BOLDFACE GCH BELOW THE FIRST UNCOUNTABLE LIMIT CARDINAL

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Abstract. If κ is an infinite cardinal, the boldface GCH at κ is the statement that κ^+ does not inject into $\mathscr{P}(\kappa)$. It will be shown here that $\omega_1 \to (\omega_1)^{\omega_1}_2$ (the strong partition property at ω_1) and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ (the ultrapower of ω_1 by the club filter on ω_1 is ω_2) implies that the boldface GCH holds at ω_n for all $n < \omega$ using combinatorial arguments. In particular, AD implies the boldface GCH holds at ω_n for all $n < \omega$.

1. Introduction

This paper will work with the Zermelo-Frankel axiom ZF for set theory (without the axiom of choice, AC). Let κ be an infinite cardinal. There is a cardinal which does not inject into $\mathscr{P}(\kappa)$. What is the smallest cardinal which does not inject into $\mathscr{P}(\kappa)$? Since κ always injects into $\mathscr{P}(\kappa)$, the smallest that this cardinal can be is κ^+, the cardinal successor of κ. Cantor showed that κ does not surject onto $\mathscr{P}(\kappa)$. Thus $|\kappa| < |\mathscr{P}(\kappa)|$. If the axiom of choice holds, then all sets are wellorderable and one must have that κ^+ injects into $\mathscr{P}(\kappa)$. Assuming the axiom of choice, the smallest cardinal which does not inject into $\mathscr{P}(\kappa)$ must be greater than κ^+. The usual generalized continuum hypothesis at κ (under AC) is the assertion that $|\mathscr{P}(\kappa)| = 2^\kappa = \kappa^+$. Assuming AC and the generalized continuum hypothesis at κ, one has that κ^{++} is the smallest cardinal which does not inject into $\mathscr{P}(\kappa)$. However, without the axiom of choice, it is potentially possible to have the most elegant answer to the above question: κ^+ is the smallest cardinal that does not inject into $\mathscr{P}(\kappa)$. Steel ([18], Theorem 8.26) calls this phenomenon the boldface GCH at κ which is the assertion that κ^+ does not inject into $\mathscr{P}(\kappa)$. Say that the boldface GCH holds below κ if the boldface GCH holds for all $\delta < \kappa$.

The boldface GCH at ω or the statement that there are no uncountable wellorderable subsets of $\mathbb{R}$ is a very important property of many nice choiceless framework for the set theoretic universe. It follows from classical regularity properties. If countable choice for $\mathbb{R}$, $\mathsf{AC}^{\mathbb{R}}_\omega$, holds and all subsets of $\mathbb{R}$ have the property of Baire, then wellordered unions of meager sets are meager. This implies $\mathbb{R}$ is not wellorderable. If in addition, all subsets of $\mathbb{R}$ have the perfect set property, then every uncountable subset of $\mathbb{R}$ cannot be wellorderable. Thus the boldface GCH at ω holds under $\mathsf{AC}^{\mathbb{R}}_\omega$ and all subsets of $\mathbb{R}$ have the property of Baire and the perfect set property. If ω_1 is measurable (there is a countably complete nonprincipal ultrafilter on ω_1), then also the boldface GCH at ω holds (see Fact 3.2). These properties are all consequences of the axiom of determinacy, AD, which states that every infinite two player game has a winning strategy for one of the two players. AD^+ is Woodin's extension of the axiom of determinacy.

The boldface GCH at ω is very important for the basic theory of determinacy. One important consequence is that if the boldface GCH at ω holds, M is an inner model of ZFC, and $\mathbb{P} \in M$ is a forcing which is countable in the real world, then in the real world, there is a generic $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over M. The existence of generics for forcings countable in the real world is used in Woodin's analysis of nice models of AD^+ as symmetric extension of their HOD-type submodels using Vopěnka forcing or ordinal definable ∞-Borel code forcing. The boldface GCH at ω synergizes well with the Baire property. For example, Woodin ([15] Theorem 5.42 Claim 2) showed that $\mathsf{AC}^{\mathbb{R}}_\omega$, the boldface GCH at ω, and all subsets of $\mathbb{R}$ have the Baire property, then for any set A, if $\mathbb{P} \in \mathrm{HOD}_{\{A\}}$ is a forcing which is countable in the real world, then there is a comeager set of $G \subseteq \mathbb{P}$ which are $\mathbb{P}$-generic over $\mathrm{HOD}_{\{A\}}$ and moreover $\mathrm{HOD}_{\{A\}}[G] = \mathrm{HOD}_{\{A,G\}}$. Recently, [2] used this observation of Woodin to show the following cardinality computations: Assume $\mathsf{AC}^{\mathbb{R}}_\omega$, all subsets of $\mathbb{R}$ have the Baire property, and the boldface GCH at ω holds, then $|{}^\omega\omega_1| < |{}^{<\omega_1}\omega_1|$, ${}^\omega\omega_1$ does not inject into $\mathbb{R} \times \mathrm{ON}$, and S_1 does not inject into ${}^\omega\omega_1$ (where $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$).

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The boldface GCH at ω and all subsets of $\mathbb{R}$ have the Baire property proves the following result: ([9] Proposition 3.5) For every $\Phi : \mathbb{R} \to \mathscr{P}(\mathrm{ON})$, there exists a comeager K and countable $\mathcal{E} \subseteq \mathscr{P}(\mathrm{ON})$ so that for all $r \in K$, there exists an $\mathcal{F} \subseteq \mathcal{E}$ so that $\Phi(r) = \bigcup \mathcal{F}$. This result is used to prove some interesting combinatorial results under AD^+. Let Θ be the supremum of the ordinals onto which $\mathbb{R}$ surjects. By [9] Lemma 3.8 and Theorem 4.3, under AD^+, if $\kappa < \Theta$ is an cardinal of uncountable cofinality, then there are no maximal almost disjoint family $\mathcal{A}$ on κ such that $\neg(|\mathcal{A}| < \mathrm{cof}(\kappa))$. More recently, the above fact was used to obtain large sets with respect to a normal measure or partition filters which are simultaneous homogeneous for many partitions. This is used in [1] to show under AD^+ that there is a four-element basis for linear ordering on $\mathbb{R} \times \kappa$ when $\kappa < \Theta$ is a regular cardinal and there is a twelve-element basis for the linear orderings on $\mathbb{R} \times \kappa$ when $\kappa < \Theta$ is a singular cardinal of uncountable cofinality.

The axiom of determinacy influences most strongly the sets which are surjective images of $\mathbb{R}$. Steel ([18] Theorem 8.26) showed that in $L(\mathbb{R})$, the boldface GCH holds below Θ. Woodin ([19] Theorem 2.16) extended these methods to show that AD^+ proves the boldface GCH holds below Θ.

The general boldface GCH plays an important role in the structure of the cardinality of sets which are nonwellorderable but linearly orderable (or equivalently, sets which are in bijection with subsets of the power set of an ordinal). If κ is a cardinal, let $\mathscr{P}_B(\kappa)$ be the set of bounded subsets of κ. By [2] and [6] Theorem 4.8, if the boldface GCH holds below κ, then $\neg(|[\kappa]^{\mathrm{cof}(\kappa)}| \leq |\mathscr{P}_B(\kappa)|)$. If κ is regular cardinal and the boldface GCH at κ holds, then $|[\kappa]^{<\kappa}| < |\mathscr{P}(\kappa)|$. Let $B(\omega, \kappa)$ be the set of all $f : \omega \to \kappa$ such that $\sup(f) < \kappa$. If $\mathrm{cof}(\kappa) > \omega$, then ${}^{\omega}\kappa = B(\omega, \kappa)$. However, [2] shows that if the boldface GCH holds below κ, then $|B(\omega, \kappa)| < |{}^{\omega}\kappa|$ if $\mathrm{cof}(\kappa) = \omega$.

Steel's and Woodin's result that the boldface GCH holds below Θ can be regarded as the first step in classifying the cardinal exponentiations below Θ. Substantial evidence from [5], [6], [8], [7], and [10] suggests that cardinal exponentiation follows a very elegant simple behavior called the ABCD Conjecture: Under AD^+, for all cardinals $\omega \leq \alpha \leq \beta < \Theta$ and $\omega \leq \gamma \leq \delta < \Theta$, $|{}^{\alpha}\beta| \leq |{}^{\gamma}\delta|$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$. Recently, [2] showed that under AD^+, if $\omega < \kappa < \Theta$ and $\epsilon < \kappa$, then $\mathscr{P}_B(\kappa)$ does not inject into ${}^{\epsilon}\mathrm{ON}$, the class of ϵ-length sequences of ordinals. By combining the latter result and the the boldface GCH below Θ, [2] proved the ABCD conjecture under AD^+.

The proof of the boldface GCH below Θ uses the inner model theory analysis of HOD. First, Steel ([17], [20], and [18] Theorem 8.26) showed that if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R}) \models$ "the boldface GCH below Θ". To show this, Steel showed that $\mathrm{HOD}^{L(\mathbb{R})} \restriction \boldsymbol{\delta}^2_1$ is a direct limit of a directed system of certain iterable mice. Woodin (as sketched in [19] Theorem 2.16) generalized this argument to show AD^+ proves the boldface GCH below Θ. To do this, one first applies Suslin-co-Suslin reflection to bring the question of the boldface GCH at some $\kappa < \Theta$ into a nice model of AD^+. Woodin then showed that a certain HOD-type submodel of this nice AD^+ model has a direct system analysis using hybrid strategy mice.

More recently, many purely combinatorial questions of determinacy have been resolved below ω_ω or the projective ordinals by classical determinacy methods to provide evidence before a general proof using inner model theory is found. The boldface GCH at ω was known by the classical regularity properties or using the fact that ω_1 is measurable. The boldface GCH at ω_1 was known by the fact that ω_2 is measurable since it is a weak partition cardinal as shown by Martin. Remarkably, it seems that Steel established the full boldface GCH below Θ without even knowing that the boldface GCH holds at ω_2 by classical determinacy arguments.

This paper will give a proof that the boldface GCH holds below ω_ω using combinatorial methods of AD. (It should be noted that by the Moschovakis coding lemma, if $\kappa < \Theta^{L(\mathbb{R})}$, the boldface GCH at κ holds in the real world if and only if $L(\mathbb{R}) \models$ "the boldface GCH holds at κ". Thus Steel's result actually implies that AD proves the boldface GCH below $\Theta^{L(\mathbb{R})}$.) The paper will work with a combinatorial principle of ω_1 which is true in AD. Let $\bigstar$ denote the following principle. (See Definition 2.10.)

$\bigstar$ For every function $f : \omega_1 \to \omega_1$, there is a Kunen function $\mathcal{K}$ which bounds f.

$\omega_1 \to_* (\omega_1)^{\omega_1}_2$ is the strong partition relation on ω_1. Martin showed that AD implies $\omega_1 \to_* (\omega_1)^{\omega_1}_2$. See Definition 2.10 for the definition of a Kunen function. Essentially, a Kunen function bounding $f : \omega_1 \to \omega_1$ is a sequence $\langle \varphi_\alpha : \alpha < \omega_1 \rangle$ such that there is a club $C \subseteq \omega_1$ so that for all $\alpha \in C$, φ_α is a surjection of α onto $f(\alpha)$. Kunen proved that AD implies every function $f : \omega_1 \to \omega_1$ has a Kunen function bounding it by defining what is known as a Kunen tree. Both of these results are important elementary consequences of AD, but this paper will only use $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. One can show that over $\omega_1 \to_* (\omega_1)^2_2$, $\bigstar$ is equivalent to $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ where $\mu^1_{\omega_1}$ is the club filter on ω_1. Kleinberg [14] studied the cardinals below ω_ω using the hypothesis that $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$. It seems that ★ is much more directly practical than $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$. AD is the only theory in which $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and ★ (or $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$) is known to hold. AD, using the method of good coding system by Martin, is the only known theory that implies the existence of a strong partition cardinal. Radin forcing was used by Mitchell ([16]) to produce a model in which the club filter $\mu^1_{\omega_1}$ is a countably complete ultrafilter and by Woodin to produce a model in which ω_1 is a weak partition cardinal ($\omega_1 \to_* (\omega_1)^\epsilon_2$ for all $\epsilon < \omega_1$). However, it seems that AD is still the only known theory in which $\mu^1_{\omega_1}$ is a countably complete ultrafilter and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$.

The main result of the paper is that $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and ★ imply the boldface GCH below ω_ω. The paper is completely self-contained. The combinatorial methods used here can be generalized using Jackson's theory of descriptions ([11]) for the projective ordinals to show the boldface GCH holds below the supremum of the projective ordinals, $\sup\{\boldsymbol{\delta}^1_n : n \in \omega\}$, and a bit beyond under AD. These methods show the boldface GCH at a level far below Θ. Only inner model theory is known to prove the boldface GCH below Θ assuming AD^+.

2. Partition Relations and Ultrapowers by Partition Filters

If X is a set and Y is a class, then ${}^XY$ is the class of all functions $f : X \to Y$. If $\epsilon \in \mathrm{ON}$ and $X \subseteq \mathrm{ON}$ is a set, then $[X]^\epsilon$ is the set of all increasing functions $f : \epsilon \to X$. If κ is a cardinal, $\epsilon \leq \kappa$ and $\gamma < \delta$, then the ordinary partition relation $\kappa \to (\kappa)^\epsilon_\gamma$ is the assertion that for all $P : [\kappa]^\epsilon \to \gamma$, there is a $\beta < \gamma$ and an $A \subseteq \kappa$ with $|A| = \kappa$ so that for all $f \in [A]^\epsilon$, $P(f) = \beta$. However one will need the correct type partition relations here since one will be primarily interested in the ultrapowers by the partition measures obtained using the correct type partition relations.

Definition 2.1. Let $\epsilon \in \mathrm{ON}$ and $f : \epsilon \to \mathrm{ON}$ be a function.

- f is discontinuous everywhere if and only if for all $\alpha < \epsilon$, $\sup(f \upharpoonright \alpha) = \sup\{f(\bar{\alpha}) : \bar{\alpha} < \alpha\} < f(\alpha)$.
- f has uniform cofinality ω if and only if there is a function $F : \epsilon \times \omega \to \mathrm{ON}$ so that for all $\alpha < \epsilon$ and $n \in \omega$, $F(\alpha, n) < F(\alpha, n+1)$ and $f(\alpha) = \sup\{F(\alpha, n) : n \in \omega\}$.
- f has the correct type if and only if f is both discontinuous everywhere and has uniform cofinality ω.

If $X \subseteq \mathrm{ON}$ and $\epsilon \in \mathrm{ON}$, then let $[X]^\epsilon_*$ denote the set of all increasing function $f : \epsilon \to X$ of the correct type.

Note that $[\kappa]^1_*$ is the set of ordinals below κ of cofinality ω.

Definition 2.2. Let κ be an uncountable cardinal, $\epsilon \leq \kappa$, and $\gamma < \kappa$. The correct type partition relation $\kappa \to_* (\kappa)^\epsilon_\gamma$ is the statement that for all $P : [\kappa]^\epsilon_* \to \gamma$, there is a $\beta < \gamma$ and a $C \subseteq \kappa$ which is a club subset of κ so that for all $f \in [C]^\epsilon_*$, $P(f) = \beta$.

If κ is an uncountable cardinal, $\epsilon \leq \kappa$, and $\gamma < \kappa$, then $\kappa \to_* (\kappa)^{<\epsilon}_\gamma$ is the statement that for all $\bar{\epsilon} < \epsilon$, $\kappa \to_* (\kappa)^{\bar{\epsilon}}_\gamma$. If κ is an uncountable cardinal, $\epsilon \leq \kappa$, and $\gamma \leq \kappa$, then $\kappa \to_* (\kappa)^\epsilon_{<\gamma}$ is the statement that for all $\bar{\gamma} < \gamma$, $\kappa \to_* (\kappa)^\epsilon_{\bar{\gamma}}$. If κ is an uncountable cardinal, $\epsilon \leq \kappa$, and $\gamma \leq \kappa$, then $\kappa \to_* (\kappa)^{<\epsilon}_{<\gamma}$ is the statement that for all $\bar{\epsilon} < \epsilon$ and $\bar{\gamma} < \gamma$, $\kappa \to_* (\kappa)^{\bar{\epsilon}}_{\bar{\gamma}}$.

If $\kappa \to_* (\kappa)^{<\kappa}_2$, then κ is called a weak partition cardinal. If $\kappa \to_* (\kappa)^\kappa_2$, then κ is called a strong partition cardinal. If $\kappa \to_* (\kappa)^\kappa_{<\kappa}$, then κ is called a very strong partition cardinal.

One can show that $\kappa \to (\kappa)^{\omega \cdot \epsilon}_\gamma$ implies $\kappa \to_* (\kappa)^\epsilon_\gamma$ and $\kappa \to_* (\kappa)^\epsilon_\gamma$ implies $\kappa \to (\kappa)^\epsilon_\gamma$ for all $\epsilon \leq \kappa$ and $\gamma < \kappa$.

Note that every function of uniform cofinality ω must take range among the limit ordinals. Thus for any cardinal κ and $1 \leq \epsilon \leq \kappa$, $[\kappa]^\epsilon_* \neq \emptyset$ requires that κ be an uncountable cardinal. Thus the notions of correct type function and the correct type partition relations are only meaningful for uncountable cardinals. Partition on ω (and notions such as the Ramsey property) can only be expressed using the ordinary partition relation.

Definition 2.3. Let κ be an uncountable cardinal and $\epsilon \leq \kappa$. Define the ϵ-exponent (correct type) partition filter μ^ϵ_κ on $[\kappa]^\epsilon_*$ by $A \in \mu^\epsilon_\kappa$ if and only if there is a club $C \subseteq \kappa$ so that $[C]^\epsilon_* \subseteq A$. Note that μ^1_κ is the ω-club filter.

If $X \subseteq \mathrm{ON}$, then let $\mathsf{enum}_X : \mathrm{ot}(X) \to X$ be the increasing enumeration of X. An ordinal γ is indecomposable if and only if for all $\alpha, \beta < \gamma$, $\alpha + \beta < \gamma$ and $\alpha \cdot \beta < \gamma$. If κ is a cardinal, $X \subseteq \kappa$, $\mathrm{ot}(X) = \kappa$, $\alpha < \kappa$, and $\gamma < \kappa$, then let $\mathsf{next}^\gamma_X(\alpha)$ be the $(1+\gamma)^{\text{th}}$-element of X greater than α.

The following results says that if $C \subseteq \kappa$ is a club, then there is a club $D \subseteq C$ which is very thin inside of C. This club is particularly useful for many constructions.

