(\vec{x}))$*.*

There is also a version of this lemma for weak 0-embeddings, but here we only consider statements φ without parameters:

Lemma 2.36. *Let* $\mathcal{N}$ *be a Q-opm, let* $\mathcal{R}$ *be an* $\mathcal{L}_0^+$*-structure and let* $\pi:\mathcal{R}\to\mathfrak{C}_0(\mathcal{N})$ *be a weak* 0*-embedding. Then:*

1. *Suppose* $\mathcal{N}$ *is type* $i\neq 3$*. Then:*
    (a) *For every* $\mathcal{L}_0^+$*-Q-formula* φ*, if* $\mathfrak{C}_0(\mathcal{N})\models\varphi$ *then* $\mathcal{R}\models\varphi$*.*
    (b) $\mathcal{R}=\mathfrak{C}_0(\mathcal{M})$ *for some type* i *Q-opm* $\mathcal{M}$*.* [27]
2. *Suppose* $\mathcal{N}$ *is type 3. Then:*
    (a) *For every* $\mathcal{L}_0^+$*-P-formula* φ*, if* $\mathfrak{C}_0(\mathcal{N})\models\varphi$ *then* $\mathcal{R}\models\varphi$*.*
    (b) *If* $\mathrm{Ult}(\mathfrak{C}_0(\mathcal{N}),F^{\mathcal{N}})$ *is wellfounded then* $\mathrm{Ult}(\mathcal{R},F^{\mathcal{R}})$ *is wellfounded.*
    (c) *If* $\mathrm{Ult}(\mathcal{R},F^{\mathcal{R}})$ *is wellfounded then* $\mathcal{R}=\mathfrak{C}_0(\mathcal{M})$ *for some type 3 Q-opm* $\mathcal{M}$*.*

The proof is routine, using Lemma 2.34 for parts 1(b) and 2(c).

Lemma 2.37. *Let* $n<\omega$ *and* $\mathcal{M}$ *be an* n*-sound Q-opm over* A *with* $\omega<\rho_n^{\mathcal{M}}$*. Let* $X\subseteq\mathfrak{C}_0(\mathcal{M})$*, let* $\mathcal{R}=\mathrm{cHull}_{n+1}^{\mathcal{M}}(A\cup X\cup\vec{p}_n^{\mathcal{M}})$ *and let* $\pi:\mathcal{R}\to\mathfrak{C}_0(\mathcal{M})$ *be the uncollapse. Then:*

1. *If either* $n>1$ *or* $\mathcal{M}$ *is non-type 3 or* $\mathrm{Ult}(\mathfrak{C}_0(\mathcal{M}),F^{\mathcal{M}})$ *is wellfounded then* $\mathcal{R}=\mathfrak{C}_0(\mathcal{N})$ *for some Q-opm* $\mathcal{N}$*.*
2. *If* $\mathcal{R}=\mathfrak{C}_0(\mathcal{N})$ *for some Q-opm* $\mathcal{N}$ *then* $\mathcal{N}$ *is* n*-sound and* π *is nearly* n*-good.*

Proof. Suppose $n=0$ and $\mathcal{M}$ is a successor. Then by Lemmas 2.34 and 2.35, it suffices to see that π is $\mathrm{r}\Sigma_1$-elementary, or in other words, that $\mathrm{rg}(\pi)\preccurlyeq_1\mathcal{M}$. But using stratification (as in part 3(3d) of Definition 2.21) we can define appropriate Σ_1 Skolem functions over $\mathcal{M}$, much as was done in Footnote 19, and thereby verify that $\mathrm{rg}(\pi)\preccurlyeq_1\mathcal{M}$.

If $n=0$ and $\mathcal{M}$ is a limit it is similar, but easier. (If $\mathcal{M}$ is type 3, then by the hypothesis in part 1, $\mathrm{Ult}(\mathfrak{C}_0(\mathcal{M}),F^{\mathcal{M}})$ is wellfounded, which implies that $\mathrm{Ult}_0(\mathcal{R},F^{\mathcal{R}})$ is wellfounded, and $\mathcal{R}=\mathfrak{C}_0(\mathcal{N})$ for a type 3 Q-opm $\mathcal{N}$.)

If $n>0$, then the proof for standard premice adapts routinely.[28] (If $\mathcal{M}$ is type 3 and $n>1$, there is $(a,f)\in\mathrm{rg}(\pi)$ such that $\nu(F^{\mathcal{M}})=[a,f]^{\mathcal{M}}_{F^{\mathcal{M}}}$, which easily gives that $\mathcal{R}=\mathfrak{C}_0(\mathcal{N})$ for a type 3 Q-opm $\mathcal{N}$, even if we don't know that $\mathrm{Ult}_0(\mathcal{M},F^{\mathcal{M}})$ is wellfounded.) $\square$

---

[27] Possibly $\mathcal{M},\mathcal{N}$ are passive and $\mathcal{N}$ is a successor but $\mathcal{M}$ a limit.

[28] The fine structural setup here is a little different from that in [4], as we have dropped the use of $u_i^{\mathcal{M}}$. See [19, §5] for calculations which deal with this difference.

Using stratifications and standard calculations, we also have:

Lemma 2.38. *Let $\mathcal{N},\mathcal{M}$ be n-sound Q-opms over A. Then:*

1. *Let $\pi:\mathfrak{C}_0(\mathcal{N})\to\mathfrak{C}_0(\mathcal{M})$ be nearly n-good. Suppose that $\mathcal{N}\notin\mathcal{M}$ and*

$$\mathcal{N}=\mathrm{cHull}^{\mathcal{N}}_{n+1}(A\cup\rho\cup\{q\}),$$

where $\rho\in\mathrm{Ord}$ and $\rho\leq\mathrm{crit}(\pi)$ and $q\in[\rho_0^{\mathcal{N}}]^{<\omega}$. Then π is n-good.

2. *If $\mathcal{N}=\mathfrak{C}_{n+1}(\mathcal{M})$ and π is the core embedding, then π is n-good.*

2.5 Operator-premice

We finally reach the ultimate notion prior to introducing an actual operator $\mathcal{F}$ from which to build our premice:

Definition 2.39. An **operator-premouse (opm)** is a soundly projecting Q-opm. For an opm $\mathcal{M}$, let $q^{\mathcal{M}}=p_1^{\mathcal{M}}\cap(\mathrm{Ord}(\mathcal{M}^-),\mathrm{Ord}(\mathcal{M}))$ (so if $\mathcal{M}$ is a limit then $q^{\mathcal{M}}=\emptyset$). $\dashv$

Definition 2.40. Let $\mathcal{M}$ be a k-sound opm over A and $q\in(\rho_k^{\mathcal{M}})^{<\omega}$. We say that $\mathcal{M}$ is $(k+1,q)$**-solid** iff for each $\alpha\in q$, letting $q'=q\backslash(\alpha+1)$ and $X=A\cup\alpha\cup q'\cup\vec{p}_k^{\mathcal{M}}$, we have $\mathrm{cHull}^{\mathcal{M}}_{k+1}(X)\in\mathcal{M}$. $\dashv$

Lemma 2.41. *Let $\mathcal{M}$ be a successor opm and $l(\mathcal{M})=\xi+1$. Let $\rho=\rho_\omega^{\mathcal{M}^-}$ and $p=p_1^{\mathcal{M}}\backslash\rho$. Then $\mathcal{M}$ is ρ-sound and $\rho_1^{\mathcal{M}}\leq\rho$ and either:*

- *$p=q^{\mathcal{M}}$ (in other words, $p_1^{\mathcal{M}}\cap[\rho,\mathrm{Ord}(\mathcal{M}^-)]=\emptyset$), or*
- *$q^{\mathcal{M}}=\emptyset$ and $p=\{\alpha\}$ for some $\alpha\in[\rho,\xi]$.*

Therefore either $\mathcal{M}$ is ω-sound and $\rho_\omega^{\mathcal{M}}=\rho=\rho_\omega^{\mathcal{M}^-}$, or there is $k<\omega$ such that $\mathcal{M}$ is k-sound and $\rho_{k+1}^{\mathcal{M}}<\rho\leq\rho_k^{\mathcal{M}}$ (so if $k>0$ then $\rho=\rho_k^{\mathcal{M}}$).

Proof. We have $\rho_1^{\mathcal{M}}\leq\rho$ as $\mathcal{M}$ is soundly projecting. If $q^{\mathcal{M}}\neq\emptyset$ then $p\cap[\rho,\mathrm{Ord}(\mathcal{M}^-)]=\emptyset$, as letting $A=cb^{\mathcal{M}}$, we have

$$\mathcal{M}^-\cup\{\mathcal{M}^-\}\subseteq\mathrm{Hull}_1^{\mathcal{M}}(A\cup\rho\cup q^{\mathcal{M}}),$$

as $X^{\mathcal{M}}$ is $\Sigma_1^{\mathcal{M}}$, and this suffices since $\mathcal{M}$ is soundly projecting. So suppose $q^{\mathcal{M}}=\emptyset$. Let r be least in $(\xi+1)^{<\omega}$ such that

$$\mathcal{M}^-\in H=\mathrm{Hull}_1^{\mathcal{M}}(A\cup\rho\cup r).$$

Note that $H=\mathcal{M}$, since $\mathcal{M}$ is soundly projecting and $q^{\mathcal{M}}=\emptyset$. So if $r=\emptyset$ then $p=\emptyset=q^{\mathcal{M}}$, so we are done. So suppose $r\neq\emptyset$.

Suppose for a contradiction that $\mathrm{card}(r)>1$ and let $\alpha_0>\alpha_1$ be the top 2 elements. Let

$$C=\mathrm{cHull}_1^{\mathcal{M}}(A\cup\alpha_1\cup\{\alpha_0\})$$

and $\pi : C \to \mathcal{M}$ the uncollapse. So $\mathrm{crit}(\pi) = \alpha_1 < \mathrm{Ord}^C$. So α_1 is an A-cardinal of C, and so $\pi(\alpha_1)$ is an A-cardinal of $\mathcal{M}$. Since $\rho \leq \alpha_1 < \pi(\alpha_1)$ and $\mathcal{M}^-$ projects to ρ, therefore $\pi(\alpha_1) > \mathrm{Ord}(\mathcal{M}^-)$. But then $\mathcal{M}^- \in \mathrm{rg}(\pi_1)$, contradicting the minimality of r.

So $r = \{\alpha_0\}$ for some α_0. It remains to verify that $\mathcal{M}$ is $(1, r)$-solid. Let

$$C = \mathrm{cHull}_1^{\mathcal{M}}(A \cup \alpha_0)$$

and $\pi : C \to \mathcal{M}$ the uncollapse. Then $\mathrm{rg}(\pi) \cap \mathrm{Ord} = \alpha_0$, because otherwise again $\mathrm{crit}(\pi)$ is an A-cardinal of C, etc. Note that because $r \neq \emptyset$, we have $P^{\mathcal{M}} = \emptyset$ (since otherwise $P^{\mathcal{M}} \subseteq \mathcal{M} \backslash \mathcal{M}^-$). Therefore $C = \mathcal{M}||\alpha_0$,[29] so $C \in \mathcal{M}$, which gives that $\mathcal{M}$ is $(1, \{\alpha_0\})$-solid, as desired. □

Lemma 2.42. *Let $\mathcal{N}$ be a successor opm and $\pi : \mathcal{M} \to \mathcal{N}$. Suppose either*

(i) π is Σ_1-elementary and $q^{\mathcal{N}} = \emptyset$, or

(ii) π is Σ_2-elementary and $q^{\mathcal{N}} \in \mathrm{rg}(\pi)$.

Then $\mathcal{M}$ is a successor opm of the same type as $\mathcal{N}$, and $\pi(q^{\mathcal{M}}) = q^{\mathcal{N}}$.

Proof. By 2.34, $\mathcal{M}$ is a Q-opm and we may assume $\mathcal{N}^- \in \mathrm{rg}(\pi)$, so $\mathcal{M}$ is a successor, $\pi(\mathcal{M}^-) = \mathcal{N}^-$, and $\mathcal{M}$ is ψ-stratified where $\mathcal{N}$ is ψ-stratified. Now

$$\mathcal{N} = \mathrm{Hull}_1^{\mathcal{N}}(\mathcal{N}^- \cup \{\mathcal{N}^-\} \cup q^{\mathcal{N}}),$$

which with Σ_1-elementarity gives

$$\mathcal{M} = \mathrm{Hull}_1^{\mathcal{M}}(\mathcal{M}^- \cup \{\mathcal{M}^-\} \cup q)$$

where $\pi(q) = q^{\mathcal{N}}$. This suffices for part (i). For part (ii) use generalized solidity witnesses to see that $\mathcal{M}$ is $(1, \bar{q})$-solid, which is enough. □

However, in the context above, if π is just Σ_1-elementary and $q^{\mathcal{N}} \neq \emptyset$, $\mathcal{M}$ might not be soundly projecting, even if $q^{\mathcal{N}} \in \mathrm{rg}(\pi)$. Such embeddings arise when we take Σ_1 hulls like in the proof of 1-solidity for $(0, \omega_1 + 1)$-iterable premice.

Lemma 2.43. *Let $n, \mathcal{M}, \mathcal{R}, \pi$ be as in Lemma 2.37, with $n > 0$, and suppose that $\mathcal{M}$ is an opm. Suppose that either $n > 1$ or $\mathcal{M}$ is non-type 3 or $\mathrm{Ult}(\mathfrak{C}_0(\mathcal{M}), F^{\mathcal{M}})$ is wellfounded. Then $\mathcal{R} = \mathfrak{C}_0(\mathcal{N})$ for an n-sound opm $\mathcal{N}$, and π is nearly n-good.*

Proof. By Lemma 2.37, we know that $\mathcal{R} = \mathfrak{C}_0(\mathcal{N})$ for some n-sound Q-opm $\mathcal{N}$ and that π is nearly n-good. So we just need to see that $\mathcal{N}$ is an opm. So we may assume that $\mathcal{N}$ is a successor, so $\mathcal{M}$ is also. Since $n > 0$, π is $\mathrm{r}\Sigma_2$-elementary, so by Lemma 2.42, it suffices to see that $q^{\mathcal{M}} \in \mathrm{rg}(\pi)$. But $q^{\mathcal{M}} \subseteq p_1^{\mathcal{M}} \in \mathrm{rg}(\pi)$, so we are done. □

---

[29] This denotes the passivization of $\mathcal{M}|\alpha_0$; that is, the passive opm $\mathcal{P}$ such that $S^{\mathcal{P}} = S^{\mathcal{M}|\alpha_0}$.

Let X be transitive. Then $X^\#$ determines naturally an opm $\mathcal{M}$ over $\hat{X}$ of length 1, so $U = \mathrm{Ult}_0(\mathcal{M}, F^{X^\#})$ is also a Q-opm over $\hat{X}$ of length 1, but U is not an opm.[30] So opm-hood is not expressible with Q-formulas. However, given a successor opm $\mathcal{N}$, we will only form ultrapowers of $\mathcal{N}$ with extenders E such that $\mathrm{crit}(E) < \mathrm{Ord}(\mathcal{N}^-)$, and under these circumstances, opm-hood is preserved. In fact, we will only form ultrapowers and fine structural hulls under further fine structural assumptions:

Definition 2.44. Let $k \leq \omega$. An opm $\mathcal{M}$ is k-**relevant** iff $\mathcal{M}$ is k-sound, and either $\mathcal{M}$ is a limit or $k = \omega$ or $\rho_{k+1}^{\mathcal{M}} < \rho_\omega^{\mathcal{M}^-}$.

A Q-opm $\mathcal{M}$ which is not an opm (so $\mathcal{M}$ is a successor) is k-**relevant** iff $k = 0$ and $\rho_1^{\mathcal{M}} < \rho_\omega^{\mathcal{M}^-}$. ⊣

For the development of the basic fine structure theory of opms, one only needs to iterate k-relevant opms (and phalanxes of such structures, and bicephali and pseudo-premice); see 2.47. For instance, the following lemma follows from 2.41:

Lemma 2.45. *Let $k < \omega$ and $\mathcal{M}$ be a k-sound operator-premouse which is not k-relevant. Then $\mathcal{M}$ is $(k+1)$-sound.*

2.6 Fine structure and iterations

Now that we have introduced operator-premice and studied how they behave under forming elementary hulls, we want to consider forming iteration trees on them. In the following lemma we establish the preservation of fine structure under degree k ultrapowers, for k-relevant opms. The proof involves a key use of stratification.

Lemma 2.46. *Let $\mathcal{M}$ be a k-relevant opm over A and E an extender over $\mathcal{M}$, weakly amenable to $\mathcal{M}$, with $\mathrm{rank}(A) < \mathrm{crit}(E) < \rho_k^{\mathcal{M}}$, and $\mathrm{crit}(E) < \rho_\omega^{\mathcal{M}^-}$ if $\mathcal{M}$ is a successor.*[31] *Let $\mathcal{N} = \mathrm{Ult}_k(\mathcal{M}, E)$ and $j = i_E^{\mathcal{M},k}$ be the ultrapower embedding. Suppose $\mathcal{N}$ is wellfounded. Then:*

1. *$\mathcal{N}$ is a k-relevant opm of the same type as $\mathcal{M}$.*
2. *$\mathcal{N}$ is a successor iff $\mathcal{M}$ is. If $\mathcal{M}$ is a successor then $j(l(\mathcal{M})) = l(\mathcal{N})$ and if $\mathcal{M}$ is ψ-stratified then $\mathcal{N}$ is ψ-stratified.*
3. *j is k-good.*
4. *For any $q \in (\rho_k^{\mathcal{M}})^{<\omega}$, if $\mathcal{M}$ is $(k+1, q)$-solid then $\mathcal{N}$ is $(k+1, j(q))$-solid.*
5. *$\rho_{k+1}^{\mathcal{N}} \leq \sup j``\rho_{k+1}^{\mathcal{M}}$.*

---

[30] U is not soundly projecting.

[31] Note that if $\mathcal{M}$ is a successor and $k > 0$, then $\rho_k^{\mathcal{M}} \leq \rho_\omega^{\mathcal{M}^-}$, but $\rho_0^{\mathcal{M}} > \rho_\omega^{\mathcal{M}^-}$, so the hypothesis that $\mathrm{crit}(E) < \rho_\omega^{\mathcal{M}^-}$ is just needed when $k = 0$.

6. *If E is close to $\mathcal{M}$ and $\mathcal{M}$ is $(k+1)$-solid then $\rho_{k+1}^{\mathcal{N}} = \sup j``\rho_{k+1}^{\mathcal{M}}$ and $p_{k+1}^{\mathcal{N}} = j(p_{k+1}^{\mathcal{M}})$ and $\mathcal{N}$ is $(k+1)$-solid.*

Proof. The fact that $\mathcal{N}$ is a Q-opm of the same type as $\mathcal{M}$ is by 2.34. Part 3 is standard and part 2 follows easily. We now verify that $\mathcal{N}$ is soundly projecting; we may assume that $\mathcal{M}, \mathcal{N}$ are successors. If $k > 0$, use elementarity and stratification. Suppose $k = 0$. Let $\rho = \rho_\omega^{\mathcal{M}^-}$ and $q = j(q^{\mathcal{M}})$. The fact that $\mathcal{N}$ is $(1,q)$-solid follows by an easy adaptation of the usual proof of preservation of the standard parameter, using stratification (where in the usual proof, one uses the natural stratification of the $\mathcal{J}$-hierarchy). So it suffices to see that $\mathcal{N} = \mathrm{Hull}_1^{\mathcal{N}}(\mathcal{N}^- \cup \{\mathcal{N}^-, q\})$. But this holds because $\mathcal{M}$ is an opm and

$$\mathcal{N} = \mathrm{Hull}_1^{\mathcal{N}}(\mathrm{rg}(j) \cup \nu_E)$$

and $\nu_E \subseteq \mathcal{N}^-$, the latter because $\mathrm{crit}(E) \leq \mathrm{Ord}(\mathcal{N}^-)$ (in fact, $\mathrm{crit}(E) < \rho_\omega^{\mathcal{N}^-}$).

Parts 4–6: If $k > 0$ the proof for standard premice works. See, for example, [4, Lemmas 4.5, 4.6], and if $\kappa < \rho_{k+1}^{\mathcal{M}}$, see [18, Corollary 2.24] and its proof (that result is formally below superstrong, but essentially the same proof works) and/or [11, Lemma 3.8***]; these arguments are related to the calculations in [4, Claim 5 of Theorem 6.2]. If $k = 0$, again use stratification to adapt the usual proof. (In the case that $l(\mathcal{M})$ is a limit, $\mathcal{M}$ is of course "stratified" by its proper segments.)

By part 5, it follows that $\mathcal{N}$ is k-relevant, completing part 1. □

We next want to define (fine, such as k-maximal, etc) *iteration trees* $\mathcal{T}$ on opms, following the general form of [25, §3.1]:

Definition 2.47. Let $k < \omega$ and let $\mathcal{M}$ be a k-sound opm. The **k-maximal iteration game $\mathcal{G}_k(\mathcal{M}, \theta)$ on $\mathcal{M}$, of length θ** is defined completely analogously to the game $\mathcal{G}_k(\mathcal{N}, \theta)$ for k-sound premice $\mathcal{N}$, as defined in [25, §3.1], except for the following differences. Let $\mathcal{T}$ be a partial play (so $\mathcal{T}$ will be a putative tree). Then:

– It is player I's responsibility that for all $\beta + 1 < \alpha + 1 < \mathrm{lh}(\mathcal{T})$, we have $\mathrm{lh}(E_\beta^{\mathcal{T}}) \leq \mathrm{lh}(E_\alpha^{\mathcal{T}})$ (as opposed to $\mathrm{lh}(E_\beta^{\mathcal{T}}) < \mathrm{lh}(E_\alpha^{\mathcal{T}})$, as is the requirement in [25]).[32]

– It is player II's responsibility that for each $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, $M_\alpha^{\mathcal{T}}$ is an opm (as opposed to a premouse, as is the usual responsibility).

The rest is as in [25, §3.1]. In particular, the game stops as soon as either player breaks a rule, and it is player I's responsibility that $E_\alpha^{\mathcal{T}} \in \mathbb{E}_+(M_\alpha^{\mathcal{T}})$ for each $\alpha + 1 < \mathrm{lh}(\mathcal{T})$. Recall here that if $\mathcal{M}$ is a successor opm and $E \in \mathbb{E}_+^{\mathcal{M}}$ then $E \in \mathbb{E}_+^{\mathcal{M}^-}$.)

A **k-maximal iteration tree on $\mathcal{M}$** is a partial play of $\mathcal{G}_k(\mathcal{M}, \infty)$ in which neither player has yet lost. A **putative k-maximal iteration tree on $\mathcal{M}$** is

---

[32] The weakening of this requirement is needed because we allow extenders of superstrong type in $\mathbb{E}_+^{\mathcal{M}}$. See Remark 2.48 for details.

as for a k-maximal iteration tree $\mathcal{T}$, except that if $\mathcal{T}$ has successor length, then the main branch $b^{\mathcal{T}}$ of $\mathcal{T}$ is allowed to drop infinitely often, and if it does not, there are no demands on the nature of the last model $M^{\mathcal{T}}_{\infty}$ (other than that it be formed via the direct limit along the branch in the usual manner).

A (k,θ)**-iteration strategy** for $\mathcal{M}$ is a winning strategy for player II in $\mathcal{G}_k(\mathcal{M},\theta)$.

The *k-maximal stack* iteration game $\mathcal{G}^*_k(\mathcal{M},\alpha,\theta)$ is defined by analogy with the game $\mathcal{G}^*_k(\mathcal{N},\alpha,\theta)$ for k-sound premice $\mathcal{N}$, essentially as defined in [22, prior to Corollary 1.10], and see also [25, §4.1].[33] As in [17, §1.1.5], the *k-optimal stack* iteration game $\mathcal{G} = \mathcal{G}^*_{\mathrm{opt},k}(\mathcal{M},\alpha,\theta)$ is defined likewise, except that we do not allow player I to drop in model or degree at the beginnings of rounds. That is, (i) round 0 of $\mathcal{G}$ is a run of $\mathcal{G}_k(\mathcal{M},\theta)$, and (ii) letting $0<\gamma<\alpha$ and $\vec{\mathcal{T}} = \langle \mathcal{T}_\beta\rangle_{\beta<\gamma}$ be the sequence of trees played in rounds $\beta<\gamma$ and $\mathcal{N} = M^{\vec{\mathcal{T}}}_{\infty}$ and $n = \deg^{\vec{\mathcal{T}}}_{\infty}$, round γ of $\mathcal{G}$ is a run of $\mathcal{G}_n(\mathcal{N},\theta)$.

A $(k,\alpha,\theta)^*$**-optimal iteration strategy** for $\mathcal{M}$ is a winning strategy for player II in $\mathcal{G}^*_{\mathrm{opt},k}(\mathcal{M},\alpha,\theta)$, and a $(k,\alpha,\theta)^*$**-iteration strategy** is likewise for $\mathcal{G}^*_k(\mathcal{M},\alpha,\theta)$. See [17, Lemma 9.8] for some basics on the connection between $(k,\alpha,\theta)^*$- and $(k,\alpha,\theta)^*$-optimal iteration strategies.

Now (k,θ)**-iterability**, $(k,\alpha,\theta)^*$**-optimal iterability**, etc, are defined by the existence of the appropriate winning strategy. $\dashv$

Remark 2.48. The requirement, in $\mathcal{G}_k(\mathcal{M},\theta)$, that $\mathrm{lh}(E^{\mathcal{T}}_\beta) \leq \mathrm{lh}(E^{\mathcal{T}}_\alpha)$ for $\beta<\alpha$, is weaker than requiring that $\mathrm{lh}(E^{\mathcal{T}}_\beta) < \mathrm{lh}(E^{\mathcal{T}}_\alpha)$, because opms may have superstrong extenders in their sequence. For example, we might have that $E^{\mathcal{T}}_0$ is type 2 and $E^{\mathcal{T}}_1$ is superstrong with $\mathrm{crit}(E^{\mathcal{T}}_1)$ the largest cardinal of $\mathcal{M}^{\mathcal{T}}_0|\mathrm{lh}(E^{\mathcal{T}}_0)$, in which case $\mathcal{M}^{\mathcal{T}}_2$ is active but $\mathrm{Ord}(\mathcal{M}^{\mathcal{T}}_2) = \mathrm{lh}(E^{\mathcal{T}}_1)$, and therefore if $E^{\mathcal{T}}_2$ is defined then $\mathrm{lh}(E^{\mathcal{T}}_2) = \mathrm{lh}(E^{\mathcal{T}}_1)$. Because $E^{\mathcal{T}}_2$ is type 2, however, if $E^{\mathcal{T}}_3$ is defined then $\mathrm{lh}(E^{\mathcal{T}}_2) < \mathrm{lh}(E^{\mathcal{T}}_3)$.

The preceding example is essentially general. It is easy to show that if $\mathcal{T}$ is k-maximal and $\alpha<\beta<\mathrm{lh}(\mathcal{T})$ then either $\mathrm{lh}(E^{\mathcal{T}}_\alpha) < \mathrm{Ord}(M^{\mathcal{T}}_\beta)$ and $\mathrm{lh}(E^{\mathcal{T}}_\alpha)$ is a cardinal of $M^{\mathcal{T}}_\beta$, or $\beta=\alpha+1$ and $\mathrm{lh}(E^{\mathcal{T}}_\alpha) = \mathrm{Ord}(M^{\mathcal{T}}_{\alpha+1})$ and $E^{\mathcal{T}}_\alpha$ is superstrong and $M^{\mathcal{T}}_{\alpha+1}$ is type 2. Therefore if $\alpha+1<\beta+1<\mathrm{lh}(\mathcal{T})$ then $\nu(E^{\mathcal{T}}_\alpha) < \nu(E^{\mathcal{T}}_\beta)$, and if $\alpha+1 \leq \beta < \mathrm{lh}(\mathcal{T})$ then $E^{\mathcal{T}}_\alpha{\restriction}\nu(E^{\mathcal{T}}_\alpha)$ is not an initial segment of any

---

[33] Here are more details. For $\gamma<\alpha$, after the first γ rounds have been played, both players having met their commitments so far, we have a γ-sequence $\vec{\mathcal{T}} = \langle\mathcal{T}_\delta\rangle_{\delta<\gamma}$ of iteration trees, with wellfounded final model $M^{\vec{\mathcal{T}}}_{\infty}$ (formed by direct limit if γ is a limit); it follows that this model is an n-sound opm where $n = \deg^{\vec{\mathcal{T}}}_{\infty}$. At the beginning of round γ, player I chooses some $(\mathcal{Q},q) \trianglelefteq (\mathcal{M}^{\vec{\mathcal{T}}}_{\infty}, n)$, and round γ is a (possibly partial) run of $\mathcal{G}_q(\mathcal{Q},\theta)$, producing a putative tree $\mathcal{T}_\gamma$. Player I is allowed to terminate this run at any stage, after producing $\mathcal{T}_\gamma$ of length some $\xi+1<\theta$. If player I wins the round of $\mathcal{G}_q(\mathcal{Q},\theta)$ at any stage before terminating, then player I wins the full run of $\mathcal{G}^*_k(\mathcal{M},\alpha,\theta)$. If player I terminates with $\mathcal{T}_\gamma$ of length $\xi+1<\theta$ and $\mathcal{T}_\gamma$ is not a win for player I in $\mathcal{G}_q(\mathcal{Q},\theta)$, then they go on to round $\gamma+1$ of $\mathcal{G}^*_k(\mathcal{M},\alpha,\theta)$, assuming $\gamma+1<\alpha$. Suppose player I never terminates at any $\xi+1<\theta$, and $\mathcal{T}_\gamma$ is not a win for player I in $\mathcal{G}_q(\mathcal{Q},\theta)$; so $\mathcal{T}_\gamma$ is an iteration tree of length θ. Then player II wins the full run of $\mathcal{G}^*_k(\mathcal{M},\alpha,\theta)$. Note that in case θ is a successor ordinal, this last rule is as in the games denoted $\mathcal{G}^*_k(\mathcal{M},\alpha,\theta)$ in [22], but differs from those denoted $\mathcal{G}_k(\mathcal{M},\alpha,\theta)$ in [25].

extender on $\mathbb{E}_+(M^{\mathcal{T}}_\beta)$.

The comparison algorithm needs to be modified slightly. Suppose we are comparing models $\mathcal{M},\mathcal{N}$, via padded k-maximal trees $\mathcal{T},\mathcal{U}$, respectively, and we have produced $\mathcal{T}\restriction(\alpha+1)$ and $\mathcal{U}\restriction(\alpha+1)$. Let γ be least such that $\mathcal{M}^{\mathcal{T}}_\alpha|\gamma \neq \mathcal{M}^{\mathcal{U}}_\alpha|\gamma$; let us assume that γ is a limit, so that this distinction is due to differing extenders indexed at γ. If only one of these models is active, then we use that active extender next. Suppose both are active. If one active extender is type 3 and one is type 2, then we use only the type 3 extender next. Otherwise we use both extenders next. With this modification, and with the remarks in the preceding paragraph, the usual proof that comparison succeeds goes through. The reason we make this modification is as follows. Suppose we use the usual process, so that if both sides are active at height γ (where the least disagreement was), we automatically use both of the disagreeing extenders. Let us use padding in the usual way for comparison. It might be that $E^{\mathcal{T}}_\alpha$ is superstrong and $E^{\mathcal{U}}_\alpha$ is type 2 (with $\mathrm{lh}(E^{\mathcal{T}}_\alpha) = \gamma = \mathrm{lh}(E^{\mathcal{U}}_\alpha)$), and the situation described in the previous paragraph occurs in $\mathcal{T}$ at that stage, so that $\gamma = \mathrm{Ord}(M^{\mathcal{T}}_{\alpha+1})$ and $M^{\mathcal{T}}_{\alpha+1}$ is active type 2. But then it might also be that $F(M^{\mathcal{T}}_{\alpha+1}) = E^{\mathcal{U}}_\alpha$. On the other hand, $\gamma < \mathrm{Ord}(M^{\mathcal{U}}_{\alpha+1})$ (since $E^{\mathcal{U}}_\alpha$ is type 2), so $E^{\mathcal{T}}_{\alpha+1} = F(M^{\mathcal{T}}_{\alpha+1}) = E^{\mathcal{U}}_\alpha$. So $\mathcal{T},\mathcal{U}$ use identical extenders at stages $\alpha+1,\alpha$ respectively, which breaks the usual comparison arguments. With the modified algorithm, we would set $E^{\mathcal{T}}_\alpha$ to be the superstrong extender (indexed at γ) and $E^{\mathcal{U}}_\alpha = \emptyset$, and if indeed $F(M^{\mathcal{T}}_{\alpha+1}) = F^{M^{\mathcal{U}}_\alpha|\gamma}$, then the comparison would terminate at stage $\alpha+1$.

Note that everything mentioned in this remark applies to standard premice with superstrong extenders, not just opms.

Lemma 2.49. *Let $\mathcal{M}$ be a k-relevant opm and $\mathcal{T}$ be a partial play of $\mathcal{G}^*_{\mathrm{opt},k}(\mathcal{M},\infty,\infty)$ of successor length, with $M^{\mathcal{T}}_\infty$ well-defined and wellfounded. Then $M^{\mathcal{T}}_\infty$ is a $\deg^{\mathcal{T}}_\infty$-relevant opm.*

Proof. Given the result for k-maximal trees $\mathcal{T}$, the generalization to stacks of the kind dealt with in the lemma is routine. For k-maximal $\mathcal{T}$, the result follows by a straightforward induction on $\mathrm{lh}(\mathcal{T})$, using Lemma 2.46, together with the following observation. Suppose $\beta+1 \in \mathscr{D}^{\mathcal{T}}_{\deg}$ and let $\alpha = \mathrm{pred}^{\mathcal{T}}(\alpha+1)$ and $\mathcal{N} = M^{*\mathcal{T}}_{\beta+1}$ and $n = \deg^{\mathcal{T}}(\beta+1)$. We claim that $M^{*\mathcal{T}}_{\beta+1}$ is n-relevant, which together with 2.46 suffices. So suppose that $\mathcal{N}$ is a successor. We have $\kappa = \mathrm{crit}(E^{\mathcal{T}}_\beta) < \nu(E^{\mathcal{T}}_\alpha)$ and $\rho^{\mathcal{N}}_{n+1} \leq \kappa < \rho^{\mathcal{N}}_n$. But $\mathrm{lh}(E^{\mathcal{T}}_\alpha) \leq \mathrm{Ord}^{\mathcal{N}^-}$, since $\mathbb{E}^{\mathcal{N}}_+$ has no elements indexed in the interval $(\mathrm{Ord}^{\mathcal{N}^-}, \mathrm{Ord}^{\mathcal{N}}]$. Since $\mathcal{N}^- \triangleleft M^{*\mathcal{T}}_{\beta+1}$, we have $\kappa < \rho^{\mathcal{N}^-}_\omega$. Therefore $\rho^{\mathcal{N}}_{n+1} \leq \kappa < \rho^{\mathcal{N}^-}_\omega$, so $\mathcal{N}$ is n-relevant. □

In 2.49, it is important that we are considering $\mathcal{G}^*_{\mathrm{opt},k}(\mathcal{M},\infty,\infty)$; $M^{\mathcal{T}}_\infty$ can fail to be $\deg^{\mathcal{T}}_\infty$-relevant for trees produced by $\mathcal{G}^*_k(\mathcal{M},\infty,\infty)$. (For example, after the first round, producing last model/degree $(M^{\mathcal{T}}_\infty, \deg^{\mathcal{T}}_\infty)$, player 1 could drop in model/degree to some $(\mathcal{Q},q) \trianglelefteq (M^{\mathcal{T}}_\infty, \deg^{\mathcal{T}}_\infty)$ such that $\mathcal{Q}$ is a successor and $\rho^{\mathcal{Q}}_{q+1} = \rho^{\mathcal{Q}^-}_\omega$, and then in the second round, use just one extender E applied to

Q with $\mathrm{crit}(E) < \rho_\omega^{Q^-}$, forming $\mathrm{Ult}_q(\mathcal{Q}, E)$. If the ultrapower map is continuous at $\rho_\omega^{Q^-}$, then $\mathrm{Ult}_q(\mathcal{Q}, E)$ is not q-relevant.)

3 $\mathcal{F}$-mice for operators $\mathcal{F}$

Operator-premice $\mathcal{M}$ are generally considered in the case that successor steps are taken by some *operator* $\mathcal{F}$ (Definition 3.4); that is, in which $\mathcal{N} = \mathcal{F}(\mathcal{N}^-)$ for each successor $\mathcal{N} \trianglelefteq \mathcal{M}$. We call such an $\mathcal{M}$ an *$\mathcal{F}$-premouse*. A key example is that of *mouse operators*, for which we have some formula φ and $\mathcal{F}(\mathcal{N}^-)$ is, roughly, the least mouse $\mathcal{R}$ over $\mathcal{N}^-$ such that either $\mathcal{R} \vDash \varphi$ or $\mathcal{R}$ projects $< \rho_\omega(\mathcal{N}^-)$ (but $\mathcal{R}$ must be coded appropriately so that $\mathcal{F}(\mathcal{N}^-)$ is an opm; see Definition 3.18 for details). One can also use the operator framework to define (iteration) *strategy mice*, although a different approach is taken in [10] (to give a more refined hierarchy).

