Partial Tower Sealing

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Abstract

The main result of this paper shows that a weak form of Tower Sealing holds in a generic extension of hod mice with a strong cardinal and a proper class of Woodin cardinals. We show Tower Sealing fails in such extensions in general. We show that this weak form of Tower Sealing (called *Partial Tower Sealing*) implies Sealing and that its consistency strength is below that of ZFC+“there is a Woodin limit of Woodin cardinals”.

1. Introduction

This paper formulates a weak form of Woodin's *Tower Sealing, Partial Tower Sealing* cf. Definition 1.1, and shows that this form of Tower Sealing implies *Sealing* under various circumstances. The main result of this paper is Theorem 1.3, which shows that Partial Tower Sealing can hold in hod mice; as a result, Partial Tower Sealing is consistent relative to the theory ZFC+“there is a Woodin limit of Woodin cardinals” (WLW).

Suppose g is a V-generic filter. Let Γ^∞_g be the class of all universally Baire sets in $V[g]$. When $V[g] = V$, we simply write Γ^∞. For a cardinal κ, we write $\mathbb{Q}_{<\kappa}$ for the countable tower forcing as defined in [Lar04, Definition 2.7.1].

Definition 1.1. Suppose there is a proper class of Woodin cardinals. Let δ be a Woodin cardinal. We say that *Partial Tower Sealing* holds at δ if whenever g is $<\delta$-generic over V and $G \subseteq \mathbb{Q}_{<\delta}$ is $V[g]$-generic, letting $j_G : V[g] \to M \subseteq V[g][G]$ be the associated generic embedding, then

1. $L(\Gamma^\infty_g) \cap \wp(\mathbb{R}) = \Gamma^\infty_g$.
2. $(\Gamma^\infty_g)^\sharp, (\Gamma^\infty_{g*G})^\sharp$ exist and there is an elementary embedding $l : L(\Gamma^\infty_{g*G}) \to L(j_G(\Gamma^\infty_g))$ such that $l \restriction \Gamma^\infty_{g*G} = \mathrm{id}$ and l is an order-preserving surjection from the class of indiscernibles of $L(\Gamma^\infty_{g*G})$ to the class of indiscernibles of $L(j_G(\Gamma^\infty_g))$.

If Partial Tower Sealing holds at δ and additionally, $\Gamma_{g*G} = j_G(\Gamma^\infty_g)$, then we say *Tower Sealing* holds at δ. ⊣

Definition 1.2. Suppose there is a proper class of Woodin cardinals. We say *Sealing* holds at a Woodin cardinal δ if the following statements hold.

1. For every $<\delta$ generic g over V, $\wp(\mathbb{R}_g) \cap L(\Gamma^\infty_g, \mathbb{R}_g) = \Gamma^\infty_g$.
2. For every $<\delta$ generic g over V, for every $<\delta$ generic h over $V[g]$, there is an elementary embedding

$$j : L(\Gamma^\infty_g, \mathbb{R}_g) \to L(\Gamma^\infty_{g*h}, \mathbb{R}_h).$$

such that for every $A \in \Gamma^\infty_g$, $j(A) = A^h$. ⊣

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We note that if clause (1) of both Definitions 1.2 and 1.1 holds, then

$$L(\Gamma^\infty_g) \vDash \mathsf{AD}^+.$$

This is by a theorem of Woodin, [Lar04, Section 3.3], which states that if there is a proper class of Woodin cardinal and if $A \in \Gamma^\infty$, then $L(A, \mathbb{R}) \vDash \mathsf{AD}^+$. Moreover, by a theorem of Steel, [Ste09], every $A \in \Gamma^\infty$ has a scale in Γ^∞, this implies that