Fact 2.4. *Let κ be an uncountable regular cardinal. Let $C \subseteq \kappa$ be a club consisting entirely of indecomposable ordinals. Let $D = \{\alpha \in C : \mathsf{enum}_C(\alpha) = \alpha\}$. Then D is a club subset of C and for any $\epsilon \in D$ and $\alpha, \beta, \gamma, \delta < \epsilon$, $\mathsf{next}_C^{\alpha \cdot \beta + \gamma}(\delta) < \epsilon$.*

Proof. D is easily seen to be closed. Let $\alpha < \kappa$. Let $\alpha_0 = \alpha + 1$. If $\alpha_n \in C$ has been defined, then let $\alpha_{n+1} = \mathsf{enum}_C(\alpha_n + 1)$. Let $\alpha_\omega = \sup\{\alpha_n : n \in \omega\}$ and note that $\alpha < \alpha_\omega \in C$ since C is a club. For all $\beta < \alpha_\omega$, there is an $n \in \omega$ so that $\beta < \alpha_n$. Thus $\mathsf{enum}_C(\beta) < \mathsf{enum}_C(\alpha_n) < \mathsf{enum}_C(\alpha_n + 1) = \alpha_{n+1} < \alpha_\omega$. Since $\{\mathsf{enum}_C(\beta) : \beta < \alpha_\omega\} \subseteq \{\gamma \in C : \gamma < \alpha_\omega\}$, $\mathrm{ot}\{\gamma \in C : \gamma < \alpha_\omega\} = \alpha_\omega$. Since $\alpha_\omega \in C$, $\mathsf{enum}_C(\alpha_\omega) = \alpha_\omega$. Thus $\alpha < \alpha_\omega$ and $\alpha_\omega \in D$. This shows that D is unbounded. Thus D is a club. Now suppose $\epsilon \in D$ and $\alpha, \beta, \gamma, \delta < \epsilon$. Since $\epsilon \in D \subseteq C$ and C consists entirely of indecomposable ordinals, ϵ is an indecomposable ordinal. Since ϵ is in particular a limit ordinal and $\epsilon = \mathsf{enum}_C(\epsilon) > \delta$, there is some $\nu < \epsilon$ so that $\delta < \mathsf{enum}_C(\nu) < \mathsf{enum}_C(\epsilon) = \epsilon$. Since ϵ is indecomposable, $\nu + \alpha \cdot \beta + \gamma < \epsilon$. Note that $\mathsf{next}_C^{\alpha \cdot \beta + \gamma}(\delta) < \mathsf{enum}_C(\nu + \alpha \cdot \beta + \gamma) < \mathsf{enum}_C(\epsilon) = \epsilon$. ☐

Fact 2.5. *Let κ be an uncountable cardinal.*

(1) $\kappa \to_* (\kappa)^2_2$ implies that κ is regular.
(2) For all $\epsilon \leq \kappa$, $\kappa \to_* (\kappa)^\epsilon_2$ implies μ^ϵ_κ is an ultrafilter.
(3) For all $\epsilon \leq \kappa$ and $\gamma < \kappa$, $\kappa \to_* (\kappa)^\epsilon_\gamma$ implies μ^ϵ_κ is a γ^+-complete ultrafilter.
(4) If $\epsilon < \kappa$, then $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$ implies $\kappa \to_* (\kappa)^\epsilon_{<\kappa}$. Thus $\kappa \to_* (\kappa)^{<\kappa}_2$ implies $\kappa \to_* (\kappa)^{<\kappa}_{<\kappa}$.

Proof. (1) Suppose κ is not regular. Let $\delta = \mathrm{cof}(\kappa) < \kappa$ and $\rho : \delta \to \kappa$ be an increasing cofinal function. Define $P : [\kappa]^2 \to 2$ by $P(\alpha, \beta) = 0$ if and only if there exists an $\eta < \delta$ so that $\alpha < \rho(\eta) < \beta$. By $\kappa \to_* (\kappa)^2_2$, let $C \subseteq \kappa$ be a club homogeneous for P. First, suppose C is homogeneous for P taking value 0. For each $\alpha < \kappa$, let $\eta_\alpha = \mathsf{enum}_C(\omega \cdot \alpha + \omega)$. For all $\alpha < \kappa$, $(\eta_\alpha, \eta_{\alpha+1}) \in [C]^2_*$. $P(\eta_\alpha, \eta_{\alpha+1}) = 0$ implies there is a $\xi < \delta$ so that $\eta_\alpha < \rho(\xi) < \eta_{\alpha+1}$. Let ξ_α be the least ξ such that $\eta_\alpha < \rho(\xi) < \eta_{\alpha+1}$. For any $\alpha < \bar{\alpha} < \kappa$, $\rho(\xi_\alpha) < \eta_{\alpha+1} \leq \eta_{\bar{\alpha}} < \rho(\xi_{\bar{\alpha}})$. Since ρ is an increasing function, this implies that $\langle \xi_\alpha : \alpha < \kappa \rangle$ is an increasing function of κ into δ which is impossible since $\delta < \kappa$. Next, suppose C is homogeneous for P taking value 1. Let α be any element of $[C]^1_*$. Since ρ is cofinal, fix $\bar{\xi} < \delta$ so that $\alpha < \rho(\bar{\xi})$. Since C is a club, let β be any element of $[C]^1_*$ so that $\rho(\bar{\xi}) < \beta$. Thus $\alpha < \rho(\bar{\xi}) < \beta$. However, $P(\alpha, \beta) = 0$ implies that there is no $\xi < \delta$ with $\alpha < \rho(\xi) < \beta$ which is contradiction. So C is not homogeneous for P which also a contradiction.

(2) Let $X \subseteq [\omega_1]^\epsilon_*$. Define $P_X : [\kappa]^\epsilon \to 2$ by $P_X(\ell) = 1$ if and only if $\ell \in X$. By $\kappa \to_* (\kappa)^\epsilon_2$, there is a club C homogeneous for P. If C is homogeneous for P taking value 1, then $[C]^\epsilon_* \subseteq X$ and hence $X \in \mu^\epsilon_\kappa$. If C is homogeneous for P taking value 0, then $[C]^\epsilon_* \subseteq [\kappa]^\epsilon_* \setminus X$ and thus $[\kappa]^\epsilon_* \setminus X \in \mu^\epsilon_\kappa$.

(3) Suppose μ^ϵ_κ is not γ^+-complete. Let $\delta < \gamma^+$ and $\langle X_\xi : \xi < \delta \rangle$ is a sequence in μ^ϵ_κ such that $\bigcap_{\xi < \delta} X_\xi \notin \mu^\epsilon_\kappa$. Let $\phi : \gamma \to \delta$ be a surjection. For $\eta < \gamma$, let $Y_\eta = X_{\phi(\eta)}$ and note that $\langle Y_\eta : \eta < \gamma \rangle$ is a sequence in μ^ϵ_κ and $\bigcap_{\eta < \gamma} Y_\eta \notin \mu^\epsilon_\kappa$. Let $C_0 \subseteq \kappa$ be a club so that $[C_0]^\epsilon_* \subseteq [\kappa]^\epsilon_* \setminus \bigcap_{\eta < \gamma} Y_\eta$. Define $P : [C_0]^\epsilon_* \to \gamma$ by $P(\ell)$ is the least $\eta < \gamma$ so that $\ell \notin Y_\eta$. By $\kappa \to_* (\kappa)^\epsilon_\gamma$, there is an $\bar{\eta} < \gamma$ and a club $C_1 \subseteq C_0$ so that for all $\ell \in [C_1]^\epsilon_*$, $P(\ell) = \bar{\eta}$. Thus $[C_1]^\epsilon_* \cap Y_{\bar{\eta}} = \emptyset$. Thus $Y_{\bar{\eta}} \notin \mu^\epsilon_\kappa$. Contradiction.

(4) Let $\gamma < \kappa$ and $P : [\kappa]^\epsilon_* \to \gamma$. If $\ell \in [\kappa]^{\epsilon+\epsilon}$, then let $\ell^0, \ell^1 \in [\kappa]^\epsilon$ be defined by $\ell^0 = \ell \restriction \epsilon$ and $\ell^1(\alpha) = \ell(\epsilon + \alpha)$. Define $Q_0 : [\kappa]^{\epsilon+\epsilon} \to 2$ by $Q_0(\ell) = 0$ if and only if $P(\ell^0) = P(\ell^1)$. By $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$, let C_0 be a club homogeneous for Q. Suppose C_0 is homogeneous for Q taking value 1. Define $Q_1 : [\kappa]^{\epsilon+\epsilon} \to 2$ by $Q_1(\ell) = 0$ if and only if $P(\ell^0) < P(\ell^1)$. By $\kappa \to_* (\kappa)^{\epsilon+\epsilon}_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for Q_1. First, suppose C_1 is homogeneous for Q_1 taking value 1. For each $n \in \omega$, let $\iota_n : \epsilon \to \kappa$ be defined by $\iota_n(\alpha) = \mathsf{enum}_{C_1}((\omega \cdot \epsilon) \cdot n + \omega \cdot \alpha + \omega)$. Let $I_n : \epsilon \times \omega \to \omega_1$ be defined by $I_n(\alpha, k) = \mathsf{enum}_{C_1}((\omega \cdot \epsilon) \cdot n + \omega \cdot \alpha + k)$. For all $n \in \omega$, ι_n is discontinuous and I_n witnesses that ι_n has uniform cofinality ω. Thus $\iota_n \in [C_1]^\epsilon_*$. Note that for all $n < \omega$, $\sup(\iota_n) < \iota_{n+1}(0)$. For each $n \in \omega$, there is an $\ell_n \in [C_1]^{\epsilon+\epsilon}_*$ so that $\ell^0_n = \iota_n$ and $\ell^1_n = \iota_{n+1}$. For each $n \in \omega$, $Q_0(\ell_n) = 1$ and $Q_1(\ell_n) = 1$ imply that $P(\iota_n) = P(\ell^0_n) > P(\ell^1_n) = P(\iota_{n+1})$. Thus $\langle P(\iota_n) : n \in \omega \rangle$ is an infinite descending sequence of ordinals which is a contradiction. Now suppose C_1 is homogeneous for Q_1 taking value 1. For each $\xi < \gamma + 1$, let $\tau_\xi(\alpha) = \mathsf{enum}_{C_1}((\omega \cdot \epsilon) \cdot \xi + \omega \cdot \alpha + \omega)$. Let $T_\xi : \epsilon \times \omega \to \omega_1$ by $T_\xi(\alpha, k) = \mathsf{enum}_{C_1}((\omega \cdot \epsilon) \cdot \xi + \omega \cdot \alpha + k)$. For each $\xi < \gamma + 1$, τ_ξ is discontinuous and has uniform cofinality ω as witnessed by T_ξ. For each $\xi_0 < \xi_1 < \gamma + 1$, there is an $\ell_{\xi_0, \xi_1} \in [C_1]^{\epsilon+\epsilon}_*$ so that $\ell^0_{\xi_0, \xi_1} = \tau_{\xi_0}$ and $\ell^1_{\xi_0, \xi_1} = \tau_{\xi_1}$. For

all $\xi_0 < \xi_1 < \gamma + 1$, $Q_0(\ell_{\xi_0,\xi_1}) = 1$ and $Q_1(\ell_{\xi_0,\xi_1}) = 0$ imply that $P(\tau_{\xi_0}) = P(\ell^0_{\xi_0,\xi_1}) < P(\ell^1_{\xi_0,\xi_1}) = P(\tau_{\xi_1})$. Thus $\langle \tau(\xi) : \xi < \gamma + 1 \rangle$ is order embedding of $\gamma + 1$ into γ which is impossible. Thus C_0 must have been homogeneous for Q_0 taking value 0. Let $\iota_0, \iota_1 \in [C_0]^\epsilon_*$. Let $\bar{\iota} \in [C_0]^\epsilon_*$ be any element such that $\max\{\sup(\iota_0), \sup(\iota_1)\} < \bar{\iota}(0)$. Then there are $\ell_0, \ell_1 \in [C_0]^\epsilon_*$ so that $\ell^0_0 = \iota_0$, $\ell^0_1 = \iota_1$ and $\ell^1_0 = \bar{\iota} = \ell^1_1$. Then $Q_0(\ell_0) = 0 = Q_1(\ell_1)$ implies that $P(\iota_0) = P(\ell^0_0) = P(\ell^1_0) = P(\bar{\iota}) = P(\ell^1_1) = P(\ell^0_1) = P(\iota_1)$. Since $\iota_0, \iota_1 \in [C_0]^\epsilon_*$ were arbitrary, one has that P is constant on $[C_0]^\epsilon_*$. ☐

Fact 2.6. *Let κ be an uncountable cardinal, $1 \leq \epsilon < \kappa$, $\delta < \epsilon$, $\kappa \rightarrow_* (\kappa)^{\delta+1+(\epsilon-\delta)}_2$, and $\kappa \rightarrow_* (\kappa)^{\epsilon-\delta}_{<\kappa}$. Let $\Phi : [\kappa]^\epsilon \rightarrow \kappa$ has the property that $\{\iota \in [\kappa]^\epsilon : \Phi(\iota) < \iota(\delta)\} \in \mu^\epsilon_\kappa$. Then there is a club $C \subseteq \kappa$ and a function $\Psi : [C]^\delta_* \rightarrow \kappa$ so that for all $\iota \in [C]^\epsilon_*$, $\Phi(\iota) = \Psi(\iota \restriction \delta)$.*

Proof. If $\ell \in [\kappa]^{\delta+1+(\epsilon-\delta)}_2$, let $\hat{\ell} \in [\omega_1]^\epsilon_*$ be defined by $\hat{\ell}(\alpha) = \ell(\alpha)$ if $\alpha < \delta$ and $\hat{\ell}(\alpha) = \ell(\delta + 1 + (\alpha - \delta))$ if $\delta \leq \alpha < \epsilon$. Let $C_0 \subseteq \kappa$ be a club consisting entirely of indecomposable ordinals so that for all $\iota \in [C_0]^\epsilon_*$, $\Phi(\iota) < \iota(\delta)$. Define $P : [C]^{\delta+1+(\epsilon-\delta)} \rightarrow 2$ by $P(\ell) = 0$ if and only if $\Phi(\hat{\ell}) < \ell(\delta)$. By $\kappa \rightarrow_* (\kappa)^{\delta+1+(\epsilon-\delta)}_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P. Let $C_2 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha\}$. Pick any $\iota \in [C_2]^\epsilon_*$. Since $\Phi(\iota) < \iota(\delta)$ because $\iota \in [C_2]^\epsilon_* \subseteq [C_0]^\epsilon_*$, $\mathsf{next}^\omega_{C_1}(\Phi(\iota)) < \iota(\delta)$ by Fact 2.4. Let $\ell \in [C_1]^{\delta+1+(\epsilon-\delta)}_*$ be such that $\hat{\ell} = \iota$ and $\ell(\delta) = \mathsf{next}^\omega_{C_1}(\Phi(\iota))$ (and note that ℓ has uniform cofinality ω since ι does and $\mathrm{cof}(\mathsf{next}^\omega_{C_1}(\Phi(\iota))) = \omega$). Since $\Phi(\hat{\ell}) = \Phi(\iota) < \mathsf{next}^\omega_{C_1}(\Phi(\iota)) = \ell(\delta)$, one has $P(\ell) = 0$. Thus C_1 is homogeneous for P taking value 0. For any $\sigma \in [C_2]^\delta_*$, let $\Phi_\sigma : [C_2 \setminus (\sup(\sigma)+1)]^{\epsilon-\delta}_* \rightarrow \kappa$ be defined by $\Phi_\sigma(\tau) = \Phi(\sigma\hat{}\tau)$. For any $\tau \in [C_2 \setminus (\sup(\sigma) + 1)]^{\epsilon-\delta}_*$, let $\ell_{\sigma,\tau} = \sigma\hat{}\langle \mathsf{next}^\omega_{C_1}(\sup(\sigma))\rangle\hat{}\tau$. Note that $\ell_{\sigma,\tau} \in [C_1]^{\delta+1+(\epsilon-\delta)}_*$, $\hat{\ell}_{\sigma,\tau} = \sigma\hat{}\tau$, and $\ell(\delta) = \mathsf{next}^\omega_{C_1}(\sup(\sigma))$. $P(\ell_{\sigma,\tau}) = 0$ implies that $\Phi_\sigma(\tau) = \Phi(\sigma\hat{}\tau) = \Phi(\hat{\ell}_{\sigma,\tau}) < \ell(\delta) = \mathsf{next}^\omega_{C_1}(\sup(\sigma))$. By $\kappa \rightarrow_* (\kappa)^{\epsilon-\delta}_{<\kappa}$, $\mu^{\epsilon-\delta}_\kappa$ is κ-complete. There is a $\gamma_\sigma < \kappa$ so that for $\mu^{\epsilon-\delta}_\kappa$-almost all τ, $\Phi_\sigma(\tau) = \gamma_\sigma$. Define $Q : [C_2]^\epsilon_* \rightarrow 2$ by $Q(\iota) = 0$ if and only if $\Phi(\iota) = \gamma_{\iota\restriction\delta}$. By $\kappa \rightarrow_* (\kappa)^\epsilon_2$, there is a club $C_3 \subseteq C_2$ which is homogeneous for Q. Pick any $\sigma \in [C_3]^\delta_*$. There is a club $D \subseteq C_3 \setminus (\sup(\sigma) + 1)$ so that for all $\tau \in [D]^{\epsilon-\delta}_*$, $\Phi_\sigma(\tau) = \gamma_\sigma$. Fix $\tau \in [D]^{\epsilon-\delta}_*$. Let $\iota = \sigma\hat{}\tau$ and note that $\iota \in [C_3]^\epsilon_*$. $\Phi(\iota) = \Phi_{\iota\restriction\delta}(\tau) = \Phi_\sigma(\tau) = \gamma_\sigma = \gamma_{\iota\restriction\delta}$. Thus $Q(\iota) = 0$. This shows that C_3 is homogeneous for Q taking value 0. Define $\Psi : [C_3]^\delta_* \rightarrow \kappa$ by $\Psi(\sigma) = \gamma_\sigma$. For any $\iota \in [C_3]^\epsilon_*$, $Q(\iota) = 0$ implies that $\Phi(\iota) = \gamma_{\iota\restriction\delta} = \Psi(\iota \restriction \delta)$. ☐

Fact 2.7. *Let κ be an uncountable cardinal satisfying $\kappa \rightarrow_* (\kappa)^2_2$. Then μ^1_κ is normal.*

Proof. Note that $\kappa \rightarrow_* (\kappa)^2_2$ implies $\kappa \rightarrow_* (\kappa)^1_{<\kappa}$ by Fact 2.5. This result now follows from Fact 2.6 with $\delta = 0$ and $\epsilon = 1$. ☐

Fact 2.8. *Suppose $\epsilon < \kappa$ and $\kappa \rightarrow_* (\kappa)^{\epsilon+1}_2$. Let $\Phi : [\kappa]^\epsilon \rightarrow \kappa$. Then there is a club $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, $\Phi(f) < \mathsf{next}^\omega_C(\sup(f))$.*

Proof. Define $P : [\kappa]^{\epsilon+1}_* \rightarrow 2$ by $P(g) = 0$ if and only if $\Phi(g \restriction \epsilon) < g(\epsilon)$. By $\kappa \rightarrow_* (\kappa)^{\epsilon+1}_2$, there is a club $C \subseteq \kappa$ which is homogeneous for P. Pick any $f \in [C]^\epsilon_*$. Let $\gamma = \mathsf{next}^\omega_C(\Phi(f))$. Let $g = f\hat{}\langle\gamma\rangle$ and note that $g \in [C]^{\epsilon+1}_*$. Since $\Phi(g \restriction \epsilon) = \Phi(f) < \mathsf{next}^\omega_C(\Phi(f)) = \gamma = g(\epsilon)$, one has that $P(g) = 0$. Since C is homogeneous for P and $g \in [C]^{\epsilon+1}_*$, one has that C is homogeneous for P taking value 0. For any $f \in [C]^\epsilon_*$, let $g_f = f\hat{}\langle\mathsf{next}^\omega_C(\sup(f))\rangle$. $P(g_f) = 0$ implies that $\Phi(f) = \Phi(g_f \restriction \epsilon) < g_f(\epsilon) = \mathsf{next}^\omega_C(\sup(f))$. ☐

Note that the ordinary partition relation $\omega \rightarrow (\omega)^n_2$ for $n \in \omega$ is the finite Ramsey theorem. For an uncountable cardinals κ, the ordinary partition relation $\kappa \rightarrow (\kappa)^2_2$ is equivalent to the weak compactness of κ which is compatible with the axiom of choice. However, the correct type partition relation $\kappa \rightarrow_* (\kappa)^2_2$ implies μ^1_κ is normal which can be used to show ${}^\omega\kappa$ is not a wellorderable set. The finite exponent correct type partition relation already seems to imply many of the consequences of the infinite exponent ordinary partition relation. If $\mathsf{AC}^\mathbb{R}_\omega$ holds and $\epsilon < \omega_1$, then a function $f : \epsilon \rightarrow \omega_1$ has uniform cofinality ω if and only if the range of f consists of limit ordinals. However if $\mu^1_{\omega_1}$ is a normal ultrafilter, then one can show that the identity function $\mathsf{id} : \omega_1 \rightarrow \omega_1$ does not have uniform cofinality ω. The notion of a correct type function is a nontrivial concept when handling functions $f : \omega_1 \rightarrow \omega_1$ which will happen frequently in this paper.