3.1 Abstract operators $\mathcal{F}$ and $\mathcal{F}$-premice

Definition 3.1. We say that X is **swo'd (self-wellordered)** iff $X = x \cup \{x, <\}$ for some transitive set x, and wellorder $<$ of x. In this situation, $<_X$ denotes the wellorder of X extending $<$, and with last two elements $x, <$. Clearly there are uniform methods of passing from a swo'd X to a wellorder of $A = \hat{X}$. Fix such a method, and for such X, A, let $<_A$ denote the resulting wellorder of A. ⊣

The domains of our operators will be *operatic domains*, which will be certain kinds of subsets of *operator backgrounds*. The set $\mathscr{H}_\kappa$ of sets hereditarily of cardinality $< \kappa$) is a basic example of an operator background:

Definition 3.2. We say that a set or class $\mathscr{B}$ is an **operator background** iff (i) $\mathscr{B}$ is transitive, rudimentarily closed and $\omega \in \mathscr{B}$, (ii) for all $x \in \mathscr{B}$ and all y, f, if $f \colon x^{<\omega} \to \mathrm{trancl}(y)$ is a surjection then $y \in \mathscr{B}$, and (iii) $\mathscr{B} \vDash \mathsf{DC}$. (So $\mathrm{Ord}(\mathscr{B}) = \mathrm{rank}(\mathscr{B})$ is a cardinal; if $\omega < \kappa \leq \mathrm{Ord}$ then $\mathscr{H}_\kappa$ is an operator background (note that this only uses ZF, since κ is an ordinal), and under ZFC these are the only operator backgrounds.) By (iii), every element of $\mathscr{B}$ has a countable elementary substructure in $\mathscr{B}$.

Let $\mathscr{B}$ be an operator background. A set C is a **cone of** $\mathscr{B}$ iff there is $a \in \mathscr{B}$ such that C is the set of all $x \in \mathscr{B}$ such that $a \in \mathcal{J}_1(\hat{x})$. With a, C as such, we say C is **the cone above** a. If $b \in \mathcal{J}_1(a)$ we say C is **above** b. A set D is a **swo'd cone of** $\mathscr{B}$ iff $D = C \cap S$, for some cone C of $\mathscr{B}$, and where S is the class of swo'd sets. Here D is **(the swo'd cone) above** a iff C is (the cone) above a. A **cone** is a cone of $\mathscr{B}$ for some operator background $\mathscr{B}$. Likewise for **swo'd cone**. ⊣

Definition 3.3. Let $\mathscr{B}$ be an operator background. For $C \subseteq \mathscr{B}$, let

$$\widehat{C} = \{\hat{Y} \mid Y \in C \wedge Y \text{ is transitive}\}.$$

An **operatic domain over** $\mathscr{B}$ is a set $D = \widehat{C} \cup P \subseteq \mathscr{B}$, where C is a cone of $\mathscr{B}$ or a swo'd cone of $\mathscr{B}$, and P is some class of $< \omega$-condensing ω-sound opms, each over some $A \in \widehat{C}$. (We do not make any closure requirements on P.) Write $C^D = C$ and $P^D = P$. Note that $\widehat{C} \cap P = \emptyset$.

An **operatic domain** is an operatic domain over some such $\mathscr{B}$. $\dashv$

We can now define *operators*:

Definition 3.4. Let $\mathscr{B}$ be an operator background. An **operator over** $\mathscr{B}$ **with domain** D is a function $\mathcal{F} : D \to \mathscr{B}$ such that (i) D is an operatic domain over $\mathscr{B}$; (ii) for all $X \in D$, $\mathcal{M} = \mathcal{F}(X)$ is a successor opm with $\mathcal{M}^- = X$ (so if $X \in \widehat{C^D}$ then $l(\mathcal{M}) = 1$ and $cb^{\mathcal{M}} = X$, whereas if $X \in P^D$ then $l(\mathcal{M}) = l(X)+1$ and $X \lhd \mathcal{M}$). Write $C^{\mathcal{F}} = C^D$ and $P^{\mathcal{F}} = P^D$. $\dashv$

Remark 3.5. The argument X to an operator should be thought of as having one of two possible types. It is a *coarse object* if $X \in \widehat{C^{\mathcal{F}}}$; it is an opm if $X \in P^{\mathcal{F}}$. Some natural operators $\mathcal{F}$ have the property that, given $\mathcal{N} \in P^{\mathcal{F}}$ (so $\widehat{\mathcal{N}} \in \widehat{C^{\mathcal{F}}}$), $\mathcal{F}(\widehat{\mathcal{N}})$ is inter-computable with $\mathcal{F}(\mathcal{N})$. But operators producing strategy mice in the "least branch" (or "least tree") hierarchy do not have this property. (For in that case, $\mathcal{F}(\mathcal{N})$ is defined by first identifying the "least tree" $\mathcal{T}_{\mathcal{N}}$ for which a branch must be added, and then adding the correct branch through that tree; this depends of course on $S^{\mathcal{N}}$, which indicates which trees have already been dealt with. On the other hand, $\widehat{\mathcal{N}}$ is treated as a coarse object, so when defining $\mathcal{F}(\widehat{\mathcal{N}})$, $S^{\mathcal{N}}$ is irrelevant, and the "least tree" $\mathcal{T}_{\widehat{\mathcal{N}}}$ chosen will likely be different from $\mathcal{T}_{\mathcal{N}}$.)

The simplest operator is essentially the rudimentary closure operator $\mathcal{J}$:

Definition 3.6. Assume DC. Let $p \in V$. Let C_p be the class of all x such that $p \in \mathcal{J}_1(\hat{x})$. Let P_p be the class of all $< \omega$-condensing ω-sound opms $\mathcal{R}$ over some $Y \in \widehat{C_p}$, with $cp^{\mathcal{R}} = p$. Then $\mathcal{J}^{\mathrm{op}}_p$ denotes the operator over V with domain $D = \widehat{C_p} \cup P_p$, where for $x \in D$, $\mathcal{J}^{\mathrm{op}}_p(x)$ is the passive successor opm $\mathcal{M}$ with universe $\mathcal{J}_1(x)$ and $\mathcal{M}^- = x$ and $cp^{\mathcal{M}} = p$.[34] (So if $x \in \widehat{C_p}$ then $l(\mathcal{M}) = 1$ and $cb^{\mathcal{M}} = x$.) Let $\mathcal{J}^{\mathrm{op}} = \mathcal{J}^{\mathrm{op}}_{\emptyset}$.

Without assuming DC, if $p \in \mathscr{H}_\kappa$, then we can define, in the same manner, the operator $\mathcal{J}^{\mathrm{op}}_{p,\mathscr{H}_\kappa}$ over $\mathscr{H}_\kappa$. We might also just write $\mathcal{J}^{\mathrm{op}}_p$ for $\mathcal{J}^{\mathrm{op}}_{p,\mathscr{H}_\kappa}$. $\dashv$

Definition 3.7 ($\mathcal{F}$-premouse)**.** For $\mathcal{F}$ an operator, an **$\mathcal{F}$-premouse ($\mathcal{F}$-pm)** is an opm $\mathcal{M}$ such that $\mathcal{N} = \mathcal{F}(\mathcal{N}^-)$ for every successor $\mathcal{N} \trianglelefteq \mathcal{M}$. $\dashv$

Let $\mathcal{M}$ be an $\mathcal{F}$-premouse, where $\mathcal{F}$ is an operator over $\mathscr{B}$. Note that $cb^{\mathcal{M}} \in \widehat{C^{\mathcal{F}}}$, as $\mathcal{M}|1 = \mathcal{F}(\mathcal{M}|0)$ and $\mathcal{M}|0 = cb^{\mathcal{M}} = \hat{x}$ for some x, and $\hat{x} \notin P^{\mathcal{F}}$. Note also that $\mathrm{Ord}(\mathcal{M}) \leq \mathrm{Ord}(\mathscr{B})$.

We now define $\mathcal{F}$-iterability for $\mathcal{F}$-premice $\mathcal{M}$, using Definition 2.47 (and hence continuing to follow [25]). The main point is that the iteration strategy

---

[34] It is easy to see that $\mathcal{M}$ is indeed an opm, so $\mathcal{J}^{\mathrm{op}}_p$ is an operator.

should produce $\mathcal{F}$-premice. One needs to be a little careful, however, because the background $\mathscr{B}$ for $\mathcal{F}$ might only be a set. To simplify things, we restrict our attention to the case that $\mathcal{M} \in \mathscr{B}$.

Definition 3.8. Let $\mathcal{F}$ be an operator over $\mathscr{B}$. Let $\mathcal{M}$ be an $\mathcal{F}$-premouse and let $\mathcal{T}$ be a putative iteration tree on $\mathcal{M}$. We say that $\mathcal{T}$ is a **putative $\mathcal{F}$-iteration tree** iff $M^{\mathcal{T}}_\alpha$ is an $\mathcal{F}$-premouse for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$. We say that $\mathcal{T}$ is an **$\mathcal{F}$-iteration tree** iff $M^{\mathcal{T}}_\alpha$ is an $\mathcal{F}$-premouse for all $\alpha + 1 \leq \mathrm{lh}(\mathcal{T})$. We may drop the "$\mathcal{F}$-" when it is clear from context.

Let $k < \omega$ and let $\mathcal{M} \in \mathscr{B}$ be a k-sound $\mathcal{F}$-premouse. Let $\theta \leq \mathrm{Ord}(\mathscr{B}) + 1$. The iteration game $\mathcal{G}^{\mathcal{F}}_k(\mathcal{M}, \theta)$ has the rules of $\mathcal{G}_k(\mathcal{M}, \theta)$, except for the following difference. Let $\mathcal{T}$ be the putative tree being produced. For $\alpha + 1 \leq \theta$, if both players meet their requirements at all stages $< \alpha$, then, in stage α, player II must first ensure that $M^{\mathcal{T}}_\alpha$ is wellfounded, and if $\alpha < \mathrm{Ord}^{\mathscr{B}}$, that $M^{\mathcal{T}}_\alpha$ is an $\mathcal{F}$-premouse. (Given this, if $\alpha + 1 < \theta$, player I then selects $E^{\mathcal{T}}_\alpha$.) Thus, if we reach stage $\mathrm{Ord}(\mathscr{B})$, then after selecting a branch, player II wins iff $M^{\mathcal{T}}_{\mathrm{Ord}(\mathscr{B})}$ is wellfounded. (We cannot in general expect $M^{\mathcal{T}}_{\mathrm{Ord}(\mathscr{B})}$ to be an $\mathcal{F}$-premouse in this situation. For example, suppose that $\mathscr{B} = \mathrm{HC}$ and $\theta = \omega_1 + 1$ and $\mathrm{lh}(\mathcal{T}) = \omega_1 + 1$. Then $M^{\mathcal{T}}_{\omega_1}$ cannot be an $\mathcal{F}$-premouse, since all $\mathcal{F}$-premice have height $\leq \omega_1$. But in applications such as comparison, we only need to know that $M^{\mathcal{T}}_{\omega_1}$ is wellfounded. So we still decide the game in favour of player II in this situation.)

Let $\lambda, \alpha \leq \mathrm{Ord}(\mathscr{B})$, and suppose that either $\mathrm{Ord}(\mathscr{B})$ is regular or $\lambda < \mathrm{Ord}(\mathscr{B})$. Let $\theta \leq \lambda + 1$. The iteration game $\mathcal{G}^{*\mathcal{F}}_k(\mathcal{M}, \alpha, \theta)$ is defined just as $\mathcal{G}^*{}_k(\mathcal{M}, \alpha, \theta)$, with the differences that (i) the rounds are runs of $\mathcal{G}^{\mathcal{F}}_q(\mathcal{Q}, \theta)$ for some $\mathcal{Q}, q$, and (ii) if α is a limit and neither player breaks any rule, and $\vec{\mathcal{T}}$ is the sequence of trees played, then player II wins iff $M^{\vec{\mathcal{T}}}_\infty$ is well-defined, wellfounded,[35] and if $\alpha < \mathrm{Ord}(\mathscr{B})$ then $M^{\vec{\mathcal{T}}}_\infty$ is an $\mathcal{F}$-premouse. (By some straightforward calculations using the restrictions on α, θ, one can see that for any $\gamma < \alpha$, if neither player has lost the game after the first γ rounds, and $\vec{\mathcal{T}}\restriction\gamma$ is the sequence of trees played thus far, then $M^{\vec{\mathcal{T}}\restriction\gamma}_\infty \in \mathscr{B}$ and $M^{\vec{\mathcal{T}}\restriction\gamma}_\infty$ is an $\mathcal{F}$-premouse, so $\mathcal{G}^{\mathcal{F}}_q(\mathcal{Q}, \theta)$ is defined for the relevant $(\mathcal{Q}, q)$. This uses the rule that if one of the rounds produces a tree of length θ, then the game terminates.)

$\mathcal{G}^{*\mathcal{F}}_{\mathrm{opt},k}(\mathcal{M}, \alpha, \theta)$ is likewise defined by analogy with $\mathcal{G}^*{}_{\mathrm{opt},k}(\mathcal{M}, \alpha, \theta)$.

An **$\mathcal{F}$-(k, θ)-iteration strategy** for $\mathcal{M}$ is a winning strategy for player II in $\mathcal{G}^{\mathcal{F}}(\mathcal{M}, k, \theta)$. An **$\mathcal{F}$-$(k, \alpha, \theta)^*$-optimal iteration strategy** for $\mathcal{M}$ is likewise for $\mathcal{G}^{*\mathcal{F}}_{\mathrm{opt}}(\mathcal{M}, k, \alpha, \theta)$. And an **$\mathcal{F}$-$(k, \alpha, \theta)^*$-iteration strategy** is likewise for $\mathcal{G}^{*\mathcal{F}}(\mathcal{M}, k, \alpha, \theta)$.

Now **$\mathcal{F}$-(k, θ)-iterability**, etc, are defined in the obvious manner. $\dashv$

3.2 Coarse condensation of operators

In order to prove that $\mathcal{F}$-premice built by background constructions are $\mathcal{F}$-iterable, we will need to know that $\mathcal{F}$ has good *condensation* properties, which

---

[35] It follows that if $\alpha = \mathrm{Ord}(\mathscr{B})$ then $M^{\vec{\mathcal{T}}}_\infty|\mathrm{Ord}(\mathscr{B})$ is an $\mathcal{F}$-premouse.

roughly demand that elementary hulls of structures $\mathcal{F}(\mathcal{M})$ should have form $\mathcal{F}(\mathcal{M}')$. (But we will also need to consider variants thereof.)

Definition 3.9. Let $\pi : \mathcal{M} \to \mathcal{N}$ be an embedding and b be transitive. We say that π is **above** b iff $b \cup \{b\} \subseteq \mathrm{dom}(\pi)$ and $\pi \restriction b \cup \{b\} = \mathrm{id}$. $\dashv$

Definition 3.10. Let $\mathcal{F}$ be an operator over $\mathscr{B}$ with domain D. Suppose C^D is the cone above some transitive $p \in \mathscr{B}$. We say that $\mathcal{F}$ **condenses coarsely** (or $\mathcal{F}$ **has coarse condensation**) **above** p iff for every successor $\mathcal{F}$-pm $\mathcal{N}$ (so $p \in \mathcal{J}_1(cb^{\mathcal{N}})$), every set-generic extension $V[G]$ of V and all $\mathcal{M}, \pi \in V[G]$, if $\mathcal{M}$ is a successor opm, $\mathcal{M}^- \in V$ and $\pi : \mathcal{M} \to \mathcal{N}$ is fully elementary and is above p, then $\mathcal{M}$ is an $\mathcal{F}$-pm (so in particular, $cb^{\mathcal{M}} \in \widehat{C^D}$ and $\mathcal{M}^- \in \mathrm{dom}(\mathcal{F})$ and $\mathcal{M} = \mathcal{F}(\mathcal{M}^-) \in V$).

We say that $\mathcal{F}$ **almost condenses coarsely above** p iff the preceding holds for $G = \emptyset$. $\dashv$

Definition 3.11. An operator $\mathcal{F}$ over $\mathscr{B}$ is **total** iff $P^{\mathcal{F}}$ includes all $< \omega$-condensing ω-sound $\mathcal{F}$-pms in $\mathscr{B}$. $\dashv$

The following lemma is a standard kind of observation:

Lemma 3.12. *Let $\mathcal{F}$ be a total operator over $\mathscr{B}$ with domain D. Suppose that C^D is the cone above some $p \in \mathrm{HC}$, and that $\mathcal{F}$ almost condenses coarsely above p. Then $\mathcal{F}$ condenses coarsely above p.*

Proof sketch. Suppose the lemma fails and let $\mathbb{P}$ be a poset, and $G \subseteq \mathbb{P}$ be V-generic, such that in $V[G]$ there is a counterexample $\pi : \mathcal{M} \to \mathcal{N}$. So $\mathcal{N} \in V$ is a successor $\mathcal{F}$-pm (so $\mathcal{N} \in \mathscr{B}$), $\mathcal{M} \in V[G]$ is a successor opm, $\mathcal{M}^- \in V$, and $\pi \in V[G]$ with $\pi : \mathcal{M} \to \mathcal{N}$ is fully elementary and is above p, but $\mathcal{M}$ is not an $\mathcal{F}$-pm. By passing to proper segments of $\mathcal{M}, \mathcal{N}$ if needed, we may assume that $\mathcal{M}^-$ is an $\mathcal{F}$-pm, and therefore that $\mathcal{M}^- \in \mathrm{dom}(\mathcal{F})$. So letting $\mathcal{M}' = \mathcal{F}(\mathcal{M}^-)$, we have $\mathcal{M} \neq \mathcal{M}'$.

Let $\mathbb{P}' = \mathrm{Col}(\omega, \mathcal{M}' \cup \mathcal{N})$. By Σ^1_1-absoluteness, we may assume that $\mathbb{P} = \mathbb{P}'$. That is, if H is $(V, \mathbb{P}')$-generic then in $\mathcal{N}, \mathcal{M}', \mathcal{M}^-, p \in \mathrm{HC}^{V[H]}$, and since in $V[G]$ there is $(\mathcal{P}, \sigma)$ such that $\mathcal{P}$ is a successor opm such that $\mathcal{P}^- = \mathcal{M}^-$ and $\mathcal{P} \neq \mathcal{M}'$ and $\sigma : \mathcal{P} \to \mathcal{N}$ is elementary and above p, there is also such a pair $(\mathcal{P}, \sigma) \in V[H]$.

Let $\varrho : \omega \to p$ be a surjection. Let $X = \mathcal{M}' \cup \mathcal{N} \cup p \cup \varrho \cup \{\mathcal{M}', \mathcal{N}, p, \varrho\}$, so $X \in \mathscr{B}$. We can fix $\eta \in \mathrm{Ord}$ such that $L_\eta(X) \models \mathsf{ZF}^{-\varepsilon}$, and in fact by condensation, taking the least such η, we have $L_\eta(X) \in \mathscr{B}$.

So $\mathbb{P} = \mathrm{Col}(\omega, \mathcal{M}' \cup \mathcal{N}) \in L_\eta(X)$ and $L_\eta(X) \models$ "It is forced by $\mathbb{P}$ that there is $(\mathcal{P}, \sigma)$ such that $\mathcal{P}$ is a successor opm with $\mathcal{P}^- = \mathcal{M}^-$ but $\mathcal{P} \neq \mathcal{M}'$ and $\sigma : \mathcal{P} \to \mathcal{N}$ is elementary and above p." Because $\mathscr{B} \models \mathsf{DC}$, we can take a countable elementary hull of $L_\eta(X)$, such that letting $\tau : L_{\bar\eta}(\bar X) \to L_\eta(X)$ be the uncollapse map, $\tau(\bar X) = X$ and $\mathrm{rg}(\tau)$ includes all relevant objects and all points in $p \cup \{p\}$. Write $\pi(\bar{\mathbb{P}}) = \mathbb{P}$, etc. Fix $g \in V$ which is $(L_{\bar\eta}(\bar X), \bar{\mathbb{P}})$-generic. Then in $L_{\bar\eta}(\bar X)[g]$, we have some opm $\mathcal{P}$ such that $\mathcal{P}^- = \bar{\mathcal{M}}^-$ but $\mathcal{P} \neq \bar{\mathcal{M}}'$, and some elementary $\sigma : \mathcal{P} \to \bar{\mathcal{N}}$ which is above $\bar p = p$. Since

$\tau{\restriction}\bar{\mathcal{N}} : \bar{\mathcal{N}} \to \mathcal{N}$ and $\tau{\restriction}\bar{\mathcal{M}}' : \bar{\mathcal{M}}' \to \mathcal{M}' = \mathcal{F}(\mathcal{M}^-)$ are elementary and above p, and $\mathcal{F}$ almost condenses coarsely above p, $\bar{\mathcal{N}}$ and $\bar{\mathcal{M}}'$ are $\mathcal{F}$-premice. But then similarly, because we have σ, $\mathcal{P}$ is an $\mathcal{F}$-premouse, so $\mathcal{P} = \mathcal{F}(\bar{\mathcal{M}}^-) = \bar{\mathcal{M}}'$, a contradiction. □

3.3 Operators which (don't) condense well

Remark 3.13. So far the only example of an operator we have formally defined is that of $\mathcal{J}$. In Definition 3.18 below, we will introduce a more general class of examples, *mouse operators*. This will be a modification of the notion *model operator* from (see [28, Definition 2.1.8], where such objects are denoted F_K). But as we describe below, the *model operators* defined in [28, 2.1.8] have a minor problem, which we rectify here. (The notion *mouse operator* as defined in [28, Definition 2.1.7] is distinct from both of these.)

We will then proceed toward the central notion of *condenses finely*, a refinement of *condenses coarsely*. This notion is based on that of *condenses well*, from [21, Definition 1.3.2] and [28, Definition 2.1.10]. We will modify *condenses well* in several respects, for multiple reasons. The main changes will be motivated by the following discussion. We can demonstrate a concrete problem with *condenses well*, at least when it is used in concert with other definitions in [28]. The following discussion uses the definitions and notation of [28, §2], without further explanation here; the terminology differs from this paper. (The remainder of this remark is for motivation only; nothing in it is needed later.)

Let K be the function $x \mapsto \mathcal{J}_2(x)$. Clearly K is a mouse operator (see [28, 2.1.7]). Let $F = F_K$ (see [28, 2.1.8]). Then we claim:

1. assuming that "nth master code" has a standard interpretation in [28, 2.1.8], F is not well-defined[36] (contrary to [28, 2.1.8, 2.1.12]), and
2. modifying the definition of F in the natural way so as to produce a (well-defined) model operator F', F' does not condense well (contrary to the spirit of [28, 2.1.12]).

Let us verify this.

The fact that F is not well-defined is just because in [28, 2.1.8], in case 2 of the definition of $F(\mathcal{M})$, the universe of $F(\mathcal{M})$ is taken to be the nth master code of $\mathcal{J}_2(\mathcal{M})|\xi$, for the relevant $n < \omega$. Here, as we are in case 2, we have $\mathcal{J}_2(\mathcal{M})|\xi = \mathcal{J}_1(\mathcal{M})$. Now it can be that $n > 0$ and $\rho(\mathcal{M}) < \mathrm{Ord}^{\mathcal{M}}$, and then the nth master code (if this is interpreted in a standard kind of fashion) has ordinal height $\rho(\mathcal{M})$, and its universe does not even include all of the universe of $\mathcal{M}$. But then it does not make sense to define $\dot{S}^{F(\mathcal{M})} = S^{\mathcal{M}} \frown \langle \mathcal{M} \rangle$, as is written in [28, 2.1.8].

So let us consider the modification of the definition of $F = F_K$, where instead of using a master code in case 2, we define

$$F'(\mathcal{M}) = (\mathcal{J}_2(\mathcal{M})|\xi; \in, \emptyset, \emptyset, \dot{S}^{\mathcal{M}} \frown \langle \mathcal{M} \rangle, \ell(\mathcal{M}) + 1, a).$$

---

[36] This is a minor point, and is easily rectified, by following the form of *mouse operator* from [21, bullet (1) after Definition 1.3.1].

(This wouldn't work for mouse operators in general, but we only consider the mouse operator K for this discussion.) In case 1, keep $F'(\mathcal{M}) = F(\mathcal{M})$ as defined in [28, 2.1.8].

Then F' is a model operator, and seems to carry the meaning intended in [28, 2.1.8]. (The adjustment in the definition brings it, moreover, in line with the definition of *mouse operator* in [21, bullet (1) after Definition 1.3.1].) But F' does not in general condense well. For clearly regular premice $\mathcal{M}$ whose ordinals are closed under "$+\omega$" can be arranged as models $\tilde{\mathcal{M}}$ with parameter $\emptyset$ (see [28, Definition 2.1.1]), such that for each $\alpha < l(\tilde{\mathcal{M}})$, $\tilde{\mathcal{M}}|(\alpha+1) = F'(\tilde{\mathcal{M}}|\alpha)$. Now let $\mathcal{M}$ be a premouse such that for some $\kappa < \mathrm{Ord}(\mathcal{M})$, κ is measurable in $\mathcal{M}$, via some measure on $\mathbb{E} = \mathbb{E}^{\mathcal{M}}$, and $\mathcal{M} \vDash$"$\lambda = \kappa^{+\kappa}$ exists", $\rho_\omega^{\mathcal{M}} = \lambda$, and $\mathcal{M} = \mathcal{J}_1(\mathcal{M}_0)$ where $\mathcal{M}_0 = \mathcal{J}_\lambda^{\mathbb{E}}$. Let $\mathcal{M}^* = \mathcal{J}(\tilde{\mathcal{M}_0})$, arranged as a model with parameter $\emptyset$ extending $\tilde{\mathcal{M}_0}$. We have $\rho_{\underset{\sim}{\omega}}^{\mathcal{M}} = \lambda = \rho(\mathcal{M}_0)$ and $\tilde{\mathcal{M}_0} \in \mathcal{M}^* \in F'(\tilde{\mathcal{M}_0})$ and $l(\mathcal{M}^*) = \lambda + 1$ and $(\mathcal{M}^*)^- = \tilde{\mathcal{M}_0}$ (see [28, Definition 2.1.3]). (We can't say $\mathcal{M}^* = \tilde{\mathcal{M}}$, because $\tilde{\mathcal{M}}$ is not defined.)

Let $E \in \mathbb{E}$ be $\mathcal{M}$-total with $\mathrm{crit}(E) = \kappa$. Let $\mathcal{N} = \mathrm{Ult}_0(\mathcal{M}, E)$ and $\pi = i_E$. Then $\rho_1^{\mathcal{N}} = \sup \pi``\lambda < \pi(\lambda)$. Let $\mathcal{N}_0 = \pi(\mathcal{M}_0)$ and $\mathcal{N}^* = \mathcal{J}_1(\tilde{\mathcal{N}_0})$, arranged as a model with parameter $\emptyset$ extending $\tilde{\mathcal{N}_0}$. Then $\rho_1(\mathcal{N}^*) < \pi(\lambda) = \rho(\tilde{\mathcal{N}_0})$, and therefore $\mathcal{N}^* = F'(\tilde{\mathcal{N}_0})$. But $\pi : \mathcal{M}^* \to \mathcal{N}^*$ is a 0-embedding (and $\pi(\tilde{\mathcal{M}_0}) = \tilde{\mathcal{N}_0}$). Since $\mathcal{M}^* \neq F(\tilde{\mathcal{M}_0})$, F' does not condense well (see [28, 2.1.10(1)]). Note that, in fact, $\mathrm{Ult}_1(\mathcal{M}, E) = \mathrm{Ult}_0(\mathcal{M}, E)$ and π is both a 0-embedding and a 1-embedding, since for all $\underset{\sim}{\mathrm{r}\Sigma}_1^{\mathcal{M}}$ functions $f : \kappa \to \mathcal{M}$ there is a measure one set $X \in \mathcal{M}$ such that $f{\restriction}X \in \mathcal{M}$. So π is also $\mathrm{r}\Sigma_2$-elementary, even though $\mathcal{M}^* \neq F'(\tilde{\mathcal{M}_0})$.)

But note that in the example above, $\mathcal{M}$ is not 0-relevant, nor k-relevant for any $k < \omega$. This motivates our focus on k-relevant opms. We now give a second example, and one in which the embedding is the kind that can arise in the proof of solidity of the standard parameter – certainly in this context we would want to make use of *condenses well*. We claim there are (consistently) mice $\mathcal{M}$, containing large cardinals, and $\rho, \alpha \in \mathrm{Ord}^{\mathcal{M}}$ such that:

- $\mathcal{M} = \mathcal{J}(\mathcal{N})$ where $\mathcal{N} = \mathcal{M}|\rho^{+\mathcal{M}}$,
- $\mathcal{M}$ is 1-sound,
- $\rho_1^{\mathcal{M}} = \rho < \alpha < \rho^{+\mathcal{M}}$,
- $p_1^{\mathcal{M}} = \{\rho^{+\mathcal{M}}, \alpha\}$, and
- letting $\mathcal{H} = \mathrm{cHull}_1^{\mathcal{M}}(\alpha \cup \{\rho^{+\mathcal{M}}\})$, we have $\rho_\omega^{\mathcal{H}} = \alpha$.

Given such $\mathcal{M}, \rho, \alpha, \mathcal{H}$, note that $\alpha = \rho^{+\mathcal{H}}$ and $\mathcal{H} = \mathcal{J}(\mathcal{M}||\alpha)$. Then $\mathcal{H}$ is a 1-solidity witness for $\mathcal{M}$, and the 0-embedding $\pi : \mathcal{H} \to \mathcal{M}$ is the one that would be used in the proof of the 1-solidity of $\mathcal{M}$. Moreover, with F' as before, "$\mathcal{M} = \mathcal{J}(\mathcal{N}) = F'(\mathcal{N})$" (since $\mathcal{M}$ projects below $\mathrm{Ord}^{\mathcal{N}}$) but "$\mathcal{H} \neq F'(\mathcal{M}||\alpha) = \mathcal{J}(\mathcal{J}(\mathcal{M}||\alpha))$". So we again have a failure of *condenses well*, and one which is arising in the context of the proof of solidity. (Of course, in the example we are already assuming 1-solidity, but the example seems to indicate that we cannot really expect to use *condenses well* in the proof of solidity for F'-mice.)

Now let us verify that such an $\mathcal{M}$ exists. Let $\mathcal{P}$ be any mouse (with large cardinals) and ρ a cardinal of $\mathcal{P}$ such that $\rho^{++\mathcal{P}} < \mathrm{Ord}^{\mathcal{P}}$. Let $\gamma = \rho^{+\mathcal{P}} + 1$. For $\alpha < \rho^{+\mathcal{P}}$ let

$$\mathcal{H}_\alpha = \mathrm{cHull}_1^{\mathcal{P}|\gamma}(\alpha \cup \{\rho^{+\mathcal{P}}\}).$$

Because $\rho_\omega^{\mathcal{P}|\gamma} = \rho^{+\mathcal{P}}$, there is α with $\rho < \alpha < \rho^{+\mathcal{P}}$ and such that the uncollapse map $\pi_\alpha : \mathcal{H}_\alpha \to \mathcal{P}|\gamma$ is fully elementary, and so $\rho_\omega^{\mathcal{H}_\alpha} = \alpha = \rho^{+\mathcal{H}_\alpha}$. (In $\mathcal{P}$, there is a club $C \subseteq \rho^{+\mathcal{P}}$ of ordinals α such that $\mathrm{crit}(\pi_\alpha) = \alpha$ and $\pi(\alpha) = \rho^{+\mathcal{P}}$. But $\mathcal{P}|\gamma = \mathrm{Hull}_1^{\mathcal{P}|\gamma}(\rho^{+\mathcal{P}} \cup \{\rho^{+\mathcal{P}}\})$, so considering Tarski-Vaught a straightforward closure argument yields a club $C' \subseteq C$ such that π_α is fully elementary for each $\alpha \in C'$.) Fix such an α. Let $\mathcal{H} = \mathcal{H}_\alpha$ and

$$\mathcal{M} = \mathrm{cHull}_1^{\mathcal{P}|\gamma}(\rho \cup \{\rho^{+\mathcal{P}}, \alpha\}).$$

We claim that $\mathcal{M}, \rho, \alpha$ are as required. For we have $\mathcal{M} \in \mathcal{P}$, which easily gives that $\rho_1^{\mathcal{M}} = \rho$. Clearly $\mathcal{M} = \mathcal{J}(\mathcal{N})$ where $\mathcal{N} = \mathcal{M}|\rho^{+\mathcal{M}}$. The 1-solidity witness associated to $\rho^{+\mathcal{M}}$ is $\mathrm{cHull}_1^{\mathcal{M}}(\rho^{+\mathcal{M}})$, which is just $\mathcal{M}|\rho^{+\mathcal{M}}$, as $\mathcal{M}|\rho^{+\mathcal{M}} \preccurlyeq_1 \mathcal{M}$, as $\mathcal{M}|\rho^{+\mathcal{M}} \models \mathsf{ZF}^-$. And the 1-solidity witness associated to α is $\mathrm{cHull}_1^{\mathcal{M}}(\alpha \cup \{\rho^{+\mathcal{M}}\})$, which is just $\mathcal{H} = \mathcal{J}(\mathcal{P}||\alpha) \in \mathcal{M}$. All of the required properties follow.

The preceding examples seem to extend to any (first-order) mouse operator K such that $\mathcal{J}(x) \in K(x)$ for all x.

To get around the problem just described, we will need to weaken the conclusion of *condenses well*, as will be seen.

Our second modification to the definition of *condenses well* is not based on a definite problem, but on a suspicion. It relates to, in the notation used in clause (2) of [28, 2.1.10], the embedding $\sigma : F(\mathcal{P}_0) \to \mathcal{M}$. In at least the basic situations in which one would want to use this clause (or its analogue in *condenses finely*), σ actually arises from something like an iteration map. But in *condenses well*, no hypothesis along these lines regarding σ is made. It seems that this could be a deficit, as it might be that $F(\mathcal{P}_0)$ is lower than $\mathcal{M}$ in the mouse order (if one can make sense of this); we might have $F(\mathcal{P}_0) \triangleleft \mathcal{M}$. Thus, it seems that in proving an operator condenses well, one might struggle to make use of the existence of σ. So, in *condenses finely*, we make stronger demands on σ.

A third change is that we do not require that $\pi \circ \sigma \in V$ (with π, σ as in [28, 2.1.10]). This is explained toward the end of 3.42.

Motivation for the remaining details will be provided by how they arise later, in our proof of the fundamental fine structural properties for $\mathcal{F}$-mice for operators $\mathcal{F}$ which condense finely, and in our proof that mouse operators condense finely. We now return to our terminology and notation. Before we can define *condenses finely*, we need to set up some terminology in order to describe the demands on σ.

3.4 Mouse operators

In this section, in Definition 3.18, we will define mouse operators, as an instance of a somewhat more general kind of operators (those of form $\mathcal{F}_G$; see 3.16). These are variants of the *model operators* of [28, 2.1.8], but in view of Remark 3.13, the details must be modified somewhat. Our definition of mouse operators will be based on *op-$\mathcal{J}$-structures*. An op-$\mathcal{J}$-structure will be used to form one step in the $\mathcal{F}_G$-hierarchy. Being a $\mathcal{J}$-structure, it has its own internal hierarchy, which will provide the stratification needed for opms:

Definition 3.14 (op-$\mathcal{J}$-structure)**.** Let $\alpha \in \mathrm{Ord}\backslash\{0\}$, and let Y be such that either $Y = \hat{Z}$ for a transitive Z, or Y is a $< \omega$-condensing ω-sound opm. Let

$$D = \mathrm{Lim} \cap [\mathrm{Ord}^Y + \omega, \mathrm{Ord}^Y + \omega\alpha)$$

and let $\vec{P} = \langle P_\beta \rangle_{\beta \in D}$ be given.

We define $\mathcal{J}_\beta^{\vec{P}}(Y)$ for $\beta \in [1, \alpha]$, if possible, by recursion on β, as follows. We set $\mathcal{J}_1^{\vec{P}}(Y) = \mathcal{J}(Y)$ and take unions at limit β. For $\beta + 1 \in [2, \alpha]$, let $R = \mathcal{J}_\beta^{\vec{P}}(Y)$ and suppose that $P =_{\mathrm{def}} P_{\mathrm{Ord}^R} \subseteq R$ and is amenable to R. In this case we define

$$\mathcal{J}_{\beta+1}^{\vec{P}}(Y) = \mathcal{J}(R, \vec{P}{\restriction}R, P).$$

Note then that by induction, $\vec{P}{\restriction}R \subseteq R$ and $\vec{P}{\restriction}R$ is amenable to R.