$$L(\Gamma^\infty) \vDash \text{“every set of reals is Suslin”}. \tag{1.1}$$

Sealing is a form of Shoenfield-type generic absoluteness for the theory of universally Baire sets. Sealing is an important hypothesis in set theory and particularly in inner model theory. [ST19] has a detailed discussion on the importance of Sealing and related topics; so we only summarize some main points here. If a large cardinal theory ϕ implies Sealing then the Inner Model Program for building canonical inner models of ϕ cannot succeed (at least with the criteria for defining “canonical inner models” as is done to date), cf [ST19, Sealing Dichotomy]. Sealing signifies a place beyond which new methodologies are needed in order to advance the Core Model Induction techniques. In particular, to obtain consistency strength beyond Sealing from strong theories such as the Proper Forcing Axiom, one needs to construct canonical subsets of Γ^∞ (third-order objects), instead of elements of Γ^∞ like what has been done before (see [ST19, Section 1] for a more detailed discussion). The consistency of Sealing was first proved by Woodin, who showed that if there is a proper class of Woodin cardinals and a supercompact cardinal κ then Sealing holds after collapsing 2^{2^κ} to be countable. Woodin's proof can be found in [Lar04]. [ST24; ST21] show that Sealing holds in hod mice and various types of hybrid mice whose existence is consistent relative to WLW, which improves significantly Woodin's result.

Woodin [Lar04, Theorem 3.4.17] also obtains the consistency of Tower Sealing from a supercompact cardinal and a proper class of Woodin cardinals. [ST24] claims Tower Sealing holds in an excellent hybrid premouse (defined in [ST24]), but this is not true. Part of this paper's motivation is to correct this, cf. Theorem 6.1. This leads us to the formulation of Partial Tower Sealing, a weak form of Tower Sealing strong enough to imply Sealing in various circumstances, cf. Theorem 1.6, and weak enough to hold in hod mice. The proof that this form of Tower Sealing holds in such hod mice is given in Theorem 1.3. It is not known whether Tower Sealing can hold in hod mice at the moment.

Theorem 1.3. *Suppose $(\mathcal{P}, \Psi)$ is an lbr hod pair or a layered hod pair such that $\mathcal{P} \vDash$“there is a strong cardinal and a proper class of Woodin cardinals”. Let κ be the least strong cardinal of $\mathcal{P}$ and $g \subset Coll(\omega, \kappa^+)$ be $\mathcal{P}$-generic. Then*

$$\mathcal{P}[g] \vDash \text{“}\forall\delta \;\; \textit{if } \delta \textit{ is Woodin, then Partial Tower Sealing holds at } \delta.\text{”}$$

Remark 1.4. In general, we cannot expect Tower Sealing to hold in generic extensions of hod mice. See Theorem 6.1.

From the hypothesis of Theorem 1.3 and recent work of the first author, we immediately obtain the following corollary.

Corollary 1.5. *Partial Tower Sealing is consistent relative to* ZFC+ *“there is a Woodin limit of Woodin cardinals”.*

The next theorem shows that Partial Tower Sealing implies Sealing holds at certain Woodin cardinals. The reader can see section 2 and section 3 for the definition of Hom^*_{g*G} and related notions.

Theorem 1.6. *Suppose δ is a Woodin cardinal which is a limit of Woodin cardinals with the property that whenever g is $<\delta$-generic, $G \subseteq \mathbb{Q}_{<\delta}$ is $V[g]$-generic, then $\Gamma^\infty_{g*G} = Hom^*_{g*G}$. Suppose Partial Tower Sealing holds at δ. Then Sealing holds at δ.*

Remark 1.7. The hypothesis used in Theorem 1.6 holds in various important situations. For example, if δ is a Woodin limit of Woodin cardinals and strong cardinals, then whenever $G \subseteq \mathbb{Q}_{<\delta}$ is V-generic, $\Gamma^\infty_G = Hom^*_G$. Also, if V is the universe of a hod mouse with a proper class of Woodin cardinals, then at every Woodin cardinal δ which is a limit of Woodin cardinals, whenever $G \subseteq \mathbb{Q}_{<\delta}$ is V-generic, $\Gamma^\infty_G = Hom^*_G$. See Section 3 for more details.

It is unclear whether Tower Sealing can hold in hod mice and whether Tower Sealing is consistent relative to ZFC+“there is a Woodin limit of Woodin cardinals”. In theorems 6.1 and 6.5, we provide further evidence that it seems very hard to force Tower Sealing to hold in hod mice.