Fact 2.9. *(Martin; [12], [11], [4], [3]) Assume* AD. $\omega_1 \rightarrow_* (\omega_1)^{\omega_1}_{<\omega_1}$.

Since AD implies $\omega_1 \to (\omega_1)^{\omega_1}_{<\omega_1}$, one has that for all $\epsilon \leq \omega_1$, $\mu^\epsilon_{\omega_1}$ are countably complete ultrafilters. Actually, AD implies there are no nonprincipal ultrafilters on ω which can be used to show any ultrafilter on any set is countably complete.

Definition 2.10. Let $\prod_{\alpha<\omega_1} \alpha = \{(\alpha,\beta) : \beta < \alpha\}$. A Kunen function is a function $\mathcal{K} : \prod_{\alpha<\omega_1} \alpha \to \omega_1$ such that for all $\alpha < \omega_1$, $\{\mathcal{K}(\alpha,\beta) : \beta < \alpha\}$ is an ordinal which will be denote $\chi^{\mathcal{K}}_\alpha$. Define $\Xi^{\mathcal{K}} : \omega_1 \to \omega_1$ by $\Xi^{\mathcal{K}}(\alpha) = \chi^{\mathcal{K}}_\alpha$. If $\beta < \omega_1$, then let $\mathcal{K}^\beta : (\omega_1 \setminus \beta + 1) \to \omega_1$ be defined by $\mathcal{K}^\beta(\alpha) = \mathcal{K}(\alpha,\beta)$.

Let $f : \omega_1 \to \omega_1$. The Kunen function $\mathcal{K}$ bounds f if and only if $\{\alpha < \omega_1 : f(\alpha) \leq \Xi^{\mathcal{K}}(\alpha)\} \in \mu^1_{\omega_1}$. The Kunen function $\mathcal{K}$ strictly bounds f if and only if $\{\alpha \in \omega_1 : f(\alpha) < \Xi^{\mathcal{K}}(\alpha)\} \in \mu^1_{\omega_1}$.

Fact 2.11. *(Kunen; [11] Lemma 4.1))* AD. *For every function $f : \omega_1 \to \omega_1$, there is a Kunen function $\mathcal{K} : \prod_{\alpha<\omega_1} \alpha \to \omega_1$ which bounds f.*

Definition 2.12. Let $\bigstar$ be the following statement.

- For any function $f : \omega_1 \to \omega_1$, there is a Kunen function $\mathcal{K} : \prod_{\alpha<\omega_1} \alpha \to \omega_1$ which bounds f.

Note that $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$ follows from AD by Fact 2.9 and Fact 2.11. The main result of the paper will be proved from the combinatorial principles $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$.

Definition 2.13. If μ is an measure on a set X. If f and g are two functions on X, then let $f \sim_\mu g$ if and only if $\{x \in X : f(x) = g(x)\} \in \mu$. If $f : X \to \mathrm{ON}$ and $g : X \to \mathrm{ON}$, then write $f <_\mu g$ if and only if $\{x \in X : f(x) < g(x)\} \in \mu$. If $f : X \to \mathrm{ON}$, then let $[f]_\mu$ be the class of all functions g with $g \sim_\mu f$. The ultrapower $\prod_X \mathrm{ON}/\mu$ is the set of $\sim_\mu$ equivalence class of functions $f : X \to \mathrm{ON}$. The ultrapower ordering on $\prod_X \mathrm{ON}/\mu$ is defined by $x \prec_\mu y$ if and only if there exists $f, g : X \to \mathrm{ON}$ so that $x = [f]_\mu$ and $y = [g]_\mu$ and $f <_\mu g$. $j_\mu : \mathrm{ON} \to \prod_X \mathrm{ON}/\mu$ is defined by $j_\mu(\alpha) = [c_\alpha]_\mu$ where $c_\alpha : X \to \{\alpha\}$ is the constant function.

If μ is a measure and $x \in j_\mu(\omega_1)$, then let $\mathsf{init}_\mu(x) = \{y \in j_\mu(\omega_1) : y \prec_\mu x\}$.

Fact 2.14. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$. $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$ implies $\bigstar$.*

Proof. $\omega_1 \to_* (\omega_1)^2_2$ implies $\mu^1_{\omega_1}$, the club filter on ω_1, is a normal ultrafilter on ω_1 by Fact 2.7. Thus $\omega_1 = [\mathsf{id}]_{\mu^1_{\omega_1}}$ where $\mathsf{id} : \omega_1 \to \omega_1$ is the identity function. Now suppose $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$. Let $f : \omega_1 \to \omega_1$ be any function with $[\mathsf{id}]_{\mu^1_{\omega_1}} \leq [f]_{\mu^1_{\omega_1}}$. Thus $\omega_1 = [\mathsf{id}]_{\mu^1_{\omega_1}} \leq [f]_{\mu^1_{\omega_1}} < j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$. Let $\mathfrak{b} : \omega_1 \to \mathsf{init}_{\mu^1_{\omega_1}}([f]_{\mu^1_{\omega_1}})$ be a bijection. Define a wellordering $\prec$ on ω_1 by $\alpha \prec \beta$ if and only if $\mathfrak{b}(\alpha) < \mathfrak{b}(\beta)$. Let $\mathcal{W} = (\omega_1, \prec)$ and note that $\mathrm{ot}(\mathcal{W}) = [f]_{\mu^1_{\omega_1}}$. For each $\alpha < \omega_1$, let $\mathcal{W}_\alpha = (\alpha, \prec\restriction \alpha)$. If $\beta < \alpha < \omega_1$, then let $\mathrm{ot}(W_\alpha, \beta)$ be the rank of β in $\mathcal{W}_\alpha$. Define $\mathcal{K} : \prod_{\alpha<\omega_1} \alpha \to \omega_1$ by $\mathcal{K}(\alpha,\beta) = \mathrm{ot}(\mathcal{W}_\alpha, \beta)$. One seeks to show that $\mathcal{K}$ is a Kunen function for f. It is clear that for all $\alpha \in \omega_1$, $\{\mathcal{K}(\alpha,\beta) : \beta < \alpha\} = \{\mathrm{ot}(W_\alpha,\beta) : \beta < \alpha\} = \mathrm{ot}(W_\alpha)$. Thus $\Xi^{\mathcal{K}}(\alpha) = \mathrm{ot}(\mathcal{W}_\alpha)$. Suppose $\eta < [f]_{\mu^1_{\omega_1}}$. Let $\xi_\eta = \mathfrak{b}^{-1}(\eta)$. Define $g_\eta : \omega_1 \setminus (\xi_\eta + 1) \to \omega_1$ by $g_\eta(\alpha) = \mathrm{ot}(W_\alpha, \xi_\eta)$. Note that for all $\alpha \in \omega_1 \setminus (\xi_\eta + 1)$, $g_\eta(\alpha) < \mathrm{ot}(W_\alpha) = \Xi^{\mathcal{K}}(\alpha)$. Define $\Psi : \mathsf{init}_{\mu^1_{\omega_1}}([f]_{\mu^1_{\omega_1}}) \to \mathsf{init}_\mu([\Xi^{\mathcal{K}}]_{\mu^1_{\omega_1}})$ by $\Psi(\eta) = [g_\eta]_{\mu^1_{\omega_1}}$. Suppose $\eta_0 < \eta_1 < [f]_{\mu^1_{\omega_1}}$. Let $\zeta = \max\{\xi_{\eta_0}, \xi_{\eta_1}\}$. For all $\alpha \in \omega_1 \setminus (\zeta + 1)$, $g_{\eta_0}(\alpha) = \mathrm{ot}(\mathcal{W}_\alpha, \xi_{\eta_0}) < \mathrm{ot}(\mathcal{W}_\alpha, \xi_{\eta_1}) = g_{\eta_1}(\alpha)$ since $\mathfrak{b}(\xi_{\eta_0}) = \eta_0 < \eta_1 < \mathfrak{b}(\xi_{\eta_1})$. Thus $\Psi(\eta_0) = [g_{\eta_0}]_{\mu^1_{\omega_1}} < [g_{\eta_1}]_{\mu^1_{\omega_1}} = \Psi(\eta_1)$. Ψ is an order embedding of $\mathsf{init}_\mu([f]_{\mu^1_{\omega_1}})$ into $\mathsf{init}_{\mu^1_{\omega_1}}([\Xi^{\mathcal{K}}])$. Thus $[f]_{\mu^1_{\omega_1}} \leq [\Xi^{\mathcal{K}}]_{\mu^1_{\omega_1}}$. This shows that $\{\alpha \in \omega_1 : f(\alpha) \leq \Xi^{\mathcal{K}}(\alpha)\} \in \mu^1_{\omega_1}$. $\mathcal{K}$ is a Kunen function bounding f. $\square$

Fact 2.15. *Assume $\omega_1 \to_* (\omega_1)^2_2$. Let $f : \omega_1 \to \omega_1$ and $\mathcal{K}$ be a Kunen function strictly bounding f. Then there is a $\gamma < \omega_1$ so that $f \sim_\mu \mathcal{K}^\gamma$.*

Proof. $\omega_1 \to_* (\omega_1)^2_2$ implies that $\mu^1_{\omega_1}$ is a normal ultrafilter by Fact 2.7. Let $A = \{\alpha \in \omega_1 : f(\alpha) < \Xi^{\mathcal{K}}(\alpha)\} \in \mu$. For each $\alpha \in \omega_1$, one has that $f(\alpha) < \Xi^{\mathcal{K}}(\alpha) = \chi^{\mathcal{K}}_\alpha = \{\mathcal{K}(\alpha,\beta) : \beta < \alpha\}$. Define $g : A \to \omega_1$ by $g(\alpha)$ is the least $\beta < \alpha$ so that $\mathcal{K}(\alpha,\beta) = f(\alpha)$. For all $\alpha \in A$, $g(\alpha) < \alpha$. Since $\mu^1_{\omega_1}$ is normal, there is a $\gamma < \omega_1$ and $B \subseteq A$ with $B \in \mu$ and $g(\alpha) = \gamma$ for all $\alpha \in B$. Thus $\mathcal{K}^\gamma(\alpha) = f(\alpha)$ for all $\alpha \in B$. $\square$

Fact 2.16. *Assume $\omega_1 \to_* (\omega_1)^2_2$. Let $f : \omega_1 \to \omega_1$ and $\mathcal{K}$ be a Kunen function bounding f. Then there is an injection $\Gamma : \mathsf{init}_{\mu^1_{\omega_1}}([f]_{\mu^1_{\omega_1}}) \to \omega_1$ so that for all $x \in \mathsf{init}_{\mu^1_{\omega_1}}([f]_{\mu^1_{\omega_1}})$, $[\mathcal{K}^{\Gamma(x)}]_{\mu^1_{\omega_1}} = x$.*

Proof. Suppose $x \prec_{\mu^1_{\omega_1}} [f]_{\mu^1_{\omega_1}}$. Let $g : \omega_1 \to \omega_1$ represent x. Then $g <_{\mu^1_{\omega_1}} f$ and hence $\mathcal{K}$ is also a Kunen function bounding g. By Fact 2.15, there is a $\gamma < \omega_1$ so that $\mathcal{K}^\gamma \sim_\mu g$. Let $\Gamma(x)$ be the least γ such that $[\mathcal{K}^\gamma]_{\mu^1_{\omega_1}} = x$. This defines the desired injection $\Gamma : [f]_{\mu^1_{\omega_1}} \to \omega_1$. $\square$

Dependent choice implies ultrapowers of ordinals are wellordering. However the existence of Kunen functions bounding functions from ω_1 to ω_1 is sufficient to show that the ultrapower of ω_1 by the finite exponent partition measures on ω_1 are wellorderings.

Fact 2.17. *Assume $\omega_1 \to_* (\omega_1)^2_2$ and $\bigstar$. The ultrapower $j_{\mu^1_{\omega_1}}(\omega_1) = \prod_{\omega_1} \omega_1/\mu^1_{\omega_1}$ is a wellordering.*

Proof. Suppose the ultrapower is not wellfounded. There is an $A \subseteq \prod_{\omega_1} \omega_1/\mu^1_{\omega_1}$ which has no minimal element according to the ultrapower ordering $\prec_{\mu^1_{\omega_1}}$. Pick any element $x \in A$. Let $f : \omega_1 \to \omega_1$ be a representative for x. Let $\mathcal{K}$ be a Kunen function bounding f. By Fact 2.16, there is an injection $\Gamma : \mathsf{init}_{\mu^1_{\omega_1}}([f]_{\mu^1_{\omega_1}}) \to \omega_1$ so that for all $y \prec_{\mu^1_{\omega_1}} [f]_{\mu^1_{\omega_1}}$, $y = [\mathcal{K}^{\Gamma(y)}]_{\mu^1_{\omega_1}}$. Let $B = \Gamma[[f]_{\mu^1_{\omega_1}}]$ be the range of Γ. Let δ_0 be the least ordinal $\delta \in B$. Suppose δ_n has been defined. Since $[f]_{\mu^1_{\omega_1}}$ has no $\prec$-least element, there is some $\delta \in B$ so that $\mathcal{K}^\delta <_{\mu^1_{\omega_1}} \mathcal{K}^{\delta_n}$. Let δ_{n+1} be the least $\delta \in B$ so that $\mathcal{K}^\delta <_{\mu^1_{\omega_1}} \mathcal{K}^{\delta_n}$. For each $n \in \omega$, $D_n = \{\alpha \in \omega_1 : \mathcal{K}^{\delta_{n+1}}(\alpha) < \mathcal{K}^{\delta_n}(\alpha)\} \in \mu^1_{\omega_1}$. Since $\mu^1_{\omega_1}$ is countably complete, $D = \bigcap_{n\in\omega} D_n \in \mu^1_{\omega_1}$ and hence nonempty. Let $\bar{\alpha} \in D$. Then $\langle \mathcal{K}^{\delta_n}(\bar{\alpha}) : n \in \omega\rangle$ is an infinite descending sequence of ordinals. Contradiction. $\square$

Fact 2.18. *Assume $\omega_1 \to_* (\omega_1)^2_2$ and $\bigstar$. $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$.*

Proof. By Fact 2.17, $j_{\mu^1_{\omega_1}}(\omega_1)$ is a wellordering. By Fact 2.16, each initial segment of $j_{\mu^1_{\omega_1}}(\omega_1)$ injects into ω_1. Thus $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$. $\square$

Fact 2.19. *Assume $\omega_1 \to_* (\omega_1)^2_2$. $\bigstar$ and $j_{\mu^1_{\omega_1}}(\omega_1) \leq \omega_2$ are equivalent.*

Proof. This follows from Fact 2.14 an Fact 2.18. $\square$

Fact 2.20. *Let κ be an uncountable cardinal and μ be a normal ultrafilter on κ containing no bounded subsets of κ. Let $q : \kappa \to \kappa$ be a function and $A \in \mu$. Then the set $B = \{\alpha \in A : (\forall \alpha' < \alpha)(q(\alpha') < \alpha)\} \in \mu$.*

Proof. Suppose not. Then $C = \kappa \setminus B \in \mu$. Let $f : C \to \kappa$ be defined by $f(\alpha)$ is the least $\alpha' < \alpha$ so that $\alpha \leq q(\alpha')$. Since μ is normal, there is a $D \subseteq C$ with $D \in \mu$ and a $\beta < \kappa$ so that for all $\alpha \in D$, $h(\alpha) = \beta$. Thus for all $\alpha \in D$, $\alpha \leq q(h(\alpha)) = q(\beta)$. This is impossible since D is an unbounded set. $\square$

Fact 2.21. *Let κ be an uncountable cardinal and μ be a normal ultrafilter on κ containing no bounded subsets of κ. If $A \in \mu$, then $\{\alpha \in A : \mathsf{enum}_A(\alpha) = \alpha\} \in \mu$.*

Proof. By applying Fact 2.20 to enum_A, the set $\bar{A} = \{\alpha \in \kappa : (\forall \alpha' < \alpha)(\mathsf{enum}_A(\alpha') < \alpha)\} \in \mu$. Let $B = A \cap \bar{A} \in \mu$. If $\alpha \in B$, then $\sup(\mathsf{enum}_A \restriction \alpha) = \alpha$ and thus $\mathsf{enum}_A(\alpha) = \alpha$ since $\alpha \in A$. So $B \subseteq \{\alpha \in A : \mathsf{enum}_A(\alpha) = \alpha\}$. $\square$