Let $\mathcal{L}_\mathcal{J}$ be the language with binary relation symbol $\dot{\in}$, predicate symbols $\dot{\vec{P}}$ and $\dot{P}$, and constant symbol $\dot{cb}$.

For Y as above, an **op-$\mathcal{J}$-structure over** Y is an amenable $\mathcal{L}_\mathcal{J}$-structure

$$\mathcal{M} = (\mathcal{J}_\alpha^{\vec{P}}(Y), \in^\mathcal{M}, \vec{P}, P, Y),$$

where $\alpha \in \mathrm{Ord}\backslash\{0\}$ and $\vec{P} = \left\langle \vec{P}_\gamma \right\rangle_{\gamma \in D}$ with domain D defined as above, $\lfloor \mathcal{M} \rfloor = \mathcal{J}_\alpha^{\vec{P}}(Y)$ is defined, $\dot{\vec{P}}^\mathcal{M} = \vec{P}$, $\dot{P}^\mathcal{M} = P$, $\dot{cb}^\mathcal{M} = Y$.

Let $\mathcal{M}$ be an op-$\mathcal{J}$-structure, and adopt the notation above. Let $l(\mathcal{M})$ denote α. For $\beta \in [1, \alpha]$ and $R = \mathcal{J}_\beta^{\vec{P}}(Y)$ and $\gamma = \mathrm{Ord}^R$, let

$$\mathcal{M}|^\mathcal{J}\gamma = (R, \in^R, \vec{P}{\restriction}R, P_\gamma, Y).$$

Write $\mathcal{N} \trianglelefteq^\mathcal{J} \mathcal{M}$ iff $\mathcal{N} = \mathcal{M}|^\mathcal{J}\gamma$ for some γ. Clearly if $\mathcal{N} \trianglelefteq^\mathcal{J} \mathcal{M}$ then $\mathcal{N}$ is an op-$\mathcal{J}$-structure over Y. Write $\mathcal{N} \triangleleft^\mathcal{J} \mathcal{M}$ iff $\mathcal{N} \trianglelefteq^\mathcal{J} \mathcal{M}$ but $\mathcal{N} \neq \mathcal{M}$.

Let $\mathcal{M}$ be an op-$\mathcal{J}$-structure. Note that $\mathcal{M}$ is pre-fine (see Definition 2.26). We define the **fine structural notions** for $\mathcal{M}$ using 2.27. ⊣

From now on we omit "$\in$" from our notation for op-$\mathcal{J}$-structures. In what follows, recall that *operator background*, *operatic domain*, $\widehat{C^D}$ and P^D were introduced in Definitions 3.2 and 3.3.

Definition 3.15 (Pre-operator)**.** Let $\mathscr{B}$ be an operator background. A **pre-operator over** $\mathscr{B}$ is a function $G : D \to \mathscr{B}$, for some operatic domain D over $\mathscr{B}$ such that for each $Y \in D$, $G(Y)$ is an op-$\mathcal{J}$-structure $\mathcal{M}$ over Y such that (i) every $\mathcal{N} \trianglelefteq \mathcal{M}$ is ω-sound, and (ii) for some $n < \omega$, $\rho_{n+1}^{\mathcal{M}} = \omega$.[37] Recalling that $D = \widehat{C^D} \cup P^D$, let $C^G = C^D$ and $P^G = P^D$. ⊣

We now want to derive an operator $\mathcal{F}_G$ from a pre-operator G. Say $\mathcal{R}$ is a sound $\mathcal{F}_G$-premouse, over some set A, and we want to define $\mathcal{F}_G(\mathcal{R})$. The initial hope is that this structure should be essentially equivalent to $G(\mathcal{R})$, but with predicates reorganized appropriately. But this might not work, for two reasons. Most importantly, the resulting structure might fail projectum amenability; that is, $G(\mathcal{R})$ might contain subsets of $\rho_\omega^{\mathcal{R}} \times A^{<\omega}$ which are not in $\mathcal{R}$. In this case, we need to first replace $G(\mathcal{R})$ with the largest $\mathcal{J}$-initial segment $G'(\mathcal{R})$ of $G(\mathcal{R})$ which does satisfy projectum amenability. And then, although $\rho_{n+1}^{G'(\mathcal{R})} = \omega$ for some $n < \omega$, we cannot expect that $\rho_1^{G'(\mathcal{R})} = \omega$. So we need to replace $G'(\mathcal{R})$ with its nth reduct, for the appropriate n, and then code this as a successor opm.

Definition 3.16 (Operator $\mathcal{F}_G$)**.** Let G be a pre-operator over an operator background $\mathscr{B}$, with domain $D = \widehat{C^D} \cup P^D$. We define a corresponding operator $\mathcal{F} = \mathcal{F}_G$, also with domain D, as follows.

Let $X \in \widehat{C^D}$ and $\mathcal{N} = G(X) = (\lfloor\mathcal{N}\rfloor, \vec{P}^{\mathcal{N}}, P^{\mathcal{N}}, X)$. Let $n < \omega$ be least such that $\rho_{n+1}^{\mathcal{N}} = \omega$, so $\mathrm{Ord}^X < \sigma$ where $\sigma = \rho_n^{\mathcal{N}}$. If $n = 0$ then let $\mathcal{M} = \mathcal{N}$. If $n > 0$ then let $\mathcal{Q} = \mathcal{N}|^{\mathcal{J}}\sigma$ and let $\mathcal{M}$ be the op-$\mathcal{J}$-structure

$$\mathcal{M} = (\lfloor\mathcal{Q}\rfloor, \vec{P}^{\mathcal{N}}\restriction\sigma, T, X),$$

where $T \subseteq \lfloor\mathcal{Q}\rfloor$ codes $\mathrm{Th}_n^{\mathcal{N}}(\lfloor\mathcal{Q}\rfloor \cup \vec{p}_n^{\mathcal{N}})$ in some uniform fashion, amenably to $\lfloor\mathcal{Q}\rfloor$, such as with mastercodes.[38] Note that in either case, $\mathcal{M} = (\lfloor\mathcal{M}\rfloor, \vec{P}^{\mathcal{M}}, P^{\mathcal{M}}, X)$ is an ω-sound op-$\mathcal{J}$-structure over X and $\rho_1^{\mathcal{M}} = \omega$. Now define $\mathcal{F}(X)$ as the hierarchical model $\mathcal{K}$ over X, of length 1, with $\lfloor\mathcal{K}\rfloor = \lfloor\mathcal{M}\rfloor$, $E^{\mathcal{K}} = \emptyset = cp^{\mathcal{K}}$,[39] and $P^{\mathcal{K}} = \{X\} \times (\vec{P}^{\mathcal{M}} \oplus P^{\mathcal{M}})$. (We use $\{X\} \times \cdots$ to ensure that $P^{\mathcal{K}} \subseteq \mathcal{K}\backslash\mathcal{K}^-$. Recall that $\mathcal{K}$ having length 1 requires that $S^{\mathcal{K}} = \emptyset$.)

Now let $\mathcal{R} \in P^D$; we define $\mathcal{F}(\mathcal{R})$. Let $A = cb^{\mathcal{R}}$ and $\rho = \rho_\omega^{\mathcal{R}}$. Let $\mathcal{P} = G(\mathcal{R})$. Let $\mathcal{N} \trianglelefteq \mathcal{P}$ be largest such if $\rho > \omega$ (so $\rho > \mathrm{rank}(A)$) then for all $\alpha < \rho$, we have $\mathcal{P}(A^{<\omega} \times \alpha^{<\omega})^{\mathcal{N}} = \mathcal{P}(A^{<\omega} \times \alpha^{<\omega})^{\mathcal{R}}$. (Such an $\mathcal{N}$ exists, since $\mathcal{J}(\mathcal{R}) = \mathcal{P}|^{\mathcal{J}}(\mathrm{Ord}^{\mathcal{R}} + \omega)$ satisfies the requirements, by choice of ρ. Note that if $\rho = \omega$

---

[37] Recall from 2.27 that $\rho_{n+1}^{\mathcal{M}} = \omega$ does not mean that there is a new subset of ω definable from parameters over $\mathcal{M}$, but just a new subset of $\omega \times (Y \cup \{Y\})^{<\omega}$.

[38] For concreteness, we take T to be the set of pairs (α, t') such that for some t, $(\vec{p}_n^{\mathcal{M}}, \alpha, t) \in T_n^{\mathcal{M}}$, and t' results from the theory t by replacing each instance of $\vec{p}_n^{\mathcal{M}}$ in statements in t with α, interpreted as a constant symbol; note that if $(\vec{p}_n^{\mathcal{M}}, \alpha, t) \in T_n^{\mathcal{M}}$ then α does not already occur as a parameter in t, and this substitution neither obscures nor creates information.

[39] A natural generalization of this definition would set $cp^{\mathcal{K}}$ to be some fixed non-empty object. For example, if one uses operators to define strategy mice, one might set $cp^{\mathcal{K}}$ to be the structure that the iteration strategy is for.

then $\mathcal{N} = \mathcal{P}$.) Let $n < \omega$ be least such that $\rho_{n+1}^{\mathcal{N}} = \omega$, so $\mathrm{Ord}^{\mathcal{R}} < \rho_n^{\mathcal{N}}$. Define $\mathcal{M}$ from $(\mathcal{N}, n)$ as in the definition of $\mathcal{F}(X)$ for $X \in \widehat{C^D}$, but with $cb^{\mathcal{M}} = \mathcal{R}$. Much as there, $\mathcal{M} = (\lfloor \mathcal{M} \rfloor, \vec{P}^{\mathcal{M}}, P^{\mathcal{M}}, \mathcal{R})$ is an ω-sound op-$\mathcal{J}$-structure over $\mathcal{R}$ and $\rho_1^{\mathcal{M}} = \omega$.

Now set $\mathcal{F}(\mathcal{R})$ to be the unique hierarchical model $\mathcal{K}$ of length $l(\mathcal{R})+1$ with $\lfloor \mathcal{K} \rfloor = \lfloor \mathcal{M} \rfloor$, $\mathcal{R} \triangleleft \mathcal{K}$ (so $S^{\mathcal{K}} = S^{\mathcal{R}} \frown \langle \mathcal{R} \rangle$), $E^{\mathcal{K}} = \emptyset$, and $P^{\mathcal{K}} = \{\mathcal{R}\} \times (\vec{P}^{\mathcal{M}} \oplus P^{\mathcal{M}})$. Let us also say that $\mathcal{F}(\mathcal{R})$ **projects early** if $\mathcal{N} \triangleleft \mathcal{P}$ (in this case, $\mathcal{F}(\mathcal{R})$ does not "reach" the full $\mathcal{P} = G(\mathcal{R})$, but just its initial segment $\mathcal{N}$). This completes the definition. $\dashv$

With notation as above, let $\mathcal{R} \in D$. Note that $\mathcal{F}(\mathcal{R})$ easily codes $G(\mathcal{R})$, *unless* $\mathcal{R} \in P^D$ and $\mathcal{F}(\mathcal{R})$ projects early. Let us verify that $\mathcal{F}_G$ is indeed an operator:

Lemma 3.17. *Let G be a pre-operator over an operator background $\mathscr{B}$, with domain D. Then $\mathcal{F}_G$ is an operator over $\mathscr{B}$. Moreover, for any $\mathcal{F}_G$-premouse $\mathcal{M}$ of length $\alpha + \omega$, for all sufficiently large $n < \omega$, $\mathcal{F}_G(\mathcal{M}|(\alpha + n))$ does not project early.*

Proof sketch. We first show that $\mathcal{F}_G$ is an operator. Let $\mathcal{F} = \mathcal{F}_G$ and $X \in D = \mathrm{dom}(\mathcal{F})$. We must verify that $\mathcal{M} = \mathcal{F}(X)$ is an opm. This follows from (i) the choice of $\lfloor \mathcal{F}(X) \rfloor$ (i.e. the choice of $\mathcal{N} \trianglelefteq G(X)$ in the definition of $\mathcal{F}(X)$, which gives, for example, projectum amenability for $\mathcal{F}(X)$), (ii) if $X \in P^D$ then X is an ω-sound opm (acceptability follows from this and projectum amenability), (iii) standard properties of $\mathcal{J}$-structures (for example, to establish stratification), and (iv) with $\mathcal{M}$ as in the definition $\mathcal{F}(X)$ (either in case $X \in \widehat{C^D}$ or in case $X \in P^D$), the fact that $\mathcal{M}$ is ω-sound and $\rho_1^{\mathcal{M}} = \omega$ (for sound projection).

Now let $\mathcal{M}$ be an $\mathcal{F}$-premouse of limit length $\alpha + \omega$. Then for all m,

$$\rho_\omega^{\mathcal{M}|(\alpha+m+1)} \leq \rho_\omega^{\mathcal{M}|(\alpha+m)},$$

because $\mathcal{M}|(\alpha + m + 1)$ is soundly projecting and $\mathcal{M}|(\alpha + m)$ is ω-sound. So if $n < \omega$ is such that $\rho_\omega^{\mathcal{M}|(\alpha+n)}$ is as small as possible, then $\mathcal{F}(\mathcal{M}|(\alpha + n))$ does not project early. $\square$

So any limit length $\mathcal{F}_G$-premouse $\mathcal{M}$ is "closed under G" in the sense that for $\in$-cofinally many $X \in \mathcal{M}$, we have $G(X) \in \mathcal{M}$.

We can now define mouse operators.

Definition 3.18. Let $\varphi \in \mathcal{L}_0$. Let $\mathscr{B}$ be an operator background. Suppose that for every transitive structure $x \in \mathscr{B}$ there is $\mathcal{M} \triangleleft \mathrm{Lp}(x)$ such that $\mathcal{M} \models \varphi$, and let $\mathcal{M}_x$ be the least such. Let $G_\varphi : \mathscr{B} \dashrightarrow \mathscr{B}$ be the pre-operator where for $x \in \mathscr{B}$ a transitive structure, $G_\varphi(\hat{x})$ is the op-$\mathcal{J}$-structure over $\hat{x}$ naturally coding $\mathcal{M}_x$, and for $x \in \mathscr{B}$ a $< \omega$-condensing ω-sound opm, $G_\varphi(x)$ is the op-$\mathcal{J}$-structure over x naturally coding $\mathcal{M}_x$. The **mouse operator** $\mathcal{F}_\varphi$ determined by φ is $\mathcal{F}_{G_\varphi}$. $\dashv$

Remark 3.19. For example, suppose that $\mathcal{M}_1^\#(X)$ is defined and fully iterable for all sets X. Then $X \mapsto \mathcal{M}_1^\#(X)$ is a pre-operator G_φ, for the obvious formula φ, and $\mathcal{F} = \mathcal{F}_\varphi$ the induced mouse operator. Let $\mathcal{M}$ be the least $\mathcal{F}$-premouse which models ZFC^-; so $\mathcal{M} \models$"Every set is countable", and letting $\eta = \mathrm{Ord}^{\mathcal{M}}$, $\rho_\omega^{\mathcal{M}|\alpha} = \omega$ for all $\alpha < \eta$, but $\rho_\omega^{\mathcal{M}} = \eta$. Note that $\mathcal{F}(\mathcal{M})$ projects early, and in fact $\rho_1^{\mathcal{J}(\mathcal{M})} = \omega$, so $\mathcal{F}(\mathcal{M})$ is $\mathcal{J}(\mathcal{M})$, reorganized as an $\mathcal{F}$-premouse. But for no $\alpha < \eta$ does $\mathcal{F}(\mathcal{M}|\alpha)$ project early (since $\rho_\omega^{\mathcal{M}|\alpha} = \omega$ already), so $\mathcal{F}(\mathcal{M}|\alpha)$ is equivalent to $\mathcal{M}_1^\#(\mathcal{M}|\alpha)$ for all $\alpha < \eta$.

There are only countably many mouse operators over a given $\mathscr{B}$, since each is determined by a formula φ. But by combining with real parameters (say specifying the base of a cone), we obtain uncountably many operators. Assuming AD in $L(\mathbb{R})$, such an operator can be used to witness the Σ_1 truths about reals in a given $\mathcal{J}_\alpha(\mathbb{R})$, and that operator is in $\mathcal{J}_{\alpha'}(\mathbb{R})$ with an α' very close to α.

3.5 Fine condensation

In this section, in Definition 3.25, we will define (almost) fine condensation. It will be the key property that ensures that copying constructions for iteration trees on $\mathcal{F}$-premice proceed in a desirable fashion; that is, if we have $\mathcal{F}$-premice $\mathcal{R}, \mathcal{S}$ and an embedding $\tau : \mathcal{R} \to \mathcal{S}$, and $\mathcal{U}$ on $\mathcal{S}$ is an $\mathcal{F}$-tree (that is, its models are $\mathcal{F}$-premice), and $\mathcal{U}$ is the copy of $\mathcal{T}$ under τ, then we would like to know that $\mathcal{T}$ is also an $\mathcal{F}$-tree. Of course, we will have the copy maps $\pi : \mathcal{M} \to \mathcal{N}$ from models $\mathcal{M}$ of $\mathcal{T}$ into models $\mathcal{N}$ of $\mathcal{U}$. (Almost) fine condensation will be applied to these copy maps, and this should allow us to conclude that $\mathcal{M}$ is an $\mathcal{F}$-pm. The property should also guarantee similar behaviour for realization maps replacing copy maps.

We will also want to apply (almost) fine condensation to maps $\pi : \mathcal{M} \to \mathcal{N}$ such as core embeddings, or hull embeddings which arise in the proof of solidity of the standard parameter, for example.

Before giving the definition, we will introduce some terminology allowing us to describe the kinds of embeddings $\pi : \mathcal{M} \to \mathcal{N}$ we want to consider.

The definition of $(z_{k+1}^{\mathcal{M}}, \zeta_{k+1}^{\mathcal{M}})$ below is a direct adaptation from [18, Definition 2.19]. The facts proved there about this notion generalize readily to the present setting, although that paper formally works below superstrong. See also [11, §3], where there is no superstrong restriction.

Definition 3.20. Let $\mathcal{M}$ be a k-sound opm. Let $\mathcal{D}$ be the class of pairs $(z, \zeta) \in [\mathrm{Ord}]^{<\omega} \times \mathrm{Ord}$ such that $\zeta \cap z = \emptyset$. For $x \in [\mathrm{Ord}]^{<\omega}$ let f_x be the decreasing enumeration of x. For $x = (z, \zeta) \in \mathcal{D}$ let $f_x = f_z {}^\frown \langle \zeta \rangle$. Order $\mathcal{D}$ by $x <^* y$ iff $f_x <_{\mathrm{lex}} f_y$, with $x <^* y$ if $f_x \subsetneq f_y$. Then $(z_{k+1}^{\mathcal{M}}, \zeta_{k+1}^{\mathcal{M}})$ denotes the $<^*$-least $(z, \zeta) \in \mathcal{D}$ such that

$$\mathrm{Th}_{k+1}^{\mathcal{M}}(cb^{\mathcal{M}} \cup z \cup \zeta) \notin \mathcal{M}.$$

The $(k+1)$**-solid-core** of $\mathcal{M}$ is

$$\mathfrak{S}_{k+1}(\mathcal{M}) = \mathrm{cHull}_{k+1}^{\mathcal{M}}(cb^{\mathcal{M}} \cup z_{k+1}^{\mathcal{M}} \cup \zeta_{k+1}^{\mathcal{M}}),$$

and the $(k+1)$**-solid-core map** $\sigma^{\mathcal{M}}_{k+1}$ is the uncollapse map. ⊣

If $\mathcal{M}$ is $(k+1)$-solid then $\mathfrak{S}_{k+1}(\mathcal{M}) = \mathfrak{C}_{k+1}(\mathcal{M})$ and $\sigma^{\mathcal{M}}_{k+1}$ is the core map. But we will need to consider the $(k+1)$-solid-core more generally, in the proof of $(k+1)$-solidity.

Iteration maps, along a portion of a branch which does not drop in model, and is at degree k, are *k-tight* embeddings (but *k-tight* is more general):

Definition 3.21. Let $k \leq \omega$, let $\mathcal{L}, \mathcal{M}$ be k-sound opms and $\sigma : \mathcal{L} \to \mathcal{M}$. We say that σ is k**-tight** iff there is $\lambda \in \mathrm{Ord}$ and a sequence $\langle \mathcal{L}_\alpha \rangle_{\alpha \leq \lambda}$ of opms such that $\mathcal{L} = \mathcal{L}_0$ and $\mathcal{M} = \mathcal{L}_\lambda$ and there is a sequence $\langle E_\alpha \rangle_{\alpha < \lambda}$ of extenders such that each E_α is weakly amenable to $\mathcal{L}_\alpha$, with $cb^{\mathcal{L}} < \mathrm{crit}(E_\alpha) < \rho_k^{\mathcal{L}_\alpha}$,

$$\mathcal{L}_{\alpha+1} = \mathrm{Ult}_k(\mathcal{L}_\alpha, E_\alpha),$$

and for limit η,

$$\mathcal{L}_\eta = \mathrm{dirlim}_{\alpha < \beta < \eta}(\mathcal{L}_\alpha, \mathcal{L}_\beta; j_{\alpha\beta})$$

where $j_{\alpha\beta} : \mathcal{L}_\alpha \to \mathcal{L}_\beta$ is the resulting ultrapower map, and $\sigma = j_{0\lambda}$. ⊣

Note that E_α is not required to be close to $\mathcal{L}_\alpha$.

Copy maps and realization maps between k-sound structures are often *k-factors*:

Definition 3.22. Let $k \leq \omega$ and $\mathcal{M}, \mathcal{N}$ be k-sound opms and p be transitive. Suppose that if $k < \omega$ then $\mathcal{M}$ is k-relevant.

We say that $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{N})$ is a k**-factor above** p iff π is a weak k-embedding above p, and if $k < \omega$ then there is a k-tight $\sigma : \mathfrak{C}_0(\mathcal{L}) \to \mathfrak{C}_0(\mathcal{M})$ such that

$$\pi \circ \sigma \circ \sigma^{\mathcal{L}}_{k+1} : \mathfrak{S}_{k+1}(\mathcal{L}) \to \mathfrak{C}_0(\mathcal{N})$$

is a near k-embedding, σ is above p, and $\mathcal{L}$ is k-relevant.

For an operator $\mathcal{F}$, a k-factor is $\mathcal{F}$**-rooted** iff either $k = \omega$ or we can take $\mathcal{L}$ to be an $\mathcal{F}$-premouse.

A k-factor is **good** iff $A =_{\mathrm{def}} cb^{\mathcal{M}} = cb^{\mathcal{N}}$ and π is above A. ⊣

An ω-factor above p is just an ω-embedding (i.e. fully elementary between ω-sound opms) above p. If $k < \omega$, then both σ and $\sigma^{\mathcal{L}}_{k+1}$, and therefore also $\sigma \circ \sigma^{\mathcal{L}}_{k+1}$, are k-good. Any near k-embedding $\pi : \mathcal{M} \to \mathcal{N}$ above p, between opms $\mathcal{M}, \mathcal{N}$, is a k-factor above p (if $k < \omega$, use $\mathcal{L} = \mathcal{M}$ and $\sigma = \mathrm{id}$), and if $\mathcal{M}$ is an $\mathcal{F}$-pm, then π is $\mathcal{F}$-rooted.

Definition 3.23. Let $\mathcal{C}$ be a successor opm and $\mathcal{M}$ a successor Q-opm with $\mathcal{C}^- = \mathcal{M}^-$. We say that $\mathcal{C}$ is a **universal hull** of $\mathcal{M}$ iff there is an above $\mathcal{C}^-$, 0-good embedding $\pi : \mathcal{C} \to \mathcal{M}$ and for every $x \in \mathcal{M}$, $\mathrm{Th}_1^{\mathcal{M}}(\mathcal{M}^- \cup \{x\})$ is $\mathrm{r}\utilde{\Sigma}_1^{\mathcal{C}}$ (after replacing x with a constant symbol). ⊣

Remark 3.24. We are now ready to define *(almost) fine condensation*. It is a variant of *condenses well* from [21] and [28]. As discussed in Remark 3.13, we need to modify that notion.

One issue that Remark 3.13 illustrates is the following: Given a Σ_1-elementary $\pi : \mathcal{M} \to \mathcal{N} = \mathcal{F}(\mathcal{N}^-)$, we should not always expect that $\mathcal{M} = \mathcal{F}(\mathcal{M}^-)$, even in the case of a mouse operator $\mathcal{F}$. However, for a mouse operator $\mathcal{F}$, the iterability of $\mathcal{N} = \mathcal{F}(\mathcal{N}^-)$ above $\mathcal{N}^-$ and the existence of π should ensure the iterability of $\mathcal{M}$ above $\mathcal{M}^-$. (Here the *iterability* we refer to is that of the ordinary mouse over $\mathcal{N}^-$ produced by $\mathcal{F}$; the *$\mathcal{F}$-iterability* of $\mathcal{N}$ above $\mathcal{N}^-$ is trivial, as $\mathbb{E}_+^{\mathcal{N}}$ is empty above $\mathrm{Ord}^{\mathcal{N}^-}$.) Secondly, the minimality of $\mathcal{N}$ above $\mathcal{N}^-$ should ensure that $\mathcal{M}$ does not strictly surpass $\mathcal{F}(\mathcal{M}^-)$. But Remark 3.13 indicates that $\mathcal{M}$ might not actually reach $\mathcal{F}(\mathcal{M}^-)$ in general, and for example, we might have $\mathcal{M} \in \mathcal{F}(\mathcal{M}^-)$. We might, for example, have that $\mathcal{M}$ is a proper segment of $\mathcal{F}(\mathcal{M}^-)$ in the hierarchy as an op-$\mathcal{J}$-structure, but there are also other possibilities.

On the other hand, we want (almost) fine condensation to hold under appropriate circumstances, and in particular, we want mouse operators to almost condense finely. So it is allowed that $\mathcal{M} \in \mathcal{F}(\mathcal{M}^-)$ in (one case of) the definition of *condenses finely*.

In Proposition 3.28, we will show that mouse operators do almost condense finely, and the proof will help to illuminate key details of the definition.

Definition 3.25. Let $\mathcal{F}$ be an operator over $\mathscr{B}$ with domain D. Suppose that C^D is the cone above some transitive $p \in \mathscr{B}$. We say that $\mathcal{F}$ **condenses finely above** p (or $\mathcal{F}$ **has fine condensation above** p) iff (i) $\mathcal{F}$ condenses coarsely above p; and (ii) Let $A, \bar{A}, \mathcal{N}, \mathcal{L} \in V$ and let $\mathcal{M}, \varphi, \sigma \in V[G]$ where G is set-generic over V. Let $k < \omega$. Suppose that:

- $p \in \mathcal{J}_1(\bar{A}) \cap \mathcal{J}_1(A)$,
- $\mathcal{L}$ is a k-sound opm over $\bar{A}$ and $\mathcal{N}$ is a k-sound opm over A,
- $\mathcal{M}$ is a Q-opm over $\bar{A}$ and if $k > 0$ then $\mathcal{M}$ is a k-sound opm,
- $\mathcal{L}, \mathcal{M}, \mathcal{N}$ each have successor length,
- $\mathcal{M}^- \in V$ and $\mathcal{L}, \mathcal{M}^-, \mathcal{N}$ are $\mathcal{F}$-premice and $\mathcal{M}^- \in \mathrm{dom}(\mathcal{F})$ (so $\mathcal{F}(\mathcal{M}^-)$ is an opm), and
- $\varphi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{N})$.

Then:

1. If $\mathcal{M}$ is a k-sound opm and either
    - φ is k-good, or
    - $\mathcal{M}$ is k-relevant and $V[G] \models$"φ is a k-factor above p, as witnessed by $(\mathcal{L}, \sigma)$",

   then either $\mathcal{M} \in \mathcal{F}(\mathcal{M}^-)$ or $\mathcal{M} = \mathcal{F}(\mathcal{M}^-)$.
2. If $k = 0$ and $\rho_1^{\mathcal{M}} \leq \mathrm{Ord}^{\mathcal{M}^-}$ and φ is 0-good (hence above p), then there is a universal hull $\mathcal{H}$ of $\mathcal{M}$ such that either $\mathcal{H} \in \mathcal{F}(\mathcal{M}^-)$ or $\mathcal{H} = \mathcal{F}(\mathcal{M}^-)$.

We say $\mathcal{F}$ **almost condenses finely above** p iff $\mathcal{F}$ almost condenses coarsely above p and condition (ii) above holds for $G = \emptyset$. $\dashv$

Recall that if $\mathcal{M}$ is a successor opm then $\rho_1^{\mathcal{M}} \leq \mathrm{Ord}^{\mathcal{M}^-}$. So in both parts 1 and 2 above, we have $\rho_1^{\mathcal{M}} \leq \mathrm{Ord}^{\mathcal{M}^-}$, but in part 2, $\mathcal{M}$ need not be an opm (although it is a Q-opm). Also note that there are cases of *condenses finely* in which we do not assume that $\mathcal{M}$ is k-relevant, though in these, φ is k-good.

Let us observe that in certain key circumstances, we can rule out the possibility that $\mathcal{M} \in \mathcal{F}(\mathcal{M}^-)$, and so the conclusion of fine condensation sharpens to $\mathcal{M} = \mathcal{F}(\mathcal{M}^-)$:

Lemma 3.26. *Let $\mathcal{F}$ be an operator. Let $k < \omega$. Let $\mathcal{N}$ be an $\mathcal{F}$-pm which is a k-sound successor opm. Let $\mathcal{M}$ be a successor Q-opm. Suppose $\mathcal{M}^-$ is an $\mathcal{F}$-pm in $\mathrm{dom}(\mathcal{F})$. If $k > 0$ then suppose also that $\mathcal{M}$ is a k-sound opm. Suppose that $\mathcal{M} = \mathfrak{C}_{k+1}(\mathcal{N})$ or $\mathcal{M}$ is k-relevant. Then $\mathcal{M} \notin \mathcal{F}(\mathcal{M}^-)$, and if $k = 0$ then there is no universal hull of $\mathcal{M}$ in $\mathcal{F}(\mathcal{M}^-)$.*

Proof. Suppose otherwise. Suppose first that $k > 0$, so $\mathcal{M} \in \mathcal{F}(\mathcal{M}^-)$. Then by projectum amenability for $\mathcal{F}(\mathcal{M}^-)$, $\mathcal{M}$ is not k-relevant. So $\mathcal{M} = \mathfrak{C}_{k+1}(\mathcal{N}) \notin \mathcal{N}$. Let $\pi : \mathcal{M} \to \mathcal{N}$ be the core map. By Lemmas 2.43 and 2.38, π is k-good, so $\pi(\mathcal{M}^-) = \mathcal{N}^-$. So since $\mathcal{M}$ is not k-relevant, $\rho_{k+1}^{\mathcal{M}} = \rho_k^{\mathcal{M}} = \rho_\omega^{\mathcal{M}^-}$, but then $\rho_{k+1}^{\mathcal{N}} = \rho_k^{\mathcal{N}}$, so $\mathcal{N}$ is $(k+1)$-sound and $\mathcal{M} = \mathcal{N} = \mathcal{F}(\mathcal{N}^-) = \mathcal{F}(\mathcal{M}^-)$, contradicting the assumption that $\mathcal{M} \in \mathcal{F}(\mathcal{M}^-)$.

So $k = 0$. (So we do not assume $\mathcal{M}$ is an opm, but it is a Q-opm.) Again by projectum amenability, $\mathcal{M}$ is not 0-relevant, so $\mathcal{M} = \mathfrak{C}_1(\mathcal{N}) \notin \mathcal{N}$. Let $\pi : \mathcal{M} \to \mathcal{N}$ be the core map. Then π is 0-good, so $\pi(\mathcal{M}^-) = \mathcal{N}^-$. So $\rho_1^{\mathcal{M}} = \rho_\omega^{\mathcal{M}^-} < \rho_0^{\mathcal{M}} = \mathrm{Ord}^{\mathcal{M}}$, and $\rho_1^{\mathcal{M}} = \rho_1^{\mathcal{N}} = \rho_\omega^{\mathcal{N}^-}$. But then since $\mathcal{N}$ is an opm and by Lemma 2.41, $\mathcal{N}$ is 1-sound, so $\mathcal{M} = \mathcal{N}$, again a contradiction. $\square$

So under the circumstances of the lemma above, if $\mathcal{M}$ is an opm, fine condensation gives the stronger conclusion that $\mathcal{M} = \mathcal{F}(\mathcal{M}^-)$. But we will need to apply fine condensation more generally, such as in the proof of solidity. Analogously to Lemma 3.12, we have:

Lemma 3.27. *Let $\mathcal{F}$ be a total operator over $\mathscr{B}$, with domain D. Suppose C^D is the cone above some transitive $p \in \mathrm{HC}$, and that $\mathcal{F}$ almost condenses finely above p. Then $\mathcal{F}$ condenses finely above p.*

Proposition 3.28. *Let $\mathcal{F}_\varphi$ be a mouse operator, as in Definition 3.18. Then $\mathcal{F}_\varphi$ almost condenses finely.*

Proof sketch. We just discuss the proof in one case, which illustrates the main points and should clarify why *almost condenses finely* is formulated as it is. Let $\mathcal{F} = \mathcal{F}_\varphi$ and let $\mathcal{N}$ be a successor $\mathcal{F}$-pm. Let $\mathcal{M}$ be a successor Q-opm with $\rho_1^{\mathcal{M}} \leq \mathrm{Ord}^{\mathcal{M}^-}$ and let $\pi : \mathcal{M} \to \mathcal{N}$ be a 0-embedding. Then we want to verify clause 2 of almost fine condensation holds with respect to $\mathcal{M}, \mathcal{N}, \pi$. So we need to see that there is a universal hull $\mathcal{H}$ of $\mathcal{M}$ such that either $\mathcal{H} \in \mathcal{F}(\mathcal{M}^-)$ or

$\mathcal{H} = \mathcal{F}(\mathcal{M}^-)$. (Clause 2 also assumes that π is 0-good, but that isn't needed here.) Note that although $\mathcal{M}$ is a Q-opm, we do not assume it is an opm.

We have $\pi(\mathcal{M}^-) = \mathcal{N}^-$. Let $\mathcal{N}^* \triangleleft \mathrm{Lp}(\mathcal{N}^-)$ be the premouse over $\mathcal{N}^-$ coded by $\mathcal{N}$. (That is, either $\rho_1^{\mathcal{N}^*} = \omega$ and $\mathrm{Ord}^{\mathcal{N}} = \mathrm{Ord}^{\mathcal{N}^*}$ and $P^{\mathcal{N}}$ encodes $\mathbb{E}_+^{\mathcal{N}^*}$ directly, or for some n such that $0 < n < \omega$, $\rho_{n+1}^{\mathcal{N}^*} = \omega < \mathrm{Ord}^{\mathcal{N}^-} < \rho_n^{\mathcal{N}^*}$ and $\mathrm{Ord}^{\mathcal{N}} = \rho_n^{\mathcal{N}^*}$ and $P^{\mathcal{N}}$ encodes $\mathrm{Th}_n^{\mathcal{N}^*}((\mathcal{N}^*|\rho_n^{\mathcal{N}^*}) \cup \{\vec{p}_n^{\mathcal{N}^*}\})$. Moreover, $\mathcal{N}^*$ has no proper segment satisfying φ, and either $\mathcal{N}^* \vDash \varphi$ or $\rho_\omega^{\mathcal{N}^-} > \omega$ and $\mathcal{N}$ projects $< \rho_\omega^{\mathcal{N}^-}$.) Let $n < \omega$ be such that $\rho_{n+1}^{\mathcal{N}^*} = \omega < \mathrm{Ord}^{\mathcal{N}^-} < \rho_n^{\mathcal{N}^*}$. By downward extension of embeddings, $\mathcal{M}$ encodes an n-sound premouse $\mathcal{M}^*$ over $\mathcal{M}^-$, and π determines an n-embedding $\pi^* : \mathcal{M}^* \to \mathcal{N}^*$ with $\pi \subseteq \pi^*$. Because $\rho_1^{\mathcal{M}} \leq \mathrm{Ord}^{\mathcal{M}^-}$, $\rho_{n+1}^{\mathcal{M}^*} \leq \mathrm{Ord}^{\mathcal{M}^-}$.

Now suppose $\mathcal{M}^*$ is $(n+1)$-sound. Then $\mathcal{M}^*$ is fully sound, as $\mathcal{M}^*$ is a premouse over $\mathcal{M}^-$, so $\mathcal{M}^* \triangleleft \mathrm{Lp}(\mathcal{M}^-)$. Moreover, $\mathcal{M}^* \trianglelefteq \mathcal{M}'$, where $\mathcal{M}'$ is the premouse over $\mathcal{M}^-$ coded by $\mathcal{F}(\mathcal{M}^-)$. For by the Σ_1-elementarity of π^*, $\mathcal{M}^*$ has no proper segment modelling φ, and if $\mathcal{M}$ has length > 0 and $\omega < \rho_\omega^{\mathcal{M}^-}$ then letting $A = \dot{c}b^{\mathcal{M}^-}$, for $\alpha < \rho_\omega^{\mathcal{M}^-}$, we have $\mathcal{P}(\alpha^{<\omega} \times A^{<\omega}) \cap \mathcal{M}^* \subseteq \mathcal{M}^-$.