The paper is organized as follows. In Section 2 we review basic notions used in this paper. In Section 3 we shows Partial Tower Sealing implies Sealing holds at certain Woodin cardinals. In Section 4, we recall the derived model representation of Γ^∞ in [ST21] and use it to prove the consistency of Partial Tower Sealing in Section 5. In Section 6 we prove Theorems 6.1 and 6.5 which show that in general, Tower Sealing fails in hod mice. In Section 7 we collect some open problems and questions related to the results of this paper.

2. Preliminaries

2.1. Homogenously Suslin and universally Baire sets

We say that a pair of trees T, S are *δ-absolutely complementing* if for any poset $\mathbb{P}$ of size $\leq \delta$, for any generic $g \subseteq \mathbb{P}$, $V[g] \vDash$ "$p[T] = \mathbb{R} - p[S]$". Similarly, we say that T, S are *$< \delta$-absolutely complementing* if for any poset $\mathbb{P}$ of size $< \delta$, for any generic $g \subseteq \mathbb{P}$, $V[g] \vDash$ "$p[T] = \mathbb{R} - p[S]$". Given a limit of Woodin cardinals ν and $g \subseteq \mathrm{Col}(\omega, < \nu)$, let

1. $\mathbb{R}^*_g = \bigcup_{\alpha<\nu} \mathbb{R}^{V[g\cap\mathrm{Col}(\omega,\alpha)]}$,
2. Hom^*_g be the set of reals $A \in V(\mathbb{R}^*_g)$ such that for some $\alpha < \nu$, there is a pair $(T, S) \in V[g \cap \mathrm{Col}(\omega, \alpha)]$ such that $V[g \cap \mathrm{Col}(\omega, \alpha)] \vDash$ "(T, S) are $< \nu$-complementing trees" and $p[T]^{V(\mathbb{R}^*_g)} = A$, and
3. the derived model associated with g be defined by: $DM(g) = L(Hom^*_g, \mathbb{R}^*_g)$.

We now recall the notions of homogeneously Suslin and universally Baire sets. Given an uncountable cardinal κ, and a set Z, $meas_\kappa(Z)$ denotes the set of all κ-additive measures on $Z^{<\omega}$. If $\mu \in meas_\kappa(Z)$, then there is a unique $n < \omega$ such that $Z^n \in \mu$ by κ-additivity; we let this $n = dim(\mu)$. If $\mu, \nu \in meas_\kappa(Z)$, we say that μ *projects to* ν if $dim(\nu) = m \leq dim(\mu) = n$ and for all $A \subseteq Z^m$,

$$A \in \nu \Leftrightarrow \{u : u \restriction m \in A\} \in \mu.$$

In this case, there is a natural embedding from the ultrapower of V by ν into the ultrapower of V by μ:

$$\pi_{\nu,\mu} : \mathrm{Ult}(V, \nu) \to \mathrm{Ult}(V, \mu)$$

defined by $\pi_{\nu,\mu}([f]_\nu) = [f^*]_\mu$ where $f^*(u) = f(u \restriction m)$ for all $u \in Z^n$. A tower of measures on Z is a sequence $\langle \mu_n : n < k\rangle$ for some $k \leq \omega$ such that for all $m \leq n < k$, $dim(\mu_n) = n$ and μ_n projects to μ_m. A tower $\langle \mu_n : n < \omega\rangle$ is *countably complete* if the direct limit of $\{\mathrm{Ult}(V, \mu_n), \pi_{\mu_m,\mu_n} : m \leq n < \omega\}$ is well-founded. We will also say that the tower $\langle \mu_n : n < \omega\rangle$ is well-founded.

Recall we identify the set of reals $\mathbb{R}$ with the Baire space ${}^\omega\omega$.

Definition 2.1. Fix an uncountable cardinal κ. A function $\bar{\mu} : \omega^{<\omega} \to meas_\kappa(Z)$ is a *κ-complete homogeneity system* with support Z if for all $s, t \in \omega^{<\omega}$, writing μ_t for $\bar{\mu}(t)$:

(a) $dom(\mu_t) = dom(t)$,

(b) $s \subseteq t \Rightarrow \mu_t$ projects to μ_s.