Fact 2.22. *(Martin) Assume κ is an uncountable cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$.*

1. *Let μ be an ultrafilter on κ such that $j_\mu(\kappa)$ is a wellordering. Then $j_\mu(\kappa)$ is a cardinal.*
2. *Let μ be a normal ultrafilter on κ which contains no bounded subsets of κ such that $j_\mu(\kappa)$ is a wellordering. Then $j_\mu(\kappa)$ is a regular cardinal.*

Proof. (1) First assume μ is an ultrafilter on an uncountable cardinal satisfying $\kappa \to_* (\kappa)^\kappa_2$ and $j_\mu(\kappa)$ is a wellordering (and thus one may assume $j_\mu(\kappa)$ is an ordinal). For the sake of contradiction, suppose $j_\mu(\kappa)$ is not a cardinal. Then there is a $\lambda < j_\mu(\kappa)$ and an injection $\Phi : j_\mu(\kappa) \to \lambda$. If $f : \kappa \to \kappa$ is a function, then let $f^0 = f(2 \cdot \alpha)$ and $f^1 = f(2 \cdot \alpha + 1)$. Define $P : [\kappa]^\kappa_* \to 2$ by $P(f) = 0$ if and only if $\Phi([f^0]_\mu) = \Phi([f^1]_\mu)$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_0 \subseteq \kappa$ which is homogeneous for P. Take any $f \in [C_0]^\kappa_*$. Note that for all $\alpha < \kappa$, $f^0(\alpha) < f^1(\alpha)$ and hence $[f^0]_\mu < [f^1]_\mu$. Since Φ is injective, $\Phi([f^0]_\mu) \neq \Phi([f^1]_\mu)$ and thus $P(f) = 1$. So C_0 is homogeneous for P taking value 1. Define $Q : [C_0]^\kappa_* \to 2$ by $Q(f) = 0$ if and only if $\Phi([f^0]_\mu) < \Phi([f^1]_\mu)$. By $\kappa \to_* (\kappa)^\kappa_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for Q. First suppose C_1 is homogeneous for Q taking value 1. For each $k \in \omega$ and $\alpha < \kappa$, let $g_k(\alpha) = \mathsf{enum}_{C_1}((\omega \cdot \omega) \cdot \alpha + \omega \cdot k + \omega)$. Note that $g_k \in [C_1]^\kappa_*$. For each $k \in \omega$, there is an $f_k \in [C_1]^\kappa_*$ so that $f^0_k = g_k$ and $f^1_k = g_{k+1}$. Then $P(f_k) = 1$ and $Q(f_k) = 1$ imply that $\Phi([g_{k+1}]_\mu) = \Phi([f^1_k]_\mu) < \Phi([f^0_k]_\mu) = \Phi([g_k]_\mu)$. Thus $\langle \Phi([g_k]_\mu) : k \in \omega\rangle$ is an infinite descending sequence in the ordinal λ which is a contradiction. Suppose C_1 is homogeneous for Q taking value 0. Since

$\lambda < j_\mu(\kappa)$, let $h : \kappa \to \kappa$ be such that $[h]_\mu = \lambda$. Define $T = \{(\alpha,\beta) : \alpha < \kappa \wedge \beta < h(\alpha)+2\}$.[1] Let $<_{\text{lex}}$ be the lexicographic ordering on T. Note that the ordertype of $(T, <_{\text{lex}})$ is κ. Let $\psi : T \to C_1$ be an order preserving function from $(T, <_{\text{lex}})$ into $(C_1, <)$ of the correct type which means the following two conditions hold:

- For all $x \in T$, $\sup\{\psi(y) : y <_{\text{lex}} x\} < \psi(x)$.
- There is a function $\Psi : T \times \omega \to \kappa$ so that for all $x \in T$ and $k \in \omega$, $\Psi(x,k) < \Psi(x,k+1)$ and $\psi(x) = \sup\{\Psi(x,k) : k \in \omega\}$.

For any function $g : \kappa \to \kappa$, define $\hat{g} : \kappa \to C_1$ by

$$\hat{g}(\alpha) = \begin{cases} \psi(\alpha, g(\alpha)) & g(\alpha) < h(\alpha)+1 \\ \psi(\alpha, 0) & \text{otherwise} \end{cases}.$$

Define $\hat{G} : \kappa \times \omega \to \kappa$ by

$$\hat{G}(\alpha, k) = \begin{cases} \Psi((\alpha, g(\alpha), k) & g(\alpha) < h(\alpha)+1 \\ \Psi((\alpha, 0), k) & \text{otherwise} \end{cases}.$$

Note that $\hat{G}$ witnesses that $\hat{g}$ has uniform cofinality ω. Since ψ is discontinuous everywhere, $\hat{g}$ is discontinuous everywhere. Thus $\hat{g} : \kappa \to C_1$ is an increasing function of the correct type and hence $\hat{g} \in [C_1]^\kappa_*$. For any $\eta < \lambda+1$, let $\delta_\eta = [\hat{g}]_\mu$ for any $g : \kappa \to \kappa$ such that $[g]_\mu = \eta$. Note that δ_η is independent of the choice of g representing η. Let $\eta_0 < \eta_1 < \lambda+1$. Let $g_0, g_1 : \kappa \to \kappa$ be such that $\eta_0 = [g_0]_\mu$ and $\eta_1 = [g_1]_\mu$. For $i \in 2$, let $\tilde{g}_0, \tilde{g}_1 \in [C_1]^\kappa_*$ be defined by

$$\tilde{g}_0(\alpha) = \begin{cases} \psi(\alpha, g_0(\alpha)) & g_0(\alpha) < g_1(\alpha) < h(\alpha)+1 \\ \psi(\alpha, 0) & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{g}_1(\alpha) = \begin{cases} \psi(\alpha, g_1(\alpha)) & g_0(\alpha) < g_1(\alpha) < h(\alpha)+1 \\ \psi(\alpha, 1) & \text{otherwise} \end{cases}$$

Note that for all $\alpha < \kappa$, $\tilde{g}_0(\alpha) < \tilde{g}_1(\alpha)$ by the definitions above and the fact that ψ is order preserving on $(T, <_{\text{lex}})$. For all $\alpha < \kappa$, $\tilde{g}_1(\alpha) < \tilde{g}_0(\alpha+1)$ since $\tilde{g}_1(\alpha) = \psi(\alpha, \xi)$ for some $\xi < h(\alpha)+2$, $\tilde{g}_0(\alpha+1) = \psi(\alpha+1, \zeta)$ for some $\zeta < h(\alpha+1)+2$, and by comparing the first coordinates since ψ is order preserving on $(T, <_{\text{lex}})$. Thus there is an $f \in [C_1]^\kappa_*$ so that $f^0 = \tilde{g}_0$ and $f^1 = \tilde{g}_1$. Since $[g_0]_\mu < [g_1]_\mu < \lambda+1$, one has that $A = \{\alpha \in \omega_1 : g_0(\alpha) < g_1(\alpha) < h(\alpha)+1\} \in \mu$. Hence for all $\alpha \in A$, $\hat{g}_0(\alpha) = \tilde{g}_0(\alpha)$ and $\hat{g}_1(\alpha) = \tilde{g}_1(\alpha)$. Thus $\delta_{\eta_0} = [\hat{g}_0]_\mu = [\tilde{g}_0]_\mu = [f^0]_\mu$ and $\delta_{\eta_1} = [\hat{g}_1]_\mu = [\tilde{g}_1]_\mu = [f^1]_\mu$. $P(f) = 1$ and $Q(f) = 0$ imply that $\delta_{\eta_0} = [f^0]_\mu < [f^1]_\mu = \delta_{\eta_1}$. Thus $\langle \delta_\eta : \eta < \lambda+1 \rangle$ is an order preserving injection of $\lambda+1$ into λ which is impossible. (Note that since $j_\mu(\kappa)$ is an ordinal, $\lambda < j_\mu(\kappa)$ is also an ordinal. For ordinals λ, $\lambda+1$ cannot inject into λ.)

(2) Now suppose μ is a normal ultrafilter on κ which does not contain any bounded subsets of κ and $j_\mu(\kappa)$ is an ordinal. For the sake of contradiction, suppose $j_\mu(\kappa)$ is not regular. Then there is an infinite cardinal $\lambda < j_\mu(\kappa)$ and an increasing map $\rho : \lambda \to j_\mu(\kappa)$. Define $V : [\kappa]^\kappa_* \to 2$ by $V(f) = 0$ if and only if there exists a $\xi < \lambda$ so that $[f^0]_\mu < \rho(\xi) < [f^1]_\mu$ (where $f^0, f^1 \in [\kappa]^\kappa$ is obtained from $f \in [\kappa]^\kappa$ as before). By $\kappa \to_* (\kappa)^\kappa_2$, there is a club C_0 homogeneous for V. First suppose C_0 is homogeneous for V taking value 0. Let $h : \kappa \to \kappa$ be such that $[h]_\mu = \lambda$. Define $W = \{(\alpha, \beta) : \beta < h(\alpha)+2\}$. As before, $(W, <_{\text{lex}})$ has ordertype κ. Let $\psi : W \to C_0$ be a order preserving function from $(W, <_{\text{lex}}) \to C_0$ of the correct type. For any $g : \kappa \to \kappa$, define $\check{g}_0, \check{g}_1 : \kappa \to \kappa$ by

$$\check{g}_0(\alpha) = \begin{cases} \psi(\alpha, g(\alpha)) & g(\alpha+1) < h(\alpha)+2 \\ \psi(0,0) & \text{otherwise} \end{cases} \quad \text{and} \quad \check{g}_1(\alpha) = \begin{cases} \psi(\alpha, g(\alpha)+1) & g(\alpha+1) < h(\alpha)+2 \\ \psi(0,1) & \text{otherwise} \end{cases}.$$

Note that for all $g : \kappa \to \kappa$, $\check{g}_0, \check{g}_1 \in [C_0]^\kappa_*$ and for all $\alpha < \kappa$, $\check{g}_0(\alpha) < \check{g}_1(\alpha) < \check{g}_0(\alpha+1)$ by arguments similar to the above. Thus there is some $f \in [C_0]^\kappa_*$ so that $f^0 = \check{g}_0$ and $f^1 = \check{g}_1$. Now suppose $\eta < \lambda+1$. Let $g : \kappa \to \kappa$ be such that $\eta = [g]_\mu$. Let $f \in [C_0]^\kappa_*$ be such that $f^0 = \check{g}_0$ and $f^1 = \check{g}_1$. $V(f) = 0$ implies that there is a $\xi < \lambda$ so that $[\check{g}_0]_\mu = [f^0]_\mu < \rho(\xi) < [f^1]_\mu = [\check{g}_1]_\mu$. Let ξ_η be the least such ξ and note that ξ_η is independent of the choice of g representing η. Now suppose $\eta_0 < \eta_1 < \lambda+1$. Let $g, p : \kappa \to \kappa$ be such that $\eta_0 = [g]_\mu$ and $\eta_1 = [p]_\mu$. Note that $B = \{\alpha \in \kappa : g(\alpha) < p(\alpha) < h(\alpha)+2\} \in \mu$. For all $\alpha \in B$, $\check{g}_1(\alpha) \leq \check{p}_0(\alpha)$. Thus $\rho(\xi_{\eta_0}) < [\check{g}_1]_\mu \leq [\check{p}_0]_\mu < \rho(\xi_{\eta_1})$. Since ρ is an increasing function, one must have that $\xi_{\eta_0} < \xi_{\eta_1}$. Thus $\langle \xi_\eta : \eta < \lambda+1 \rangle$ is an order preserving injection of $\lambda+1$ into λ. This is impossible.

[1]The purpose for adding 2 rather than 1 is to ensure that $(\alpha, 0), (\alpha, 1) \in T$ for all $\alpha < \kappa$.

Now suppose C_0 is homogeneous for V taking value 1. Let $g_0 \in [C_0]^\kappa_*$. Since $\rho : \lambda \to j_\mu(\kappa)$ is cofinal, there is some $\bar{\xi}$ so that $\rho(\bar{\xi}) > [g_0]_\mu$. Since $j_\mu(C_0)$ is order isomorphic to $j_\mu(\kappa)$, let $g_1 \in [C_0]^\kappa_*$ be so that $[g_1]_\mu > \rho(\bar{\xi})$. Since g_0 is an increasing function, for all $\alpha < \kappa$, there is an α' so that $g_1(\alpha) < g_0(\alpha')$. Let $q(\alpha)$ be the least α' so that $g_1(\alpha) < g_0(\alpha')$. Since $[g_0]_\mu < \rho(\bar{\xi}) < [g_1]_\mu$, let $A_0 \in \mu$ be such that for all $\alpha \in A_0$, $g_0(\alpha) < g_1(\alpha)$. Let $A_1 = \{\alpha \in A_0 : (\forall \alpha' < \alpha)(q(\alpha') < \alpha)\}$ and note that $A_1 \in \mu$ by Fact 2.20. Define $f : \kappa \to \kappa$ by $f(2\cdot\alpha) = g_0(\mathsf{enum}_{A_1}(\alpha))$ and $f(2\cdot\alpha+1) = g_1(\mathsf{enum}_{A_1}(\alpha))$ for all $\alpha < \kappa$. Note that for all $\alpha < \kappa$, $f(2\cdot\alpha) = g_0(\mathsf{enum}_{A_1}(\alpha)) < g_1(\mathsf{enum}_{A_1}(\alpha)) = f(2\cdot\alpha+1)$ since $g_0(\gamma) < g_1(\gamma)$ for all $\gamma \in A_1$. Note also that for all $\alpha < \kappa$, $f(2\cdot\alpha+1) = g_1(\mathsf{enum}_{A_1}(\alpha)) < g_0(q(\mathsf{enum}_{A_1}(\alpha))) < g_0(\mathsf{enum}_{A_1}(\alpha+1)) = f(2\cdot(\alpha+1))$ by the definition of q and A_1. This shows that $f : \kappa \to \kappa$ is an increasing function. It is clear that $f \in [C_0]^\kappa_*$ since $g_0, g_1 \in [C_0]^\kappa_*$. Let $A_2 = \{\alpha \in A_1 : \mathsf{enum}_{A_1}(\alpha) = \alpha\}$ which belongs to μ by Fact 2.21. For all $i \in 2$ and $\alpha \in A_2$, $f^i(\alpha) = g_i(\mathsf{enum}_{A_1}(\alpha)) = g_i(\alpha)$. Thus $[f^i]_\mu = [g_i]_\mu$ for both $i \in 2$. $V(f) = 1$ implies that there is no $\xi < \lambda$ with $[g_0]_\mu = [f^0]_\mu < \rho(\xi) < [f^1]_\mu = [g_1]_\mu$. This is contradiction since $[g_0]_\mu < \rho(\bar{\xi}) < [g_1]_\mu$. It has been shown that V has no homogeneous club which violates $\kappa \to_* (\kappa)^\kappa_2$. □

Fact 2.23. *(Martin) Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and ★. Then $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ and ω_2 is a regular cardinal.*

Proof. By Fact 2.17, $j_{\mu^1_{\omega_1}}(\omega_1)$ is a wellordering. By Fact 2.16, each initial segment of $j_{\mu^1_{\omega_1}}(\omega_1)$ injects into ω_1. Thus $\omega_1 = [\mathsf{id}]_{\mu^1_{\omega_1}} < j_{\mu^1_{\omega_1}}(\omega_1) \leq (\omega_1)^+ = \omega_2$. Since Fact 2.22 implies $j_{\mu^1_{\omega_1}}(\omega_1)$ must be a cardinal, one has that $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$. Since $\mu^1_{\omega_1}$ is a normal ultrafilter by Fact 2.7, Fact 2.22 also implies that $\omega_2 = j_{\mu^1_{\omega_1}}(\omega_1)$ is regular. □

Fact 2.24. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$. ★ and $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ are equivalent.*

Proof. This follow from Fact 2.14 and Fact 2.23 □

Next, one will show that for all $1 \leq n < \omega$, $j_{\mu^n_{\omega_1}}(\omega_1) = \omega_{n+1}$ and $\mathrm{cof}(\omega_{n+1}) = \omega_2$ (without assuming any form of dependent choice or even countable choice).

Fact 2.25. *Assume $\omega_1 \to_* (\omega_1)^{n+1}_2$. For each $1 \leq n < \omega$ and $g : \omega_1 \to \omega_1$, let $\hat{\Sigma}_n(g) : [\omega_1]^n \to \omega_1$ be defined by $\hat{\Sigma}(g)(\iota) = g(\iota(n-1))$. Let $\Sigma_n : j_{\mu^1_{\omega_1}}(\omega_1) \to j_{\mu^n_{\omega_1}}(\omega_1)$ be defined by $\Sigma_n([g]_{\mu^1_{\omega_1}}) = [\hat{\Sigma}(g)]_{\mu^n_{\omega_1}}$. Then $\Sigma_n : j_{\mu^1_{\omega_1}}(\omega_1) \to j_{\mu^n_{\omega_1}}(\omega_1)$ is cofinal.*

Proof. Suppose $x \in j_{\mu^n_{\omega_1}}(\omega_1)$. Let $f : [\omega_1]^n \to \omega_1$ represent x. Define $P_f : [\omega_1]^{n+1} \to 2$ by $P_f(\ell) = 0$ if and only if $f(\ell \restriction n) < \ell(n)$. By $\omega_1 \to_* (\omega_1)^{n+1}_2$, there is a club $C \subseteq \omega_1$ which is homogeneous for P_f. Pick any $\iota \in [C]^n_*$. Let $\gamma \in [C]^1_*$ be such that $f(\iota) < \gamma$. Let $\ell = \iota\hat{\ }\langle\gamma\rangle$ and note that $\ell \in [C]^{n+1}_*$. Then $P_f(\ell) = 0$ since $f(\ell \restriction n) = f(\iota) < \gamma = \ell(n)$. This shows that C is homogeneous for P_f taking value 0. Let $g : \omega_1 \to \omega_1$ be defined by $g(\alpha) = \mathsf{next}^\omega_C(\alpha)$. Let $\iota \in [C]^n_*$ and let $\ell_\iota = \iota\hat{\ }\langle g(\iota(n-1))\rangle$. Note that $\ell_\iota \in [C]^{n+1}_*$ and thus $P_f(\ell_\iota) = 0$. This implies that $f(\iota) = f(\ell_\iota \restriction n) < \ell_\iota(n) = g(\iota(n-1)) = \hat{\Sigma}_n(g)(\iota)$. Since $\iota \in [C]^n_*$ was arbitrary, it has been shown that $x = [f]_{\mu^n_{\omega_1}} \prec_{\mu^n_{n-1}} [\hat{\Sigma}_n(g)]_{\mu^n_{\omega_1}} = \Sigma_n([g]_{\mu^1_{\omega_1}})$. □

Fact 2.26. *Assume $1 \leq m < n < \omega$ and $\omega_1 \to_* (\omega_1)^n_2$. There is an order embedding of $j_{\mu^m_{\omega_1}}(\omega_1)$ into a proper initial segment of $j_{\mu^n_{\omega_1}}(\omega_1)$.*

Proof. If $f : [\omega_1]^m \to \omega_1$, then let $\hat{f} : [\omega_1]^n \to \omega_1$ be defined by $\hat{f}(\ell) = f(\ell \restriction m)$. Define $\Psi : j_{\mu^m_{\omega_1}}(\omega_1) \to j_{\mu^n_{\omega_1}}(\omega_1)$ by $\Psi(x) = [\hat{f}]_{\mu^n_{\omega_1}}$ where $f : [\omega_1]^m \to \omega_1$ represents x. One can check that Ψ is well defined independent of the choice of representative f for x and is an order embedding.