Now suppose instead that $\mathcal{M}^*$ is not $(n+1)$-sound. Let $\mathcal{H}^* = \mathfrak{C}_{n+1}(\mathcal{M}^*)$. Then $\mathcal{H}^* \trianglelefteq \mathcal{M}'$, where $\mathcal{M}'$ is as before, and the n^{th} master code $\mathcal{H}$ of $\mathcal{H}^*$ is a universal hull of $\mathcal{M}$, and either $\mathcal{H} \in \mathcal{F}(\mathcal{M}^-)$ or $\mathcal{H} = \mathcal{F}(\mathcal{M}^-)$, as required. □

Note we makes significant use here of the assumption that $\rho_1^{\mathcal{M}} \leq \mathrm{Ord}(\mathcal{M}^-)$.

3.6 Copying and realization

We next want to consider the copying construction and how it relates to operators $\mathcal{F}$ with fine condensation. As discussed in [25], [15] and [14], even for standard mice, the copying construction is complicated by type 3 premice $\mathcal{M}$, because one must handle segments $\mathcal{N} \triangleleft \mathcal{M}$ such that $\mathcal{N} \ntrianglelefteq \mathfrak{C}_0(\mathcal{M})$, but the fine structural maps $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{R})$ only act directly on $\mathfrak{C}_0(\mathcal{M})$. We first make some preparations in this regard. The following notions are from [15] and [11]:

Definition 3.29. Let $\mathcal{M}$ be an opm. If $\mathcal{M}$ is not type 3 then $\mathcal{M}^\uparrow$ denotes $\mathcal{M}$. If $\mathcal{M}$ is type 3 and $\kappa = \mu^{\mathcal{M}}$ then $\mathcal{M}^\uparrow$ dentes $\mathrm{Ult}_0(\mathcal{M}|\kappa^{+\mathcal{M}}, F^{\mathcal{M}})$.

For $\pi : \mathcal{M} \to \mathcal{N}$, a Σ_0-elementary embedding between opms of the same type, we define $\mathrm{Shift}(\pi) : \mathcal{M}^\uparrow \to \mathcal{N}^\uparrow$ as follows. If $\mathcal{M}$ is not type 3 then $\mathrm{Shift}(\pi) = \pi$. If $\mathcal{M}$ is type 3 then $\mathrm{Shift}(\pi)$ is the embedding induced via the Shift Lemma by π.

If $\mathcal{M}$ is not type 3, we say that π is *ν-preserving*, not *ν-high* and not *ν-low*. Suppose $\mathcal{M}$ is type 3. Then we say that π is *ν-preserving* iff $\mathrm{Shift}(\pi)(\nu(F^{\mathcal{M}})) = \nu(F^{\mathcal{N}})$, *$\nu$-high* iff $\mathrm{Shift}(\pi)(\nu(F^{\mathcal{M}})) > \nu(F^{\mathcal{N}})$, and *$\nu$-low* iff $\mathrm{Shift}(\pi)(\nu(F^{\mathcal{M}})) < \nu(F^{\mathcal{N}})$. ⊣

Remark 3.30. Elementarity considerations show that if π is $\mathrm{r}\Sigma_1$-elementary, it is not ν-low, and if π is $\mathrm{r}\Sigma_2$-elementary, then it is ν-preserving; see [15].

Lemma 3.31. *Let $\mathcal{F}$ be an operator above b which almost condenses coarsely above b. Let $A \in \widehat{C_{\mathcal{F}}}$. Let $\mathcal{N}$ be a type 3 $\mathcal{F}$-pm over A such that $\mathcal{N}^{\uparrow}$ is an $\mathcal{F}$-pm. Let $\pi : \mathcal{R} \to \mathfrak{C}_0(\mathcal{N})$ be a weak 0-embedding which is above b. Then $\mathcal{R} = \mathfrak{C}_0(\mathcal{M})$ for some $\mathcal{F}$-pm $\mathcal{M}$.*

Proof. Because π is a weak 0-embedding, $E = E^{\mathcal{R}}$ is an extender over $\mathcal{R}$. So we can define $\mathcal{R}^{\uparrow}$ and $\mathrm{Shift}(\pi) : \mathcal{R}^{\uparrow} \to \mathcal{N}^{\uparrow}$ as in 3.29. By almost coarse condensation, $\mathcal{R}^{\uparrow}$ is an $\mathcal{F}$-pm, which yields the desired conclusion. $\square$

Given an iteration tree $\mathcal{U}$ on an opm $\mathcal{M}$, and given $\mathcal{N} \trianglelefteq \mathcal{M}$, the next definition sets up notation for embeddings on $\mathcal{N}$ induced by the iteration maps of $\mathcal{U}$.

Definition 3.32. Let $\mathcal{U}$ be a k-maximal tree on an opm $\mathcal{M}$, and let $\mathcal{N} \trianglelefteq \mathcal{M}$. If $\mathcal{N} \triangleleft \mathcal{M}$ then let $\langle \mathcal{N}_i \rangle_{i<k}$ be the model dropdown sequence of $\mathcal{N}$ in $\mathcal{M}$. (That is, $\mathcal{N}_0 = \mathcal{N}$, for each $i+1 < k$, $\mathcal{N}_{i+1}$ is the least $\mathcal{N}' \triangleleft \mathcal{M}$ such that $\rho_\omega^{\mathcal{N}'} < \rho_\omega^{\mathcal{N}_i}$, and $\rho_\omega^{\mathcal{N}_{k-1}}$ is an $\mathcal{M}$-cardinal.) We say that $\mathcal{N}$ is *$\mathcal{M}$-stable* iff:

- If $\mathcal{M}$ is active type 3 then either $\mathcal{N} = \mathcal{M}$ or $\mathcal{N} \triangleleft \mathcal{M}^{\mathrm{sq}}$ (hence $\mathcal{N}_k \triangleleft \mathcal{M}^{\mathrm{sq}}$).
- If $\mathcal{N} \triangleleft \mathcal{M}$ then for all $i+1 < k$, if $\mathcal{N}_{i+1}$ is active type 3 then $\mathcal{N}_i \triangleleft (\mathcal{N}_{i+1})^{\mathrm{sq}}$.

Suppose $\mathcal{N}$ is $\mathcal{M}$-stable. Let $\alpha < \mathrm{lh}(\mathcal{U})$. Let us say for the moment that $(\mathcal{U}, \alpha)$ is *good* iff either

1. $\mathcal{N} = \mathcal{M}$ and $[0, \alpha]^{\mathcal{U}} \cap \mathscr{D}^{\mathcal{U}} = \emptyset$, or
2. there are ordinals $\gamma_0 \leq^{\mathcal{U}} \delta_0 = \gamma_1 \leq^{\mathcal{U}} \delta_1 = \gamma_2 \leq^{\mathcal{U}} \delta_2 \ldots \delta_{k-1} = \gamma_k \leq^{\mathcal{U}} \delta_k = \alpha$ such that:
    (a) $\gamma_0 = 0$ and $[\gamma_0, \delta_0]^{\mathcal{U}} \cap \mathscr{D}^{\mathcal{U}} = \emptyset$, and
    (b) for each $i \in (0, k]$, if $\gamma_i < \delta_i$ then:
        i. $(\gamma_i, \delta_i]^{\mathcal{U}} \cap \mathscr{D}^{\mathcal{U}} = \{\varepsilon_i\}$ where $\mathrm{pred}^{\mathcal{U}}(\varepsilon_i) = \gamma_i$ (and $\varepsilon_i \leq^{\mathcal{U}} \delta_i$),
        ii. $\mathcal{N}_{k-i} \in \mathrm{dom}(j)$ where $j = i^{*\mathcal{U}}_{\varepsilon_{i-1}\delta_{i-1}} \circ i^{*\mathcal{U}}_{\varepsilon_{i-2}\delta_{i-2}} \circ \ldots \circ i^{*\mathcal{U}}_{\varepsilon_1\delta_1} \circ i^{\mathcal{U}}_{0\delta_0}$, and $M^{*\mathcal{U}}_{\varepsilon_i} = j(\mathcal{N}_{k-i})$.

If $(\mathcal{U}, \alpha)$ is good then we define $M^{\mathcal{U}}_{\mathcal{N},\alpha} \trianglelefteq M^{\mathcal{U}}_\alpha$ and

$$i^{\mathcal{U}}_{\mathcal{N},0\alpha} : \mathfrak{C}_0(\mathcal{N}) \to \mathfrak{C}_0(M^{\mathcal{U}}_{\mathcal{N},\alpha})$$

as follows. Let γ_k, δ_k be as above. If $\gamma_k < \delta_k$ then set $M^{\mathcal{U}}_{\mathcal{N},\alpha} = M^{\mathcal{U}}_\alpha$ and $i^{\mathcal{U}}_{\mathcal{N},0\alpha} = i^{*\mathcal{U}}_{\varepsilon_k\delta_k} \circ \ldots \circ i^{*\mathcal{U}}_{\varepsilon_1\delta_1} \circ i^{\mathcal{U}}_{0\delta_0}$. If $\gamma_k = \delta_k$ then set $M^{\mathcal{U}}_{\mathcal{N},\alpha} = j(\mathcal{N})$, where $j = i^{*\mathcal{U}}_{\varepsilon_{k-1}\delta_{k-1}} \circ \ldots \circ i^{*\mathcal{U}}_{\varepsilon_1\delta_1} \circ i^{\mathcal{U}}_{0\delta_0}$, and set $i^{\mathcal{U}}_{\mathcal{N},0\alpha} = j{\upharpoonright}\mathfrak{C}_0(\mathcal{N})$.

Now we say that $[0, \beta]^{\mathcal{U}}$ is *$\mathcal{N}$-bounded* iff there is $\alpha \leq^{\mathcal{U}} \beta$ such that $[0, \alpha]^{\mathcal{U}}$ is good and if $\alpha <^{\mathcal{U}} \beta$ then letting $\varepsilon \leq^{\mathcal{U}} \beta$ be such that $\mathrm{pred}^{\mathcal{U}}(\varepsilon) = \alpha$, we have $M^{*\mathcal{U}}_\varepsilon \triangleleft M^{\mathcal{U}}_{\mathcal{N},\alpha}$. If $[0, \beta]^{\mathcal{U}}$ is $\mathcal{N}$-bounded, as witnessed by α, we say that $[0, \beta]^{\mathcal{U}}$ **drops below the image of $\mathcal{N}$** iff β is not good; that is, $\alpha < \beta$.

We say that $\mathcal{U}$ is *$\mathcal{N}$-bounded* iff β is $\mathcal{N}$-bounded for all $\beta < \mathrm{lh}(\mathcal{U})$. $\dashv$

The following lemma is an instance of some very related material in [18, §2] (for example, [18, Lemma 2.27]), [11, §6], [15, §7] and the preprint [13, §5]. It shows that when we define an embedding $\tau : \mathrm{Ult}_k(\mathcal{R}, F^{\mathcal{M}}) \to \mathrm{Ult}_k(\mathcal{S}, F^{\mathcal{N}})$ via the Shift Lemma from a given embedding $\pi : \mathfrak{C}_0(\mathcal{R}) \to \mathfrak{C}_0(\mathcal{S})$, if π is ν-preserving, so is τ.

Lemma 3.33. *Let $\mathcal{R}, \mathcal{S}$ be type 3 k-sound opms, where $k < \omega$, and $\pi : \mathfrak{C}_0(\mathcal{R}) \to \mathfrak{C}_0(\mathcal{S})$ a ν-preserving weak k-embedding. Let $\mathcal{M}, \mathcal{N}$ be active $\mathcal{F}$-premice and $\psi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{N})$ a weak 0-embedding. Let $\kappa = \mathrm{crit}(F^{\mathcal{M}})$. Suppose $\mathcal{R}||\kappa^{+\mathcal{R}} = \mathcal{M}|\kappa^{+\mathcal{M}}$ and $\mathcal{S}||\psi(\kappa)^{+\mathcal{S}} = \mathcal{N}|\psi(\kappa)^{+\mathcal{N}}$ and $\pi{\restriction}(\mathcal{R}||\kappa^{+\mathcal{R}}) = \psi{\restriction}(\mathcal{M}|\kappa^{+\mathcal{M}})$. Suppose $\kappa < \rho_k^{\mathcal{R}}$. Suppose π is ν-preserving. Let*

$$\tau : \mathrm{Ult}_k(\mathcal{R}, F^{\mathcal{M}}) \to \mathrm{Ult}_k(\mathcal{S}, F^{\mathcal{N}})$$

by the Shift Lemma map induced by π, ψ. Then τ is ν-preserving.

Proof. This kind of argument is given, for example, in [11, §6.1]. If the ultrapower maps $i^{\mathcal{R},k}_{F^{\mathcal{M}}}$ and $i^{\mathcal{S},k}_{F^{\mathcal{N}}}$ are both ν-preserving, commutativity ($\tau \circ i^{\mathcal{R},k}_{F^{\mathcal{M}}} = i^{\mathcal{S},k}_{F^{\mathcal{N}}} \circ \pi$) gives the desired result. If $k > 0$ then these ultrapower maps are indeed ν-preserving, by Remark 3.30).

So suppose $k = 0$.

Let $\mu = \mathrm{crit}(F^{\mathcal{R}})$. We have $R^{\uparrow} = \mathrm{Ult}_0(\mathcal{R}|\mu^{+\mathcal{R}}, F^{\mathcal{R}})$. Note that $F^{\mathcal{M}}$ is also an extender over $\mathcal{R}^{\uparrow}$. Let $U' = \mathrm{Ult}_0(\mathcal{R}^{\uparrow}, F^{\mathcal{M}})$ and let $j' : \mathcal{R}^{\uparrow} \to U'$ be the ultrapower map. Then a standard calculation shows that

$$U' = U^{\uparrow} = \mathrm{Ult}_0(U|j(\mu)^{+U}, F^U)$$

and $j = j'{\restriction}\mathfrak{C}_0(\mathcal{R})$, and $j' \circ i^{\mathcal{R}|\mu^{+\mathcal{R}},0}_{F^{\mathcal{R}}} = i^{U|j(\mu)^{+\mathcal{R}},0}_{F^U} \circ j$.

Now let $a, f \in \mathfrak{C}_0(\mathcal{R})$ be such that $\nu(F^{\mathcal{R}}) = [a, f]^{\mathcal{R},0}_{F^{\mathcal{R}}}$. Then we may assume $f \in \mathcal{R}|\mu^{+\mathcal{R}}$ and we have $\nu(F^{\mathcal{R}}) = [a, f]^{\mathcal{R}|\mu^{+\mathcal{R}},0}_{F^{\mathcal{R}}} = i^{\mathcal{R}|\mu^{+\mathcal{R}},0}_{F^{\mathcal{R}}}(f)(a)$. So

$$\begin{aligned}
j'(\nu(F^{\mathcal{R}})) &= j'(i^{\mathcal{R}|\mu^{+\mathcal{R}},0}_{F^{\mathcal{R}}}(f)(a)) \\
&= j'(i^{\mathcal{R}|\mu^{+\mathcal{R}},0}_{F^{\mathcal{R}}}(f))(j'(a)) \\
&= i^{U|j(\mu)^{+U},0}_{F^U}(j(f))(j(a)) \\
&= [j(a), j(f)]^{U|j(\mu)^{+U},0}_{F^U} = [j(a), j(f)]^{U,0}_{F^U}.
\end{aligned} \tag{3.1}$$

Now $\nu(F^U) = \sup j``\nu(F^{\mathcal{R}})$. So by line (3.1), if j' is continuous at $\nu(F^{\mathcal{R}})$ then j is ν-preserving, and if j' is discontinuous at $\nu(F^{\mathcal{R}})$ then j is ν-high.

But j' is continuous at $\nu(F^{\mathcal{R}})$ iff $\mathrm{cof}^{\mathcal{R}}(\nu(F^{\mathcal{R}})) \neq \kappa$. And $\mathrm{cof}^{\mathcal{R}}(\nu(F^{\mathcal{R}})) = \kappa$ iff $\mathrm{cof}^{\mathcal{S}}(\nu(F^{\mathcal{S}})) = \pi(\kappa) = \psi(\kappa)$, since π is ν-preserving. So if $\mathrm{cof}^{\mathcal{R}}(\nu(F^{\mathcal{R}})) \neq \kappa$, then j is ν-preserving, and similarly, so is $i^{\mathcal{S},0}_{F^{\mathcal{N}}}$, and so as remarked earlier, it follows that τ is also ν-preserving, as desired.

So it just remains to consider the case that $\mathrm{cof}^{\mathcal{R}}(\nu(F^{\mathcal{R}})) = \kappa$ (so j' is discontinuous at $\nu(F^{\mathcal{R}})$), and so $\mathrm{cof}^{\mathcal{S}}(F^{\mathcal{S}}) = \pi(\kappa)$ (and $i^{\mathcal{S},0}_{F^{\mathcal{N}}}$ is discontinuous at

$\nu(F^{\mathcal{S}})$). Let $f \in \mathcal{R}$ be such that $f : \kappa \to \nu(F^{\mathcal{R}})$ is continuous, strictly increasing and $\sup f``\kappa = \nu(F^{\mathcal{R}})$. Then note that since $\nu(F^U) = \sup j``\nu(F^{\mathcal{R}})$ and $\kappa = \mathrm{crit}(j) = \mathrm{crit}(j')$, we have $j'(f){\restriction}\kappa = j \circ f : \kappa \to \nu(F^U)$ and $j'(f){\restriction}\kappa$ is continuous, strictly increasing and $\sup j'(f)``\kappa = \nu(F^U)$. Now let $g : [\mu]^{<\omega} \to \mathcal{R}|\mu^{+\mathcal{R}}$ and $a \in [\nu(F^{\mathcal{R}})]^{<\omega}$ be such that $f = i^{\mathcal{R}|\mu^{+\mathcal{R}},0}_{F^{\mathcal{R}}}(g)(a)$ and $\kappa \in a$. Say $\kappa = \alpha_i$ where $a = \{\alpha_0, \ldots, \alpha_{k-1}\}$ and $\alpha_0 < \ldots < \alpha_{k-1}$. Then define $g' : [\mu]^k \to \mathcal{R}|\mu^{+\mathcal{R}}$ to be $g'(u) = \sup g(u)``u_i$, where $u = \{u_0, \ldots, u_{k-1}\}$ and $u_0 < \ldots < u_{k-1}$. Then note that $\nu(F^{\mathcal{R}}) = [a, g']^{\mathcal{R},0}_{F^{\mathcal{R}}}$. Let $h' \in U$ be the function $h' : [j(\mu)]^{k+1} \to U|j(\mu)^{+U}$ with

$$h'(u) = \sup(j(g')(u \backslash \{u_i\}))``u_i,$$

where $u = \{u_0, \ldots, u_k\}$ and $u_0 < \ldots < u_k$. Note here that $j(g')$ has domain $[j(\mu)]^k$, whereas h' has domain $[j(\mu)]^{k+1}$, and if $u = \{u_0, \ldots, u_k\}$ as above, then

$$h'(u) = \sup(j(g')(\{u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k\}))``u_i.$$

Then $\nu(F^U) = [j(a) \cup \{\kappa\}, h']^{U,0}_{F^U}$. For note that $|a| = k$ and κ is the ith element of a, but $j(a) \cap [\kappa, j(\kappa)) = \emptyset$, and $j(\kappa)$ is the ith element of $j(a)$, so κ is the ith element of $j(a) \cup \{\kappa\}$.

Since π is ν-preserving, $\nu(F^{\mathcal{S}}) = [\pi(a), \pi(g')]^{\mathcal{S},0}_{F^{\mathcal{S}}}$ and $[\pi(a), \pi(g)]^{\mathcal{S},0}_{F^{\mathcal{S}}}$ is a function $f^* : \pi(\kappa) \to \nu(F^{\mathcal{S}})$ which is continuous and strictly increasing, and $\nu(F^{\mathcal{S}}) = \sup f^*``\pi(\kappa)$. So it now easily follows that $\nu(F^{\mathcal{S}}) = [\tau(j(a)) \cup \{\tau(\kappa)\}, \tau(h')]^{U^*,0}_{F^{U^*}}$, where $U^* = \mathrm{Ult}_0(\mathcal{S}, F^{\mathcal{N}})$, so τ is ν-preserving, as desired. $\square$

We now verify that fine condensation for $\mathcal{F}$ ensures that the copying construction proceeds smoothly for relevant $\mathcal{F}$-premice. The indexing function ι in the following lemma need not be the identity, because of the possibility of ν-high copy embeddings $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{N})$ between type 3 premice $\mathcal{M}, \mathcal{N}$.

Lemma 3.34. *Let $\mathcal{F}$ be an operator above b which almost condenses finely above b. Let $\bar{A}, A \in \widehat{C_{\mathcal{F}}}$. Let $j \leq \omega$ and let $\mathcal{Q}$ be a j-sound $\mathcal{F}$-premouse over A. Let $(\mathcal{N}, k) \trianglelefteq (\mathcal{Q}, j)$ be such that $\mathcal{N}$ is $\mathcal{Q}$-stable* (see 3.32). *Let $\mathcal{M}$ be a k-relevant $\mathcal{F}$-pm over $\bar{A}$. Let $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{N})$ be an $\mathcal{F}$-rooted k-factor above b. Let $\Sigma_{\mathcal{Q}}$ be an $\mathcal{F}$-$(j, \omega_1 + 1)$-strategy for $\mathcal{Q}$. Then there is an $\mathcal{F}$-$(k, \omega_1 + 1)$-strategy $\Sigma_{\mathcal{M}}$ for $\mathcal{M}$ such that trees $\mathcal{T}$ via $\Sigma_{\mathcal{M}}$ lift to trees $\mathcal{U}$ via $\Sigma_{\mathcal{Q}}$. In fact, for such pairs $(\mathcal{T}, \mathcal{U})$, there is $\iota : \mathrm{lh}(\mathcal{T}) \to \mathrm{lh}(\mathcal{U})$ such that for each $\alpha < \mathrm{lh}(\mathcal{T})$, there is $(N^{\mathcal{U}}_\alpha, \pi_\alpha)$ such that:*

1. *$(N^{\mathcal{U}}_\alpha, \deg^{\mathcal{T}}_\alpha) \trianglelefteq (M^{\mathcal{U}}_{\iota(\alpha)}, \deg^{\mathcal{U}}_{\iota(\alpha)})$.*
2. *$\pi_\alpha : \mathfrak{C}_0(M^{\mathcal{T}}_\alpha) \to \mathfrak{C}_0(N^{\mathcal{U}}_\alpha)$ is an $\mathcal{F}$-rooted $\deg^{\mathcal{T}}_\alpha$-factor which is above b.*
3. *If π is good then π_α is good.*
4. *$\mathcal{U}$ is $\mathcal{N}$-bounded.*
5. *$[0, \alpha]^{\mathcal{T}} \cap \mathscr{D}^{\mathcal{T}} \neq \emptyset$ iff $[0, \iota(\alpha)]^{\mathcal{U}}$ drops below the image of $\mathcal{N}$.*

6. *If* $[0,\alpha]^{\mathcal{T}} \cap \mathscr{D}^{\mathcal{T}} = \emptyset$ *then* $N^{\mathcal{U}}_\alpha = M^{\mathcal{U}}_{\mathcal{N},\iota(\alpha)}$ *and*

$$\pi_\alpha \circ i^{\mathcal{T}}_{0\alpha} = i^{\mathcal{U},I}_{\mathcal{N},0,\iota(\alpha)} \circ \pi. \tag{3.2}$$

7. *If* $[0,\alpha]^{\mathcal{T}} \cap \mathscr{D}^{\mathcal{T}} \neq \emptyset$ *then* $N^{\mathcal{U}}_\alpha = M^{\mathcal{U}}_{\iota(\alpha)}$, $\deg^{\mathcal{T}}_\alpha = \deg^{\mathcal{U}}_{\iota(\alpha)}$ *and* π_α *is a* ν*-preserving near* $\deg^{\mathcal{T}}_\alpha$*-embedding.*

8. *If* $(\mathcal{N},k) = (\mathcal{Q},j)$ *and* π *is a near* k*-embedding then* $N^{\mathcal{U}}_\alpha = M^{\mathcal{U}}_{\iota(\alpha)}$, $\deg^{\mathcal{T}}_\alpha = \deg^{\mathcal{U}}_{\iota(\alpha)}$ *and* π_α *is a near* $\deg^{\mathcal{T}}_\alpha$*-embedding, and if* π *is also* ν*-preserving then so is* π_α*.*

The previous paragraph also holds with "$(j,\omega_1,\omega_1+1)^*$*-optimal" replacing "*$(j,\omega_1+1)$*" and "*$(k,\omega_1,\omega_1+1)^*$*-optimal" replacing "*$(k,\omega_1+1)$*".*

Proof. We just sketch the proof, for the case that $\mathcal{T}$ is k-maximal. It is mostly the standard copying construction, augmented with propagation of near embeddings (using the methods of the proof of [8, Lemma 1.3]) and the standard extra details dealing with type 3 premice (see [25], [15], [14]). Because of how we handle type 3 premice, the tree orders of $\mathcal{T}$ and $\mathcal{U}$ need not be identical, and the indexing map ι can fail to be the identity.

The construction is by recursion on $\mathrm{lh}(\mathcal{T})$. Suppose we have determined $\mathcal{T}\restriction(\alpha+1)$, $\iota(\alpha)$, $\mathcal{U}\restriction(\iota(\alpha)+1)$ and all the other objects mentioned in the lemma, satisfying the properties there, in particular with $N^{\mathcal{U}}_\alpha \trianglelefteq M^{\mathcal{U}}_{\iota(\alpha)}$ and $\pi_\alpha : \mathfrak{C}_0(M^{\mathcal{T}}_\alpha) \to \mathfrak{C}_0(N^{\mathcal{U}}_\alpha)$. We now want to proceed to $\mathcal{T}\restriction(\alpha+2)$, etc.

Suppose first that π_α is non-ν-high or $E^{\mathcal{T}}_\alpha = F(M^{\mathcal{T}}_\alpha)$ or $\mathrm{lh}(E^{\mathcal{T}}_\alpha) < \rho_0(M^{\mathcal{T}}_\alpha)$. Then we set $\iota(\alpha+1) = \iota(\alpha)+1$ and set $E^{\mathcal{U}}_{\iota(\alpha)}$ to be:

- $F(M^{\mathcal{U}}_{\mathcal{N},\iota(\alpha)})$, if $E^{\mathcal{T}}_\alpha = F(M^{\mathcal{T}}_\alpha)$ and π_α is non-ν-low,
- $F(M^{\mathcal{U}}_{\mathcal{N},\iota(\alpha)})\restriction\mathrm{Shift}(\pi_\alpha)(\nu(E^{\mathcal{T}}_\alpha))$, if $E^{\mathcal{T}}_\alpha = F(M^{\mathcal{T}}_\alpha)$ and π_α is ν-low,
- $\pi_\alpha(E^{\mathcal{T}}_\alpha)$, if $\mathrm{lh}(E^{\mathcal{T}}_\alpha) < \rho_0(M^{\mathcal{T}}_\alpha)$,
- $\mathrm{Shift}(\pi_\alpha)(E^{\mathcal{T}}_\alpha)$, if $\rho_0(M^{\mathcal{T}}_\alpha) < \mathrm{lh}(E^{\mathcal{T}}_\alpha) < \mathrm{Ord}(M^{\mathcal{T}}_\alpha)$.

Now suppose instead that π_α is ν-high and $\rho_0(M^{\mathcal{T}}_\alpha) < \mathrm{lh}(E^{\mathcal{T}}_\alpha) < \mathrm{Ord}(M^{\mathcal{T}}_\alpha)$. (So by induction with part 7, $[0,\alpha]^{\mathcal{T}} \cap \mathscr{D}^{\mathcal{T}} = \emptyset$ and $\mathcal{M}$ is active type 3.) In this case let us say that α is an *insertion stage*. We set $\iota(\alpha+1) = \iota(\alpha)+2$, and set $E^{\mathcal{U}}_{\iota(\alpha)} = F(M^{\mathcal{U}}_{\mathcal{N},\iota(\alpha)})$ and $E^{\mathcal{U}}_{\iota(\alpha)+1} = \mathrm{Shift}(\pi_\alpha)(E^{\mathcal{T}}_\alpha)$.

Let $\beta = \mathrm{pred}^{\mathcal{T}}(\alpha+1)$ and $\beta' = \mathrm{pred}^{\mathcal{U}}(\iota(\alpha+1))$ (determined k- and j-maximality respectively). Then $\beta' = \iota(\beta)$, unless β was an insertion stage and $\nu(F(M^{\mathcal{T}}_\beta)) \leq \mathrm{crit}(E^{\mathcal{T}}_\alpha)$, in which case $\beta' = \iota(\beta)+1$ and $\alpha+1 \in \mathscr{D}^{\mathcal{T}}$ and $\iota(\alpha+1) \in \mathscr{D}^{\mathcal{U}}$. We (can and do) define $\pi_{\alpha+1}$ via the Shift Lemma from $\pi_{\beta'}, \pi_\alpha$ as usual.

For limit ordinals α, $\iota(\alpha) = \sup_{\beta<\alpha}\iota(\beta)$, and $[0,\alpha)^{\mathcal{T}}$ is the unique cofinal branch of $\mathcal{T}\restriction\alpha$ such that for some $\beta <^{\mathcal{T}} \alpha$, we have $\iota``[\beta,\alpha)^{\mathcal{T}} \subseteq [0,\iota(\alpha))^{\mathcal{U}}$. We omit the remaining details of the definitions, which are routine.

Now let us observe that for each α, π_α is an $\mathcal{F}$-rooted $\deg^{\mathcal{T}}_\alpha$-factor above b (see Definition 3.22); that is, that there is a $\deg^{\mathcal{T}}_\alpha$-sound $\mathcal{F}$-pm $\mathcal{L}$ and a $\deg^{\mathcal{T}}_\alpha$-tight $\sigma : \mathcal{L} \to M^{\mathcal{T}}_\alpha$ such that $\pi_\alpha \circ \sigma \circ \sigma^{\mathcal{L}}_{\deg^{\mathcal{T}}_\alpha+1} : \mathfrak{S}_{k+1}(\mathcal{L}) \to \mathfrak{C}_0(N^{\mathcal{U}}_{\iota(\alpha)})$ is a near k-embedding. For given this, fine condensation, together with Lemmas 2.49 (to see $M^{\mathcal{T}}_\alpha$ is $\deg^{\mathcal{T}}_\alpha$-relevant) and 3.31, give that $M^{\mathcal{T}}_\alpha$ is an $\mathcal{F}$-pm. (If $M^{\mathcal{T}}_\alpha$ might be type 3 (that is, if $N^{\mathcal{U}}_\alpha$ is type 3), then 3.31 applies, because $(N^{\mathcal{U}}_\alpha)^\uparrow$ is an $\mathcal{F}$-pm, since we can extend $\mathcal{U}{\restriction}(\iota(\alpha)+1)$ to a tree $\mathcal{U}'$, setting $E^{\mathcal{U}'}_{\iota(\alpha)} = F(N^{\mathcal{U}}_\alpha)$.)

Fix $(\mathcal{L}_0, \sigma_0)$ witnessing that π is an $\mathcal{F}$-rooted k-factor; so $\sigma_0 : \mathcal{L}_0 \to \mathcal{M}$ is k-tight and $\pi \circ \sigma_0 \circ \sigma^{\mathcal{L}_0}_{k+1} : \mathfrak{S}_{k+1}(\mathcal{L}_0) \to \mathfrak{C}_0(\mathcal{N})$ is a near k-embedding.

Case 3.35. $[0,\alpha]^{\mathcal{T}}$ does not drop in model in $\mathcal{T}$.

In this case, it is routine to verify that $[0,\iota(\alpha)]^{\mathcal{U}}$ does not drop below the image of $\mathcal{N}$, π_α is a weak $\deg^{\mathcal{T}}_\alpha$-embedding and line (3.2) holds.

If $\deg^{\mathcal{T}}_\alpha = k$ then $(\mathcal{L}_0, \sigma)$ witnesses the fact that π_α is an $\mathcal{F}$-rooted k-factor above b, where $\sigma = i^{\mathcal{T}}_{0\alpha} \circ \sigma_0$, because $i^{\mathcal{U}}_{\mathcal{N},0,\iota(\alpha)}$ and $\pi \circ \sigma_0 \circ \sigma^{\mathcal{L}_0}_{k+1}$ are near k-embeddings, and $\pi_\alpha \circ i^{\mathcal{T}}_{0\alpha} = i^{\mathcal{U}}_{\mathcal{N},0,\iota(\alpha)} \circ \pi$.

Now suppose that $[0,\alpha]^{\mathcal{T}}$ drops in degree and let $n = \deg^{\mathcal{T}}_\alpha$. Then letting $\mathcal{L} = \mathfrak{C}_{n+1}(M^{\mathcal{T}}_\alpha)$ and $\sigma : \mathfrak{C}_0(\mathcal{L}) \to \mathfrak{C}_0(M^{\mathcal{T}}_\alpha)$ be the core embedding, $(\mathcal{L}, \sigma)$ witnesses the fact that π_α is an $\mathcal{F}$-rooted n-factor above b (we have $\mathfrak{S}_{n+1}(\mathcal{L}) = \mathcal{L}$ and $\sigma^{\mathcal{L}}_{n+1} = \mathrm{id}$). The fact that $\pi_\alpha \circ \sigma$ is a near n-embedding is because $\pi_\alpha \circ \sigma = i^{\mathcal{U}}_{\mathcal{N},\iota(\xi),\iota(\alpha)} \circ \pi_\xi$, π_ξ is a weak $(n+1)$-embedding, and $i^{\mathcal{U}}_{\mathcal{N},\iota(\xi),\iota(\alpha)}$ a near n-embedding.

Now consider part 8. Suppose that $(\mathcal{N}, k) = (\mathcal{Q}, j)$ and π is a near k-embedding. Clearly $N^{\mathcal{U}}_\alpha = M^{\mathcal{U}}_{\iota(\alpha)}$. The fact that π_α is a near $\deg^{\mathcal{T}}_\alpha$-embedding (in fact for every $\beta \leq \alpha$, π_β is a near $\deg^{\mathcal{T}}_\beta$-embedding) is by [8, Lemma 1.3], or more literally, its proof. However, note that that proof is inductive on α, and one should also maintain as part of the induction that $\deg^{\mathcal{T}}_\beta = \deg^{\mathcal{U}}_{\iota(\beta)}$ for every $\beta \leq \alpha$. In order to see that $\deg^{\mathcal{T}}_{\beta+1} = \deg^{\mathcal{U}}_{\iota(\beta+1)}$, letting $\gamma = \mathrm{pred}^{\mathcal{T}}(\beta+1)$, if $\beta+1 \notin \mathscr{D}^{\mathcal{T}}$, one uses the fact that π_γ is a near $\deg^{\mathcal{T}}_\gamma$-embedding and $\deg^{\mathcal{T}}_\gamma = \deg^{\mathcal{U}}_{\iota(\gamma)}$. Finally, if π is ν-preserving, then using Lemma 3.33, one easily shows inductively that π_β is ν-preserving for every $\beta \leq \alpha$.

Case 3.36. $[0,\alpha]^{\mathcal{T}}$ drops in model in $\mathcal{T}$.

It is straightforward to see that $[0,\iota(\alpha)]^{\mathcal{U}}$ drops below the image of $\mathcal{N}$ and that $N^{\mathcal{U}}_\alpha = M^{\mathcal{U}}_{\iota(\alpha)}$. The fact that π_α is an $\mathcal{F}$-rooted $\deg^{\mathcal{T}}_\alpha$-factor is almost the same as in the dropping degree case above. The fact that π_α is in fact a near $\deg^{\mathcal{T}}_\alpha$-embedding and $\deg^{\mathcal{T}}_\alpha = \deg^{\mathcal{U}}_{\iota(\alpha)}$ follows much as before, though this time it is not quite as directly by [8, Lemma 1.3] itself, but from an examination of its proof; one observes that the inductive argument used in [8] can simply be done above any node in $\mathcal{T}$ at which there is a drop in model, instead of having to start at the root node 0.[40] (Similar arguments were also used in [14].) And the fact that π_α is ν-preserving is proved similarly, by an induction above any node in $\mathcal{T}$ at which there is a drop in model. □

---

[40] Note that we are not assuming that π itself is a near embedding in this case; it is just that above any drop in model in $\mathcal{T}$, we get near embeddings.

Definition 3.37. Let $\mathcal{N}$ be an $\mathcal{F}$-pm and $k \leq \omega$. Then $\mathcal{N}$ is **$\mathcal{F}$-k-fine** iff for each $j \leq k$:

- $\mathfrak{C}_j(\mathcal{N})$ is a j-solid $\mathcal{F}$-pm,
- if $j < k$ then $\mathfrak{C}_j(\mathcal{N})$ is $(j+1)$-universal,
- if $k = \omega$ then $\mathfrak{C}_\omega(\mathcal{N})$ is $< \omega$-condensing. $\dashv$

We next consider background constructions building $\mathcal{F}$-mice.