Let $g : [\omega_1]^n \to \omega_1$ be defined by $g(\ell) = \ell(m)$. The claim is that the range of ψ is below $[g]_{\mu^n_{\omega_1}}$. Let $f : [\omega_1]^m \to \omega_1$. Let $P_f : [\omega_1]^{m+1} \to \omega_1$ be defined by $P_f(\sigma) = 0$ if and only if $f(\sigma \restriction m) < \sigma(m)$. By $\omega_1 \to (\omega_1)^{m+1}_2$, let $C_0 \subseteq \omega_1$ be a club homogeneous for P_f. Pick $\iota \in [C_0]^m_*$. Let $\sigma = \iota\hat{\ }\langle \mathsf{next}^\omega_{C_0}(f(\iota))\rangle$ and note that $\sigma \in [C_0]^{m+1}_*$. Since $f(\sigma \restriction m) = f(\iota) < \mathsf{next}^\omega_{C_0}(f(\iota)) = \sigma(m)$, one has that $P_f(\sigma) = 0$. Thus C_0 is a homogeneous for P_f taking value 0. For any $\iota \in [C_0]^m_*$, let $\sigma_\iota = \iota\hat{\ }\langle \mathsf{next}^\omega_{C_0}(\iota(m-1))\rangle$. $P_f(\sigma_\iota) = 0$ implies that $f(\iota) < \mathsf{next}^\omega_{C_0}(\iota(m-1))$. Let $C_1 = \{\alpha \in C_0 : \mathsf{enum}_{C_0}(\alpha) = \alpha\}$. Let $\ell \in [C_1]^n$. One has $\hat{f}(\ell) = f(\ell \restriction m) < \mathsf{next}^\omega_{C_0}(\ell(m-1)) < \ell(m) = g(\ell)$ since $\ell(m) \in C_1$ and using Fact 2.4. Thus $\Psi([f]_{\mu^m_{\omega_1}}) < [g]_{\mu^n_{\omega_1}}$. This shows that Ψ maps $j_{\mu^m_{\omega_1}}(\omega_1)$ into an initial segment of $j_{\mu^n_{\omega_1}}(\omega_1)$. □

Definition 2.27. Suppose $f : [\omega_1]^n \to \omega_1$. For each $1 \leq k \leq n$, define $I_f^k : [\omega_1]^k \to \omega_1$ by $I_f^k(\sigma) = \sup\{f(\tau\hat{\ }\sigma) : \tau \in [\omega_1]^{n-k} \wedge \sup(\tau) < \sigma(0)\}$. (Note that $I_f^n = f$.)

Note that if $f, g : [\omega_1]^n \to \omega_1$ with $[f]_{\mu_{\omega_1}^n} \preceq_{\mu_{\omega_1}^n} [g]_{\mu_{\omega_1}^n}$, then it is not necessarily true that $[I_f^1]_{\mu_{\omega_1}^1} \leq [I_g^1]_{\mu_{\omega_1}^1}$. However, one has the following.

Fact 2.28. *Suppose $1 \leq n < \omega_1$, $f, g : [\omega_1]^n \to \omega_1$ with $[f]_{\mu_{\omega_1}^n} \preceq_{\mu_{\omega_1}^n} [g]_{\mu_{\omega_1}^n}$. Then there is a $\bar{f} : [\omega_1]^n \to \omega_1$ so that $[\bar{f}]_{\mu_{\omega_1}^n} = [f]_{\mu_{\omega_1}^n}$ and for all $\alpha < \omega_1$, $I_{\bar{f}}^1(\alpha) \leq I_g^1(\alpha)$.*

Proof. Since $[f]_{\mu_{\omega_1}^n} \preceq_{\mu_{\omega_1}^n} [g]_{\mu_{\omega_1}^n}$, $A = \{\ell \in [\omega_1]^n : f(\ell) \leq g(\ell)\} \in \mu_{\omega_1}^n$. Define $\bar{f} : [\omega_1]^n \to \omega_1$ by

$$\bar{f}(\ell) = \begin{cases} f(\ell) & \ell \in A \\ 0 & \ell \notin A \end{cases}.$$

Note that $[f]_{\mu_{\omega_1}^n} = [\bar{f}]_{\mu_{\omega_1}^n}$. For all $\alpha \in \omega_1$, $I_{\bar{f}}^1(\alpha) = \sup\{\bar{f}(\ell) : \ell \in [\omega_1]^n \wedge \ell(n-1) = \alpha\} = \sup\{\bar{f}(\ell) : \ell \in A \wedge \ell(n-1) = \alpha\} \leq \sup\{g(\ell) : \ell \in A \wedge \ell(n-1) = \alpha\} \leq \sup\{g(\ell) : \ell \in [\omega_1]^n \wedge \ell(n-1) = \alpha\} = I_g^1(\alpha)$ where the first inequality uses the fact that $f(\ell) \leq g(\ell)$ for all $\ell \in A$. $\square$

Definition 2.29. Suppose $1 \leq n < \omega$, $\mathcal{K}$ is a Kunen function, and $h : [\omega_1]^n \to \omega_1$. Define $\mathcal{K}^{n,h} : [\omega_1]^{n+1} \to \omega_1$ by $\mathcal{K}^{n,h}(\ell) = \mathcal{K}(\ell(n), h(\ell \restriction n))$ when $h(\ell \restriction n) < \ell(n)$ and $\mathcal{K}^{n,h}(\ell) = 0$ otherwise.

Fact 2.30. *Assume $\omega_1 \to_* (\omega_1)_2^{\omega_1}$ and $\bigstar$. For all $1 \leq n < \omega$, $j_{\mu_{\omega_1}^n}(\omega_1) = \omega_{n+1}$ and $\operatorname{cof}(\omega_{n+1}) = \omega_2$.*

Proof. Suppose $1 \leq n < \omega$ and the following has been shown:

(1) $j_{\mu_{\omega_1}^n}(\omega_1) = \omega_{n+1}$ and $\operatorname{cof}(\omega_{n+1}) = \omega_2$.

(2) If $A \subseteq \omega_{n+1}$ with $|A| \leq \omega_1$, there is a function Σ so that for all $\alpha \in A$, $\Sigma(\alpha) : [\omega_1]_*^n \to \omega_1$ and $[\Sigma(\alpha)]_{\mu_{\omega_1}^n} = \alpha$.[2]

For $n = 1$, both properties have been shown. (1) is Fact 2.23. To see (2), suppose $A \subseteq \omega_2$ with $|A| \leq \omega_1$. Since ω_2 is regular, $\sup(A) < \omega_2$. Let $f : \omega_1 \to \omega_1$ represent $\sup(A)$ and $\mathcal{K}$ be a Kunen function bounding f. By Fact 2.16, there is a function Γ so that for all $\alpha < \sup(A)$, $\alpha = [\mathcal{K}^{\Gamma(\alpha)}]_{\mu_{\omega_1}^1}$. For $\alpha \in A$, let $\Sigma(\alpha) = \mathcal{K}^{\Gamma(\alpha)}$.

Now suppose the two properties have been established at n. One seeks to establish the two properties at $n+1$.

First, one will show that $j_{\mu_{\omega_1}^{n+1}}(\omega_1)$ is wellfounded. Suppose $X \subseteq j_{\mu_{\omega_1}^{n+1}}(\omega_1)$ has no $\prec_{\mu_{\omega_1}^{n+1}}$-minimal elements. Pick $x \in X$ and let $f : [\omega_1]^{n+1} \to \omega_1$ represent x. Let $\mathcal{K}$ be a Kunen function bounding $I_f^1 : \omega_1 \to \omega_1$. For any $y \prec_{\mu_{\omega_1}^{n+1}} x$, use Fact 2.28 to pick a $g : [\omega_1]^{n+1} \to \omega_1$ which represents y and $I_g^1 \leq_{\mu_{\omega_1}^1} I_f^1$. Thus $\mathcal{K}$ is also a Kunen function which bounds I_g^1. Let $A_g = \{\alpha \in \omega_1 : I_g^1(\alpha) < \Xi^{\mathcal{K}}(\alpha)\}$ and note that $A_g \in \mu_{\omega_1}^1$. For any $\ell \in [A_g]_*^{n+1}$, $g(\ell) \leq I_g^1(\ell(n)) < \Xi^{\mathcal{K}}(\ell(n))$. Let $\hat{h}(\ell)$ be the least ordinal $\gamma < \ell(n)$ so that $g(\ell) = \mathcal{K}(\ell(n), \gamma)$. Since $\hat{h}(\ell) < \ell(n)$ for $\mu_{\omega_1}^{n+1}$-almost all ℓ, Fact 2.6 implies there is an $h : [\omega_1]^n \to \omega_1$ so that for $\mu_{\omega_1}^{n+1}$-almost all ℓ, $\hat{h}(\ell) = h(\ell \restriction n)$. By (1) at n, one has that $[h]_{\mu_{\omega_1}^n} \in j_{\mu_{\omega_1}^n}(\omega_1) = \omega_{n+1}$. It has been shown that for all $y \prec_{\mu_{\omega_1}^{n+1}} x$, there is an ordinal $\delta < \omega_{n+1}$ so that for any $h : [\omega_1]^n \to \omega_1$ representing δ, the function $\mathcal{K}^{n,h} : [\omega_1]^{n+1} \to \omega_1$ defined (in Definition 2.29) by $\mathcal{K}^{n,h}(\ell) = \mathcal{K}(\ell(n), h(\ell \restriction n))$ represents y. Let δ_y be the least such δ for y. Let $B = \{\delta_y : y \in X \wedge y \prec_{\mu_{\omega_1}^{n+1}} x\}$ and note that $B \subseteq \omega_{n+1}$. Let δ_0 be the least member of B according to the usual ordering on ω_{n+1}. Suppose δ_k has been found and let y_k be the element of X represented by $\mathcal{K}^{n,h}$ for any $h : [\omega_1]^n \to \omega_1$ representing δ_k. Since X has no minimal element, there is some δ so that for any $h : [\omega_1]^n \to \omega_1$ representing δ, the function $\mathcal{K}^{n,h}$ represents an element of X which is $\prec_{\mu_{\omega_1}^{n+1}}$ below y_k. Let δ_{k+1} be the least such ordinal δ. This defines a sequence $\langle \delta_k : k \in \omega \rangle$ of ordinals below ω_{n+1} and corresponding sequence $\langle y_k : k \in \omega \rangle$ in X. Note that for all $k \in \omega$, $y_{k+1} \prec_{\mu_{\omega_1}^{n+1}} y_k$. Since $|\{\delta_k : k \in \omega\}| = \omega < \omega_1$, property (2) at n gives a sequence $\langle h_k : k \in \omega \rangle$ so that $[h_n]_{\mu_{\omega_1}^n} = \delta_k$. For each $n \in \omega$, let $E_k = \{\ell \in [\omega_1]^{n+1} : \mathcal{K}^{n,h_{k+1}}(\ell) < \mathcal{K}^{n,h_k}(\ell)\}$ and note that $E_k \in \mu_{\omega_1}^{n+1}$ since $y_{k+1} \prec_{\mu_{\omega_1}^{n+1}} y_k$. Since $\omega_1 \to_* (\omega_1)_2^{2n+2}$ implies $\mu_{\omega_1}^{n+1}$ is countably complete, $E = \bigcap_{k \in \omega} E_k \in \mu_{\omega_1}^{n+1}$ and hence nonempty. Let $\ell \in E$.

[2] For this proof, one only needs the result for $|A| \leq \omega$. However, the proof is no different for A with $|A| \leq \omega_1$. There are other applications which require the stronger form.

Then $\langle \mathcal{K}^{n,h_k}(\ell) : k \in \omega\rangle$ is an infinite descending sequence of ordinals which is a contradiction. It has been shown that $j_{\mu^{n+1}_{\omega_1}}(\omega_1)$ is a wellordering.

Next, one will show that $j_{\mu^{n+1}_{\omega_1}}(\omega_1) \leq \omega_{n+2}$. Let $x \in j_{\mu^{n+1}_{\omega_1}}(\omega_1)$, $f : [\omega_1]^{n+1} \to \omega_1$ represent x, and $\mathcal{K}$ be a Kunen function bounding I^1_f. The argument above showed that for any $y \prec_{\mu^{n+1}_{\omega_1}} x$, there is an ordinal $\delta < \omega_{n+1}$ so that for any $h : [\omega_1]^n \to \omega_1$ which represents δ, $\mathcal{K}^{n,h} : [\omega_1]^{n+1} \to \omega_1$ represents y. Let δ_y be the least such δ for y. The map $\Upsilon : \mathsf{init}_{\mu^{n+1}_{\omega_1}}(x) \to \omega_{n+1}$ defined by $\Upsilon(y) = \delta_y$ is an injection. So $\mathsf{init}_{\mu^{n+1}_{\omega_1}}(x)$ has cardinality less than or equal to ω_{n+1}. Since $x \in j_{\mu^{n+1}_{\omega_1}}(\omega_1)$ was arbitrary, this shows that $j_{\mu^{n+1}_{\omega_1}}(\omega_1) \leq (\omega_{n+1})^+ = \omega_{n+2}$.

By Fact 2.26, $\omega_{n+1} = j_{\mu^n_{\omega_1}}(\omega_1) < j_{\mu^{n+1}_{\omega_1}}(\omega_1) \leq \omega_{n+2}$. Since Fact 2.22 implies $j_{\mu^{n+1}_{\omega_1}}(\omega_1)$ is a cardinal, one has that $j_{\mu^{n+1}_{\omega_1}}(\omega_1) = \omega_{n+2}$. $\mathrm{cof}(\omega_{n+2}) = j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ by Fact 2.25. Property (1) at $n+1$ has been established.

Now to establish property (2) at $n+1$. Suppose $A \subseteq \omega_{n+2}$ with $|A| \leq \omega_1$. Since $\mathrm{cof}(\omega_{n+2}) = \omega_2$, one has that $\sup(A) < \omega_{n+2}$. Let $f : [\omega_1]^{n+1} \to \omega_1$ represent $\sup(A)$ and let $\mathcal{K}$ be a Kunen function bounding I^1_f. As argued above, there is a sequence $\langle \delta_\alpha : \alpha \in A\rangle$ in ω_{n+1} with the property that for all $\alpha \in A$, for any $h : [\omega_1]^n \to \omega_1$ representing δ_α, $\mathcal{K}^{n,h}$ represents α. The set $\{\delta_\alpha : \alpha \in A\}$ is a subset of ω_{n+1} of cardinality less than or equal to ω_1. By property (2) at n, there is a sequence $\langle h_\alpha : \alpha \in A\rangle$ so that $h_\alpha : [\omega_1]^n \to \omega_1$ represents δ_α. Then $\langle \mathcal{K}^{n,h_\alpha} : \alpha \in A\rangle$ has the property that for all $\alpha \in A$, $\mathcal{K}^{n,h_\alpha}$ represents α. This verifies property (2) at $n+2$.

By induction, this completes the proof. ☐

As mentioned in the footnote, the proof of Fact 2.30 actually showed that one can find representatives for ω_1-many elements of $j_{\mu^n_{\omega_1}}(\omega_1)$. Although this is not needed in the proof of Fact 2.30 or anywhere else in this paper, this is a very important instance of choice that is required for many combinatorial results below ω_ω. For example, it is needed to show ω_2 is a weak partition cardinal. This fact proved within the proof of Fact 2.30 is explicitly isolated below.

Fact 2.31. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. Let $1 \leq n < \omega$ and $A \subseteq j_{\mu^n_{\omega_1}}(\omega_1) = \omega_{n+1}$ with $|A| \leq \omega_1$. There is a function Γ on A so that for all $\alpha \in A$, $\Gamma(\alpha) : [\omega_1]^n \to \omega_1$ and $\alpha = [\Gamma(\alpha)]_{\mu^n_{\omega_1}}$.*

[3] shows that these combinatorial methods using the Kunen functions can show that $j_{\mu^\epsilon_{\omega_1}}(\omega_\omega)$ is wellfounded and even show that $j_{\mu^\epsilon_{\omega_1}}(\omega_\omega) < \omega_{\omega+1}$ for all $\epsilon < \omega_1$. However this seems to be the limit of the purely combinatorial methods. These methods cannot be used to calculate $j_{\mu^\epsilon_{\omega_1}}(\omega_{\omega+1})$ for $\epsilon < \omega_1$. These combinatorial methods have no influence on the ultrapower by the strong partition measure $\mu^{\omega_1}_{\omega_1}$. Using Martin's good coding system for ${}^{\omega\cdot\epsilon}\omega_1$ for $\epsilon < \omega_1$ to make complexity calculations, [3] showed that $j_{\mu^\epsilon_{\omega_1}}(\omega_{\omega+1}) = \omega_{\omega+1}$ for all $\epsilon < \omega_1$. Calculating the ultrapowers by the strong partition measure on ω_1 is an important question concerning the strong partition property. [3] showed that $j_{\mu^{\omega_1}_{\omega_1}}(\omega_1)$ is wellfounded in AD alone by using Martin's good coding system for ${}^{\omega_1}\omega_1$ to bring the ultrapower into $L(\mathbb{R})$ to apply a result of Kechris [13] which states that AD implies $L(\mathbb{R}) \models \mathsf{DC}$. The first step in understanding the ultrapower by the strong partition measure on ω_1 was completed in [3] by answering a question of Goldberg that $j_{\mu^{\omega_1}_{\omega_1}}(\omega_1) < \omega_{\omega+1}$.