Definition 3.38. Let $\mathcal{F}$ be an operator over $\mathscr{B}$. Let $A \in \widehat{C_\mathcal{F}}$ and $\chi \leq \mathrm{Ord}(\mathscr{B})+1$. An **$L^\mathcal{F}[\mathbb{E}, A]$-construction (of length χ)** is a sequence $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha<\chi}$ such that for all $\alpha < \chi$:

- $\mathcal{N}_0 = \mathcal{F}(A)$ and $\mathcal{N}_\alpha$ is an $\mathcal{F}$-pm over A.
- If α is a limit then $\mathcal{N}_\alpha = \liminf_{\beta<\alpha} \mathcal{N}_\beta$.
- If $\alpha+1 < \chi$ then either (i) $\mathcal{N}_{\alpha+1}$ is E-active and $\mathcal{N}_{\alpha+1}||\mathrm{Ord}(\mathcal{N}_{\alpha+1}) = \mathcal{N}_\alpha$, or (ii) $\mathcal{N}_\alpha$ is $\mathcal{F}$-ω-fine and $\mathcal{N}_{\alpha+1} = \mathcal{F}(\mathfrak{C}_\omega(\mathcal{N}_\alpha))$. $\dashv$

We will now explain how fine condensation for $\mathcal{F}$ leads to the $\mathcal{F}$-iterability of substructures $\mathcal{R}$ of $\mathcal{F}$-pms built by background construction. The basic engine behind this is the realizability of iterates of $\mathcal{R}$ back into models of the construction.

Definition 3.39. Let $\mathcal{F}$ be an operator above b which almost condenses finely. Let $\bar{A}, A \in \widehat{C_\mathcal{F}}$. Let $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha\leq\lambda}$ be an $L^\mathcal{F}[\mathbb{E}, A]$-construction. Let $k \leq \omega$ and suppose that $\mathcal{N}_\lambda$ is $\mathcal{F}$-k-fine. Let $\mathcal{R}$ be a k-sound $\mathcal{F}$-pm over $\bar{A}$ and $\pi : \mathfrak{C}_0(\mathcal{R}) \to \mathfrak{C}_k(\mathcal{N}_\lambda)$ be an above b weak k-embedding. Let $\mathcal{T}$ be a putative $\mathcal{F}$-tree on $\mathcal{R}$, with $\deg_0^\mathcal{T} = k$. We say that $\mathcal{T}$ is **$(\pi, \mathbb{C})$-realizable above** b iff for every $\alpha < \mathrm{lh}(\mathcal{T})$, $\beta = \mathrm{base}^\mathcal{T}(\alpha)$ exists (that is, $[0, \alpha]^\mathcal{T}$ drops only finitely often) and letting $m = \deg_\alpha^\mathcal{T}$, there are ζ, τ such that:

- $(\zeta, m) \leq_{\mathrm{lex}} (\lambda, k)$,
- if $[0, \alpha]^\mathcal{T}$ does not drop in model or degree then $\zeta = \lambda$ and $\tau = \pi$,
- if $[0, \alpha]^\mathcal{T}$ drops in model or degree then $\tau \colon \mathfrak{C}_0(M_\beta^{*\mathcal{T}}) \to \mathfrak{C}_m(\mathcal{N}_\zeta)$ is a near m-embedding above b,
- if $M_\beta^{*\mathcal{T}}$ is not type 3 then there is a weak m-embedding $\varphi : \mathfrak{C}_0(M_\alpha^\mathcal{T}) \to \mathfrak{C}_m(\mathcal{N}_\zeta)$ such that $\varphi \circ i_{\beta\alpha}^{*\mathcal{T}} = \tau$.
- if $M_\beta^{*\mathcal{T}}$ is type 3 then there is a weak m-embedding $\varphi : \mathcal{S} \to \mathfrak{C}_m(N_\zeta)$ such that $\varphi \circ i_{\beta\alpha}^{*\mathcal{T}} = \tau$, where $\mathcal{S}$ is "$(M_\alpha^\mathcal{T})^{\mathrm{sq}}$".[41] $\dashv$

---

[41] $(M_\alpha^\mathcal{T})^{\mathrm{sq}}$ might not make literal sense, if say $M_\alpha^\mathcal{T}$ is not wellfounded. By "$(M_\alpha^\mathcal{T})^{\mathrm{sq}}$" we mean that either $\alpha = \xi + 1$ and $\mathcal{S} = \mathrm{Ult}_m((M_\alpha^{*\mathcal{T}})^{\mathrm{sq}}, E_\xi^\mathcal{T})$ (formed without unsquashing), or α is a limit and $\mathcal{S}$ is the direct limit of the structures $(M_\xi^\mathcal{T})^{\mathrm{sq}}$ for $\xi \in [\beta, \alpha)_\mathcal{T}$, under the iteration maps.

Definition 3.40. A **putative $\mathcal{F}$-(k,θ)-iteration strategy** for a k-sound $\mathcal{F}$-pm $\mathcal{N}$ is a function Σ such that for every k-maximal $\mathcal{F}$-tree $\mathcal{T}$ on $\mathcal{N}$, with $\mathcal{T}$ via Σ and $\mathrm{lh}(\mathcal{T}) < \theta$ a limit, $\Sigma(\mathcal{T})$ is a $\mathcal{T}$-cofinal branch. $\dashv$

Lemma 3.41. *Let $\mathcal{F}$ be an operator above b which almost condenses finely above b. Let $\bar{A}, A \in \widehat{C_{\mathcal{F}}}$. Let $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha<\chi}$ be an $L^{\mathcal{F}}[\mathbb{E}, A]$-construction. Suppose that $(\mathcal{N}_\alpha)^{\uparrow}$ is an $\mathcal{F}$-pm for each $\alpha < \chi$. Let $\lambda < \chi$ and $k \leq \omega$ be such that $\mathcal{N}_\lambda$ is $\mathcal{F}$-k-fine, and let $\mathcal{S} = \mathfrak{C}_k(\mathcal{N}_\lambda)$. Let $\mathcal{R}$ be a k-relevant $\mathcal{F}$-pm over $\bar{A}$. Let $\pi : \mathfrak{C}_0(\mathcal{R}) \to \mathfrak{C}_0(\mathcal{S})$ be an $\mathcal{F}$-rooted k-factor above b. Let Σ be either:*

- *a putative $\mathcal{F}$-$(k, \omega_1 + 1)$-iteration strategy for $\mathcal{R}$, or*
- *a putative $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)^*$-optimal iteration strategy for $\mathcal{R}$.*

Suppose that every putative $\mathcal{F}$-tree via Σ is $(\pi, \mathbb{C})$-realizable above b. Then Σ is an $\mathcal{F}$-$(k, \omega_1 + 1)$, or $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)^*$-optimal, iteration strategy.

Proof. The argument is almost that used for Lemma 3.34, using the $(\pi, \mathbb{C})$-realizability maps in place of copy maps. The hypothesis that each $(\mathcal{N}_\alpha)^{\uparrow}$ is an $\mathcal{F}$-pm is used to see that Lemma 3.31 applies where needed. We leave the details to the reader. $\square$

The above proof does not work with $(k, \omega_1, \omega_1 + 1)^*$-*optimal* replaced by $(k, \omega_1, \omega_1 + 1)^*$, because of the reliance on m-relevance in connection with fine condensation.

Remark 3.42. We digress to mention a key application of the extra strength that *condenses finely* has compared to *almost condenses finely*; this essentially comes from [20], such as in [20, §2]. Adopt the assumptions and notation of the first paragraph of 3.41. Assume further that $\mathcal{F}$ condenses finely (not just almost), $\mathscr{B} = V$ and $\mathcal{F}$ is total. For an $\mathcal{F}$-premouse $\mathcal{M}$, say that $\mathcal{M}$ is **$\mathcal{F}$-full** iff there is no $\alpha \in \mathrm{Ord}$ such that $\mathcal{F}^\alpha(\mathcal{M})$ projects $< \mathrm{Ord}(\mathcal{M})$.[42] Assume also that there is no $\mathcal{F}$-full $\mathcal{M}$ such that $\mathrm{Ord}(\mathcal{M})$ is Woodin in $\mathcal{F}^{\mathrm{Ord}}(\mathcal{M})$. Let κ be a cardinal. Suppose that $\mathcal{R} \vDash$"there is no Woodin cardinal" and every k-maximal putative $\mathcal{F}$-tree $\mathcal{T}$ on $\mathcal{R}$ of length $\leq \kappa$ is such that in some set-generic extension $V[G]$, either $\mathcal{T}$ is $(\pi, \mathbb{C})$-realizable, or there is a limit $\lambda \leq \mathrm{lh}(\mathcal{T})$ and a $\mathcal{T}\!\upharpoonright\!\lambda$-cofinal branch c such that c is $\mathcal{T}$-maximal and $(\mathcal{T}\!\upharpoonright\!\lambda)\,^\frown c$ is $(\pi, \mathbb{C})$-realizable. Then $\mathcal{R}$ is $\mathcal{F}$-$(k, \kappa + 1)$-iterable, via the strategy guided by Q-structures of the form $\mathcal{F}^\alpha(M(\mathcal{T}))$ for some $\alpha \in \mathrm{Ord}$.[43] This follows by a straightforward adaptation of the proof for standard premice (cf. [20]), where $\mathcal{F} = \mathcal{J}$. In the argument one needs to apply *condenses finely* to embeddings φ, σ, when $\varphi \circ \sigma \notin V$. Here σ is an iteration map arising from $\mathcal{T}$ (and possibly c), with codomain $M^{\mathcal{T}}_\infty$ (or $M^{\mathcal{T}}_c$), and φ embeds the codomain of φ into some model of $\mathbb{C}$. We can only expect $\varphi \circ \sigma \in V$ if the realized branch does not drop in model or degree (indeed, in

---

[42] Here $\mathcal{F}^\alpha(\mathcal{M})$ is the unique $\mathcal{F}$-pm $\mathcal{N}$ such that $\mathcal{M} \trianglelefteq \mathcal{N}$ and $l(\mathcal{N}) = l(\mathcal{M}) + \alpha$ and $\mathcal{N}|\beta$ is E-passive for every $\beta \in (l(\mathcal{M}), l(\mathcal{N})]$.

[43] It might be that the Q-structure satisfies "$\delta(\mathcal{T})$ is not Woodin", but in this case, $\alpha = \beta + 1$ for some β and $\mathcal{F}^\beta(M(\mathcal{T}))$ satisfies "$\delta(\mathcal{T})$ is Woodin".

the latter case, $\varphi \circ \sigma = \pi$), or if all relevant objects are countable. We use fine condensation to see that the Q-structure $Q \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ (where $\mathcal{M}(\mathcal{T}) \in V$ and $b \in V[G]$) is in fact of the form $\mathcal{F}^{\alpha}(\mathcal{M}(\mathcal{T}))$; the hypothesis of *condenses finely* that $\mathcal{M}^- \in V$ holds where needed since $\mathcal{M}(\mathcal{T}) \in V$, and so $\mathcal{F}^{\alpha}(\mathcal{M}(\mathcal{T})) \in V$ for each α; Q has no extenders in its sequence above $\delta(\mathcal{T})$ by the smallness assumption.

From now on we will only deal with *almost condenses finely*.

3.7 Weak Dodd-Jensen

The weak Dodd-Jensen property is defined as the obvious adaptation of the usual one (see [5]):

Definition 3.43. Let $k \leq \omega$ and $\mathcal{M}$ be a countable k-relevant opm.

We say that $(\mathcal{T}, \mathcal{Q}, \pi)$ is **$(\mathcal{M}, k)$-large** iff $\mathcal{T}$ is a run of $\mathcal{G}_{\mathrm{opt}}^{\mathcal{F}}(\mathcal{M}, k, \omega_1, \omega_1)$ of countable successor length, in which neither player has lost, $\mathcal{Q} \trianglelefteq M_\infty^{\mathcal{T}}$ and $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{Q})$ is a nearly k-good embedding.[44]

Let Σ be an iteration strategy for $\mathcal{M}$. Let $\vec{\alpha} = \langle \alpha_n \rangle_{n<\omega}$ enumerate $\mathrm{Ord}^{\mathcal{M}}$. We say that Σ has the **k-weak Dodd-Jensen (DJ) property for $\vec{\alpha}$** iff for all $(\mathcal{M}, k)$-large $(\mathcal{T}, \mathcal{Q}, \pi)$ with $\mathcal{T}$ via Σ, we have $\mathcal{Q} = M_\infty^{\mathcal{T}}$, $b^{\mathcal{T}}$ does not drop in model (hence, nor degree), and

$$i^{\mathcal{T}} \restriction \mathrm{Ord}^{\mathcal{M}} \leq_{\mathrm{lex}}^{\vec{\alpha}} \pi \restriction \mathrm{Ord}^{\mathcal{M}}$$

(that is, either $i^{\mathcal{T}} \restriction \mathrm{Ord}^{\mathcal{M}} = \pi \restriction \mathrm{Ord}^{\mathcal{M}}$, or $i^{\mathcal{T}}(\alpha_n) < \pi(\alpha_n)$ where $n < \omega$ is least such that $i^{\mathcal{T}}(\alpha_n) \neq \pi(\alpha_n)$). $\dashv$

Note that in the context above, if $i^{\mathcal{T}} \restriction \mathrm{Ord}^{\mathcal{M}} = \pi \restriction \mathrm{Ord}^{\mathcal{M}}$ then $i^{\mathcal{T}} = \pi$, because $i^{\mathcal{T}}, \pi$ are both nearly 0-good, and $\mathcal{M} = \mathrm{Hull}_1^{\mathcal{M}}(cb^{\mathcal{M}} \cup \mathrm{Ord}^{\mathcal{M}})$.

Following [5], one can convert given strategies for stacks of trees on a countable mouse into strategies with weak DJ:

Lemma 3.44. *Assume* $\mathsf{DC}_{\mathbb{R}}$. *Let $\mathcal{F}$ be an operator above $b \in \mathrm{HC}$ which almost condenses finely above b. Let $A \in \widehat{C_{\mathcal{F}}}$. Let $\mathcal{M} \in \mathrm{HC}$ be an $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)^*$-optimally iterable k-relevant $\mathcal{F}$-pm. Let $\vec{\alpha} = \langle \alpha_n \rangle_{n<\omega}$ enumerate $\mathrm{Ord}^{\mathcal{M}}$. Then there is an $\mathcal{F}$-$(k, \omega_1, \omega_1 + 1)^*$-optimal strategy for $\mathcal{M}$ with the k-weak DJ property for $\vec{\alpha}$.*

Proof Sketch. The proof is like the usual one (see [5]), using one minor observation: Suppose $\mathcal{T}$ is a run of $\mathcal{G}_{\mathrm{opt}}^{\mathcal{F}}(\mathcal{M}, k, \omega_1, \omega_1)$ of countable successor length, $\mathcal{Q} \triangleleft M_\infty^{\mathcal{T}}$ and $\pi : \mathfrak{C}_0(\mathcal{M}) \to \mathfrak{C}_0(\mathcal{Q})$ is a near k-embedding, but $\mathcal{Q}$ is not $M_\infty^{\mathcal{T}}$-stable. In this case we can't use Lemma 3.34 to copy trees on $\mathcal{M}$ to trees on $M_\infty^{\mathcal{T}}$ via π, which is superficially a problem for the proof of the lemma. But because $\mathcal{Q}$ is $M_\infty^{\mathcal{T}}$-unstable, there is $E \in \mathbb{E}_+^{M_\infty^{\mathcal{T}}}$ such that $\mathcal{Q} \triangleleft M_\infty^{\mathcal{T}} | \mathrm{lh}(E)$. Let E

---

[44] So $\mathcal{Q}$ is k-sound; the rules of $\mathcal{G}_{\mathrm{opt}}^{\mathcal{F}}$ therefore imply that if $\mathcal{Q} = M_\infty^{\mathcal{T}}$ then $\deg_\infty^{\mathcal{T}} \geq k$. So we do not need to explicitly stipulate that $\deg_\infty^{\mathcal{T}} \geq k$, unlike in [14].

be least such. Then note that $\mathcal{Q} \triangleleft M_\infty^{\mathcal{T}\,^\frown\langle E\rangle}$ and $\mathcal{Q}$ is $M_\infty^{\mathcal{T}\,^\frown\langle E\rangle}$-stable. (Here $\mathcal{T}\,^\frown\langle E\rangle$ is the run of the game for which $\mathcal{T}$ constitutes the first α rounds (for some α), followed by 1 more round, which is the tree on $M_\infty^{\mathcal{T}}$ which only uses E.) So we can apply Lemma 3.34 to π, $\mathcal{Q}$ and $M_\infty^{\mathcal{T}\,^\frown\langle E\rangle}$, and this is enough for the proof. □

3.8 Solidity and condensation

In this final section we prove some of the basic fine structural facts (solidity, etc) hold for iterable $\mathcal{F}$-mice, assuming $\mathcal{F}$ condenses finely. The proof will be heavily based on the corresponding proofs as presented in the union of [4], [25] and [9]. Beyond extra details in connection with operator mice, which are relatively minor, we need to handle some details which arise with superstrong extenders, which did not arise in the papers just mentioned.

We also take the opportunity to discuss a couple of elements of the proof which are not made explicit in [4], [25], [9]. These elements are also relevant for standard premice, not just operator premice. They deal with two issues.

First, [4, Lemma 6.1.5] (on closeness of extenders) establishes that if $\mathcal{T}$ is a k-maximal tree on a k-sound premouse $\mathcal{M}$, then for every $\alpha+1 < \mathrm{lh}(\mathcal{T})$, $E^{\mathcal{T}}_\alpha$ is close to $M^{*\mathcal{T}}_{\alpha+1}$. The proof of solidity, etc, involves trees on phalanxes, to which this lemma does not literally apply, though the arguments analysing the comparisons seem to assume that the lemma does apply to them. So we discuss the generalization of the lemma to trees on phalanxes which arise in the proof. This kind of thing has also been discussed in [14] and [11], for example.

Second, and somewhat more importantly, we discuss why weak Dodd-Jensen is enough to rule out certain situations in the analysis of comparisons, in which it might not be entirely obvious that it is enough, and which are not addressed explicitly in [4], [25], [9].

The main issue regarding weak Dodd-Jensen arises in the following situation. Suppose that $\mathcal{M}$ is a k-sound, $(k,\omega_1,\omega_1+1)^*$-iterable premouse, and we want to prove that $\mathcal{M}$ is $(k+1)$-solid. Let $\alpha \in p^{\mathcal{M}}_{k+1}$. Let $\mathcal{W}$ be the $(k+1)$-solidity witness

$$\mathcal{W} = \mathrm{cHull}^{\mathcal{M}}_{k+1}(\alpha \cup \{\vec{p}^{\,\mathcal{M}}_k, p^{\mathcal{M}}_{k+1}\backslash(\alpha+1)\})$$

at α, and $\pi : \mathcal{W} \to \mathcal{M}$ the uncollapse map. We need to see that $\mathcal{W} \in \mathcal{M}$. Suppose that α is not an $\mathcal{M}$-cardinal. Let $\kappa = \mathrm{card}^{\mathcal{M}}(\alpha)$. Let $\mathcal{R} \triangleleft \mathcal{M}$ be least such that $\rho^{\mathcal{R}}_\omega = \kappa$ and $\alpha \leq \mathrm{Ord}^{\mathcal{R}}$. The proof given in [25] that $\mathcal{W} \in \mathcal{M}$ proceeds by comparing the phalanx $\mathfrak{P} = ((\mathcal{M}, <\kappa), (\mathcal{R}, \kappa), \mathcal{W})$ versus $\mathcal{M}$, producing trees $\bar{\mathcal{T}}$ on $\mathfrak{P}$ and $\mathcal{U}$ on $\mathcal{M}$. An iteration strategy Σ for $\mathcal{M}$ with the weak Dodd-Jensen property with respect to some enumeration of $\mathcal{M}$ is used to form $\mathcal{U}$, and $\bar{\mathcal{T}}$ is formed by simultaneously lifting $\bar{\mathcal{T}}$ to a tree $\mathcal{T}$ on $\mathcal{M}$ via Σ. Let r be such that $\rho^{\mathcal{R}}_{r+1} = \kappa < \rho^{\mathcal{R}}_r$. The tree $\bar{\mathcal{T}}$ is (k,r,k)-maximal, meaning that in $\mathfrak{P}$, $\mathcal{M}$ is at degree k, $\mathcal{R}$ is at degree r, and $\mathcal{H}$ at degree k. The main issue does not arise in the *anomalous* case, i.e. when $\mathcal{R}$ is active type 3 with $\mathrm{Ord}^{\mathcal{R}} = \alpha$, so we will ignore this case, and therefore we have $\kappa < \rho^{\mathcal{R}}_0$, so $r < \omega$ is well-defined.

Now the process for forming $\mathcal{T}$ in this context at a step for which $\mathrm{crit}(E^{\bar{\mathcal{T}}}_\eta) = \kappa$ is not made clear in [4].[45] The process is, however, clarified in [9]:[46] we lift $M^{\bar{\mathcal{T}}}_{\eta+1}$ to $i^{*\mathcal{T}}_{\eta+1}(\mathcal{R})$, defining $\pi_{\eta+1} : M^{\bar{\mathcal{T}}}_{\eta+1} \to i^{*\mathcal{T}}_{\eta+1}(\mathcal{R})$; there is a natural definition for this map $\pi_{\eta+1}$, analogous to Shift Lemma maps. Then $\pi_{\eta+1}$ is a weak r-embedding, but not clear that it is a near r-embedding; that is, not clear that it is $\mathrm{r}\Sigma_{r+1}$-elementary. In fact, if there is an $\underset{\sim}{\mathrm{r}\Sigma}{}^{\mathcal{R}}_r$-definable function $f : \kappa \to \rho^{\mathcal{R}}_r$, and κ is the least such ordinal, then $\pi_{\eta+1}$ is *not* a near r-embedding, as then $\pi_{\eta+1}(\rho_r^{M^{\bar{\mathcal{T}}}_{\eta+1}}) < \rho_r^{i^{*\mathcal{T}}_{\eta+1}(\mathcal{R})}$.

We write $M^{\bar{\mathcal{T}}}_\infty$ and $M^{\mathcal{U}}_\infty$ for the last models of $\bar{\mathcal{T}}$ and $\mathcal{U}$ respectively. Suppose $M^{\bar{\mathcal{T}}}_\infty = M^{\mathcal{U}}_\infty$ and neither $b^{\bar{\mathcal{T}}}$ nor $b^{\mathcal{U}}$ drops in model, $b^{\mathcal{U}}$ does not drop in degree, and $b^{\bar{\mathcal{T}}}$ is above $\mathcal{R}$ (but maybe $b^{\bar{\mathcal{T}}}$ drops in degree strictly below r), so the first extender used along $b^{\bar{\mathcal{T}}}$ is some $E^{\bar{\mathcal{T}}}_\eta$ as above. If $b^{\bar{\mathcal{T}}}$ does not drop in degree (strictly below r), then one can use some standard fine structural arguments to reach a contradiction. But if it drops in degree (and it follows that it drops to degree exactly $k < r$), then standard fine structural arguments do not seem to suffice, and the most obvious tool appears to be weak Dodd-Jensen. But for this, we need to know that $\pi_\infty \circ i^{\mathcal{U}}$ is a near k-embedding.

While [4] appears to ignore this situation, [25] and [9] do not say much further regarding its analysis outside of the anomalous case, which we are presently ignoring. In [25], it is mentioned briefly that [8, 1.3] shows the copying construction gives rise to near k-embeddings (see [25, Remark 4.3], and also [4, §4.3], immediately following the definition of $(\mathcal{M}, k)$*-large*). But the results in [8] themselves do not really suffice themselves to cover all cases, especially the one described above. We will show in Claim 3.45.7 below, at least under some contradictory assumptions which we are free to make,[47] that in the above situation, π_∞ is a near k-embedding, and hence so is $\pi_\infty \circ i^{\mathcal{U}}$. The proof will involve an extension of the methods of [8]. Part of these calculations were used, for example, in [14] and [11]. But in the most subtle case, they were not, and the argument we give to handle it seems to be new.

There are actually multiple options available to either complete the proof as it is, or to modify the proof somewhat and thereby prove solidity slightly differently:

1. Instead of formulating the weak Dodd-Jensen property for near k-embeddings, formulate it for k-lifting embeddings (or maybe cardinal-preserving k-lifting embeddings), as defined in [16]. The copying construction routinely

---

[45] In fact, it is ill-defined, because on p. 47 of [4], clause (4) of Definition 5.0.6 requires $\mathcal{P}(\kappa) \cap \mathcal{M}^*_{\alpha+1} = \mathcal{P}(\kappa) \cap \mathcal{N}$, whereas on p. 78, in Subcase A of Case 2, it is required that $\bar{\mathcal{P}}^*_{\eta+1} = \mathcal{P}^*_{\eta+1}$. But in case $\kappa = \mathrm{crit}(E^{\bar{\mathcal{T}}}_\eta) = \mathrm{crit}(E^{\mathcal{T}}_\eta)$ (where $\kappa^{+\mathcal{R}} = \alpha$), then $E^{\bar{\mathcal{T}}}_\eta$ measures only $\mathcal{P}(\kappa) \cap \mathcal{R}$, so is not $\mathcal{M}$-total, but $E^{\mathcal{T}}_\eta$ *is* $\mathcal{M}$-total, so according to Definition 5.0.6, in order for $\mathcal{T}$ to be an iteration tree, we cannot set $\mathcal{P}^*_{\eta+1} = \bar{\mathcal{P}}^*_{\eta+1}$.

[46] See p. 728 of [9], within the proof of Theorem 3.3, in the paragraph beginning "There is one wrinkle in the copying argument".

[47] We will assume that $\mathcal{W} \notin \mathcal{M}$, that $\mathcal{M}$ is $(k+1)$-solid with respect to $p^{\mathcal{M}}_{k+1} \backslash (\alpha+1)$, and that the enumeration of $\mathcal{M}$ we use for specifying weak Dodd-Jensen begins with $p^{\mathcal{M}}_{k+1} \backslash (\alpha+1)$, in descending order.

yields such embeddings.

2. Use the results of [18] on (z,ζ)-preservation (and their generalizations in [11] to the superstrong level) and some of the arguments from [11]; these make the troublesome appeals to weak Dodd-Jensen unnecessary.

3. We have not thought carefully about the following option, but it seems it should work: instead of formulating the weak Dodd-Jensen property with respect to stacks in which the rounds are each n-maximal for the relevant n, consider stacks in which the rounds can be "weakly n-maximal", in which degrees of nodes in the tree need not be taken as large as possible. Then we would be free to enforce agreement over degrees of nodes in $\bar{\mathcal{T}}$ (as above) with corresponding nodes in $\mathcal{T}$ at certain points of the construction, thus avoiding the difficulties we encounter in the proof of Claim 3.45.7 later.

   A variant of this, which should also work, would be to allow the equation "$\mathcal{P}^*_{\eta+1} = \bar{\mathcal{P}}^*_{\eta+1}$" on page 78 of [4] (see Footnote 45). So the lift tree $\mathcal{T}$ would apply $E^{\mathcal{T}}_\eta$ to a model with strictly fewer subsets of its critical point than what is measured by $E^{\mathcal{T}}_\eta$. But then the resulting map $\pi_{\eta+1}$ should in fact be a $\deg^{\bar{\mathcal{T}}}_{\eta+1}$-embedding, and it seems this should also remove the problem.

Option 1 is probably mathematically the most natural option. But it involves a significant change to the general setup for the proof, and seems it might require using a different iteration strategy than that selected in [25] (since we need one that has the resulting Dodd-Jensen property). Option 3 also involves changes to the general development, and selection of iteration strategy. Option 2 does not involve such changes to the setup, so can be used to show that the original comparison argument (as in [25] and [9]) works. But it involves a significant change in method for analysing aspects of the comparison, and new ideas for this.

We wanted to keep the setup of the original proof the same, excluding the alternatives 1 and 3, and as far as possible, to stick to the same basic methods of the original proof. The proof we give achieves the latter better than option 2. It also gives some information which does not come out of the other methods. Let us begin.

We now state the central result of the paper – the fundamental fine structural facts for $\mathcal{F}$-mice. An **$\mathcal{F}$-pseudo-premouse** is just the $\mathcal{F}$- version of a pseudo-premouse (see [4, §10]), an **$\mathcal{F}$-bicephalus** is that of a bicephalus (see [4, §9]), and the **$\mathcal{F}$-iterability** of such structures is defined in the obvious manner. Likewise the definition of **$\mathcal{F}$-iterability** for phalanxes of $\mathcal{F}$-pms.

Theorem 3.45. *Let $\mathcal{F}$ be an operator above $b \in \mathrm{HC}$, over $\mathscr{B}$, which almost condenses finely. Then:*

1. *For $k < \omega$, every k-sound, $\mathcal{F}$-$(k,\omega_1,\omega_1+1)^*$-optimally iterable $\mathcal{F}$-premouse is $\mathcal{F}$-$(k+1)$-fine.*

2. *Every ω-sound, $\mathcal{F}$-$(\omega, \omega_1, \omega_1+1)^*$-optimally iterable $\mathcal{F}$-premouse is $<\omega$-condensing.*

3. *Every $\mathcal{F}$-$(0, \omega_1, \omega_1+1)^*$-optimally iterable $\mathcal{F}$-pseudo-premouse is an $\mathcal{F}$-premouse.*

4. *There is no non-trivial $\mathcal{F}$-$(0, \omega_1, \omega_1+1)^*$-optimally iterable $\mathcal{F}$-bicephalus.*

Proof sketch. The proof is heavily based on that for standard premice, as given by the combination of [25], [9] and [4], and with which the reader should be reasonably familiar. But we will give quite a detailed proof of the solidity aspect of part 1, in order that we can describe the new features which arise for operator-premice and in particular superstrong extenders,[48] and also in order to discuss the details in the situation described in the introduction to this section.[49] We will omit some more routine calculations. We also sketch enough of the rest of parts 1 and 2, focusing on aspects new for operator-premice, that combined with [25], [9], [4] and the proof of solidity we give, one obtains a complete proof of parts 1 and 2. Part 3 involves similar modifications to the standard proof, and part 4 is an immediate transcription.

Part 1: Let $\mathcal{M}$ be a k-sound, $\mathcal{F}$-$(k, \omega_1, \omega_1+1)^*$-optimally iterable $\mathcal{F}$-premouse over $A \in \widehat{C_{\mathcal{F}}}$. We may assume that $\rho_{k+1}^{\mathcal{M}} < \rho_k^{\mathcal{M}}$, and by Lemma 2.45, that $\mathcal{M}$ is k-relevant. We may assume that $\mathcal{M}$ is countable (otherwise we can replace $\mathcal{M}$ with a countable elementary substructure, because $\mathcal{F}$ almost condenses coarsely above $b \in \mathrm{HC}$ and $\mathscr{B} \models \mathsf{DC}$).

Let Σ_0 be an $\mathcal{F}$-$(k, \omega_1, \omega_1+1)^*$-optimal iteration strategy for $\mathcal{M}$. We would like to use Lemma 3.44, but that lemma assumes $\mathsf{DC}_{\mathbb{R}}$. But we may assume $\mathsf{DC}_{\mathbb{R}}$. For we can work in $W = L^{\mathcal{F}, \Sigma_0}[x]$, where $x \in \mathbb{R}$ codes $\mathcal{M}$. (The hypotheses of the theorem hold in W regarding $b, A, \mathcal{M}, \mathcal{F}^W, \Sigma_0^W, \mathscr{B}^W$, where $\mathscr{B}^W, \mathcal{F}^W, \Sigma_0^W$ are the restrictions of $\mathscr{B}, \mathcal{F}, \Sigma_0$ to W.)

Now using 3.44, let Σ be an $\mathcal{F}$-(k, ω_1+1) iteration strategy for $\mathcal{M}$ with the weak DJ property for some enumeration of $\mathrm{Ord}^{\mathcal{M}}$.

We will first establish $(k+1)$-universality and that $\mathcal{C} = \mathfrak{C}_{k+1}(\mathcal{M})$ is an $\mathcal{F}$-pm. For this, let us assume for better focus that $\mathcal{M}$ is a successor, since the limit case is easier and much closer to the standard proof. Let $\pi : \mathcal{C} \to \mathcal{M}$ be the core map.

First suppose $k = 0$, and consider 1-universality. Because π is 0-good and by 2.36, $\mathcal{C}$ is a Q-opm, $\mathcal{C}$ is a successor and $\pi(\mathcal{C}^-) = \mathcal{M}^-$. By fine condensation and 3.26, $\mathcal{H} = \mathcal{F}(\mathcal{C}^-)$ is a universal hull of $\mathcal{C}$, as witnessed by $\sigma : \mathcal{H} \to \mathcal{C}$.

---

[48] Various other papers have dealt in various ways with fine structure for superstrong extenders, for example [29], [16], [11], [23], [12]. The extra considerations that we need to handle superstrong extenders are very similar to some of those which appear in those papers. But because we are working with Mitchell-Steel indexing and generally following the original proof setup from [25], [9], [4], we can't directly cite those works.

[49] The reader who has not gone through the rest of this paper, and is just interested in the proof of solidity for standard premice and the issues mentioned in the introduction, should be able to read the proof, ignoring the details specifically regarding $\mathcal{F}$-premice. In that case, the structure $\mathcal{H}$ we introduce is just $\mathcal{W}$, and the uncollapse map $\sigma : \mathcal{H} \to \mathcal{W}$ is just the identity.

We claim that $\mathcal{C}$ is 0-relevant. For suppose otherwise, so $\rho_1^{\mathcal{C}} = \rho_\omega^{\mathcal{C}^-}$. But $\rho_1^{\mathcal{C}} = \rho_1^{\mathcal{M}}$ and $\mathcal{M}$ is 0-relevant, so $\rho_1^{\mathcal{M}} < \rho_\omega^{\mathcal{M}^-}$. Since π is 0-good, $\pi(\mathcal{C}^-) = \mathcal{M}^-$ and $\pi(\rho_1^{\mathcal{C}}) = \pi(\rho_\omega^{\mathcal{C}^-}) = \rho_\omega^{\mathcal{M}^-}$. So by $< \omega$-condensation for $\mathcal{M}^-$ and since $\rho_\omega^{\mathcal{C}^-} = \rho_1^{\mathcal{M}}$ is an $\mathcal{M}$-cardinal, we have $\mathcal{C}^- \triangleleft \mathcal{M}^-$, so $\mathcal{H} = \mathcal{F}(\mathcal{C}^-) \triangleleft \mathcal{M}^-$ also. But $\mathcal{H}$ is a universal hull of $\mathcal{C}$, and therefore $\mathcal{C} \in \mathcal{M}$ also, contradicting the fact that $\mathcal{C} = \mathfrak{C}_1(\mathcal{M})$.

So letting $\rho = \rho_1^{\mathcal{M}}$, we have $\rho = \rho_1^{\mathcal{C}} < \rho_\omega^{\mathcal{C}^-}$. Since $\mathcal{H}^- = \mathcal{C}^-$, therefore $\mathcal{C}|\rho^{+\mathcal{C}} = \mathcal{H}|\rho^{+\mathcal{H}}$. So it suffices to see that $\mathcal{M}|\rho^{+\mathcal{M}} = \mathcal{H}|\rho^{+\mathcal{H}}$.

The phalanx $\mathfrak{P} = ((\mathcal{M}, <\rho), \mathcal{H})$ is $\mathcal{F}$-$((0,0), \omega_1 + 1)$-maximally iterable.[50] Moreover, we get an $\mathcal{F}$-$((0,0), \omega_1+1)$-iteration strategy for $\mathfrak{P}$ by lifting to 0-maximal trees on $\mathcal{M}$ via Σ. This is proved by using $\pi \circ \sigma$ to lift $\mathcal{H}$ to $\mathcal{M}$ and the identity to lift $\mathcal{M}$ to $\mathcal{M}$, combining the usual methods for lifting trees on phalanxes (as in [4] and [25]), handling various details much as in the proof of Lemma 3.34, in particular to see that the strategy is indeed an $\mathcal{F}$-strategy. We can therefore compare $\mathfrak{P}$ with $\mathcal{M}$. The analysis of the comparison is mostly routine, using weak DJ to rule out various possibilities. The only, small, difference is when $b^{\mathcal{T}}$ is above $\mathcal{H}$ without drop and $M_\infty^{\mathcal{T}} \trianglelefteq M_\infty^{\mathcal{U}}$. Because $\mathcal{H}$ is a universal hull of $\mathcal{C} = \mathfrak{C}_1(\mathcal{M})$, this implies that $b^{\mathcal{U}}$ does not drop and $M_\infty^{\mathcal{T}} = M_\infty^{\mathcal{U}}$; now deduce that $\mathcal{M}||\rho^{+\mathcal{M}} = \mathcal{H}||\rho^{+\mathcal{H}}$ as usual, completing the proof.[51]

We now show that $\mathcal{C} = \mathcal{H}$, and therefore that $\mathcal{C}$ is an $\mathcal{F}$-pm. Recall that $\mathcal{H}$ is an $\mathcal{F}$-pm, $\mathcal{H}^- = \mathcal{C}^-$, $\sigma : \mathcal{H} \to \mathcal{C}$ is a 0-embedding which is above $\mathcal{H}^-$, and every $\underset{\sim}{\mathrm{r}\Sigma}_1^{\mathcal{H}}$ subset of $\mathcal{H}^- = \mathcal{C}^-$ is $\underset{\sim}{\mathrm{r}\Sigma}_1^{\mathcal{H}}$. Therefore $\rho_1^{\mathcal{H}} = \rho = \rho_1^{\mathcal{C}} < \rho_\omega^{\mathcal{H}^-}$, and since $\mathcal{H}$ is $\mathcal{H}^-$-sound, also $p_1^{\mathcal{C}} \leq \sigma(p_1^{\mathcal{H}})$. Recall from Definition 2.39 that $q^{\mathcal{H}} = p_1^{\mathcal{H}} \cap (\mathrm{Ord}^{\mathcal{H}^-}, \mathrm{Ord}^{\mathcal{H}})$, and likewise for $q^{\mathcal{C}}$. Since $\mathcal{H}$ is $(1, q^{\mathcal{H}})$-solid, $\mathcal{C}$ is $(1, \sigma(q^{\mathcal{H}}))$-solid (since by stratification, σ preserves solidity), so $\sigma(q^{\mathcal{H}}) \trianglelefteq p_1^{\mathcal{C}}$. And since σ is above $\mathcal{C}^-$ and $\mathcal{H}|\rho^{+\mathcal{H}} = \mathcal{C}|\rho^{+\mathcal{C}}$, it follows that $\sigma(p_1^{\mathcal{H}}) = p_1^{\mathcal{C}}$. But by 1-universality, $\pi(p_1^{\mathcal{C}}) = p_1^{\mathcal{M}}$, so $\mathcal{C} = \mathrm{Hull}_1^{\mathcal{C}}(A \cup \rho \cup p_1^{\mathcal{C}})$, so $\mathcal{H} = \mathcal{C}$ and $\sigma = \mathrm{id}$, completing the proof.

Now suppose $k > 0$. Then $\mathcal{C} = \mathfrak{C}_{k+1}(\mathcal{M})$ is an opm by 2.42, and is k-relevant as $\rho_{k+1}^{\mathcal{C}} < \rho_k^{\mathcal{C}} \leq \rho_\omega^{\mathcal{C}^-}$. So by fine condensation and 3.26, $\mathcal{C} = \mathcal{F}(\mathcal{C}^-)$ is an $\mathcal{F}$-pm. The rest is a simplification of the argument for $k = 0$.

Now consider $(k+1)$-solidity. Here we will not assume that $\mathcal{M}$ is a successor, since there are some subtleties we want to deal with explicitly, which do not arise in the successor case. Let $q = p_{k+1}^{\mathcal{M}}$, $i < \mathrm{lh}(q)$, $\mathcal{W} = \mathcal{W}_{k+1}^{\mathcal{M}}(q_i)$ be the $(k+1)$-solidity witness for $\mathcal{M}$ at q_i (see 2.27), and $\sigma : \mathcal{W} \to \mathcal{M}$ the uncollapse map. We have $\rho_{k+1}^{\mathcal{W}} \leq \mu$ where $\mu = \mathrm{crit}(\sigma) = q_i$. By 2.38 we may assume that σ is k-good, so:

- $\mathcal{W}$ is a k-sound Q-opm,
- if $\mathcal{M}$ is a successor then $\sigma(\mathcal{W}^-) = \mathcal{M}^-$, and

---

[50] A $(k_0, k_1, \ldots, k)$-maximal tree on a phalanx $((M_0, \rho_0), (M_1, \rho_1), \ldots, H)$, is one formed according to the usual rules for k-maximal trees, except that an extender E with $\rho_{i-1} \leq \mathrm{crit}(E) < \rho_i$ (where $\rho_{-1} = 0$) is applied to M_i, at degree k_i.

[51] There are some minor details involved here which are not explicitly discussed in [25], etc, and which we have not mentioned. We will cover such details in the solidity to proof to follow.

- if $\mathcal{M}$ is a limit or $k > 0$ then $\mathcal{W}$ is an opm.

We may also assume that $\mathcal{W}^{\mathcal{M}}_{k+1}(q_j) \in \mathcal{M}$ for each $j < i$, or in other words,

$$\mathcal{M} \text{ is } (k+1)\text{-solid with respect to } q{\restriction}i. \tag{3.3}$$

Suppose for the moment that $\mathcal{M}$ is a successor. Then by 2.41 we may assume that $\mu < \rho_\omega^{\mathcal{M}^-}$, so $\mu \leq \rho_\omega^{\mathcal{W}^-}$. Suppose $\mu = \rho_\omega^{\mathcal{W}^-}$. Then since $\mathcal{M}^-$ is $<\omega$-condensing, either $\mathcal{W}^- \trianglelefteq \mathcal{M}^-$ or $\mathcal{M}|\mu$ is active and $\mathcal{W}^- \trianglelefteq \mathrm{Ult}(\mathcal{M}|\mu, F^{\mathcal{M}|\mu})$, and in either case, $\mathcal{F}(\mathcal{W}^-) \in \mathcal{M}^-$. But by the fine condensation of $\mathcal{F}$, $\mathcal{W}$ is computable from $\mathcal{F}(\mathcal{W}^-)$, and so $\mathcal{W} \in \mathcal{M}^-$, which suffices. (That is, if $\mathcal{W} \in \mathcal{F}(\mathcal{W}^-)$ or $\mathcal{W} = \mathcal{F}(\mathcal{W}^-)$, we are done. But otherwise, by fine condensation, $k = 0$ and there is a universal hull $\mathcal{H}$ of $\mathcal{W}$ such that $\mathcal{H} \in \mathcal{F}(\mathcal{W}^-)$ or $\mathcal{H} = \mathcal{F}(\mathcal{W}^-)$, and so $\mathcal{H} \in \mathcal{M}^-$. But $\mathcal{W} = \mathrm{Hull}_1^{\mathcal{W}}(\mathcal{W}^- \cup \{x\})$ for some $x \in \mathcal{W}^-$, so working in $\mathcal{M}^-$, we can recover $\mathcal{W}$ from $\mathcal{H}$.) So we may assume that $\mu < \rho_\omega^{\mathcal{W}^-}$, so $\mathcal{W}$ is k-relevant, so $\mathcal{W} \notin \mathcal{F}(\mathcal{W}^-)$ and if $k = 0$ then $\mathcal{W}$ has no universal hull in $\mathcal{F}(\mathcal{W}^-)$. If $k = 0$, let $\mathcal{H} = \mathcal{F}(\mathcal{W}^-)$; by fine condensation, $\mathcal{H}$ is an $\mathcal{F}$-pm, and is a universal hull of $\mathcal{W}$; let $\sigma' : \mathcal{H} \to \mathcal{W}$ be a map witnessing this, with $\sigma' = \mathrm{id}$ if $\mathcal{H} = \mathcal{W}$. If $k > 0$ then $\mathcal{W}$ is an opm, so by fine condensation, $\mathcal{W} = \mathcal{F}(\mathcal{W}^-)$ is an $\mathcal{F}$-pm. If $k > 0$, let $\mathcal{H} = \mathcal{W}$.

Now if $\mathcal{M}$ is in fact a limit, then so is $\mathcal{W}$; in this case let $\mathcal{H} = \mathcal{W}$.

We now return to the general case (successor or limit). We just need to see that $\mathcal{H} \in \mathcal{M}$. (For if $\mathcal{W} \neq \mathcal{H}$ then $\mathcal{M}$ is a sucessor, $\mu < \rho_\omega^{\mathcal{W}^-}$, $k = 0$ and $\mathcal{W} = \mathrm{Hull}_1^{\mathcal{W}}(\mu \cup \{x\})$ for some $x \in \mathcal{W}$, and since $\mathcal{H}$ is a universal hull, $\mathrm{Th}_{\Sigma_1}^{\mathcal{W}}(\mu \cup \{x\}) \in \mathcal{J}(\mathcal{H}) \subseteq \mathcal{M}$.) So let us assume from now on, for simplicity, that

$$\mathcal{H} \notin \mathcal{M};$$

we will derive a contradiction. It easily follows that

$$\mu = \kappa^{+\mathcal{H}} < \mathrm{Ord}^{\mathcal{H}}.$$

If $\mathcal{H} = \mathcal{W}$ let $\sigma' = \mathrm{id} : \mathcal{H} \to \mathcal{H}$. Let $\pi = \sigma \circ \sigma' : \mathcal{H} \to \mathcal{M}$. Note that $\mathrm{crit}(\pi) = \mu$.

Let us also assume from now on, for focus, that μ is not an $\mathcal{M}$-cardinal, since the $\mathcal{M}$-cardinal case is easier and more routine. So we have $\mu = \kappa^{+\mathcal{H}} = \kappa^{+\mathcal{W}}$ for some $\mathcal{M}$-cardinal κ. Let $\mathcal{R} \triangleleft \mathcal{M}$ be least such that $\mu \leq \mathrm{Ord}^{\mathcal{R}}$ and $\rho_\omega^{\mathcal{R}} = \kappa$. Let

$$\mathfrak{P} = ((\mathcal{M}, <\kappa), (\mathcal{R}, \kappa), \mathcal{H}).$$

Then $\mathfrak{P}$ is (k, r, k)-maximally $\mathcal{F}$-iterable, where r is least such that $\rho_{r+1}^{\mathcal{R}} = \kappa$, by lifting to k-maximal trees $\mathcal{V}$ on $\mathcal{M}$ (possibly $r = -1$, which holds iff $\mathcal{R}$ is active type 3 with $\mu = \mathrm{Ord}^{\mathcal{R}}$). In fact, fix an enumeration e of $\mathcal{M}$ in ordertype ω, with $e{\restriction}i = q{\restriction}i$ (recall $q = p_{k+1}^{\mathcal{M}}$ and $q_i = \mu$). Fix a $(k, \omega_1 + 1)$-iteration strategy Σ for $\mathcal{M}$ with weak DJ with respect to e. Then we will form $\mathcal{V}$ according to Σ.

Note here that since $\mathcal{R} \triangleleft \mathcal{M}$ and $\rho_\omega^{\mathcal{R}} = \rho_{r+1}^{\mathcal{R}} = \kappa$ is an $\mathcal{M}$-cardinal, the model dropdown sequence of $\mathcal{R}$ (see 3.32) contains only one element, $\mathcal{R}$ itself. Moreover, $\mathcal{R}$ is $\mathcal{M}$-stable (see 3.32), since $\kappa < \rho_0^{\mathcal{M}}$. So the usual methods for lifting trees on phalanxes (as in [25], [9] and [4]), combined with the methods in the proof of Lemma 3.34, do work here. In particular, note that we are in the situation described in [9, p. 728, proof of Theorem 3.3, paragraph "There is one wrinkle in the copying argument..."], and we use that method mentioned there, defining $\pi_{\eta+1} : M^{\mathcal{T}}_{\eta+1} \to \mathcal{R}' = M^{\mathcal{V}}_{\mathcal{R},\iota(\eta+1)} \triangleleft M^{\mathcal{V}}_{\iota(\eta+1)}$ when $\mathrm{crit}(E^{\mathcal{T}}_\eta) = \kappa$. Note that $\kappa' = i^{\mathcal{V}}_{0,\iota(\eta+1)}(\kappa)$ is an $M^{\mathcal{T}}_{\iota(\eta+1)}$-cardinal and $\rho^{\mathcal{R}'}_{r+1} = \kappa' < \rho^{\mathcal{R}'}_r$. As discussed in the analogous situation in [11, Proof of Claim 9, §14.2***], letting $E^{\mathcal{V}}_{\eta'}$ be the copy of $E^{\mathcal{T}}_\eta$, if $E^{\mathcal{V}}_{\eta'}$ is superstrong, then there is a further small wrinkle. For in this case, $E^{\mathcal{T}}_\eta$ is also superstrong and $\pi_{\eta+1}(\lambda(E^{\mathcal{T}}_\eta)) = \kappa' < (\kappa')^{+\mathcal{R}'} = \pi_{\eta+1}(\mathrm{lh}(E^{\mathcal{T}}_\eta))$, and by commutativity, it is easy enough to see that $\pi_{\eta+1}$ is non-ν-high, so $E^{\mathcal{V}}_{\eta'+1}$ is the $\pi_{\eta+1}$-copy of $E^{\mathcal{T}}_{\eta+1}$, and $\mathrm{lh}(E^{\mathcal{T}}_\eta) < \mathrm{lh}(E^{\mathcal{T}}_{\eta+1})$, so $(\kappa')^{+\mathcal{R}'} < \mathrm{lh}(E^{\mathcal{V}}_{\eta'+1}) \leq \mathrm{Ord}^{\mathcal{R}'} < \mathrm{lh}(E^{\mathcal{V}}_{\eta'})$, and in particular, $\mathrm{lh}(E^{\mathcal{V}}_{\eta'+1}) < \mathrm{lh}(E^{\mathcal{V}}_{\eta'})$. So $\mathcal{V}$ itself is not in fact a k-maximal tree; it is *essentially-k-maximal* in the sense of [11, §6.2***]. But this does not actually cause a problem, because one can easily replace $\mathcal{V}$ with an essentially equivalent k-maximal tree $\mathcal{V}'$, in particular with $M^{\mathcal{V}'}_\infty = M^{\mathcal{V}}_\infty$ and corresponding iteration maps. In $\mathcal{V}'$, $E^{\mathcal{V}}_{\eta'}$ does not get used; instead, the next extender used in $\mathcal{V}'$ is $E^{\mathcal{V}}_{\eta'+1}$, unless that extender is also superstrong with critical point κ, in which case $\mathrm{lh}(E^{\mathcal{V}}_{\eta'+2}) < \mathrm{lh}(E^{\mathcal{V}}_{\eta'+1})$, etc, producing a (finitely long) strictly decreasing sequence. Then the next extender used in $\mathcal{V}'$ is simply the next one used in $\mathcal{V}$ which is *not* superstrong with critical point κ, if there is one, or otherwise the last extender used in $\mathcal{V}$. See [11, §6.2***] for more details. We will continue to work with $\mathcal{V}$ itself, not $\mathcal{V}'$, however.

Let $(\mathcal{T},\mathcal{U})$ be the successful comparison of $(\mathfrak{P},\mathcal{M})$ and let $\mathcal{V}$ be the lift of $\mathcal{T}$ to a tree on $\mathcal{M}$, formed with $\mathcal{T}$ and $\mathcal{V}$ according to Σ. In order that we can make use of standard fine structural preservation arguments (like [4, Lemma 4.5]), we want to know:

Claim 3.45.1. *$E^{\mathcal{T}}_\alpha$ is close to $M^{*\mathcal{T}}_{\alpha+1}$ for every $\alpha+1 < \mathrm{lh}(\mathcal{T})$.*

Recall here we are assuming that $\mathcal{H} \notin \mathcal{M}$; this will be used in the proof of the claim. Since $\mathfrak{P}$ is a phalanx, the claim does not literally follow from [4, Lemma 6.1.5], but needs a slight variant of that argument. Such arguments were also given in [14] and [11, Proof of Claim 10, §14.2***], but we include an argument here for better self-containment. We will follow an inductive proof much as in [4, 6.1.5], and the reader should have that proof in mind; we will point out the key differences.

Proof of Claim. Let $b \in [\nu(E^{\mathcal{T}}_\alpha)]^{<\omega}$ and consider the measure $(E^{\mathcal{T}}_\alpha)_b$. We must see that

$$(E^{\mathcal{T}}_\alpha)_b \text{ is } \utilde{\mathrm{r}\Sigma}_1^{M^{*\mathcal{T}}_{\alpha+1}}. \tag{3.4}$$

Suppose first that $(E^{\mathcal{T}}_\alpha)_b \in M^{\mathcal{T}}_\alpha$. Things are as usual by induction as usual unless $\mathrm{pred}^{\mathcal{T}}(\alpha+1) = \mathcal{M}$ or $\mathrm{pred}^{\mathcal{T}}(\alpha+1) = \mathcal{R}$.

Suppose $\mathrm{pred}^{\mathcal{T}}(\alpha+1) = \mathcal{M}$, and so $M^{*\mathcal{T}}_{\alpha+1} = \mathcal{M}$ (as $E^{\mathcal{T}}_\alpha$ is $\mathcal{M}$-total in this case). If $\mathrm{root}^{\mathcal{T}}(\alpha) = \mathcal{M}$ then line (3.4) is shown as usual. If $\mathrm{root}^{\mathcal{T}}(\alpha) = \mathcal{H}$ then we get as usual that $(E^{\mathcal{T}}_\alpha)_b$ is $\utilde{\mathrm{r}\Sigma}_1^{\mathcal{H}}$, but then since $\pi : \mathcal{H} \to \mathcal{M}$ is Σ_1-elementary and $\mathrm{crit}(\pi)=\mu$, it follows that $(E^{\mathcal{T}}_\alpha)_b$ is $\mathrm{r}\utilde{\Sigma}_1^{\mathcal{M}}$ also. And if $\mathrm{root}^{\mathcal{T}}(\alpha)=\mathcal{R}$ then it is similar, but this time because $\mathcal{R}\in\mathcal{M}$.

So suppose that $\mathrm{pred}^{\mathcal{T}}(\alpha+1)=\mathcal{R}$, so $M^{*\mathcal{T}}_{\alpha+1}=\mathcal{R}$ and $\mathrm{crit}(E^{\mathcal{T}}_\alpha)=\kappa$. Note then that since $\kappa^{+\mathcal{H}}=\mu\leq\mathrm{lh}(E^{\mathcal{T}}_0)$ and $\mu<\mathrm{Ord}^{\mathcal{H}}$ and $E^{\mathcal{T}}_0\in\mathbb{E}^{\mathcal{H}}_+$, in fact $\kappa^{+\mathcal{H}}<\mathrm{lh}(E^{\mathcal{T}}_0)$, and so $(E^{\mathcal{T}}_\alpha)_b\in\mathcal{H}$. Since also $(E^{\mathcal{T}}_\alpha)_b\subseteq\kappa^{+\mathcal{H}}=\mathrm{crit}(\pi)$, we can fix $\mathcal{Y}\triangleleft\mathcal{H}$ such that $\rho^{\mathcal{Y}}_\omega=\kappa^{+\mathcal{H}}$ and $(E^{\mathcal{T}}_\alpha)_b\in\mathcal{Y}$. By condensation applied to $\pi\restriction\mathcal{Y}:\mathcal{Y}\to\pi(\mathcal{Y})$, either $\mathcal{M}|\kappa^{+\mathcal{H}}$ is passive and $\mathcal{Y}\triangleleft\mathcal{M}$, or $\mathcal{M}|\kappa^{+\mathcal{H}}$ is active with extender F and $\mathcal{Y}\triangleleft\mathrm{Ult}(\mathcal{M}|\mathrm{lh}(F),F)$. But if $\mathcal{M}|\kappa^{+\mathcal{H}}$ is passive, then since $\rho^{\mathcal{R}}_\omega=\kappa<\kappa^{+\mathcal{H}}=\rho^{\mathcal{Y}}_\omega$, we must have $\mathcal{Y}\triangleleft\mathcal{R}$, so $(E^{\mathcal{T}}_\alpha)_b\in\mathcal{R}$. And if $\mathcal{M}|\kappa^{+\mathcal{H}}$ is active with F, then $\mathcal{R}=\mathcal{M}|\mathrm{lh}(F)$, so $\mathcal{Y}\triangleleft\mathrm{Ult}(\mathcal{R},F^{\mathcal{R}})$, so $(E^{\mathcal{T}}_\alpha)_b$ is $\mathrm{r}\utilde{\Sigma}_1^{\mathcal{R}}$. This establishes line (3.4) in this case.

Now suppose that $(E^{\mathcal{T}}_\alpha)_b\notin M^{\mathcal{T}}_\alpha$. So $E^{\mathcal{T}}_\alpha=F(M^{\mathcal{T}}_\alpha)$ and $M^{\mathcal{T}}_\alpha$ is active type 1 or 2, and $\rho_1^{M^{\mathcal{T}}_\alpha}\leq\theta^{+\mathcal{H}}$ where $\theta=\mathrm{crit}(E^{\mathcal{T}}_\alpha)$. If $\mathrm{crit}(E^{\mathcal{T}}_\alpha)>\kappa$, then one can argue essentially as in [4, 6.1.5], so suppose $\mathrm{crit}(E^{\mathcal{T}}_\alpha)\leq\kappa$. Suppose first that $\mathrm{crit}(E^{\mathcal{T}}_\alpha)<\kappa$, so $\mathrm{pred}^{\mathcal{T}}(\alpha+1)=\mathcal{M}$ and $M^{*\mathcal{T}}_{\alpha+1}=\mathcal{M}$ and $\rho_1^{M^{\mathcal{T}}_\alpha}\leq\kappa$. Letting $\xi=\mathrm{root}^{\mathcal{T}}(\alpha)$, the argument in [4] therefore shows that $(\xi,\alpha]^{\mathcal{T}}$ does not drop in model, and that $\mathrm{crit}(E^{\mathcal{T}}_\alpha)<\mathrm{crit}(i^{\mathcal{T}}_{\xi\alpha})$. By induction, all extenders used along $(\xi,\alpha]^{\mathcal{T}}$ are close to their target model, and it follows that $(E^{\mathcal{T}}_\alpha)_b$ is $\mathrm{r}\utilde{\Sigma}_1^{\mathcal{N}}$, where $\mathcal{N}=M^{\mathcal{T}}_\xi$. Like before, since $\mathcal{R}\in\mathcal{M}$ and $\pi:\mathcal{H}\to\mathcal{M}$ is Σ_1-elementary, this suffices to give line (3.4).

So now suppose that $\mathrm{crit}(E^{\mathcal{T}}_\alpha)=\kappa$, so $\mathrm{pred}^{\mathcal{T}}(\alpha+1)=\mathcal{R}$. Supposing $\mathcal{T}$ drops in model in $(\xi,\alpha]^{\mathcal{T}}$, let $\gamma+1$ be the last such drop. Then as in [4], $\kappa<\mathrm{crit}(i^{*\mathcal{T}}_{\gamma+1,\alpha})$, so $\mathrm{pred}^{\mathcal{T}}(\gamma+1)\neq\mathcal{M}$ and $\mathrm{pred}^{\mathcal{T}}(\gamma+1)\neq\mathcal{R}$, and $(E^{\mathcal{T}}_\alpha)_b$ is $\mathrm{r}\utilde{\Sigma}_1^{M^{*\mathcal{T}}_{\gamma+1}}$. So $(E^{\mathcal{T}}_\alpha)_b\in M^{\mathcal{T}}_\delta$ where $\mathrm{pred}^{\mathcal{T}}(\gamma+1)=\delta$, but then $(E^{\mathcal{T}}_\alpha)_b\in\mathcal{H}$, and we can deduce that $(E^{\mathcal{T}}_\alpha)_b$ is $\mathrm{r}\utilde{\Sigma}_1^{\mathcal{R}}$ as before. So suppose $\mathcal{T}$ does not drop in model in $(\xi,\alpha]^{\mathcal{T}}$. Then $\kappa<\mathrm{crit}(i^{\mathcal{T}}_{\xi\alpha})$, and so $\xi=\mathcal{H}$. Since $M^{\mathcal{T}}_\alpha$ is active type 1 or 2 and $\kappa=\mathrm{crit}(E^{\mathcal{T}}_\alpha)=\mathrm{crit}(F(M^{\mathcal{T}}_\alpha))$, $\mathcal{H}$ is therefore also active type 1 or 2 and $\kappa=\mathrm{crit}(F^{\mathcal{H}})$, so $\mathcal{M}$ is active type 1 or 2 and $\kappa=\mathrm{crit}(F^{\mathcal{M}})$. But $\mathcal{H}$ is determined by $F^{\mathcal{H}}$ and $\mathcal{H}|\kappa^{+\mathcal{H}}$, and $F^{\mathcal{H}}$ is finitely generated (that is, generated by finitely many generators), since $\mathcal{H}=\mathrm{Hull}_1^{\mathcal{H}}(\mu\cup\{x\})$ for some finite x. And $\kappa^{+\mathcal{H}}=\mu<\kappa^{+\mathcal{M}}$, so $F^{\mathcal{H}}\in\mathcal{M}$, so $\mathcal{H}\in\mathcal{M}$, a contradiction. $\square$

Claim 3.45.2. *For all $\delta<\mathrm{lh}(\mathcal{T})$, if either δ is above $\mathcal{H}$ or above $\mathcal{M}$ or $(\mathrm{root}^{\mathcal{T}}(\delta),\delta]^{\mathcal{T}}$ drops in model then $\deg^{\mathcal{T}}_\delta=\deg^{\mathcal{V}}_{\iota(\delta)}$ and $\pi_\delta:M^{\mathcal{T}}_\delta\to M^{\mathcal{V}}_{\iota(\delta)}$ is a near $\deg^{\mathcal{T}}_\delta$-embedding.*

(Note here that if δ is above $\mathcal{R}$ and $(\mathcal{R},\delta]^{\mathcal{T}}$ drops in model then $[0,\iota(\xi+1)]^{\mathcal{V}}$ drops below the image of $\mathcal{R}$, so $M^{\mathcal{V}}_{\mathcal{R},\iota(\xi+1)}=M^{\mathcal{V}}_{\iota(\xi+1)}$.)

Proof. If δ is above $\mathcal{M}$ and there is no drop in model or degree in $(\mathcal{M},\delta]^{\mathcal{T}}$ then $(0,\iota(\delta)]^{\mathcal{V}}$ also does not drop in model or degree, and π_δ is in fact a $\deg^{\mathcal{T}}_\delta$-embedding, by its commutativity with the iteration maps. This applies in particular in the case that δ is a successor and $\mathrm{pred}^{\mathcal{T}}(\delta+1)=\mathcal{M}$, since $\kappa<\mu<\rho^{\mathcal{M}}_k$. And since $\mathcal{H}\notin\mathcal{M}$, $\pi``\rho^{\mathcal{H}}_k$ is cofinal in $\rho^{\mathcal{M}}_k$, so similar remarks apply to the case that δ is above $\mathcal{H}$ and there is no drop in model or degree in $(\mathcal{H},\delta]^{\mathcal{T}}$. In the remaining cases, the claim now follows from an inspection of the proof of [8, Lemma 1.3]. □

We now begin to analyse the comparison. We will make use of the claim implicitly, in the usual fashion.

Claim 3.45.3. *$M^{\mathcal{T}}_\infty = M^{\mathcal{U}}_\infty$ and $b^{\mathcal{U}}$ does not drop in model or degree.*

Proof. We can't have $M^{\mathcal{U}}_\infty \triangleleft M^{\mathcal{T}}_\infty$, by weak DJ. Suppose $M^{\mathcal{T}}_\infty \triangleleft M^{\mathcal{U}}_\infty$. Then by weak DJ, $b^{\mathcal{T}}$ is not above $\mathcal{M}$, so it is above $\mathcal{R}$ or $\mathcal{H}$. And $M^{\mathcal{T}}_\infty$ is sound, which implies $\mathcal{T}$ is trivial and $M^{\mathcal{T}}_\infty = \mathcal{H}$. But then it follows that $\mathcal{H} \in \mathcal{M}$, a contradiction. So $M^{\mathcal{T}}_\infty = M^{\mathcal{U}}_\infty$.

Suppose $b^{\mathcal{U}}$ drops in model or degree; so $b^{\mathcal{T}}$ does not drop in model or degree. If $b^{\mathcal{T}}$ is above $\mathcal{M}$, then $M^{\mathcal{T}}_\infty$ is k-sound, so $b^{\mathcal{U}}$ must drop in model, and then $i^{\mathcal{T}}_{\mathcal{M}\infty} : \mathcal{M} \to M^{\mathcal{T}}_\infty = M^{\mathcal{U}}_\infty$ is nearly k-good, contradicting weak DJ (as $b^{\mathcal{U}}$ drops). If $b^{\mathcal{T}}$ is above $\mathcal{R}$, then because $\mathcal{R}$ is sound, it is just as if $b^{\mathcal{T}}$ drops in model, and so we reach the usual contradiction via compatible extenders. So $b^{\mathcal{T}}$ is above $\mathcal{H}$, but then the usual calculations yield that $\mathcal{H} \in \mathcal{M}$, a contradiction. □

Claim 3.45.4. *$b^{\mathcal{T}}$ is not above $\mathcal{M}$.*

Proof. Suppose $b^{\mathcal{T}}$ is above $\mathcal{M}$. If $b^{\mathcal{T}}$ does not drop in model or degree, then by weak DJ, $i^{\mathcal{T}} = i^{\mathcal{U}}$, giving the usual contradiction to comparison. So $b^{\mathcal{T}}$ drops in model or degree, and (since $M^{\mathcal{T}}_\infty = M^{\mathcal{U}}_\infty$ is k-sound) therefore in model, and hence $b^{\mathcal{V}}$ also drops in model. By Claim 3.45.2, $\pi_\infty : M^{\mathcal{T}}_\infty \to M^{\mathcal{V}}_\infty$ is a near $\deg^{\mathcal{T}}_\infty$-embedding, and since $\deg^{\mathcal{T}}_\infty \geq k$ (as $M^{\mathcal{U}}_\infty$ is k-sound), therefore π_∞ is a near k-embedding. But then $\pi_\infty \circ i^{\mathcal{U}} : \mathcal{M} \to M^{\mathcal{V}}_\infty$ is a near k-embedding, contradicting weak DJ. □

Claim 3.45.5. *$b^{\mathcal{T}}$ is not above $\mathcal{H}$.*

Proof. Suppose it is above $\mathcal{H}$. Just as in the proof of Claim 3.45.4, it does not drop in model or degree. By Claim 3.45.2, $\pi_\infty : M^{\mathcal{T}}_\infty \to M^{\mathcal{V}}_\infty$ is a near k-embedding. So note that by our choice of Σ and enumeration of e (with $e{\restriction}i = q{\restriction}i$, where $q = p^{\mathcal{M}}_{k+1}$), letting $\pi(\bar{q}) = q{\restriction}i$, we have $i^{\mathcal{T}}_{\mathcal{H}\infty}(\bar{q}) \leq i^{\mathcal{U}}(q{\restriction}i)$. (For $\pi_\infty(i^{\mathcal{T}}_{\mathcal{H}\infty}(\bar{q})) = i^{\mathcal{V}}(\pi(\bar{q})) = i^{\mathcal{V}}(q{\restriction}i) \leq \pi_\infty(i^{\mathcal{U}}(q{\restriction}i))$, by weak DJ.)

On the other hand, $i^{\mathcal{U}}(q{\restriction}i) \leq i^{\mathcal{T}}_{\mathcal{H}\infty}(\bar{q})$. For by our inductive hypothesis (3.3), $\mathcal{M}$ is $(k+1)$-solid with respect to $q{\restriction}i$, so $M^{\mathcal{U}}_\infty$ is $(k+1)$-solid with respect to $i^{\mathcal{U}}(q{\restriction}i)$, so if $i^{\mathcal{T}}_{\mathcal{H}\infty}(\bar{q}) < i^{\mathcal{U}}(q{\restriction}i)$ then

$$\mathrm{Th}^{M^{\mathcal{U}}_\infty}_{k+1}(\mu \cup \{\vec{p}^{\,M^{\mathcal{U}}_\infty}_k, i^{\mathcal{T}}_{\mathcal{H}\infty}(\bar{q})\}) \in M^{\mathcal{U}}_\infty,$$

but $\mathcal{H}$ and $M^{\mathcal{U}}_\infty$ have the same subsets of μ, so then

$$\mathrm{Th}^{\mathcal{H}}_{k+1}(\mu \cup \{\vec{p}^{\,\mathcal{H}}_k, \bar{q}\}) \in \mathcal{H},$$

whereas $\mathcal{H} = \mathrm{Hull}^{\mathcal{H}}_{k+1}(\mu \cup \{\vec{p}^{\,\mathcal{H}}_k, \bar{q}\})$, which is impossible.

So we have established that $i^{\mathcal{T}}_{\mathcal{H}\infty}(\bar{q}) = i^{\mathcal{U}}(q{\restriction}i)$.

Now if $\rho^{\mathcal{M}}_{k+1} \leq \mathrm{crit}(i^{\mathcal{U}})$ then $\rho^{\mathcal{M}}_{k+1} = \rho^{M^{\mathcal{U}}_\infty}_{k+1} = \rho^{M^{\mathcal{T}}_\infty}_{k+1} = \rho^{\mathcal{H}}_{k+1} \leq \mathrm{crit}(i^{\mathcal{T}})$, and $t = \mathrm{Th}^{\mathcal{M}}_{k+1}(\rho^{\mathcal{M}}_{k+1} \cup \{\vec{p}^{\mathcal{M}}_{k+1}\})$ is $\utilde{\mathrm{r}\Sigma}^{M^{\mathcal{U}}_\infty}_{k+1}$, hence $\utilde{\mathrm{r}\Sigma}^{\mathcal{H}}_{k+1}$, and $\utilde{\mathrm{r}\Sigma}^{\mathcal{W}}_{k+1}$, which contradicts the minimality of $p^{\mathcal{M}}_{k+1}$. So $\mathrm{crit}(i^{\mathcal{U}}) < \rho^{\mathcal{M}}_{k+1}$. A straightforward calculation shows that $\sup i^{\mathcal{U}} \text{“} \rho^{\mathcal{M}}_{k+1} \leq \rho^{M^{\mathcal{U}}_\infty}_{k+1}$.[52] But $\rho^{\mathcal{H}}_{k+1} \leq \mu \leq \mathrm{crit}(i^{\mathcal{T}})$, and it follows that $\rho^{\mathcal{H}}_{k+1} = \mu = \mathrm{lh}(E^{\mathcal{U}}_0)$ and $E^{\mathcal{U}}_0$ is superstrong, and $\mathrm{crit}(E^{\mathcal{U}}_0)^{+\mathcal{M}} = \rho^{\mathcal{M}}_{k+1}$, and $\rho^{M^{\mathcal{U}}_1}_{k+1} = \mu$.[53] But since $q_i = \mu \in p^{\mathcal{M}}_{k+1}$, we have

$$u = \mathrm{Th}^{\mathcal{M}}_{k+1}(\rho^{\mathcal{M}}_{k+1} \cup \{\vec{p}^{\mathcal{M}}_k, q{\restriction}i\}) \in \mathcal{M}.$$

So $i^{\mathcal{U}}(u) \in M^{\mathcal{U}}_\infty$, and the usual arguments with solidity witnesses (and preservation of the standard parameter under iteration maps) show that from $i^{\mathcal{U}}(u)$, we can recover

$$u' = \mathrm{Th}^{M^{\mathcal{U}}_\infty}_{k+1}(\mu \cup \{\vec{p}^{M^{\mathcal{U}}_\infty}_k, i^{\mathcal{U}}(q{\restriction}i)\}) \in M^{\mathcal{U}}_\infty.$$

But then since $i^{\mathcal{T}}_{\mathcal{H}\infty}(\bar{q}) = i^{\mathcal{U}}(q{\restriction}i)$, we again get that $\mathrm{Th}^{\mathcal{H}}_{k+1}(\mu \cup \{\vec{p}^{\mathcal{H}}_k, \bar{q}\}) \in \mathcal{H}$, again a contradiction. ☐

Claim 3.45.6. *$b^{\mathcal{T}}$ is above $\mathcal{R}$, does not drop in model, $0 \leq k = \mathrm{deg}^{\mathcal{T}}_\infty < r$.*

Proof. $b^{\mathcal{T}}$ is above $\mathcal{R}$ by Claims 3.45.4 and 3.45.5. Now if $b^{\mathcal{T}}$ drops in *model*, then by Claim 3.45.2, π_∞ is nearly k-good, so we can apply weak DJ for a contradiction. We have $r \geq 0$, since if $r = -1$ then $F^{M^{\mathcal{T}}_\infty}$ fails the ISC, and hence $M^{\mathcal{T}}_\infty$ is not an opm. Since $M^{\mathcal{T}}_\infty = M^{\mathcal{U}}_\infty$ is k-sound, $k \leq \ell \leq r$ where $\ell = \mathrm{deg}^{\mathcal{T}}_\infty$. The final copy map $\pi_\infty : M^{\mathcal{T}}_\infty \to M^{\mathcal{V}}_{\mathcal{R},\infty}$ is a weak ℓ-embedding. If $k < \ell$ then π_∞ is a near k-embedding, and so $\pi_\infty \circ i^{\mathcal{U}} : \mathcal{M} \to M^{\mathcal{V}}_{\mathcal{R},\infty}$ is also a near k-embedding, and either $M^{\mathcal{V}}_{\mathcal{R},\infty} \triangleleft M^{\mathcal{V}}_\infty$ or $b^{\mathcal{V}}$ drops in model, contradicting weak DJ. So $k = \ell \leq r$.