Fact 2.32. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. Let $1 \leq n < \omega$. If $\delta \in \omega_{n+1} \setminus \omega_n$, then there is a function $f : [\omega_1]^n \to \omega_1$ such that $[f]_{\mu^n_{\omega_1}} = \delta$ with the property that for all $\iota_0, \iota_1 \in [\omega_1]^n_*$, if $\iota_0(n-1) < \iota_1(n-1)$, then $f(\iota_0) < f(\iota_1)$.*

Proof. Let $f_0 : [\omega_1]^n \to \omega_1$ be any representative for δ with respect to $\mu^n_{\omega_1}$. Let $A_0 = \{\iota \in [\omega_1]^n_* : f(\iota) \geq \iota(n-1)\}$. One must have that $A_0 \in \mu^n_{\omega_1}$ since otherwise Fact 2.6 implies there is a function $g : [\omega_1]^{n-1} \to \omega_1$ so that for $\mu^n_{\omega_1}$-almost all ℓ, $f_0(\ell) = g(\ell \restriction n-1)$. This would imply that $\delta = [f_0]_{\mu^n_{\omega_1}} = [g]_{\mu^{n-1}_{\omega_1}} \in j_{\mu^{n-1}_{\omega_1}}(\omega_1) = \omega_n$ which contradicts the assumption that $\delta \in \omega_{n+1} \setminus \omega_n$. Let $C_0 \subseteq \omega_1$ be a club so that $[C_0]^n_* \subseteq A_0$. Define $P : [C_0]^{n+1}_* \to 2$ by $P(\ell) = 0$ if and only if $f_0(\ell \restriction n) < \ell(n)$. Let $C_1 \subseteq C_0$ be homogeneous for P using $\omega_1 \to_* (\omega_1)^{n+1}_2$. Pick any $\iota \in [C_1]^n_*$. Let $\ell = \iota\hat{\ }\langle \mathsf{next}^\omega_{C_1}(f(\iota))\rangle$ and note that $\ell \in [C_1]^{n+1}_*$. Then $f(\ell \restriction n) = f(\iota) < \mathsf{next}^\omega_{C_1}(f(\iota)) = \ell(n+1)$. Thus $P(\ell) = 0$ and hence C_1 must be homogeneous for P taking value 0. For any $\iota \in [C_1]^n_*$, let $\ell_\iota = \iota\hat{\ }\langle \mathsf{next}^\omega_{C_0}(\iota(n-1))\rangle$. $P(\ell_\iota) = 0$ implies that $f(\iota) = f(\ell_\iota \restriction n) < \ell_\iota(n) = \mathsf{next}^\omega_{C_1}(\iota(n-1))$. Let $C_2 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha\}$. Note that if $\ell \in [\omega_1]^n_*$, then $\mathsf{enum}_{C_2} \circ \ell \in [C_2]^n_*$. Define $f_1 : [\omega_1]^n_* \to \omega_1$ by $f_1(\iota) = f_0(\mathsf{enum}_{C_2} \circ \iota)$. Now suppose $\iota_0, \iota_1 \in [\omega_1]^n_*$ with $\iota_0(n-1) < \iota_1(n-1)$. By the observations above and the definition of C_2, one has that $f_1(\iota_0) = f_0(\mathsf{enum}_{C_2} \circ \iota_0) < \mathsf{next}^\omega_{C_1}(\mathsf{enum}_{C_2}(\iota_0(n-1))) < (\mathsf{enum}_{C_2} \circ \iota_1)(n-1) \leq f_0(\mathsf{enum}_{C_2} \circ \iota_1) = f_1(\iota_1)$. Let $C_3 = \{\alpha \in C_2 : \mathsf{enum}_{C_2}(\alpha) = \alpha\}$. For all $\iota \in [C_3]^n_*$, one has that $\mathsf{enum}_{C_2} \circ \iota = \iota$. Thus $\delta = [f_0]_{\mu^n_{\omega_1}} = [f_1]_{\mu^n_{\omega_1}}$. f_1 is the representative of δ with the desired properties. $\square$

Definition 2.33. Let $\sqsubset_n$ be the reverse lexicographic ordering on $[\omega_1]^n$ defined as follows: Let $<_{\text{lex}}$ be the lexicographic ordering on n-tuples. If $\iota \in [\omega_1]^n$ (so ι is an increasing function), let $\iota^* \in {}^n\omega_1$ be defined by $\iota^*(k) = \iota(n-1-k)$. Define $\sqsubset_n$ on $[\omega_1]^n_*$ by $\iota \sqsubset_n \ell$ if and only if $\iota^* <_{\text{lex}} \ell^*$. (Even more explicitly, let $\alpha_0 < ... < \alpha_{n-1} < \omega_1$ and $\beta_0 < ... < \beta_{n-1} < \omega_1$. $(\alpha_0, ..., \alpha_{n-1}) \sqsubset_n (\beta_0, ..., \beta_{n-1})$ if and only if $(\alpha_{n-1}, \alpha_{n-2}, ..., \alpha_0) <_{\text{lex}} (\beta_{n-1}, \beta_{n-2}, ..., \beta_0)$.)

A function $f : [\omega_1]^n \to \omega_1$ has type n if and only if the following hold:

- f is order preserving from $([\omega_1]^n, \sqsubset_n)$ into the usual ordinal ordering $(\omega_1, <)$.
- (Discontinuous everywhere) For any $\ell \in [\omega_1]^n$, $\sup\{f(\iota) : \iota \sqsubset_n \ell\} < f(\ell)$.
- (Uniform cofinality ω) There is a function $F : [\omega_1]^n \times \omega \to \omega_1$ so that for all $\ell \in [\omega_1]^n$ and $k \in \omega$, $F(\ell, k) < F(\ell, k+1)$ and $f(\ell) = \sup\{F(\ell, k) : k \in \omega\}$.

Note that a function $f : \omega_1 \to \omega_1$ has type 1 if and only if it is an increasing function of the correct type (that is, $f \in [\omega_1]^{\omega_1}_*$).

Define $\mathfrak{B}_{n+1} \subseteq \omega_{n+1}$ to be the set of $\delta \in \omega_{n+1}$ so that there is a function $f : [\omega_1]^n \to \omega_1$ of type n with $\delta = [f]_{\mu^n_{\omega_1}}$. If $C \subseteq \omega_1$, then let $\mathfrak{B}^C_{n+1}$ be the set of $\delta \in \omega_{n+1}$ so that there is a function $f : [\omega_1]^n \to C$ of type n with $\delta = [f]_{\mu^n_{\omega_1}}$.

Definition 2.34. Let $V_n = \{(\alpha_{n-1}, ..., \alpha_0, \gamma) \in {}^{n+1}\omega_1 : \alpha_0 < \alpha_1 < ... < \alpha_{n-1} \wedge \gamma < \alpha_{n-1}\}$. Let $\ll$ be the lexicographic ordering on V_n. Let $\mathcal{V}_n = (V_n, \ll)$ which is a wellordering of ordertype ω_1. A function $\phi : V_n \to \omega_1$ has the correct type if and only if the following holds:

- ϕ is order preserving from $\mathcal{V}_n$ into $(\omega_1, <)$, the usual ordering on ω_1.
- ϕ is discontinuous everywhere: for all $x \in V_n$, $\phi(x) > \sup\{\phi(y) : y \ll x\}$.
- ϕ has uniform cofinality ω: there is a function $\Phi : V_n \times \omega \to \omega_1$ with the property that for all $x \in V$ and $k \in \omega$, $\Phi(x, k) < \Phi(x, k+1)$ and $\phi(x) = \sup\{\Phi(x, k) : k \in \omega\}$.

Fact 2.35. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. Let $C \subseteq \omega_1$ be a club. There is an order embedding of ω_{n+1} into $\mathfrak{B}^C_{n+1}$.*

Proof. Let $\phi : V_n \to C$ be a function of the correct type. It is clear that the order type of $\omega_{n+1} \setminus \omega_n$ is ω_{n+1}. One will define an order embedding of $\omega_{n+1} \setminus \omega_n$ into $\mathfrak{B}^C_{n+1}$. Let $\delta \in \omega_{n+1} \setminus \omega_n$. By Fact 2.32, there is an $f : [\omega_1]^n_* \to \omega_1$ so that $\delta = [f]_{\mu^n_{\omega_1}}$ and has the property that for all $\iota_0, \iota_1 \in [\omega_1]^n_*$, if $\iota_0(n-1) < \iota_1(n-1)$, then $f(\iota_0) < f(\iota_1)$. Suppose $\alpha_0 < \alpha_1$ are two limit ordinals. Let γ_0, γ_1 be such that $\alpha_0 < \gamma_0 < \gamma_1 < \alpha_1$. For $i \in 2$, let $\iota_{\gamma_i} = (0, 1, ..., n-2, \gamma_i)$. By the property of f, one has that $f(\iota_{\gamma_0}) < f(\iota_{\gamma_1})$. Hence $I^1_f(\alpha_0) \leq f(\iota_{\gamma_0}) < f(\iota_{\gamma_1}) < I^1_f(\alpha_1)$. So I^1_f is an increasing function on the limit ordinals below ω_1. This implies that $(I^1_f(\iota(n-1)), I^1_f(\iota(n-2)), ..., I^1_f(\iota(0)), f(\iota)) \in V_n$ when $\iota \in [\omega_1]^n_*$. Let $\hat{f} : [\omega_1]^n_* \to \omega_1$ be defined by $\hat{f}(\iota) = \phi(I^1_f(\iota(n-1)), I^1_f(\iota(n-2)), ..., I^1_f(\iota(0)), f(\iota))$. Define $\Psi : (\omega_{n+1} \setminus \omega_n) \to \mathfrak{B}^C_{n+1}$ by $\Psi(\delta) = [\hat{f}]_{\mu^n_{\omega_1}}$ for any $f : [\omega_1]^n \to \omega_1$ such that $\delta = [f]_{\mu^n_{\omega_1}}$ and for all $\iota_0, \iota_1 \in [\omega_1]^n$, if $\iota_0(n-1) < \iota_1(n-1)$, then $f(\iota_0) < f(\iota_1)$. Ψ is well defined independent of the choice of such f representing δ. Suppose $\iota_0 \sqsubset_n \iota_1$. If $k < n$ is largest such that $\iota_0(k) \neq \iota_1(k)$, then $\iota_0(k) < \iota_1(k)$. By the obervation above, $I^1_f(\iota_0(j)) = I^1_f(\iota_1(j))$ for all $k < j < n$ and $I^1_f(\iota_0(k)) < I^1_f(\iota_1(k))$. Since ϕ is order preserving on $\mathcal{V}_n$, one has that $\hat{f}(\iota_0) < \hat{f}(\iota_1)$. This shows that $\hat{f}$ is order preserving on $([\omega_1]^n_*, \sqsubset_n)$. $\hat{f}$ is discontinuous everywhere and has uniform cofinality ω since ϕ is discontinuous everywhere and has uniform cofinality ω. So $\hat{f}$ has type n. This shows that Ψ does map into $\mathfrak{B}^C_{n+1}$. Suppose $\delta_0, \delta_1 \in \omega_{n+1} \setminus \omega_n$ and $\delta_0 < \delta_1$. Let $g_0, g_1 : [\omega_1]^n \to \omega_1$ represent δ_0 and δ_1, respectively, with the necessary properties stated above. Let $D \subseteq \omega_1$ be a club so that for all $\iota \in [D]^n_*$, $g_0(\iota) < g_1(\iota)$. For $i \in 2$, define $f_i : [\omega_1]^n \to \omega_1$ by $f_i(\ell) = g_i(\mathsf{enum}_D \circ \ell)$. Let $\tilde{D} = \{\alpha \in D : \mathsf{enum}_D(\alpha) = \alpha\}$. Note that for all $\ell \in \tilde{D}$ and $i \in 2$, $f_i(\ell) = g_i(\ell)$. Thus $\delta_i = [g_i]_{\mu^n_{\omega_1}} = [f_i]_{\mu^n_{\omega_1}}$ and f_i still has the necessary properties. Moreover, for all $\ell \in [\omega_1]^n$, $f_0(\ell) < f_1(\ell)$. Thus for all $\alpha \in [\omega_1]^n$, $I^1_{f_0}(\alpha) \leq I^1_{f_1}(\alpha)$. So for all $\iota \in [\omega_1]^n$,

$(I^1_{f_0}(\iota(n-1)),...,I^1_{f_0}(\iota(0)),f_0(\iota)) \ll (I^1_{f_1}(\iota(n-1)),...,I^1_{f_1}(\iota(0)),f_1(\iota))$ and therefore $\hat{f}_0(\iota) < \hat{f}_1(\iota)$. This shows that $\Psi(\delta_0) = [\hat{f}_0]_{\mu^n_{\omega_1}} < [\hat{f}_1]_{\mu^n_{\omega_1}} = \Psi(\delta_1)$. This shows that $\Psi : (\omega_{n+1} \setminus \omega_n) \to \mathfrak{B}^C_{n+1}$ is an order preserving injection. □

Definition 2.36. Suppose $1 \leq n < \omega$ and $\delta \in \mathfrak{B}_{n+1}$. For any $1 \leq k \leq n$, let $\mathcal{I}^k_\delta = [I^k_f]_{\mu^k_{\omega_1}}$ for all $f : [\omega_1]^n \to \omega_1$ of type n such that $[f]_{\mu^n_{\omega_1}} = \delta$. (Note that $\mathcal{I}^k_\delta$ is independent of the choice of f representing δ but f must be a function of type n.)

Fact 2.37. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. Let $1 \leq n < \omega$, $C \subseteq \omega_1$ be a club, and $\phi : V_{n+1} \to C$ be a function of the correct type. Let $\psi : [\omega_1]^n \to \omega_1$ be defined by $\psi(\alpha_0,...,\alpha_{n-1}) = \sup\{\phi(\alpha_{n-1},...,\alpha_0,\zeta,0) : \zeta < \alpha_0\}$. Let $\delta = [\psi]_{\mu^n_{\omega_1}}$. The set $|\{\eta < \mathfrak{B}^C_{n+2} : \mathcal{I}^n_\eta = \delta\}| = |\omega_{n+1}|$.*

Proof. Let $A = \{\eta < \mathfrak{B}^C_{n+2} : \mathcal{I}^n_\eta = \delta\}$. Let $\nu \in \omega_{n+1}$. Let $f : [\omega_1]^n \to \omega_1$ be such that $[f]_{\mu^n_{\omega_1}} = \nu$. By Fact 2.8, there is a club $D_0 \subseteq \omega_1$ so that for all $\ell \in [D]^n_*$, $f(\ell) < \mathsf{next}^\omega_{D_0}(\ell(n-1))$. Let $D_1 = \{\alpha \in D_0 : \mathsf{enum}_{D_0}(\alpha) = \alpha\}$. For all $\ell \in [D_1]^{n+1}$, $f(\ell \upharpoonright n) < \mathsf{next}^\omega_{D_0}(\ell(n-1)) < \ell(n)$ using Fact 2.4. Define a function $\hat{f} : [D_1]^{n+1} \to C$ by

$$\hat{f}(\alpha_0,...,\alpha_n) = \phi(\alpha_n,...,\alpha_0,f(\alpha_0,...,\alpha_{n-1})).$$

Note that this is well defined since $(\alpha_n,..,\alpha_0,f(\alpha_0,...,\alpha_{n-1})) \in V_{n+1}$ by the property of D_1. Let $\tilde{f} : [\omega_1]^{n+1} \to C$ be defined by $\tilde{f}(\ell) = \hat{f}(\mathsf{enum}_{D_1} \circ \ell)$. Let $D_2 = \{\alpha \in D_1 : \mathsf{enum}_{D_1}(\alpha) = \alpha\}$. Note that for all $\ell \in [D_2]^{n+1}_*$, $\ell = \mathsf{enum}_{D_1} \circ \ell$. Thus $[\hat{f}]_{\mu^{n+1}_{\omega_1}} = [\tilde{f}]_{\mu^{n+1}_{\omega_1}}$. Note that $\tilde{f}$ has type $n+1$ since ϕ has the correct type from $\mathcal{V}_{n+1}$ into $(C,<)$. So $[\tilde{f}]_{\mu^{n+1}_{\omega_1}} \in \mathfrak{B}^C_{n+2}$. Observe that for all $(\alpha_0,...,\alpha_{n-1}) \in [D_2]^n_*$,

$$I^n_{\tilde{f}}(\alpha_0,...,\alpha_{n-1}) = \sup\{\tilde{f}(\zeta,\alpha_0,...,\alpha_{n-1}) : \zeta < \alpha_0\} = \sup\{\hat{f}(\zeta,\alpha_0,...,\alpha_{n-1}) : \zeta < \alpha_0\}$$

$$= \sup\{\phi(\alpha_{n-1},...,\alpha_0,\zeta,f(\alpha_0,...,\alpha_{n-1})) : \zeta < \alpha_0\} = \sup\{\phi(\alpha_{n-1},...,\alpha_0,\zeta,0) : \zeta < \alpha_0\} = \psi(\alpha_0,...,\alpha_{n-1})$$

Let $\Upsilon(\eta) = [\tilde{f}]_{\mu^{n+1}_{\omega_1}} = [\hat{f}]_{\mu^{n+1}_{\omega_1}}$. By the above discussion, $\Upsilon(\eta) \in \mathfrak{B}^C_{n+2}$ and $\mathcal{I}^n_{\Upsilon(\eta)} = [I^n_{\tilde{f}}]_{\mu^n_{\omega_1}} = [\psi]_{\mu^n_{\omega_1}} = \delta$. Thus $\Upsilon : \omega_{n+1} \to A$. Suppose $\eta_0 < \eta_1$. Let $f_0, f_1 : [\omega_1]^n \to \omega_1$ be such that $\eta_0 = [f_0]_{\mu^n_{\omega_1}}$ and $\eta_1 = [f_1]_{\mu^n_{\omega_1}}$. For $\mu^n_{\omega_1}$-almost all ℓ, $f_0(\ell) < f_1(\ell)$. Thus for $\mu^{n+1}_{\omega_1}$-almost all ι, $\hat{f}_0(\iota) < \hat{f}_1(\iota)$. Thus $\Upsilon : \omega_{n+1} \to A$ is order preserving and hence an injection. Thus $|A| = |\omega_{n+1}|$. □

3. Boldface GCH below ω_ω

Definition 3.1. Let κ be a cardinal. The boldface GCH holds at κ if and only if there is no injection of κ^+ into $\mathscr{P}(\kappa)$. The boldface GCH below κ is the statement that for all $\delta < \kappa$, the boldface GCH holds at δ.

Fact 3.2. *Let κ be a cardinal and $\delta < \kappa$. If there is a δ^+-complete nonprincipal ultrafilter on κ, then there is no injection of κ into $\mathscr{P}(\delta)$.*

Proof. Let μ be a δ^+-complete nonprincipal ultrafilter on κ. Suppose $\langle A_\alpha : \alpha < \kappa\rangle$ is an injection of κ into $\mathscr{P}(\delta)$. For each $\xi < \delta$, let $E^0_\xi = \{\alpha \in \kappa : \xi \notin A_\alpha\}$ and $E^1_\xi = \{\alpha \in \kappa : \xi \in A_\alpha\}$. Since μ is an ultrafilter, there is a unique $i_\xi \in 2$ so that $E^{i_\xi}_\xi \in \mu$. Let $E = \bigcap_{\xi<\delta} E^{i_\xi}_\xi$ and note that $E \in \mu$ since μ is δ^+-complete. Since μ is nonprincipal, let $\alpha_0, \alpha_1 \in E$ with $\alpha_0 \neq \alpha_1$. For all $\xi < \delta$, since $\alpha_0, \alpha_1 \in E \subseteq E^{i_\xi}_\xi$, $\xi \in A_{\alpha_0}$ if and only if $i_\xi = 1$ if and only if $\xi \in A_{\alpha_1}$. Thus $A_{\alpha_0} = A_{\alpha_1}$. This contradicts the injectiveness of $\langle A_\alpha : \alpha < \kappa\rangle$. □

Fact 3.3. *If $\kappa \to_* (\kappa)^2_2$, then there is no injection of κ into $\mathscr{P}(\delta)$ for any $\delta < \kappa$.*

Proof. $\kappa \to_* (\kappa)^2_2$ implies that μ^1_κ is a κ-complete ultrafilter by Fact 2.5. The result follow from Fact 3.2. □

Fact 3.4. *$\omega_1 \to_* (\omega_1)^2_2$ implies the boldface GCH at ω.*

Martin [12] (and Kleinberg [14]) (also see [4]) showed that $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$ imply that ω_2 is a weak partition cardinal (satisfies $\omega_2 \to (\omega_2)^{<\omega_2}_2$). Thus under these assumptions, Fact 3.3 implies that the boldface GCH holds at ω_1. Although the proof of the weak partition property on ω_2 can be shown by similar techniques used here, it is very enlightening to see a direct proof of the boldface GCH at ω_1 to motivate the proof of the boldface GCH at ω_n for $2 \leq n < \omega$.

Definition 3.5. Let $U_1 = \{(0,0)\} \cup \{(1,\alpha,i) : \alpha < \omega_1 \wedge i < 2\}$. Let $\ll_1$ be the lexicographic ordering on U_1. Let $\mathcal{U}_1 = (U_1, \ll_1)$. Note that $(0,0)$ is the minimal element of $\mathcal{U}_1$. Note that the ordertype of $\mathcal{U}_1$ is ω_1. Suppose $F : U^1 \to \omega_1$. Define $F^0, F^1 : \omega_1 \to \omega_1$ and $F^2 \in \omega_1$ by $F^0(\alpha) = F(1,\alpha,0)$, $F^1(\alpha) = F(1,\alpha,1)$, and $F^2 = F(0,0)$. (Note that F^2 is just a countable ordinal.) A function $F : U_1 \to \omega_1$ has the correct type if and only if the following two conditions hold.

- F is discontinuous everywhere: For all $x \in U_1$, $\sup\{F(y) : y \ll_1 x\} < F(x)$.
- F has uniform cofinality ω: There is a function $\mathcal{F} : U_1 \times \omega \to \omega$ so that for all $x \in U_1$ and $k \in \omega$, $\mathcal{F}(x,k) < \mathcal{F}(x,k+1)$ and $F(x) = \sup\{\mathcal{F}(x,k) : k \in \omega\}$.

If $F : U_1 \to \omega_1$ has the correct type, then $F^0, F^1 : \omega_1 \to \omega_1$ are functions of the correct type and $\mathrm{cof}(F^2) = \omega$. If $X \subseteq \omega_1$, then let $[X]^{\mathcal{U}_1}_*$ be the set of all $F : U_1 \to X$ of the correct type and order preserving between $\mathcal{U}_1$ and $(X, <)$, where $<$ is the usual ordinal ordering.

Lemma 3.6. *Assume $\omega_1 \to_* (\omega_1)^{n+1}_2$. Let $f_0, f_1 : \omega_1 \to \omega_1$ be functions of type 1 such that $[f_0]_{\mu^1_{\omega_1}} < [f_1]_{\mu^1_{\omega_1}}$ and let $\delta \in [\omega_1]^1_*$ (i.e. is a limit ordinal). Then there is an $F \in [\omega_1]^{\mathcal{U}_1}_*$ so that $[F^0]_{\mu^1_{\omega_1}} = [f_0]_{\mu^1_{\omega_1}}$, $[F^1]_{\mu^1_{\omega_1}} = [f_1]_{\mu^1_{\omega_1}}$, $F^2 = \delta$, $F^0[\omega_1] \subseteq f_0[\omega_1]$, and $F^1[\omega_1] \subseteq f_1[\omega_1]$.*

Proof. Since f_0 and f_1 are of type 1, they are both increasing, discontinuous, and have uniform cofinality ω. Let $G_0 : \omega_1 \times \omega \to \omega_1$ witness that f_0 has uniform cofinality ω and let $G_1 : \omega_1 \times \omega \to \omega_1$ witness that f_1 has uniform cofinality ω. Since $\delta < \omega_1$ is a limit ordinal, let $\rho : \omega \to \delta$ be an increasing cofinal function. For each $\alpha < \omega_1$, let $h(\alpha)$ be the least element $\bar{\alpha} \in \omega_1$ so that $f_1(\alpha) < f_0(\bar{\alpha})$. Since $[f_0]_{\mu^1_{\omega_1}} < [f_1]_{\mu^1_{\omega_1}}$, there is a club $C_0 \subseteq \omega_1$ so that for all $\alpha \in [C_0]^1_*$, $f_0(\alpha) < f_1(\alpha)$. Let $C_1 = \{\alpha \in C_0 : (\forall \alpha' < \alpha)(h(\alpha') < \alpha)\}$. C_1 is a club subset of C_0. One may assume $\delta < \min(C_1)$. For notational simplicity, let $\mathfrak{e} = \mathsf{enum}_{[C_1]^1_*}$. Define $F : U_1 \to \omega_1$ by $F(0,0) = \delta$, and $F(1,\alpha,i) = f_i(\mathfrak{e}(\alpha))$ for $i < 2$ and $\alpha < \omega_1$. Fix $\alpha < \omega_1$. Note that $F(1,\alpha,0) = f_0(\mathfrak{e}(\alpha)) < f_1(\mathfrak{e}(\alpha)) = F(1,\alpha,1)$ by the property of the club C_0 and the fact that $\mathfrak{e}(\alpha) \in [C_1]^1_*$. $F(1,\alpha,1) = f_1(\mathfrak{e}(\alpha)) < f_0(h(\mathfrak{e}(\alpha))) < f_0(\mathfrak{e}(\alpha+1)) = F(1,\alpha+1,0)$ since the first inequality comes from the property of h and the second inequality comes from the property of C_1. This shows that F is order preserving from $\mathcal{U}_1$ into the usual ordering on ω_1. The elements of $\mathcal{U}_1$ of limit rank take the form $(1,\alpha,0)$ where α is a limit ordinal. Note that $\sup\{F(x) : x \ll_1 (1,\alpha,0)\} = \sup\{F(1,\alpha',0) : \alpha' < \alpha\} = \sup(f_0 \restriction \mathfrak{e}(\alpha) < f_0(\mathfrak{e}(\alpha)) < F(1,\alpha,0)$ using the discontinuity of f_0. This shows that F is discontinuous everywhere. Let $\mathcal{F} : U_1 \times \omega \to \omega_1$ be defined by

$$\mathcal{F}(x,k) = \begin{cases} \rho(k) & x = (0,0) \\ G_i(\mathfrak{e}(\alpha), k) & x = (1,\alpha,i) \end{cases}$$

$\mathcal{F}$ witnesses that F has uniform cofinality ω. Thus F has the correct type. By construction, it is clear that $F^0[\omega_1] \subseteq f_0[\omega_1]$ and $F^1[\omega_1] \subseteq f_1[\omega_1]$. Let $C_2 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha\}$ which is a club subset of C_1. For all $\alpha \in C_2$ and $i \in 2$, $F(1,\alpha,i) = f_i(\mathfrak{e}(\alpha)) = f_i(\mathsf{enum}_{[C_1]^1_*}(\alpha)) = f_i(\alpha)$ since $\mathsf{enum}_{C_1}(\alpha) = \alpha$ implies that $\mathrm{ot}(\{\bar{\alpha} \in C_1 : \bar{\alpha} < \alpha \wedge \mathrm{cof}(\bar{\alpha}) = \omega\}) = \alpha$ and thus $\mathsf{enum}_{[C_1]^1_*}(\alpha) = \alpha$. This shows that for all $i < 2$, $[F^i]_{\mu^1_{\omega_1}} = [f_i]_{\mu^1_{\omega_1}}$. ☐

Theorem 3.7. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. Then the boldface GCH holds at ω_1.*

Proof. By Fact 3.4, one has that the boldface GCH holds at ω. Suppose the boldface GCH at ω_1 fails. Let $\langle A_\eta : \eta < \omega_2 \rangle$ be an injection of ω_2 into $\mathscr{P}(\omega_1)$. Define $P : [\omega_1]^{\mathcal{U}_1}_* \to 2$ by $P(F) = 0$ if and only if $\min(A_{[F^0]_{\mu^1_{\omega_1}}} \triangle A_{[F^1]_{\mu^1_{\omega_1}}}) < F^2$ (where $\triangle$ refers to symmetric difference). Since $\mathcal{U}_1$ has ordertype ω_1, $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ implies there is a club $C \subseteq \omega_1$ homogeneous for P. Pick any $f_0, f_1 : \omega_1 \to C$ of type 1 so that $[f_0]_{\mu^1_{\omega_1}} < [f_1]_{\mu^1_{\omega_1}}$. Since $\langle A_\eta : \eta < \omega_2 \rangle$ is an injection, $A_{[f_0]_{\mu^1_{\omega_1}}} \neq A_{[f_1]_{\mu^1_{\omega_1}}}$ and thus $A_{[f_0]_{\mu^1_{\omega_1}}} \triangle A_{[f_1]_{\mu^1_{\omega_1}}} \neq \emptyset$. Let $\delta \in [C]^1_*$ be such that $\min(A_{[f_0]_{\mu^1_{\omega_1}}} \triangle A_{[f_1]_{\mu^1_{\omega_1}}}) < \delta$. By Lemma 3.6, there is an $F \in [\omega_1]^{\mathcal{U}_1}_*$ so that $[F^0]_{\mu^1_{\omega_1}} = [f_0]_{\mu^1_{\omega_1}}$, $[F^1]_{\mu^1_{\omega_1}} = [f_1]_{\mu^1_{\omega_1}}$, $F^2 = \delta$, $F^0[\omega_1] \subseteq f_0[\omega_1] \subseteq C$, and $F^1[\omega_1] \subseteq f_1[\omega_1] \subseteq C$. Thus $F \in [C]^{\mathcal{U}_1}_*$. Thus $\min(A_{[F^0]_{\mu^1_{\omega_1}}} \triangle A_{[F^1]_{\mu^1_{\omega_1}}}) < F^2$ implies that $P(F) = 0$. This shows that C must be homogeneous for P taking value 0. Fix a $\delta \in [C]^1_*$. By Fact 2.35, ω_2 order embeds into $\mathfrak{B}^C_2$. Pick any $\nu \in \mathfrak{B}^C_2$ so that $\mathfrak{B}^C_2 \restriction \nu = \{\eta \in \mathfrak{B}^C_2 : \eta < \nu\}$ has cardinality ω_1. Let $\eta_0, \eta_1 \in \mathfrak{B}^C_2 \restriction \nu$ with $\eta_0 \neq \eta_1$. Without loss of generality, suppose $\eta_0 < \eta_1$. Let $f_0, f_1 : \omega_1 \to C$ be functions of type 1 so that $\eta_0 = [f_0]_{\mu^1_{\omega_1}}$ and

$\eta_1 = [f_1]_{\mu^1_{\omega_1}}$. By Lemma 3.6, there is an $F \in [C]^{\mathcal{U}_1}_*$ so that $[F_0]_{\mu^1_{\omega_1}} = [f_0]_{\mu^1_{\omega_1}} = \eta_0$, $[F^1]_{\mu^1_{\omega_1}} = [f_1]_{\mu^1_{\omega_1}} = \eta_1$, and $F^2 = \delta$. Thus $P(F) = 0$ implies that $\min(A_{\eta_0} \triangle A_{\eta_1}) < \delta$. This implies that the function $\Phi : \mathfrak{B}^C_2 \restriction \nu \to \mathscr{P}(\delta)$ defined by $\Phi(\eta) = A_\eta \cap \delta$ is an injection. Since $\delta < \omega_1$ implies $|\mathscr{P}(\delta)| = |\mathscr{P}(\omega)|$ and $|\mathfrak{B}^C_2 \restriction \nu| = \omega_1$, one has an injection of ω_1 into $\mathscr{P}(\omega)$. This violates the boldface GCH at ω. $\square$

Definition 3.8. Let $2 \leq n < \omega$. Let $U_n = \{(\alpha_{n-1}, 0, \alpha_{n-2}, ..., \alpha_0, i) : \alpha_0 < ... < \alpha_{n-1} < \omega_1 \wedge i < 2\} \cup \{(\alpha, 1) : n-1 \leq \alpha < \omega_1\}$. Let $\ll_n$ be the lexicographic ordering on U_n. Let $\mathcal{U}_n = (U_n, \ll_n)$. Observe that $\mathrm{ot}(\mathcal{U}_n) = \omega_1$. Suppose $F : U_n \to \omega_1$. Define $F^0, F^1 : [\omega_1]^n \to \omega_1$ and $F^2 : \omega_1 \to \omega_1$ by $F^0(\iota) = F(\iota(n-1), 0, \iota(n-2), ..., \iota(0), 0)$, $F^1(\iota) = F(\iota(n-1), 0, \iota(n-2), ..., \iota(0), 1)$, and $F^2(\alpha) = F(\alpha, 1)$. A function $F : U_n \to \omega_1$ has the correct type if and only if the following two conditions hold:

- F is discontinuous everywhere: For all $x \in U_n$, $\sup\{F(y) : y \ll_n x\} < F(x)$.
- F has uniform cofinality ω: There is a function $\mathcal{F} : U_n \times \omega \to \omega_1$ so that for all $x \in U_n$ and $k \in \omega$, $\mathcal{F}(x, k) < \mathcal{F}(x, k+1)$ and $F(x) = \sup\{\mathcal{F}(x, k) : k \in \omega\}$.

Note that if $F : U_n \to \omega_1$ has the correct type, then $F^0, F^1 : [\omega_1]^n \to \omega_1$ has type n, $[F^0]_{\mu^n_{\omega_1}} < [F^1]_{\mu^n_{\omega_1}}$, and $F^2 : \omega_1 \to \omega_1$ has the correct type. If $X \subseteq \omega_1$, then let $[X]^{\mathcal{U}_n}_*$ be the set of all $F : U_n \to X$ of the correct type and order preserving between $\mathcal{U}_n$ and $(X, <)$ with the usual ordering.

Lemma 3.9. *Suppose $2 \leq n < \omega$. Assume $\omega_1 \to_* (\omega_1)^{n+1}_2$. Let $f_0, f_1 : [\omega_1]^n \to \omega_1$ and $f_2 : \omega_1 \to \omega_1$ be functions with the following properties.*

- *f_0 and f_1 have type n. f_2 has type 1.*
- *$[I^{n-1}_{f_0}]_{\mu^{n-1}_{\omega_1}} = [I^{n-1}_{f_1}]_{\mu^{n-1}_{\omega_1}}$ and $[f_0]_{\mu^n_{\omega_1}} < [f_1]_{\mu^n_{\omega_1}}$.*
- *$[I^1_{f_0}]_{\mu^n_{\omega_1}} = [I^1_{f_1}]_{\mu^n_{\omega_1}} < [f_2]_{\mu^1_{\omega_1}}$.*

Then there is an $F \in [\omega_1]^{\mathcal{U}_n}_*$ with the following properties.

- *$[F^0]_{\mu^n_{\omega_1}} = [f_0]_{\mu^n_{\omega_1}}$, $[F^1]_{\mu^n_{\omega_1}} = [f_1]_{\mu^n_{\omega_1}}$, and $[F^2]_{\mu^1_{\omega_1}} = [f_2]_{\mu^1_{\omega_1}}$,*
- *$F^0[[\omega_1]^n] \subseteq f_0[[\omega_1]^n]$, $F^1[[\omega_1]^n] \subseteq f_1[[\omega_1]^n]$, and $F^2[\omega_1] \subseteq f_2[\omega_1]$.*

Proof. Let $C_0 \subseteq \omega_1$ be a club with the following properties:

(1) For all $\ell \in [\omega_1]^n_*$, $I^{n-1}_{f_0}(\ell) = I^{n-1}_{f_1}(\ell)$.
(2) For all $\ell \in [\omega_1]^n_*$, $f_0(\ell) < f_1(\ell)$.
(3) For all $\alpha \in C_0$, $I^1_{f_0}(\alpha) = I^1_{f_1}(\alpha) < f_2(\alpha)$.

If $\ell \in [\omega_1]^{n+1}$, let $\ell^0, \ell^1 \in [\omega_1]^n$ be defined by $\ell^1(k) = \ell(k+1)$ and

$$\ell^0(k) = \begin{cases} \ell(0) & k = 0 \\ \ell(k+1) & 0 < k < n \end{cases}.$$

Define $P : [C_0]^{n+1} \to 2$ by $P(\ell) = 0$ if and only if $f_1(\ell^0) < f_0(\ell^1)$. By $\omega_1 \to_* (\omega_1)^{n+1}_2$, there is a club $C_1 \subseteq C_0$ which is homogeneous for P. Let $C_2 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha\}$. Pick $(\alpha_0, \alpha_1, ..., \alpha_{n-1}) \in [C_2]^n_*$. Since α_1 is a limit point of C_1 and $I^{n-1}_{f_0}(\alpha_1, ..., \alpha_{n-1}) = I^{n-1}_{f_1}(\alpha_1, ..., \alpha_{n-1})$ because $(\alpha_1, ..., \alpha_{n-1}) \in [C_2]^{n-1}_* \subseteq [C_0]^{n-1}_*$, there must be some $\gamma \in C_1$ so that $\alpha_0 < \gamma < \alpha_1$ and $f_1(\alpha_0, ..., \alpha_{n-1}) < f_0(\gamma, \alpha_1, ..., \alpha_{n-1})$. Pick $\ell \in [C_1]^{n+1}_*$ so that $\ell^0 = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$ and $\ell^1 = (\gamma, \alpha_1, ..., \alpha_{n-1})$. Then $P(\ell) = 0$ since $f_1(\ell^0) = f_1(\alpha_0, ..., \alpha_{n-1}) < f_0(\gamma, \alpha_1, ..., \alpha_{n-1}) = f_0(\ell^1)$. This shows that C_1 is homogeneous for P taking value 0. Since f_0 and f_1 have type n, $I^1_{f_0}$ and $I^1_{f_1}$ are increasing functions. For any $\alpha \in \omega_1$, there is some $\gamma \in C_1$ so that $f_2(\alpha) < I^1_{f_0}(\gamma) = I^1_{f_1}(\gamma)$. Let $h : \omega_1 \to C_1$ be defined by $h(\alpha)$ is the least such $\gamma \in C_1$. Let $C_3 = \{\alpha \in C_1 : \mathsf{enum}_{C_1}(\alpha) = \alpha \wedge (\forall \alpha' < \alpha)(h(\alpha') < \alpha)\}$ which is a club subset of C_1.