Now if $k = \ell = r$ then some fairly standard fine structural calculations give a contradiction: We have $\rho^{\mathcal{R}}_{k+1} = \kappa < \rho^{\mathcal{R}}_k$, and as $\mathrm{crit}(i^{\mathcal{T}}) = \kappa$. Using closeness, [4, Lemma 4.5] (adapted to our context) now gives that $\rho^{M^{\mathcal{T}}_\infty}_{k+1} = \kappa$ and $i^{\mathcal{T}}(p^{\mathcal{R}}_{k+1}) = p^{M^{\mathcal{T}}_\infty}_{k+1}$. As earlier, $\sup i^{\mathcal{U}} \text{“} \rho^{\mathcal{M}}_{k+1} \leq \rho^{M^{\mathcal{U}}_\infty}_{k+1} = \rho^{M^{\mathcal{T}}_\infty}_{k+1} = \kappa$. But since $\kappa < \mathrm{lh}(E^{\mathcal{U}}_\alpha)$ for all $\alpha + 1 < \mathrm{lh}(\mathcal{U})$, it follows that $\kappa \leq \mathrm{crit}(i^{\mathcal{U}})$, and so in fact $\kappa = \rho^{\mathcal{M}}_{k+1}$. But now as usual (as in the proof of [4, Lemma 4.5]), it follows that $\mathrm{Th}^{\mathcal{M}}_{\mathrm{r}\Sigma_{k+1}}(\rho^{\mathcal{M}}_{k+1} \cup \vec{p}^{\mathcal{M}}_{k+1})$ is definable from parameters over $\mathcal{R}$, and hence an element of $\mathcal{M}$, a contradiction. ☐

Now π_∞ can't be a near k-embedding, since otherwise $\pi_\infty \circ i^{\mathcal{U}} : \mathcal{M} \to M^{\mathcal{V}}_{\mathcal{R},\infty}$ is a near k-embedding, and as $\mathcal{R} \triangleleft \mathcal{M}$, this contradicts weak DJ. So the following claim reaches a contradiction, completing the proof of solidity:

---

[52] In fact $\rho^{M^{\mathcal{U}}_\infty}_{k+1} = \sup i^{\mathcal{U}} \text{“} \rho^{\mathcal{M}}_{k+1}$, by [11, Lemma 3.8***] or the methods of proof of [18, Corollary 2.24], but as discussed at the start of §3.8, we are avoiding using those results in this proof.

[53] In fact, letting $t' = \mathrm{Th}^{M^{\mathcal{U}}_1}_{k+1}(\mu \cup \{i^{\mathcal{T}}_{01}(\vec{p}^{\mathcal{M}}_{k+1})\})$, then $t' \notin M^{\mathcal{U}}_1$, for otherwise, $t' \in \mathcal{M}$, and $E^{\mathcal{U}}_0 \in \mathcal{M}$, but from t' and $E^{\mathcal{U}}_0$, one can easily compute t, so $t \in M$, a contradiction. Here t was defined above; a statement φ is in t iff $i_{E^{\mathcal{U}}_0}(\varphi) \in t'$.

Claim 3.45.7. *π_∞ is a near k-embedding.*

Proof. The proof will be a variant of the proof of [8, Lemma 1.3]. Let α_0 be least such that $\alpha_0+1 \in b^{\mathcal{T}}$, so $M^{*\mathcal{T}}_{\alpha_0+1} = \mathcal{R}$ and $\deg^{\mathcal{T}}_{\alpha_0+1} = r$. Since $k = \ell < r$ where $\ell = \deg^{\mathcal{T}}_\infty$, there is $\alpha_1+1 \in (\alpha_0+1, \infty]^{\mathcal{T}}$ such that, letting $\beta_1 = \mathrm{pred}^{\mathcal{T}}(\alpha_1+1)$, we have $\deg^{\mathcal{T}}_{\beta_1} = r$ and $k \leq \deg^{\mathcal{T}}_{\alpha_1+1} < r$. Let $\pi_\gamma : M^{\mathcal{T}}_\gamma \to M^{\mathcal{V}}_{\mathcal{R},\iota(\gamma)}$ be the copy map; we have $M^{\mathcal{V}}_{\mathcal{R},\iota(\gamma)} \trianglelefteq M^{\mathcal{V}}_{\iota(\gamma)}$. Then π_{β_1} is a weak r-embedding, and so a near $(r-1)$-embedding. Note moreover that $i^{\mathcal{T}}_{\mathcal{R},\beta_1}(\kappa) < \rho_r^{M^{\mathcal{T}}_{\beta_1}} \leq \mathrm{crit}(E^{\mathcal{T}}_{\alpha_1})$, so $\pi_{\beta_1}(i^{\mathcal{T}}_{\mathcal{R},\beta_1}(\kappa)) = i^{\mathcal{U}}_{\mathcal{R},0,\iota(\beta_1)}(\kappa) < \pi_{\beta_1}(\mathrm{crit}(E^{\mathcal{T}}_{\alpha_1}))$. So either $M^{\mathcal{V}}_{\iota(\beta_1)} = M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}$ or $\mathcal{V}$ drops in model at $\iota(\alpha_1+1)$ and $M^{\mathcal{V}}_{\iota(\alpha_1+1)} = M^{\mathcal{V}}_{\mathcal{R},\iota(\alpha_1+1)}$.

If $\rho_r^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}} \leq \pi_{\beta_1}(\mathrm{crit}(E^{\mathcal{T}}_{\alpha_1}))$ then $\deg^{\mathcal{T}}_{\alpha_1+1} = \deg^{\mathcal{V}}_{\alpha_1+1}$, and because π_{β_1} is a near $\deg^{\mathcal{T}}_{\alpha_1+1}$-embedding, an inspection of the proof of [8, Lemma 1.3] shows that for all $\xi \in [\alpha_1+1, \infty]^{\mathcal{T}}$, we have $\deg^{\mathcal{T}}_\xi = \deg^{\mathcal{V}}_{\iota(\xi)}$ and $\pi_\xi : M^{\mathcal{T}}_\xi \to M^{\mathcal{V}}_{\iota(\xi)} = M^{\mathcal{V}}_{\mathcal{R},\iota(\xi)}$ is a near $\deg^{\mathcal{T}}_\xi$-embedding, so π_∞ is a near k-embedding, as desired.

So suppose from now on that $\pi_{\beta_1}(\mathrm{crit}(E^{\mathcal{T}}_{\alpha_1})) < \rho_r^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}}$, and so $\deg^{\mathcal{V}}_{\iota(\alpha_1+1)} = r$ (since $\rho^{\mathcal{R}}_{r+1} = \kappa$, we have $\deg^{\mathcal{V}}_{\iota(\alpha_1+1)} \leq r$). Note then that $\mathrm{crit}(E^{\mathcal{T}}_{\alpha_1}) < \rho_{r-1}^{M^{\mathcal{T}}_{\beta_1}}$, since either $\rho_{r-1}^{M^{\mathcal{T}}_{\beta_1}} = \rho_0^{M^{\mathcal{T}}_{\beta_1}}$ and $\rho_{r-1}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}} = \rho_0^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}}$, or $\pi_{\beta_1}(\rho_{r-1}^{M^{\mathcal{T}}_{\beta_1}}) = \rho_{r-1}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}}$. So $\deg^{\mathcal{T}}_{\alpha_1+1} = r-1$.

For r-sound opm $\mathcal{N}$ and $\rho \leq \rho_0^{\mathcal{N}}$, the *weak $\mathrm{r}\utilde{\Sigma}_r^{\mathcal{N}}$-cofinality of $\rho^{\mathcal{N}}_{r-1}$*, denoted $\mathrm{wcof}^{\mathrm{r}\utilde{\Sigma}_r^{\mathcal{N}}}(\rho^{\mathcal{N}}_{r-1})$, is the least $\theta \leq \rho_r^{\mathcal{N}}$ such that there is $x \in \mathfrak{C}_0(\mathcal{N})$ with $\rho^{\mathcal{N}}_{r-1} \cap \mathrm{Hull}_r^{\mathcal{N}}(\theta \cup \{x\})$ cofinal in $\rho^{\mathcal{N}}_{r-1}$.

A degree r iteration map $j : \mathcal{N} \to \mathcal{N}'$ preserves $\mathrm{wcof}^{\mathrm{r}\utilde{\Sigma}_r}(\rho^{\mathcal{N}}_{r-1})$, to the extent that if $\theta = \mathrm{wcof}^{\mathrm{r}\utilde{\Sigma}_r^{\mathcal{N}}}(\rho^{\mathcal{N}}_{r-1}) < \rho_r^{\mathcal{N}}$ then $\mathrm{wcof}^{\mathrm{r}\utilde{\Sigma}_r^{\mathcal{N}'}}(\rho^{\mathcal{N}'}_{r-1}) = j(\theta)$, and if $\theta = \rho_r^{\mathcal{N}}$ then $\mathrm{wcof}^{\mathrm{r}\utilde{\Sigma}_r^{\mathcal{N}'}}(\rho^{\mathcal{N}'}_{r-1}) = \rho_r^{\mathcal{N}'}$ (but it might be that $j(\rho_r^{\mathcal{N}}) > \rho_r^{\mathcal{N}'}$). Moreover, say $\mathcal{X}$ is an iteration tree and $\mathcal{N} = M^{*\mathcal{X}}_{\alpha+1}$ and $\mathcal{N}' = M^{\mathcal{X}}_\infty$ and $j = i^{*\mathcal{X}}_{\alpha+1,\infty}$, where $(\alpha+1,\infty]^{\mathcal{X}} \cap \mathscr{D}^{\mathcal{X}} = \emptyset$ and $\deg^{\mathcal{X}}_{\alpha+1} = \deg^{\mathcal{X}}_\infty = r$. Then the following are equivalent:

- j is discontinuous at $\rho^{\mathcal{N}}_{r-1}$,
- $\theta < \rho_n^{\mathcal{N}}$ and j is discontinuous at θ,
- $\theta < \rho_n^{\mathcal{N}}$ and either $\theta = \mathrm{crit}(j)$ or there is γ such that $\alpha+1 \leq^{\mathcal{X}} \gamma <^{\mathcal{X}} \infty$ and $i^{*\mathcal{X}}_{\alpha+1,\gamma}(\theta) = \mathrm{crit}(i^{\mathcal{X}}_{\gamma,\infty})$.

These facts follow from some straightforward arguments; there are very similar calculations (and more) in [11, §6.1].

Let $\beta' \leq^{\mathcal{T}} \beta_1$ be largest such that $M^{\mathcal{V}}_{\mathcal{R},\beta'} \triangleleft M^{\mathcal{V}}_{\iota(\beta')}$. So $i^{\mathcal{V}}_{\mathcal{R},0,\iota(\beta')} : \mathcal{R} \to M^{\mathcal{V}}_{\mathcal{R},\beta'}$ is fully elementary and $\pi_{\beta'} \circ i^{\mathcal{T}}_{\mathcal{R},\beta'} = i^{\mathcal{V}}_{\mathcal{R},0,\iota(\beta')}$. Note that β' is the least $\beta'' \leq^{\mathcal{T}} \beta_1$ such that $\mathrm{crit}(i^{\mathcal{T}}_{\beta'',\infty}) \geq i^{\mathcal{T}}_{0\beta''}(\kappa)$. Let $\beta_2 \in (\beta_1, \infty]^{\mathcal{T}}$ be largest such that either $\beta_2 = \infty$ or $\deg^{\mathcal{V}}_{\iota(\beta_2)} = r$. So $\beta' \leq^{\mathcal{T}} \beta_1 <^{\mathcal{T}} \beta_2$ and $i^{\mathcal{V}}_{\mathcal{R},\iota(\beta'),\iota(\beta_2)}$ is a degree r

iteration map, to which we can apply the preceding remarks on preservation of weak $\utilde{\mathrm{r}\Sigma}_r$-cofinality. We are interested here in both $i^{\mathcal{V}}_{\mathcal{R},\iota(\beta'),\iota(\beta_1)}$ and $i^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1),\iota(\beta_2)}$.

Subclaim 3.45.7.1. *$j = i^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1),\iota(\beta_2)}$ is continuous at $\rho = \rho_{r-1}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}}$.*

Proof. Suppose first that $\mathrm{wcof}^{\utilde{\mathrm{r}\Sigma}_r^{\mathcal{R}}}(\rho^{\mathcal{R}}_{r-1}) = \rho^{\mathcal{R}}_r$. Then by the elementarity of $i^{\mathcal{V}}_{\mathcal{R},0,\iota(\beta')}$, $\mathrm{wcof}^{\utilde{\mathrm{r}\Sigma}_r^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta')}}}(\rho_{r-1}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta')}}) = \rho_r^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta')}}$. But then by the preceding remarks on preservation of weak $\utilde{\mathrm{r}\Sigma}_r$-cofinality, $\mathrm{wcof}^{\utilde{\mathrm{r}\Sigma}_r^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}}}(\rho_{r-1}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}}) = \rho_r^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}}$, and also by those remarks, it follows that j is continuous at ρ.

Now suppose otherwise, so $\theta = \mathrm{wcof}^{\utilde{\mathrm{r}\Sigma}_r^{\mathcal{R}}}(\rho^{\mathcal{R}}_{r-1}) < \rho^{\mathcal{R}}_r$. By full elementarity, $i^{\mathcal{V}}_{\mathcal{R},0,\iota(\beta')}(\theta) = \mathrm{wcof}^{\utilde{\mathrm{r}\Sigma}_r^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta')}}}(\rho_{r-1}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta')}})$, and so by the remarks, $i^{\mathcal{V}}_{\mathcal{R},0,\iota(\beta_1)}(\theta) = \mathrm{wcof}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}}(\rho_{r-1}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}})$. But by commutativity, $i^{\mathcal{V}}_{\mathcal{R},0,\iota(\beta_1)}(\theta) = \pi_{\beta_1}(i^{\mathcal{T}}_{\mathcal{R},\beta_1}(\theta)) < \sup \pi_{\beta_1}``\rho_r^{M^{\mathcal{T}}_{\beta_1}}$. But $\mathrm{crit}(j) \geq \sup \pi_{\beta_1}``\rho_r^{M^{\mathcal{T}}_{\beta_1}}$, so again by the remarks, j is continuous at ρ, as desired. $\square$

Now recall that $\mathrm{pred}^{\mathcal{T}}(\alpha_1+1) = \beta_1$ and $\deg^{\mathcal{T}}_{\beta_1} = r$ (so π_{β_1} is a near $(r-1)$-embedding), but $\deg^{\mathcal{T}}_{\alpha_1+1} = r-1$, whereas $\deg^{\mathcal{V}}_{\alpha_1+1} = r$.

Subclaim 3.45.7.2. *π_{α_1+1} is a near $(r-1)$-embedding.*

Proof. Since π_{α_1+1} is a weak $(r-1)$-embedding, we just have to verify $\mathrm{r}\Sigma_{(r-1)+1}$-elementarity. The proof is much as in the proof of [8, Lemma 1.3]. Let $b \in [\nu(E^{\mathcal{T}}_{\alpha_1})]^{<\omega}$ and let $f : [\mathrm{crit}(E^{\mathcal{T}}_{\alpha_1})]^{<\omega} \to \mathfrak{C}_0(M^{\mathcal{T}}_{\beta_1})$ be $\utilde{\mathrm{r}\Sigma}_{r-1}^{M^{\mathcal{T}}_{\beta_1}}$. Say f is so defined from the parameter $q \in \mathfrak{C}_0(M^{\mathcal{T}}_{\beta_1})$, and write this as $f = f_q$. Let φ be an $\mathrm{r}\Sigma_{(r-1)+1}$ formula. We want to see that $M^{\mathcal{T}}_{\alpha_1} \models \varphi([b, f_q])$ iff $M^{\mathcal{V}}_{\mathcal{R},\iota(\alpha_1+1)} \models \varphi([\psi(b), f_{\pi_{\beta_1}(q)}])$, where $\psi : M^{\mathcal{T}}_{\alpha_1}|\mathrm{lh}(E^{\mathcal{T}}_{\alpha_1}) \to \mathcal{N}$ is the relevant extender lifting map. As in [8], we can find some $p \in \mathfrak{C}_0(M^{\mathcal{T}}_{\beta_1})$ such that $(E^{\mathcal{T}}_{\alpha_1})_b$ is $\mathrm{r}\Sigma_1^{M^{\mathcal{T}}_{\beta_1}}(\{b\})$, via a certain Σ_1 formula ϱ, and such that $(F^N)_{\psi(b)}$ is $\mathrm{r}\Sigma_1^{M^{\mathcal{V}}_{\iota(\beta_1)}}(\{\pi_{\beta_1}(p)\})$, via the same formula ϱ. Say $\varphi(u)$ is the formula "there is $t \in T_{r-1}$ such that $\tau(t,u)$", where τ is some Σ_1 formula; this assumes $r-1 > 0$, but if $r-1 = 0$, then one uses the usual kind of variant. Then $M^{\mathcal{T}}_{\alpha_1+1} \models \varphi([b, f_q])$ iff $M^{\mathcal{T}}_{\beta_1} \models \varphi'(p, q, \vec{p}_{r-1}^{M^{\mathcal{T}}_{\beta_1}})$, where $\varphi'(p, q, \vec{p}_{r-1}^{M^{\mathcal{T}}_{\beta_1}})$ asserts "there is $t \in T_{r-1}$ such that t is a theory in parameters $\alpha \cup \{p, q, \vec{p}_{r-1}^{M^{\mathcal{T}}_{\beta_1}}\}$ for some $\alpha < \rho_{r-1}^{M^{\mathcal{T}}_{\beta_1}}$, and there is some $X \in (E^{\mathcal{T}}_{\alpha_1})_b$, such that for all $x \in X$, t codes a sub-theory t' (of the appropriate form for elements of T_{r-1}, and with truth corresponding to truth exhibited directly in t) and t exhibits that $\tau(t', f_q(x))$ holds". By the $\mathrm{r}\Sigma_r$-elementarity of π_{β_1}, this holds iff $M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)} \models \varphi'(\pi_{\beta_1}(p), \pi_{\beta_1}(q), \vec{p}_{r-1}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}})$, and note that by choice of

p, q, and because $i^{*\mathcal{V}}_{\iota(\alpha_1+1)}$ is continuous at $\rho_{r-1}^{M^{\mathcal{V}}_{\mathcal{R},\iota(\beta_1)}}$ (by Subclaim 3.45.7.1), this holds iff $M^{\mathcal{V}}_{\mathcal{R},\iota(\alpha_1+1)} \models \varphi([\psi(b), f_{\pi_{\beta_1}(q)}])$, so π_{α_1+1} is a near $(r-1)$-embedding, as desired. □

Generalizing the previous argument directly, we have:

Subclaim 3.45.7.3. *For each* $\xi \in [\alpha_1+1, \beta_2]^{\mathcal{T}}$, π_ξ *is a near* $(r-1)$*-embedding.*

But now for nodes $\xi \in b^{\mathcal{T}}$ beyond β_2, we can argue just as before: an inspection of the proof of [8, Lemma 1.3] shows that $\deg^{\mathcal{T}}_\xi = \deg^{\mathcal{V}}_{\iota(\xi)}$ and π_ξ is a near $\deg^{\mathcal{T}}_\xi$-embedding for each such ξ. So π_∞ is a near k-embedding, a completing the proof of the claim. □

As mentioned just prior to Claim 3.45.7, the claim yields a contradiction, completing the proof of solidity.

Now consider part 2 of the theorem, regarding condensation. Let $k < \omega$ and let $\mathcal{H}$ be a $(k+1)$-sound potential opm which is soundly projecting. Let $\pi : \mathcal{H} \to \mathcal{M}$ be nearly k-good, with $\rho = \rho^{\mathcal{H}}_{k+1} < \rho^{\mathcal{M}}_{k+1}$. Then $\mathcal{H}$ is in fact an opm. Let us assume that $\mathcal{H}, \mathcal{M}$ are both successors, so $\pi(\mathcal{H}^-) = \mathcal{M}^-$. By fine condensation of $\mathcal{F}$, $\mathcal{H}^-$ is an $\mathcal{F}$-pm, and either $\mathcal{H} \in \mathcal{F}(\mathcal{H}^-)$ or $\mathcal{H} = \mathcal{F}(\mathcal{H}^-)$. If $\mathcal{H}$ is not k-relevant then the result follows from the fact that $\mathcal{M}^-$ is $< \omega$-condensing and $\mathcal{H}^-$ is an $\mathcal{F}$-pm. So assume $\mathcal{H}$ is k-relevant, so $\mathcal{H} = \mathcal{F}(\mathcal{H}^-)$.

We now use ω-weak DJ and the usual phalanx comparison argument to reach the desired conclusion. Say $\mathfrak{P} = ((\mathcal{M}, <\rho), \mathcal{H})$ is the phalanx. Then $\mathfrak{P}$ is $\mathcal{F}$-$((\omega, k), \omega_1+1)$-iterable, lifting to $\mathcal{F}$-(ω, ω)-maximal trees $\mathcal{V}$ on $\mathcal{M}$. (It could be that $\mathcal{M}$ is not k-relevant. So we want to keep the degrees of nodes of $\mathcal{V}$ at ω where possible, to ensure that each $M^{\mathcal{V}}_\alpha$ is an $\mathcal{F}$-pm.) Suppose $\mathcal{T}$ is non-trivial. Because $k < \omega$, if $M^{\mathcal{T}}_\infty$ is above $\mathcal{H}$ without drop in model or degree, π_∞ need only be a weak k-embedding. But in this case, $M^{\mathcal{T}}_\infty$ is not ω-sound, which implies $M^{\mathcal{U}}_\infty \triangleleft M^{\mathcal{T}}_\infty$, which contradicts ω-weak DJ. The rest is routine. □

Acknowledgements

First author partly funded by the Austrian Science Fund (FWF) [10.55776/Y1498].

Index

References

[1] Dominik Adolf, Grigor Sargsyan, Nam Trang, Trevor Wilson, and Martin Zeman. Ideals and strong axioms of determinacy. *Journal of the American Mathematical Society*, 37(4):1203–1273, 2024.

[2] Daniel Busche and Ralf Schindler. The strength of choiceless patterns of singular and weakly compact cardinals. *Annals of Pure and Applied Logic*, 159(1-2):198–248, 2009.

[3] Donald Martin and John R. Steel. Iteration trees. *Journal of the American Mathematical Society*, 7(1):1–73, January 1989.

[4] William J. Mitchell and John R. Steel. *Fine structure and iteration trees*, volume 3 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1994.

[5] Itay Neeman and John Steel. A weak Dodd-Jensen lemma. *Journal of Symbolic Logic*, 64(3):1285–1294, 1999.

[6] Grigor Sargsyan and Nam Trang. Non-tame mice from tame failures of the unique branch hypothesis. *Canadian Journal of Mathematics*, 66(4):903–923, 2014.

[7] Grigor Sargsyan and Nam Trang. Tame failures of the unique branch hypothesis and models of $\mathsf{AD}_{\mathbb{R}}+\theta$ is regular. *Journal of Mathematical Logic*, 16(02):1650007, 2016.

[8] E. Schimmerling and J. R. Steel. Fine structure for tame inner models. *The Journal of Symbolic Logic*, 61(2):621–639, 1996.

[9] Ralf-Dieter Schindler, John Steel, and Martin Zeman. Deconstructing inner model theory. *The Journal of Symbolic Logic*, 67(2):721–736, 2002.

[10] F. Schlutzenberg and N. Trang. Scales in hybrid mice over $\mathbb{R}$. arXiv:1210.7258v4.

[11] Farmer Schlutzenberg. Fine structure from normal iterability. arXiv:2011.10037v3.

[12] Farmer Schlutzenberg. The initial segment condition for κ^+-supercompactness. arXiv:2306.13827.

[13] Farmer Schlutzenberg. Mouse scales. arXiv:2310.19764.

[14] Farmer Schlutzenberg. Reconstructing copying and condensation. Notes available at https://sites.google.com/site/schlutzenberg/home-1/research/papers-and-preprints.

[15] Farmer Schlutzenberg. *Measures in mice*. PhD thesis, University of California, Berkeley, 2007. arXiv:1301.4702.

[16] Farmer Schlutzenberg. A premouse inheriting strong cardinals from V. *Annals of Pure and Applied Logic*, 171(9), 2020.

[17] Farmer Schlutzenberg. Iterability for (transfinite) stacks. *Journal of Mathematical Logic*, 21(2), 2021.

[18] Farmer Schlutzenberg. The definability of $\mathbb{E}$ in self-iterable mice. *Annals of Pure and Applied Logic*, 174(2), 2023.

[19] Farmer Schlutzenberg. The definability of the extender sequence $\mathbb{E}$ from $\mathbb{E} \upharpoonright \aleph_1$ in $L[\mathbb{E}]$. *The Journal of Symbolic Logic*, 89(2):427–459, 2024.

[20] J. R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1996.

[21] J. R. Steel and R. D. Schindler. The core model induction. Unpublished notes, available at `https://ivv5hpp.uni-muenster.de/u/rds/`.

[22] John Steel. Core models with more Woodin cardinals. *The Journal of Symbolic Logic*, 67(3):1197–1226, 2002.

[23] John Steel and Itay Neeman. Fine structure for plus one premice. Unpublished notes available at `https://math.berkeley.edu/~steel`.

[24] John R. Steel. PFA implies $\mathsf{AD}^{L(\mathbb{R})}$. *Journal of Symbolic Logic*, 70(4):1255–1296, 2005.

[25] John R. Steel. An outline of inner model theory. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1595–1684. Springer, Dordrecht, 2010.

[26] Nam Trang. PFA and guessing models. *Israel Journal of Mathematics*, 215(2):607–667, 2016.

[27] Nam Trang and Trevor M Wilson. Determinacy from strong compactness of ω_1. *Annals of Pure and Applied Logic*, 172(6):102944, 2021.

[28] Trevor Miles Wilson. *Contributions to Descriptive Inner Model Theory*. PhD thesis, University of California, Berkeley, 2012. ProQuest ID:Wilson_berkeley_0028E_13013, `https://escholarship.org/uc/item/8fg2x6cr`.

[29] Martin Zeman. *Inner models and large cardinals*, volume 5 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2002.

4 Comments to referee

Thank you for your detailed response and long list of recommendations. They have been a very helpful guide in improving the paper. We hope we have been able to do that. And of course we apologise for the exceptionally long time lapse between receiving the report and doing the new version.

The most significant changes between the originally submitted version of the paper and this one are expository, along the lines suggested by the referee, such as adding more introduction, motivation and context in various places, and removing unnecessary definitions. There is a significant mathematical change to the proof of solidity in Theorem 3.45. This change is explained in the comments below, especially comment 31 in §4.3. Because of part of this change, we were able to replace what was called "k-simple weak Dodd-Jensen" with the traditional "weak Dodd-Jensen". We also simplified the discussion of copying iteration trees. There are other minor mathematical changes which are described in the comments.

We have tried to highlight the more significant changes in red/blue. Blue highlighting indicates a change that was made primarily in connection with one of your comments, whereas red indicates a change that was primarily of our own volition.

4.1 Responses to referee's general comments

(a) *Some fundamental definitions are not included, or included only partially, and the reader is referred to other papers. As this is one of the two papers where the very basics of operator mice are developed and which seem to become standard references in this area and most likely also the first contacts with the subject for a newcommer to the inner model theory at these levels, it is crucial that the paper is self-contained when it comes to fundamental definitions. Besides, some of the definitions in other papers the authors refer to contain mistakes, which will only confuse the reader.*

We have made the paper much more self-contained, adding such things.

(b) *On the other hand, there are several cases where the authors introduce formal definitions in connection with quite unimportant notions: These notions are then typically used once or twice, and this happens in the text immediately following the definitions only. What is the point of defining those notions if they are not used later in the text?*

We have removed various definitions which were not used much.

At some points, however, it is simply done to break otherwise longer discussions into logical components. We believe that this should help the reader more quickly identify these components.

For example, we have retained Definition 2.26 ("pre-fine"), because it helps to separate logically and visually the hypotheses listed there from the discussion of fine structural notions in Definition 2.27. What was indeed missing in the original version of the paper in the last paragraph of Definition 3.14, when we made the second use of “pre-fine” in the paper, was a reference in the form “(see Definition 2.26)”, so that the reader could easily see where to look up the definition of “pre-fine”; we have now added such a reference.

Aside from making it easier to parse what is going on in the interaction of Definitions 2.26 and 2.27, since “pre-fine” is in fact referred to a second time in the paper (in Definition 3.14), we believe that separating it as a definition eases reading further, since it is made simply clear and explicit in Defin 3.14 that we are referring to exactly the same hypotheses as those used earlier (in “pre-fine”).

(c) *Another issue I wonder about is, that the authors tend to re-name the existing standard notions or change the existing standard notation in a kind of trivial way, like permuting the parameters, putting some of them as a subscript/superscript etc. What is the point of this? Why not to use the standard notation that already is in the literature and has been used for decades?*

We have tried to change notation to match well established standard notation where such exists, in particular that for iteration games.

(d) *There are also several occasions where the authors introduce certain notion or discuss some situation withoug giving any explanation/motivation, or some explanation/motivation comes only several pages later, without prior hint to it. I think this should be addressed, for obvious reasons that the reader would be reading though many pages having no clue what is going on. If the explanation/motivation is given later then the authors should mention this when right away and better give a reference where the reader can locate the explanation/motivation.*

Agreed, a lot of things needed better introduction/motivation. We have tried to address this, adding a lot more such material. Also, we have moved the section on mouse operators back earlier, near the definition of “condenses finely”, as it provides a good example of operators and fine condensation, and should help the reader get a better feel for “condenses finely”.

(e) *Because of the vast number of new notions and a huge amount of notation defined, it would be very helpful for the reader to add an index of defined notions and notation with links to pages where these definitions occur, as any serious reader will need to go back-and-forth frequently.*

Done

(f) *The authors should mark the end of each definition and each proof. Sometimes it is difficult to determine where the end is.*

We have improved the marking of ends of definitions/proofs.

4.2 Responses to referee's detailed comments

1. *This is a minor point, but I think the use of the term "latter" on p.2 line 28, then on line 32 in "latter problem" and again on p.3 line 40 is confusing. It may be helpful to formulate the text more clearly, as it is not clear what the term "latter" refers to.*

   Fixed.

2. *p.3 lines 42-43: If the proof is omitted, the authors should give the exact reference where in the other paper the proof can be find, i.e., give the number of the theorem/lemma/etc.*

   There should no longer be parts omitted which can only be found in unpublished articles, and the citations have been improved.

3. *p.3 lines 44-46: See my remarks in (c) in the introduction to this report.*

4. *p.5 line 74: "under ZF": Should not it be "in ZF"*

   Changed to "in ZF".

5. *p.8 footnote 4: It may be a good idea to include this in the main text, rather than making it a footnote.*

   Changed it so (Page 7)

6. *p.8 line 138: I think the text should read "...elements of $A \cup \{A\}$ will be in the relevant hulls"*

   Changed it so (Page 7)

7. *p.9 lines 152-4: At this level of generality, is it clear that the hulls are elementary enough to guarantee that the relation $\in$ is extensional on these hulls?*

   Ah, right. This has been adjusted accordingly. (Page 7)

8. *p.9. line 162: I think it would be helpful to give some explanation/motivation what is the point of introducing the coarse parameter. In Section 2, there is no use of the coarse that would give a slightest hint what the use of it may be; the notion only occurs in some other definitions that focus on other issues. The first time the coarse parameter occurs in the text where one can get some vague idea about its purpose is Definition 3.6, alhough it does not make the things much clearer either. Some kind of explanation is in Definition 3.39 (bottom of the page) and Footnote 34, but again the general motivation/intention is not given. See also my remak (d) in the introduction to this report.*

   Motivation and an example has been added at the start of §2, page 7.

9. *p.10 footnotes: I think the contents of the footnotes is important enough that it should be included in the main text rather than in footnotes.*

One of the footnotes has been removed and its content moved to the introduction of §2, on page 7. The other footnote has been converted into Remark 2.6.

10. *p.10 It would be a good idea to include an explanation why the models of interest have soundly projecting proper levels.*

    We added some discussion just after the definition. Is this the kind of thing you meant? (Just after Def 2.7.)

11. *p.11 the last paragraph and p.12 Lemma 2.8: The the proof that these maps exit and have a uniform Σ_1-definition is based on the fact that the model in question is stratified; at least the proof I know. Stratification is used in Jensen's proof for models of the form $L[A]$, and also Steel's proof for models of the form $L(\mathbb{R})$ and $K(\mathbb{R})$. Jensen uses stratification by hierarchy S^A_α; Steel similarly stratification by hierarchy $S_\alpha(\mathbb{R})$ in his paper "Scales in $L(\mathbb{R})$". I am not aware of a proof that avoids using some stratification, but if the authors see such a proof they should either write the proof down or give a reference. The notion of adequate model does not require stratification, however. Thus, if the authors are unable to provide a proof or a reference as I suggest above, they should revise this part of the text appropriately.*

    We have added a proof. See proof of Lemma 2.10, page 10.

12. *p.12 line 218: When we talk about ρ being relevant, do we mean that ρ is a cardinal in the given structure, or just an ordinal?*

    "Relevant" applies to any ordinal $< \mathrm{Ord}(\mathcal{M})$. (Definition 2.13)

13. *p.15 Definition 2.19 clause 2 on E-goodness: Definition 2.19 is a fundamental definition as the entire rest of the paper is based on it. Therefore all clauses in this definition should be spelled out explicitly. It may be o.k. if the reference given were standard, but the reference [12] (the numbering in the paper) is not standard. It is a bad idea to give a reference to a personal website, as it is not clear when the website may be moved to a new unknown location or the source would be removed from that website. Besides, and this is even more serious issue, Definition 2.2.1 in [12] involves a significant error in its treatment of premouse axioms, precisely the coherence axiom, which only brings even more confusion into the presentation of the material. I would like to stress one more time, that a fundamental definition like Definition 2.19 should be spelled out in its entirety, especially since this is the only place (perhaps with the exception of the authors' paper "Scales in hybrid mice over $\mathbb{R}$") where this definition occurs at this level of generality.*

    We have added the full definition. See Def 2.21, clause 2.

14. *p.15 Definition 2.19 clause 3(b): "a map" should read "and a map"*

    Fixed, page 15

15. *p.15 Definition 2.19 clause 3(c): It looks like Σ_1-ordinal generation follows from soundly projecting for proper initial levels of the model. Some discussion connecting these notions and giving a bigger picture would be helpful here.*

    I don't follow what you mean. If $\mathcal{M}$ is an adequate model (hence all its proper segments are soundly projecting) and is a successor, why should Σ_1-ordinal generation hold for $\mathcal{M}$?

    We have expanded Remark 2.20, filling in some more information.

16. *p.17 line 305: The term "super-small" is not used in the paper, in fact, its definition is the only occurrence of this term in the paper. Therefore I do not see any point in defining it. See my comment (b) in the introduction. Besides, I feel that the choice of terminology here is unfortunate. The term "super-small" as defined in the paper defines premice which are actually quite large, as it allows anything below a locally superstrong cardinal. On the other hand, the expression "super-small" evokes an impression that the object in question is really small.*

    Fixed; it has been removed.

17. *p.17 Lemma 2.22: This is a minor point. According to the Definition 2.2. of a hull,* $\mathrm{Hull}^N_\Gamma(X)$ *is formed using all finite sequences of elements of* X*. Thus, to conform to the notation from Definition 2.2, one should write "*$\mathrm{Hull}_1^{\widetilde{\mathcal{M}}_\alpha}(A \cup \mathrm{Ord}(\widetilde{\mathcal{M}}_\alpha))$*" in place of …*[same formula but $A^{<\omega}$ instead of A]*… This should be also corrected an several other places in the later parts of the text.*

    Done; e.g. Lemma 2.24

18. *p.17 Definition 2.23. The notion "pre-fine" occurs precisely twice in the text after Definition 2.23. The first occurrence is in Definition 2.24 which comes immediately below Definition 2.23, so the clauses of Definition 2.23 can as well be made an assumption in Definition 2.24. The next occurrence is on p.62 on line 1216. By that time, the reader will forget what "pre-fine" means, and will have a hard time to find its definition, especially if he reads a printout. In any case, the use of the term on p.62 is marginal. I think it makes no sense to define the notion "pre-fine"; it only adds to the vast number of notions and and makes the notational/terminological mist even denser. Instead it would be just easier to refer to the appropriate place earlier in the paper. See also my comment (b) in the introduction to this report.*

    I don't really agree with removing the definition in this case; I think it is preferable to retain the definition of "pre-fine" add the comment "(see Definition 2.26)" following the second reference to "pre-fine" in Definition 3.14. This is discussed in more detail in my comment (b) (which regards your comment (b)).