For notational simplicity, let $\mathfrak{e} = \mathsf{enum}_{[C_3]^1_*}$. Define $F : U_n \to \omega_1$ by $F(\alpha_{n-1}, 0, ..., \alpha_1, i) = f_i(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n-1}))$ for all $(\alpha_0, .., \alpha_{n-1}) \in [\omega_1]^n$ and $i \in 2$ and $F(\alpha, 1) = f_2(\mathfrak{e}(\alpha))$ for all $\alpha < \omega_1$.

First, one will show that F is an order preserving map from $\mathcal{U}_n$ into the usual ordering on ω_1. Suppose $x, y \in U_n$ and $x \ll_n y$. One seeks to show $F(x) < F(y)$.

- Suppose $x = (\alpha, 1)$ and $y = (\beta, 1)$ with $\alpha < \beta$:

$$F(x) = F(\alpha, 1) = f_2(\mathfrak{e}(\alpha)) < f_2(\mathfrak{e}(\beta)) = F(\beta, 1) = F(y)$$

since f_2 has type 1.

- Suppose $x = (\alpha, 1)$ and $y = (\beta_{n-1}, 0, \beta_{n-2}, ..., \beta_0, i)$ for some $i < 2$ and $\beta_0 < ... < \beta_{n-1}$ with $\alpha < \beta_{n-1}$: Note that
$$F(x) = F(\alpha, 1) = f_2(\mathfrak{e}(\alpha)) < I^1_{f_i}(h(\mathfrak{e}(\alpha)))$$
$$< f_i(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n-1})) = F(\beta_{n-1}, 0, \beta_{n-2}, ..., \beta_0, i) = F(y).$$
The first inequality comes from the definition of h, the second inequality comes from $h(\mathfrak{e}(\alpha)) < \mathfrak{e}(\beta_{n-1})$ by the definition of $\mathfrak{e}(\beta_{n-1}) \in C_3$, and the last inequality comes from f_i having type n.
- Suppose $x = (\alpha_{n-1}, 0, \alpha_{n-2}, ..., \alpha_0, i)$ and $y = (\beta, 1)$ for some $i < 2$ and $\alpha_0 < ... < \alpha_{n-1}$ with $\alpha_{n-1} \leq \beta$:
$$F(x) = F(\alpha_{n-1}, 0, \alpha_{n-1}, ..., \alpha_0, i) = f_i(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n-1})) \leq I^1_{f_i}(\mathfrak{e}(\alpha_{n-1}))$$
$$< f_2(\mathfrak{e}(\alpha_{n-1})) \leq f_2(\mathfrak{e}(\beta)) = F(\beta, 1) = F(y).$$
The first inequality comes from the definition of $I^1_{f_i}$, the second inequality comes from property (3) of the club C_0, and the third inequality comes from the fact that f_2 has type 1.
- Suppose $x = (\alpha_{n-1}, 0, \alpha_{n-1}, ..., \alpha_0, i)$ and $y = (\beta_{n-1}, 0, \beta_{n-1}, ..., \beta_0, j)$ for some $i, j \in 2$, $\alpha_0 < ... < \alpha_{n-1}$, $\beta_0 < ... < \beta_{n-1}$, and there is some $k > 0$ so that $\alpha_k < \beta_k$ and for all $k < k' < n$, $\alpha_{k'} = \beta_{k'}$:
$$F(x) = F(\alpha_{n-1}, 0, \alpha_{n-1}, ..., \alpha_0, i) = f_i(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n-1})) \leq I^{n-k}_{f_i}(\mathfrak{e}(\alpha_k), ..., \mathfrak{e}(\alpha_{n-1}))$$
$$= I^{n-k}_{f_j}(\mathfrak{e}(\alpha_k), ..., \mathfrak{e}(\alpha_{n-1})) < I^{n-k}_{f_j}(\mathsf{next}_{C_1}(\mathfrak{e}(\alpha_k)), \mathfrak{e}(\alpha_{k+1}), ..., \mathfrak{e}(\alpha_{n-1}))$$
$$= I^{n-k}_{f_j}(\mathsf{next}_{C_1}(\mathfrak{e}(\alpha_k)), \mathfrak{e}(\beta_{k+1}), ..., \mathfrak{e}(\beta_{n-1})) < f_j(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n-1}))$$
$$= F(\beta_{n-1}, 0, \beta_{n-1}, ..., \beta_0, j) = F(y).$$
- Suppose $x = (\alpha_{n-1}, 0, \alpha_{n-1}, ..., \alpha_0, 0)$ and $y = (\beta_{n-1}, 0, \beta_{n-1}, ..., \beta_0, 1)$ for $\alpha_0 < ... < \alpha_{n-1}$ and $\beta_0 < ... < \beta_{n-1}$ such that for all $0 < k < n$, $\alpha_k = \beta_k$ and $\alpha_0 \leq \beta_0$:
$$F(x) = F(\alpha_{n-1}, 0, \alpha_{n-1}, ..., \alpha_0, 0) = f_0(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n-1})) < f_1(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n-1}))$$
$$\leq f_1(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n-1})) = F(\beta_{n-1}, 0, \beta_{n-2}, ..., \beta_0, 1) = F(y).$$
- Suppose $x = (\alpha_{n-1}, 0, \alpha_{n-1}, ..., \alpha_0, 1)$ and $y = (\beta_{n-1}, 0, \beta_{n-1}, ..., \beta_0, 0)$ for $\alpha_0 < ... < \alpha_{n-1}$ and $\beta_0 < ... < \beta_{n-1}$ such that for all $0 < k < n$, $\alpha_k = \beta_k$ and $\alpha_0 < \beta_0$: Let $\ell = (\mathfrak{e}(\alpha_0), \mathfrak{e}(\beta_0), \mathfrak{e}(\beta_1), ..., \mathfrak{e}(\beta_{n-1})) = (\mathfrak{e}(\alpha_0), \mathfrak{e}(\beta_0), \mathfrak{e}(\alpha_1), ..., \mathfrak{e}(\alpha_{n-1}))$. In the notation above, $\ell^0 = (\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n-1}))$ and $\ell^1 = (\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n-1}))$. $P(\ell) = 0$ implies that $f_1(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n-1})) = f_1(\ell^0) < f_0(\ell^1) = f_0(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n-1}))$. So we have the following.
$$F(x) = F(\alpha_{n-1}, 0, \alpha_{n-2}, ..., \alpha_0, 1) = f_1(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n-1}))$$
$$< f_0(\mathfrak{e}(\beta_0), ..., \mathfrak{e}(\beta_{n-1})) = F(\beta_{n-1}, 0, \beta_{n-2}, ..., \beta_0, 0) = F(y)$$

This shows that F is order preserving.

Next one will show that F is discontinuous everywhere. Suppose $x \in U_n$ has limit rank in $\ll_n$.

- Suppose $x = (\alpha, 1)$ for some $\alpha \in \omega_1$. Then $\sup(F \restriction x) = \sup\{F(\alpha, 0, \alpha_{n-2}, ..., \alpha_0, i) : i \in 2 \wedge \alpha_0 < ... < \alpha_{n-1} < \alpha\} = \sup\{F(\alpha, 0, \alpha_{n-2}, ..., \alpha_0, 0) : \alpha_0 < ... < \alpha_{n-2} < \alpha\} = \sup\{f_0(\mathfrak{e}(\alpha_0), \mathfrak{e}(\alpha_1), ..., \mathfrak{e}(\alpha_{n-2}), \mathfrak{e}(\alpha)) : \alpha_0 < \alpha_1 < ... < \alpha_{n-2} < \alpha\} = I^1_{f_0}(\mathfrak{e}(\alpha)) < f_2(\mathfrak{e}(\alpha)) = F(x)$.
- Suppose $x = (\alpha_{n-1}, 0, \alpha_{n-2}, ..., \alpha_0, 0)$ and has limit rank. Then $\sup(F \restriction x) = \sup\{f_0(\ell) : \ell \sqsubset_n (\alpha_0, ..., \alpha_{n-1})\} < f_0(\alpha_0, ..., \alpha_{n-1}) = F(x)$ using the discontinuity of f_0.

This shows that F is discontinuous everywhere.

Let $G_0, G_1 : [\omega_1]^n \times \omega \to \omega_1$ witness that f_0 and f_1 have uniform cofinality ω. Let $G_2 : \omega_1 \times \omega \to \omega_1$ witness that f_2 has uniform cofinality ω. Define $\mathcal{F} : U_n \times \omega \to \omega_1$ be defined as follows.
$$\mathcal{F}(x, k) = \begin{cases} G_2(\mathfrak{e}(\alpha), k) & x = (\alpha, 1) \\ G_i((\mathfrak{e}(\alpha_0), ..., \mathfrak{e}(\alpha_{n-1})), k) & x = (\alpha_{n-1}, 0, \alpha_{n-2}, ..., \alpha_0, i) \wedge i \in 2 \end{cases}.$$
$\mathcal{F}$ witness that F has uniform cofinality ω. It has been shown that F is a function of the correct type.

It is clear from the construction that $F^0[[\omega_1]^n] \subseteq f_0[[\omega_1]^n]$, $F^1[[\omega_1]^n] \subseteq f_1[[\omega_1]^n]$, and $F^2[\omega_1] \subseteq f_2[\omega_1]$. Let $C_4 = \{\alpha \in C_3 : \mathsf{enum}_{C_3}(\alpha) = \alpha\}$ which is a club subset of C_3. For all $\alpha \in [C_4]^1_*$, $\alpha = \mathsf{enum}_{C_3}(\alpha) = \mathsf{enum}_{[C_3]^1_*}(\alpha) = \mathfrak{e}(\alpha)$. For all $(\alpha_0, ..., \alpha_{n-1}) \in [C_4]^n$ and $i \in 2$, $F^i(\alpha_0, ..., \alpha_{n-1}) = f_i(\mathfrak{e}(\alpha_0), ..., \mathfrak{e}_{\alpha_{n-1}}) =$

$f_i(\alpha_0, ..., \alpha_{n-1})$. For all $\alpha \in [C_3]^1_*$, $F^2(\alpha) = f_2(\mathfrak{e}(\alpha)) = f_2(\alpha)$. This shows that $[F^1]_{\mu^n_{\omega_1}} = [f_0]_{\mu^n_{\omega_1}}$, $[F^1]_{\mu^n_{\omega_1}} = [f_1]_{\mu^n_{\omega_1}}$, and $[F^2]_{\mu^1_{\omega_1}} = [f_2]_{\mu^1_{\omega_1}}$. $\square$

Theorem 3.10. *Assume $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ and $\bigstar$. The boldface* GCH *holds below ω_ω.*

Proof. The boldface GCH at ω_n for all $n < \omega$ will be shown by induction. For $n = 0$, the boldface GCH at ω has already been shown by Fact 3.4. For $n = 1$, the boldface GCH at ω_1 has already been shown by Theorem 3.7. Suppose $n > 1$ and the boldface GCH has been shown at ω_{n-1}. Suppose for the sake of contradiction, the boldface GCH at ω_n fails. Let $\langle A_\eta : \eta < \omega_{n+1} \rangle$ be an injection of ω_{n+1} into $\mathscr{P}(\omega_n)$. Recall $\mathcal{U}_n = (U_n, \ll_n)$ from Definition 3.8. By Fact 2.30, $\mathrm{cof}(\omega_{n+1}) = \omega_2$ for all $1 \leq n < \omega$. Fix $\rho : \omega_2 \to \omega_n$ be an increasing cofinal map. Define $P : [\omega_1]^{\mathcal{U}_n} \to 2$ by $P(F) = 0$ if and only if $\min(A_{[F^0]_{\mu^n_{\omega_1}}} \triangle A_{[F^1]_{\mu^n_{\omega_1}}}) < \rho([F^2]_{\mu^1_{\omega_1}})$, where $\triangle$ refers to the symmetric difference. (Note that here one is using the fact that $j_{\mu^1_{\omega_1}}(\omega_1) = \omega_2$ and $j_{\mu^n_{\omega_1}}(\omega_1) = \omega_{n+1}$ established in Fact 2.30.) Since $\mathrm{ot}(\mathcal{U}_n) = \omega_1$, $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ implies there is a club $C \subseteq \omega_1$ which is homogeneous for P. Let $f_0, f_1 : [\omega_1]^n \to C$ be any two function of type n with $[f_0]_{\mu^n_{\omega_1}} < [f_1]_{\mu^n_{\omega_1}}$ and $[I^{n-1}_{f_0}]_{\mu^{n-1}_{\omega_1}} = [I^{n-1}_{f_1}]_{\mu^{n-1}_{\omega_1}}$. Since $\langle A_\eta : \eta < \omega_{n+1} \rangle$ is an injection, $A_{[f_0]_{\mu^n_{\omega_1}}} \neq A_{[f_1]_{\mu^n_{\omega_1}}}$ and thus $A_{[f_0]_{\mu^n_{\omega_1}}} \triangle A_{[f_1]_{\mu^n_{\omega_1}}} \neq \emptyset$. Let $f_2 : \omega_1 \to C$ be any function of type 1 so that $\rho([f_2]_{\mu^n_{\omega_1}}) > \min(A_{[f_0]_{\mu^n_{\omega_1}}} \triangle A_{[f_1]_{\mu^n_{\omega_1}}})$ and $[f_2]_{\mu^1_{\omega_1}} > [I^1_{f_0}]_{\mu^1_{\omega_1}} = [I^1_{f_1}]_{\mu^1_{\omega_1}}$. By Lemma 3.9, there is an $F \in [\omega_1]^{\mathcal{U}_n}_*$ so that $[F^0]_{\mu^n_{\omega_1}} = [f_0]_{\mu^n_{\omega_1}}$, $[F^1]_{\mu^n_{\omega_1}} = [f_1]_{\mu^n_{\omega_1}}$, $[F^2]_{\mu^1_{\omega_1}} = [f_2]_{\mu^n_{\omega_1}}$, $F^0[[\omega_1]^n] \subseteq f_0[[\omega_1{}^n]] \subseteq C$, $F^1[[\omega_1]^n] \subseteq f_1[[\omega_1]^n] \subseteq C$, and $F^2[\omega_1] \subseteq f_2[\omega_1] \subseteq C$. Thus $F \in [C]^{\mathcal{U}_n}_*$. Then $\rho([F^2]_{\mu^n_{\omega_1}}) > \min(A_{[F^0]_{\mu^n_{\omega_1}}} \triangle A_{[F^1]_{\mu^n_{\omega_1}}})$ implies that $P(F) = 0$. This shows that C must be homogeneous for P taking value 0. Pick any $\phi : V_n \to C$ of the correct type from $\mathcal{V}_n$ into $(C, <)$ (where recall V_n is defined in Definition 2.34). By Fact 2.37, there is a $\chi < \omega_n$ so that $E_\chi = \{\eta \in \mathfrak{B}^C_{n+1} : I^{n-1}_\eta = \chi\}$ has cardinality ω_n. Let $g : \omega_1 \to C$ be any function of type 1 so that $[g]_{\mu^1_{\omega_1}} > \mathcal{I}^1_\chi$. Let $\epsilon = [g]_{\mu^1_{\omega_1}}$. Suppose $\eta_0, \eta_1 \in E_\chi$ and $\eta_0 \neq \eta_1$. Without loss of generality, suppose $\eta_0 < \eta_1$. Let $f_0, f_2 : [\omega_1]^n \to C$ be functions of type n so that $[f_0]_{\mu^n_{\omega_1}} = \eta_0$ and $[f_1]^{\mu^n_{\omega_1}} = \eta_1$. By definition of $\eta_0, \eta_1 \in E_\chi$, $[I^{n-1}_{f_i}]_{\mu^{n-1}_{\omega_1}} = \chi$ and $[I^1_{f_i}]_{\mu^1_{\omega_1}} = \mathcal{I}^1_\chi < [g]_{\mu^1_{\omega_1}} = \epsilon$ for both $i \in 2$. By Lemma 3.9, there is an $F \in [C]^{\mathcal{U}_n}_*$ so that $[F^0]_{\mu^n_{\omega_1}} = [f_0]_{\mu^n_{\omega_1}} = \eta_0$, $[F^1]_{\mu^n_{\omega_1}} = [f_1]^{\mu^n_{\omega_1}} = \eta_1$, and $[F^2]_{\mu^1_{\omega_1}} = [g]_{\mu^1_{\omega_1}} = \epsilon$. By $P(F) = 0$, one has that $\min(A_{\eta_0} \triangle A_{\eta_1}) < \rho(\epsilon)$. This shows that the function $\Upsilon : E_\chi \to \mathscr{P}(\rho(\epsilon))$ defined by $\Upsilon(\eta) = A_\eta \cap \rho(\epsilon)$ is an injection. Since $|E_\chi| = \omega_n$ and $|\mathscr{P}(\rho(\epsilon))| = |\mathscr{P}(\omega_{n-1})|$ because $\rho(\epsilon) < \omega_n$, Υ induces an injection of ω_n into $\mathscr{P}(\omega_{n-1})$ which violates the inductive assumption that the boldface GCH holds at ω_{n-1}. $\square$

Under AD, $\omega_{\omega+1} = \boldsymbol{\delta}^1_3$ and there is a $\omega_{\omega+1}$-complete nonprincipal ultrafilter on $\omega_{\omega+1}$. Thus the boldface GCH holds at ω_ω by Fact 3.2. The combinatorial methods used here can be generalized with Jackson's theory of descriptions ([11]) for the projective ordinals to show that the boldface GCH holds below the supremum of the projective ordinals, $\sup\{\boldsymbol{\delta}^1_n : n \in \omega\}$, assuming AD. Jackson's theory can go slightly beyond the projective ordinals but not all the way through Θ. The inner model theoretic techniques of Steel and Woodin are the only known methods to prove the boldface GCH below Θ under AD^+.

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