19. *p.18 line 328: replace “$\rho = \rho^{\mathcal{N}}_{n+1}$” with “$\rho^{\mathcal{N}}_{n+1}$”, otherwise the definition of $\rho^{\mathcal{N}}_{n+1}$ is confusing. Then replace the occurrence of ρ in the displayed formula between lines 332 and 333 with “$\rho^{\mathcal{N}}_{n+1}$”.*

    Done; see Definition 2.27

20. *p.18 Definition 2.24. It would make more sense to move the sentence on line 331 to the beginning of the paragraph starting on line 327, as this sentence contains a reference to the footnote 12 which reminds what* $\mathrm{r}\Sigma_{n+1}$ *is. The reason here is that the notion “*$\mathrm{r}\Sigma_{n+1}$*” is already used on line 329.*

    This has now been organized as a correct (though somewhat informal) induction, mentionining the (somewhat informally stated) inductive hypotheses in the 3rd paragraph of the definition, just prior to defining $\rho^{\mathcal{N}}_{n+1}$. Note that it is now $\mathrm{r}\Sigma_{n+2}$ that gets defined at the end. See Definition 2.27.

21. *p.18 footnote 12: I think it is a good idea to recall the definition of “*$\mathrm{r}\Sigma_{n+1}$*”; I think it would also be good to recall the definition of “*$\mathrm{Th}^{\mathcal{N}}_{\mathrm{r}\Sigma_{n+1}}(X)$*”.*

    The definition of $\mathrm{r}\Sigma_1$ is given in the 2nd paragraph of Definition 2.27, and the definition of $\mathrm{r}\Sigma_{n+2}$ (for $n < \omega$) now appears more explicitly in the main text, at the end of Definition 2.27. The definition of $\mathrm{Th}^{\mathcal{M}}_{\Gamma}(X)$ now appears at the end of §1.1, page 6.

22. *p.21 Definition 2.27. I think the authors should to include the definitions of weak/near embedding to make the paper self-contained and also that the reader better see how they compare to the definition of 0-weak embedding. I think this is important, as these notions are used throughout the paper and reference [8] in the paper is not a standard reference.*

    Done, Def 2.30

23. *p.21 Definition 2.28: (a) the assumption “$\rho < \rho^{\mathcal{N}}_{k+1}$” is too restrictive, as many applications of condensation require larger $\rho^{\mathcal{N}}_{k+1}$.*
    (b) lines 381 and 382: The sentence starting with “If $M|\rho$ is E-passive...”: Replace each occurrence of “$\mathcal{M}$” with “$\mathcal{N}$”, three times in total.
    (c) If $\rho < \mathrm{crit}(\pi)$ then one should set $Q = \mathcal{N}$ and not $Q = \mathrm{Ult}(\mathcal{N}|\rho, F^{\mathcal{N}|\rho})$.
    (b) The requirement on $\mathcal{M}$ being $(k+1)$-sound is also too restrictive, as in practice one often has to deal with M which is sound above $\mathrm{crit}(\pi)$, which makes a difference if $\rho^{\mathcal{M}}_{n+1} < \mathrm{crit}(\pi)$.

    (See Definition 2.31.)

    Re (a) and the second (b) (presumably it was intended to be called (d)): “$< \omega$-condensing” is not intended to be a comprehensive statement of condensation. It just covers some simpler cases which come up in the basic fine structure proofs. It is, moreover, of simple enough complexity that it is preserved by the relevant hulls and iteration maps. I have added a comment to this effect just prior to the definition.

    Re the first (b): done.

    Re (c): If $\rho < \mathrm{crit}(\pi)$ then $\mathcal{N}|\rho$ is E-passive, so $Q = \mathcal{N}$ already.

24. *p.21 line 385: I think this is false. It may happen that* $\mathcal{M}^- \models \mathsf{ZFC}^-$ *and* $\rho_\omega^{\mathcal{M}} = \omega$*, even in the case of pure extender mice.*

    The point is that this doesn't happen here. I have added some more explanation to this effect (paragraph following Definition 2.31, page 20).

25. *p.22 Definition 2.30: The same definition, just with different number, occurs on page 10 (bottom of the page).*

    Fixed; this was changed to a reminder (immediately prior to Lemma 2.34)

26. *p.22 Lemma 2.31: The authors should say that the formulas* $\varphi_{0,\psi}$ *and* $\varphi_{4,\psi}$ *are obtained uniformly/recursively from* ψ*.*

    Done; Lemma 2.34

27. *p.23 Lemma 2.32: One cannot give a lemma in an official text this way. You need to formulate the lemma.*

    I added this; see Lemma 2.35

28. *p.25 Lemma 2.45: What is* q*?*

    It looks to me like "2.45" was supposed to be "2.35" (since there is no 2.45 on page 25, but there is a 2.35, and it has an undefined "q").

    Fixed: I defined q (Lemma 2.38) (I also separated the two statements in the lemma, into two parts, 1 and 2).

29. *p.26 Lemma 2.38 line 474: Although the inequality* $\rho_{k+1}^{\mathcal{M}} < \rho_\omega^{M^-} \le \rho_k^{\mathcal{M}}$ *is technically correct, it is a bit confusing/misleading to write it this way in this context. It would be better to replace the inequality* $\le$ *on the right with* $\rho_\omega^{\mathcal{M}^-} = \rho_k^{\mathcal{M}}$*.*

    But if $k = 0$ then $\rho_\omega^{\mathcal{M}^-} < \rho_k^{\mathcal{M}}$. We added a comment pointing out that if $k > 0$ then $\rho_\omega^{\mathcal{M}^-} = \rho_k^{\mathcal{M}}$. (And also state that $\rho_1^{\mathcal{M}} \le \rho_\omega^{\mathcal{M}^-}$.) Lemma 2.41

30. *p.28 Lemma 2.42: (a) If* $\mathcal{M}$ *is a successor and* $k > 0$ *then the condition "*$\mathrm{crit}(E) < \rho_\omega^{\mathcal{M}^-}$*" follows from "*$\mathrm{crit}(E) < \rho_k^{\mathcal{M}}$*", so it may be a good idea to add a remark along these lines.*
    (b) Clause (5): If $\rho_{k+1}^{\mathcal{M}} \le \mathrm{crit}(E)$ *then I agree that weak amenability of* E *is sufficient to guarantee (5). However, if* $\rho_{k+1}^{\mathcal{M}} > \mathrm{crit}(E)$ *then I do not see how weak amenability helps – rather, I can see how to obtain (5) from the assumption that* E *is close to* $\mathcal{M}$*, or, more precisely, that each measure of* E *is* $\utilde{\Sigma}_1$*-definable over* $\mathcal{M}$*. I also checked the argument in Claim 5 in Theorem 6.2 in "Fine Structure and Iteration Trees", the authors refer to, but that argument also uses* $\utilde{\Sigma}_1$*-definability of measures of* E *over* $\mathcal{M}$*. If the authors see how to run the argument without the assumption on* $\utilde{\Sigma}_1$*-definability of measures of* E *over* $\mathcal{M}$ *they should give the argument or provide a reference where such an argument occurs. Otherwise they should revise the statement of Lemma 2.42.*

    (a) Done (statement of Lemma 2.46)

(b) For the case that $\kappa < \rho_{k+1}$, the most specific reference was "[5, §2, (p, ρ)-preservation]" in the original submission (as written in the last paragraph of the proof, for parts 4–6). A more precise reference can now be made (as the relevant paper has since been published). It is [18, Corollary 2.24] and its proof, although this version is formally below superstrong. It's the same argument allowing superstrong, and this can be seen in (as of yet unpublished) [11, Lemma 3.8]. The arguments are related to those for [4, Theorem 6.2]. I updated the citations.

31. *p.29 Definition 2.43: It is not clear to me what the authors mean by "except that" (two occurrences). In the case of pure extender premice, extenders used on the trees are chosen the same way as required in Definition 2.43. In the case of putative iteration trees, the definition on "Outline of Inner Model Theory" makes no requirements on the last model of the tree, either. So there do not seem to be any differences between the notions defined in Definition 2.43 and standard definitions in literature.*

    We should have made it more explicit, that the "iteration trees" here are along the lines of those considered in [25, §3.1] (so not, for example, coarser variants such as those considered in [3]). But the first paragraph of Definition 2.47 was supposed to be a slightly informal, to give the basic point, before becoming more precise after that. So, instead, there is now a brief remark just prior to the definition about the intentions, and we have clarified the definition in general. The first "except that" was just saying that the models $M^{\mathcal{T}}_\alpha$ are required to be opms, instead of premice. (Cf. the "typical run of $\mathcal{G}_k(\mathcal{M}, \theta)$" described just after [25, Definition 3.3], in particular the second bulleted point there, "*premice* $\mathcal{M}_\alpha$ for $\alpha < \theta$,...". So the first "except that" means that that word "premice" is replaced with "opms".)

    I think that in the description of putative iteration trees in the first paragraph, the word "likewise" was intended to be comparing putative trees on opms with *(actual) trees on opms*, and *not* intended to be comparing putative trees on opms with *putative trees on premice*. So, I think it was supposed to mean that a putative iteration tree on an opm is as an iteration tree on an opm (in the sense described in the preceding sentences), except that there are no demands on $M^{\mathcal{T}}_\infty$ when $\mathcal{T}$ has successor length. Anyway, we have rewritten the description.

32. *p.29 lines 545-550, the issue of replacing the requirement* $\mathrm{lh}(E^{\mathcal{T}}_\alpha) < \mathrm{lh}(E^{\mathcal{T}}_\beta)$ *with the weaker* $\mathrm{lh}(E^{\mathcal{T}}_\alpha) \leq \mathrm{lh}(E^{\mathcal{T}}_\beta)$*: The relaxed requirement* $\mathrm{lh}(E^{\mathcal{T}}_\alpha) \leq \mathrm{lh}(E^{\mathcal{T}}_\beta)$ *is needed because of the presence of superstrong extenders in the extender sequences of opm's which the authors allow. I think this should be mentioned right when the relaxed requirement on extender indices is introduced, and the reader should be referred to Remark 2.44 where the authors give a detailed discussion of the situation, and which comes somewhat later, so that the reader is not puzzled what the point of this modification in the definition of the iteration game is. See also my comment (d) in the*

introduction to this paper. Now, I do not see the point of introducing the notation "$G^{\mathcal{M}}(k,\theta)$", which only permutes the parameters in the standard notation $G_k(\mathcal{M},\theta)$ coming from Steel's "Outline of Inner Model Theory". The iteration game the authors consider is the same as the standard one, except they extend it to a broader class of models. The two games are identical when restricted to premice without locally superstrong extenders as in "Outline of Inner Model Theory". Introducing unnecessary notation only contributes to the notational mist; see my remark (c) in the introduction to this report. Similarly regarding the game $G^{\mathcal{M}}(k,\alpha,\theta)$.

Re the extender lengths: done.

Re the game notation: done. (Definition 2.47)

33. *p.30 line 561 what is the point of introducing the notion "k-stack-maximal"? The notion is used only twice in the paper: in Lemma 2.45 and then on the line immediately following the proof of the lemma. I think one can just state the assumptions of the lemma explicitly and will save not only another unimportant notion but also some room. See my comment (b) in the introduction to this paper.*

    We probably thought it made Lemma 2.49 and its proof a little easier to read. But we have removed the definition, and edited Lemma 2.49 accordingly.

34. *p.31 lines 579-586. I think the authors should explain why the comparison algorithm needs to be modified. This is again because of the presence of superstrong extenders, which may cause that the authors do not stress. I think they should explain the situation in detail: That actually it does not matter in which order the extenders are applied, except in the case where say we have two comparison trees $\mathcal{T}$ and $\mathcal{U}$, the extender $E^{\mathcal{T}}_\alpha$ is superstrong, and $M^{\mathcal{T}}_{\alpha+1}$ has the same height as the index of $E^{\mathcal{T}}_\alpha$ and the top extender of $M^{\mathcal{T}}_{\alpha+1}$ is of type 2, and is equal to $E^{\mathcal{U}}_\alpha$, in which case the comparison terminates. But applying the exende $E^{\mathcal{U}}_\alpha$ at step α would cause that $E^{\mathcal{U}}_\alpha$ and $E^{\mathcal{T}}_{\alpha+1}$ are two identical extenders, which would ruin the usual comparison argument.*

    Done (Remark 2.48)

35. *p.32 lines 592-593: It would be helpful to explain in what sense would Lemma 2.45 fail if $\mathcal{T}$ were not k-stack-maximal: Would $M^{\mathcal{T}}_\infty$ fail to be $\deg^{\mathcal{T}}(\infty)$-relevant? Or something else?*

    Done (following Lemma 2.49)

36. *p.32 Definition 3.1: The term "explicitly swo'd" has not been defined but is used on p.32 line 604 and again on p.33 line 617. Was it the intention that Definition 3.1 defines the term "explicitly swo'd" ?*

    The term "explicitly" has been removed.

37. *p.33 line 622: what does "possibly" mean here? Do you want it to be swo'd or not?*

Fixed (Def 3.3)

38. *p.35 line 656: What is the point of introducing the term "well-putative"? It is used precisely once in the paper, namely on line 664. One can as well say "putative F-iteration tree with well-founded last model" here, in order to avoid adding to the terminological mist. See my comment (b) in the introduction to this paper.*

Removed it (Definition 3.8).

39. *p.35 footnotes 24 and 25. I think the contents of these footnotes is important enough that they should be included in the main text. Regarding Footnote 25, it would be good to add it to the paragraph on lines 660-666, as it touches relevant aspects concerning iteration games for normal iteration trees.*

We added the footnotes to the main text. But the second footnote (what was 25) refers specifically to stacks, not normal trees. So it has been added to the paragraph on stacks. Did you mean that something similar should be added to the paragraph on nomal trees? But that seems pretty clear, particularly in light of what was footnote 24. (Def 3.8)

40. *p.37 proof of Lemma 3.12. I think the proof should be written in more details, as the sketch omits several points.*
(a) "We may easily assume": Say why. This is not much longer than saying "We may easily assume". Keep in mind that your potential reader is someone with no prior experience with operator mice/F-mice, and this is a paper targeted to such a reader. You want to make clear which property is used for what purpose and how.
(b) I don't see any reason to refer to Σ_1^1-absoluteness, as the objects that exist in a generic extension obviously exist in the larger generic extension via the collapse. Maybe you want to refer to Δ_0-absoluteness between transitive models? But this is a triviality that may be omitted.
(c) The transitive rud closed set X: I think you need to require more of X. First, I don't believe you can define forcing relation over a general transitive rud closed structures. This requires a recursive definition on $\in$, for instance to define "$p \Vdash \dot{x} = \dot{y}$", so you need to work in a model which is able to carry out such a recursion, say X admissible, or stratified, or so. Second, you want the model X to know about your operator F, so you may need to add F as a predicate and make sure $F \cap X$ is defined on all objects you need, or add a large enough fragment of F to X as an element. This leads to another point in the proof which is not mentioned in the text; see (d) below.
(d) Now you have your map $\sigma : \bar{X} \to X$ where $\bar{X}$ is transitive. Then F collapses to some $\bar{F}$, so if $g \in V$ is your generic over $\bar{X}$, your model $\bar{X}[g]$ satisfies the statement: $\bar{\mathcal{M}} \neq \bar{F}(\bar{\mathcal{M}}^-)$. Of course, by applying almost

condensation coarsely you conclude that $\bar{\mathcal{M}} = F(\bar{\mathcal{M}}^-)$*, but how does this lead to contradiction? It looks like you need to prove that* $\bar{F} \subseteq F$ *or something along these lines. Since you are working in a general setup, this should be addressed and written up appropriately, instead of saying that everything goes "easily". Notice that the proof of Lemma 3.12, as written up, is not instructive at all, as this is just a standard absoluteness argument, and, as written, it does not address any specifics of the presence of* F*.*

See Lemma 3.12 proof:

(a) Done.

(b) The point is that we can take a *small* forcing, not a larger one; in particular, one in $\mathscr{B}$. We can use Σ^1_1-absoluteness to get such an embedding in the extension via that forcing, since the relevant objects have been made countable.

(c) Well, one can define the relevant forcing relation much more locally, without assuming KP. But we don't need that here. So we have adjusted the proof to avoid any of that.

(d) The model doesn't need to have any version of the operator F; it's enough that it has the output of F on the one input, M^-. We have made it more explicit.

41. *p40 line 761: "easy to find* α*": I think this requires more creativity than the rest of the argument. Instead of "easy to find* α*" say explicitly that for instance* $\alpha = X \cap \rho^{+P}$ *where* X *is a fully elementary hull of* ρ *constructed in* $P||\rho^{++P}$*, or something similar.*

    Done (something similar). Page 36

42. *p.40 line 767: I think acceptability of* $\mathcal{M}$ *is more relevant here than the fact that* $\mathcal{M}||\rho^{+\mathcal{M}} \models \mathsf{ZFC}^-$*.*

    Why? By ZFC^- in $\mathcal{M}|\rho^{+\mathcal{M}}$, we have $\mathcal{M}|\rho^{\mathcal{M}} \preccurlyeq_1 \mathcal{M}$, which is all we want. Acceptability doesn't seem to be enough by itself...Do you mean condensation? But that is a simliar argument to the ZFC^- argument.

43. *p.40 line 774: "the second change": Recall the context. At this point the reader may have forgotten where you started.*

    Done. Page 36

44. *p.44 Definition 3.18: If* $\mathcal{M}$ *is an opm then it is soundly projecting, and therefore* $\rho_1^{\mathcal{M}} \leq o(\mathcal{M}^-)$*. So it looks like the two clauses in the conclusion both discuss structures* $\mathcal{M}$ *such that* $\rho_1^{\mathcal{M}} \leq o(\mathcal{M}^-)$*, just that in the latter case* $\mathcal{M}$ *may not be opm. Is that correct? In any case, it would be helpful to add a little discussion after the definition that would help the reader to understand what is going on.*

    Correct; a remark to this effect has been added immediately after the definition (Definition 3.25)

45. *p.44 Lemma 3.19 and its proof: The lemma and its proof are stated in a confusing way. First, the statement of the lemma begins with "Let $k, \mathcal{M}, G, \mathcal{N}$, etc, be as in 3.18." So I understand that the map $\varphi : \mathcal{N} \to \mathcal{N}$ is also as in 3.18. But on line 852 it says "...; let $\varphi : \mathcal{N} \to \mathcal{M}$ be the core map." So are we not in the situation of Definition 3.18? Or we are in that situation, and the map φ turns out to be the core map? The authors should make clear what the context here is.*

    line 854: The reference to 2.41 assumes that $\mathcal{N}$ is k-sound. But no assumption on k-soundness is made in the statement of Lemma 3.19, and in Definition 3.18 either. So we may well have the situation that $\rho_k^{\mathcal{N}} = \rho_{k+1}^{\mathcal{N}}$.

    Definition 3.25, Lemma 3.26:

    Re φ: It doesn't matter about the relationship to the φ of the proof. So it has been renamed to π.

    Re k-soundness: the k-soundness has now been made explicit in Definition 3.25, and the statement and proof of Lemma 3.26 has been generally rewritten, simplified and clarified, and had a minor correction (that $\mathcal{M}^-$ should be assumed to be in $\mathrm{dom}(\mathcal{F})$).

46. *p.45 lines 858-859: Are you applying fine condensation for F? If so then the statement of Lemma 3.19 should include the assumption that F is finely condensing. Also, it should be mentioned on lines 858-859 explicitly that this is the spot in the proof where fine condensation of F is used.*

    Fine condensation is *not* being assumed. Thus, the way the lemma was stated might have been confusing. The lemma has been rewritten, both its statement and proof. It should be clearer now. (Lemma 3.26)

47. *p.45 Definition 3.20. This definition seems to be out of place. It would be more natural to move it immediately above Lemma 3.25.*

    This definition has been removed, as it wasn't really necessary, and seemed to just add definitional clutter.

48. *p.45 Lemma 3.21: It would be a good idea to check if anything needs to be added in the proof of the lemma, given my comment 40 for Lemma 3.12.*

    Hopefully the proof of Lemma 3.12 is clearer now (and there was the small correction to its proof regararding the base a of the cone being in HC, which has also been made for Lemma 3.27; see our comments 12 and 21).

49. *p.50 line 985: I do not see the point of defining the term "tenable" It is used precisely once, namely on p.52 in Lemma 3.31.*

    Done; it has been removed.

50. *p.51 Definition 3.29: add "and" after "ground".*

    This has been rewritten somewhat anyway (in particular, "almost fine ground" has been removed as a definition). Definition 3.39.

51. *p.52 line 1004: What is the point of introducing the term "weakly realizable"? It is used precisely once, namely on p.53 line 1031. Also, the term "weakly realizable" has the option of realizable branch in the case the tree in question is of limit length, whereas the notion of realizable tree does not have this option. Is this the intention? Also notice that "b" has two meanings here: both the parameter in the fine gound as well as a cofinal branch throught the tree.*

    Done. I removed the definition; it is just now part of the text of Remark 3.42. And the "b" has been changed to "c". Yes, it is the intention that one can have a branch at some limit stage realized. But it had missed the hypothesis that c is $\mathcal{T}$-maximal, which has been added.

52. *p.53 lines 1035-1036: Say briefly what* φ, σ *are.*

    Done (Remark 3.42)

53. *p.54 line 1045: I also doubt the term "relevant tree" is useful. It is only used in the proof of Lemma 3.34.*

    It has been removed.

54. *p.56 Definition 3.35: I think* $A = cb^{\mathcal{M}}$ *should be included in the hull, as the projecta as well as* $\vec{p}_n^{\mathcal{M}}$ *are defined relative to* A.

    This definition has been shifted to 2.27. In 2.27 it (now) includes A in the hull.

55. *p.57 line 1114: You already said on line 1100 that you may assume* $\mathcal{M}$ *is* k*-relevant.*

    Removed it (proof of Theorem 3.45, page 56)

56. *p.58 line 1136: Recall what* q^H *is! (Refer to the page or so.) The parameter* q *has not been used for a long time and by now the reader would have hard time to find out what it is! Finally I had to do a search, but it is hard to do a search for symbol with superscripts in a pdf file, especially since in the definition it appears as* $q^{\mathcal{M}}$. *So in the end I had to guess where the definition would be and do a search for "q", which is really awkward for obvious reasons.*

    Done. (Proof of Theorem 3.45, page 57.)

57. *p.59 line 1146: Is it not the case that* $\mu \leq \rho_\omega^{\mathcal{W}^-}$ *– simply because* μ *is a cardinal in* $\mathcal{W}$*?*

    I don't follow what you mean. $\mathcal{W}$ can have all sorts of cardinals $>$ $\mathrm{Ord}^{\mathcal{W}^-}$... Proof of Theorem 3.45, page 58

58. *p.63 Footnote 33: I find the footnote incomprehensible. What does the statement "*t' *results from* t *by replacing* $\vec{p}_n^{\mathcal{M}}$ *with* $\mathcal{R}$*" say? And what is* $\mathcal{R}$*?*

    Fixed. The "$\mathcal{R}$" was a typo. Footnote 38

59. *p.64 line 1239: $\rho^{\mathcal{N}}$ is computed relative to $\mathcal{R}$. So if n is the least such that $\rho^{\mathcal{N}}_{n+1} = \omega$ then it follows by definition that $\rho^{\mathcal{N}}_n > o(\mathcal{R})$.*

    Well, it didn't say "least" before, hence the "and $\mathrm{Ord}^{\mathcal{R}} < \rho^{\mathcal{N}}_n$". But it has now been modified to say "least n such that $\rho_{n+1} = \omega$", in two places; Definition 3.16.

60. *p.64 line 1240: In the case $X \in \widehat{C^D}$ you do not say what $cb^{\mathcal{K}}$ is. In the current case you demand that $cb^{\mathcal{K}}$ is $\mathcal{R}$, but it seems to me that $cb^{\mathcal{K}}$ should be A, as that is what the definition of a hierarchical model requires.*

    It is X, because it defines it as a "hierarchical model over X" (which was just another way of saying that $cb = X$). (Definition 3.16)

61. *p.64 Lemma 3.40: What does the term "project early" mean? I did the search, but did not find the definition.*

    The definition of "projects early" has been added to the end of Definition 3.16.

62. *p.65 lines 1262-1263: I am confused by this remark. $\mathcal{F}_G(X)$ is an initial segment of $G(X)$, so why is it the case that $G(X) \in \mathcal{M}$ for $\in$-cofinally many $X \in \mathcal{M}$? Maybe the confusion comes because I don't know what "projects early" means.*

    Hopefully with "projects early" defined, this is clarified. The point is that for all large enough $n < \omega$, $\mathcal{F}_G(\mathcal{M}|(\alpha+n))$“ = ”$G(\mathcal{M}|(\alpha+n))$. (After proof of Lemma 3.17.)

4.3 Further comments

Further comments on the more significant changes in this version; some minor changes have not been listed here:

1. Definition 2.4 has been somewhat reformulated for much better clarity, but the content is still the same. (In the original version, the definition was implicitly by recursion, by referring to hierarchical models in the definition of hierarchical model. This recursion has been removed.) In connection with this change, Lemma 2.5 has been added (this lemma used to be built into the definition of hierarchical model).

2. In Definition 2.23, we added the definition of $\mathcal{M}||\alpha$, which seemed to be missing from the earlier version.

3. We added condition (iv) to "pre-fine" (Definition 2.26), because in Definition 2.27, we had claimed that "If $A \subseteq X$ then the hull is indeed extensional, as then $\mathrm{Hull}_1^{\mathcal{N}}(X) \preccurlyeq_1 \mathcal{N}$", and that this was supposed to follow from pre-fineness, in particular condition (iii). But I now don't see that that was actually enough; the "stratification" in the new condition (iv) lets one define Σ_1 Skolem functions in the usual sort of manner, which together with (iii) does now ensure that $\mathrm{Hull}_1^{\mathcal{N}}(X) \preccurlyeq_1 \mathcal{N}$ given $A \subseteq X$.

4. Definition 2.30: In the definition of weak k-embedding, the original version had omitted the clause that X is cofinal in $\rho_k^{\mathcal{M}}$; this has been corrected. (Clearly without that clause, we could have just taken $X = \emptyset$, so the X would have been pointless.)

5. I added the explicit definitions of $\mathcal{L}_0^+$-*Q-formula* and $\mathcal{L}_0^+$-*P-formula*, whereas it used to say that these were as in [4]. These have also been moved to Definition 2.33, which is immediately prior to where these formulas are actually used for the first time in the text. However, I changed the definition of $\mathcal{L}_0^+$-*P-formula*; now instead of following [4, Definition 3.1.4], the definition is just the usual notion of "Q-formula". I don't see why the particular definition [4, Definition 3.1.4] was used in [4]. So I opted for the simpler notion.

6. Added part 2(b) to Lemma 2.36 (and added the part numbers to the statement).

7. In the proof of Lemma 2.37, I removed the argument for $\mathrm{r}\Sigma_1$-elementarity of the hull, since such an argument was really used implicitly earlier in Definition 2.27, and I have now added that argument explicitly in Footnote 19, to which the proof of this lemma now refers. I also cleaned up the lemma statement and proof somewhat.

8. Added Lemma 2.43.

9. In Definition 2.47, the terminology "max" in $\mathcal{G}_{\max}$ was changed to "optimal", in line with [17]. This is particularly relevant in Lemma 2.49, which needed that player 1 was not making such artificial drops. Thus, we have modified the statement/proof of this lemma, by adding the word "optimal" where needed, bringing it into line with the modified definitions.

   Also, in Footnote 22 on page 30 of the original submission, the last sentence of the footnote asserted "If round γ is won by player II and the run produces a tree of length θ, then the run of $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$ is won by player II." This notation was wrong; with this rule, the game is usually denoted $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)^*$ (see e.g. [22]). Moreover, this variant is the more natural of the two. So we have modified the definitions to use the superscript $*$ notation, here and also in Definition 3.8 etc.

10. Definition 3.6 had an implicit assumption of DC in V in the original submission, so we made this explicit. We also added the version restricted to $\mathscr{H}_\kappa$, which does not assume DC in V.

11. Definition 3.10 didn't quite make sense in the original submission, because it ignored the issue of $cb^{\mathcal{M}}$ being in the domain of $\mathcal{F}$. Thus, I have corrected this, discussing the base of the cone C^D where D is the domain of $\mathcal{F}$. Also, the relationship between the "p" mentioned here and the base of the cone was not made explicit. One could define things keeping these distinct, which was probably formally the case in the original version of

the paper, because the base of the cone was not mentioned. But doing this would just add more clutter, having two parameters, instead of just one. And doing so probably wouldn't add any real value. So preferring readability, we have just identified these two objects.

12. Lemma 3.12 was incorrect in the original submission, because it also ignored the base of the cone for the domain of $\mathcal{F}$, which should be assumed to be in HC. This has been corrected.

13. I noticed that in [28], the definition of F_J, for a mouse operator J, also has a problem which was not addressed in the original submission. This is now included in the discussion in Remark 3.13; this is point 1 of the claim just after "defining" $F = F_K$ (that "definition" doesn't actually work). The remainder is then adapated to a modified operator F'.

14. Remark 3.13: After showing that F' does not condense well, in the original version there was a parenthetical remark regarding using $\mathrm{Ult}_1(\mathcal{M}, E)$ in place of $\mathrm{Ult}_0(\mathcal{M}, E)$, and hence getting a more elementary embedding. But actually, these two ultrapowers are equivalent in the present situation, so this is now pointed out instead.

15. In Def 3.14, I added the hypothesis "$<\omega$-condensing", since we only applied it in this case. (This is just for uniformity with other things really; it's not actually important here.)

16. The material on mouse operators (that is, the definition of pre-operators (now Def 3.15), defining the operator $\mathcal{F}_G$ from a pre-operator (now Def 3.16), the proof that $\mathcal{F}_G$ is indeed an operator (now Lemma 3.17), the definition of mouse operator (now Def 3.18), and the proof of fine condensation for mouse operators (now Prop 3.28), was all right at the end of the original version of the paper. It made much more sense to move it back earlier, as it helps provide motivation and intuition for the definitions of operators and fine condensation. So this has been done.

17. In Def 3.16, in paragraph starting "Now let $\mathcal{R} \in P^D$", added hypothesis "if $\rho > \omega$", which was missing before. (This is the projectum of the largest proper segment, and it is only possible for some segment of the operator application to project $< \rho$ if $\rho > \omega$, which is what is being considered here.)

18. In the proof of Lemma 3.17, first paragraph, item (iv), there was a typo: the "$\mathcal{P}$" that was there was incorrect; it has been changed to "$\mathcal{M}$". ($\mathcal{P}$ was something else on the way to producing $\mathcal{M}$. We needn't have $\rho_1^{\mathcal{P}} = \omega$, for example; we have $\rho_1^{\mathcal{M}} = \omega$ since $\mathcal{M}$ is essentially the master code at the level that next projects $\leq \mathrm{Ord}^{\mathcal{M}^-}$, but is also a structure over $\mathcal{M}^-$, so in fact $\rho_1^{\mathcal{M}} = \omega$.)

19. Definition 3.18: just changed the notation from G to G_φ, to make the dependence on φ explicit.

20. Def 3.22: Added hypothesis that $\mathcal{M}$ is k-relevant. This was implicitly there before, but maybe not obvious, so we made it explicit. (For if $\pi : \mathcal{M} \to \mathcal{N}$ is a k-factor and $k < \omega$, then there must be a k-tight $\sigma : \mathcal{L} \to \mathcal{M}$ and $\mathcal{L}$ must be k-relevant. But this implies that $\mathcal{M}$ is k-relevant.) (It was also already an explicit hypothesis in the clause of *condenses finely* which assumes a k-factor.)

21. Definition 3.25 had a problem analogous to that of Definition 3.10, mentioned in comment 11 above. This has been corrected. Likewise for Lemma 3.27, analogous to Lemma 3.12, mentioned in comment 12.

22. Definition 3.25 and Lemma 3.26 were both missing the assumption that $\mathcal{M}^- \in \mathrm{dom}(\mathcal{F})$, which has been added to both.

23. Lemma 3.26 was not stated really correctly. Also, we noticed that it has a simpler and more elementary proof than what was the original version. So we rewrote the both the lemma statement and proof. Also broke the proof into two cases ($k > 0$ and $k = 0$) for better readability.

24. Prop 3.28: The proof has been expanded a little, for more clarity.

25. In the original submission, the paper cited [14] for some of its calculations. The expectation at that point was that [14] would also be published. But this probably isn't the plan now. So some of that cited material has been shifted here. The main material is in §3.6 on the copying construction, in relation with ν-preservation of copy maps (and non-ν-preservation), and how this relates to copying, and also propagation of near k-embeddings under copying – results of [8] and variants of those results.

    Also, the material in §3.6 section was modified and simplified in another way. In the original version there was the definition of *insert set* and *I-reordering*. We have removed these and added instead Definition 3.32, which leads to a less general copying process, but things are simpler. The less general copying process suffices; the proof of weak Dodd-Jensen just makes a simple observation to show this. (See comments 29 and 31 regarding the change from k-simple to weak DJ.)

    The changes in these connections are:

    (a) Definition 3.29: We changed the notation for the shifted embedding from $\pi^\uparrow$ to $\mathrm{Shift}(\pi)$, for consistency with [11]. We also added the definitions *ν-preserving*, *ν-high* and *ν-low*.

    (b) Added Remark 3.30.

    (c) Lemma 3.33 was added; this deals with propagation of ν-preservation via the copying construction; in the original version this was cited from [14].

    (d) In Lemma 3.34, we have added the restriction that $\mathcal{N}$ is $\mathcal{Q}$-stable. This simplifies the copying process substantially, and suffices for our purposes, because of a small change in the proof of (now weak instead of simple) Dodd-Jensen.

  Parts 7 and 8 have been added to the lemma. These properties were mentioned in the proof of the lemma in the original submission, so for better clarity, we have added them to the actual lemma statement.

  (e) The proof of Lemma 3.34 has also been rewritten:

    i. Since "insert set" and "$<^{\mathcal{U},I}$" have been removed from the paper, and replaced with 3.32. the copying construction had to be accordingly modified.

    ii. Parts of the proof, particularly parts 7 and 8, were cited to [14] in the original submission. This cited material has been added to this paper, since [14] is no longer planned for publication.

    iii. Some more details have been added for better clarity.

26. Lemma 3.31: Changed the hypothesis from almost fine condensation to almost coarse condensation, since that was all the proof used (as was already remarked after the proof, in the original submission).

27. Def 3.39: The assumption that the branch $[0,\alpha]^{\mathcal{T}}$ drops only finitely often was made explicit.

28. In Remark 3.42, the hypothesis that $\mathcal{R} \vDash$"there is no Woodin cardinal" was missing, and has been added (it guarantees there is always a Q-structure $Q \trianglelefteq M_b^{\mathcal{T}}$ for k-maximal trees $\mathcal{T}$ on $\mathcal{R}$; the existence of a Q-structure seemed to have been implicitly assumed in the original version).

    Also the assumption that c is $\mathcal{T}$-maximal had been omitted, and has now been added.

29. Definition 3.43: We have replaced "k-simple DJ" with the standard notion "weak DJ". This is because we ended up changing the proof of solidity etc. See comment 31.

30. Lemma 3.44 has been replaced with the proof of weak Dodd-Jensen, for which we can mostly cite [5], modulo the details with copying as discussed in the proof.

31. Theorem 3.45: The proof has been expanded and rewritten extensively, and some discussion added in the section prior to the proof, for the following reasons:

    (a) Regarding issues to do with operator mice, some details were added for better clarity.

    (b) There were some details of the proof in connection with superstrong extenders which were ignored in our original submission. We have added these. They arise in the proof of Claim 3.45.5 and the details of the copying process when $E^{\mathcal{V}}_{\eta'}$ is superstrong on page 59.

(c) The original version of the paper cited [14] for the details of how to handle some cases. But as that paper is no longer intended for publication, we have included all relevant details here. Part of those details handled aspects of the proof of solidity which do not seem to be fully dealt with in the combination of [4], [25] and [9]. Moreover, the argument in our original submission changed the setup for the proof significantly from that in [4], [25], [9], primarily by working with k-simple Dodd-Jensen instead of weak Dodd-Jensen (thus possibly changing what iteration strategy we need to work with), and also by using results from [18], which were not used in the proof in [4], [25], [9]. And aside from k-simple DJ being different from weak DJ, k-simple DJ is also more complicated.

We ended up finding an argument which uses the (standard) weak DJ, and does not need to change the setup of the proof from [25] and [9], and does not use the results of [18]; so it just fills in some calculations in the proof of [25] and [9]. We have discussed these things at the beginning of the section, prior to the theorem statement.

The increased length of the proof is primarily due to incorporating these arguments.

(d) Remark: From the phrase "Now if $\mathcal{M}$ is in fact a limit..." on page 58 onward, nearly all of the rest of the proof of solidity has been modified, so for ease of reading, we haven't put in any red highlighting (it would have been mostly red). This is so until the end of the proof of solidity, which is page 65, where the proof of part 2 starts. (The proof of part 2 is just the last two paragraphs of the paper, and is basically as it was in the original submission.)