Often times, we will not specify the support Z; instead, we just say $\bar{\mu}$ is a κ-complete homogeneity system.

A set $A \subseteq \mathbb{R}$ is *κ-homogeneous* iff there is a κ-complete homogeneity system $\bar{\mu}$ such that

$$A = S_\mu =_{def} \{x : \bar{\mu}_x \text{ is countably complete}\}.$$

A is homogeneous if it is κ-homogeneous for all κ. Let Hom_∞ be the collection of all homogeneous sets. ⊣

Definition 2.2. $A \subseteq \mathbb{R}$ is *κ-universally Baire* if there are trees $T, U \subseteq (\omega \times ON)^{<\omega}$ that are κ-absolutely complemented, i.e. $A = p[T] = \mathbb{R}\backslash p[U]$ and whenever $\mathbb{P}$ is a forcing such that $|\mathbb{P}| \leq \kappa$ and $g \subseteq \mathbb{P}$ is V-generic, in $V[g]$, $p[T] = \mathbb{R}\backslash p[U]$. In this case, we let $A_g = p[T]$ be the canonical interpretation of A in $V[g]$.

A is *universally Baire* if A is κ-universally Baire for all κ. Let Γ^∞ be the collection of all universally Baire sets. ⊣

We remark that if A is κ-universally Baire as witnessed by pairs (T_1, U_1) and (T_2, U_2) and $\mathbb{P}$ such that $|\mathbb{P}| \leq \kappa$ and $g \subset \mathbb{P}$ is V-generic, then $A_g = p[T_1] = p[T_2]$, i.e. A_g does not depend on the choice of absolutely complemented trees that witness A is κ-universally Baire. A similar remark applies to κ-homogeneously Suslin sets; in other words, if $A = S_{\bar{\mu}}$ where the measures in $\bar{\mu}$ are κ-complete, then for any $<$-κ generic g, the canonical interpretation A_g is defined as

$$(S_{\bar{\mu}})_g = \{x \in \mathbb{R}^{V[g]} : \bar{\mu}_x \text{ is countably complete in } V[g]\}.$$

Suppose there is a proper class of Woodin cardinals. The following are some standard results about universally Baire sets we will use throughout our paper. The proof of these results can be found in [Ste09].

(I) $\mathrm{Hom}_\infty = \Gamma^\infty$.

(II) For any $A \in \Gamma^\infty$, $L(A, \mathbb{R}) \vDash \mathsf{AD}^+$; furthermore, given such an A, there is a $B \in \Gamma^\infty$ such that $B \notin L(A, \mathbb{R})$ and $A \in L(B, \mathbb{R})$. In fact, $A^\sharp$ is an example of such a B.

(III) Suppose $A \in \Gamma^\infty$. Let B be the code for the first order theory with real parameters of the structure $(HC, \in, A)$ (under some reasonable coding of HC by reals). Then $B \in \Gamma^\infty$ and if g is V-generic for some forcing, then in $V[g]$, $B_g \in \Gamma^\infty$ is the code for the first order theory with real parameters of $(HC^{V[g]}, \in, A_g)$.

(IV) Every set in Γ^∞ has a scale in Γ^∞.

Under the same hypothesis, the results above also imply that

- Γ^∞ is closed under Wadge reducibility,
- if $A \in \Gamma^\infty$, then $\neg A \in \Gamma^\infty$,
- if $A \in \Gamma^\infty$ and g is V-generic for some forcing, then there is an elementary embedding $j : L(A, \mathbb{R}) \to L(A_g, \mathbb{R}_g)$, where $\mathbb{R}_g = \mathbb{R}^{V[g]}$.

2.2. Hod mice

Suppose $(\mathcal{P}, \Psi)$ is a hod pair in the sense of [Ste22] and that $\mathcal{P}$ has a proper class of Woodin cardinals. We recall some properties of iteration strategies of certain countable elementary substructures $X \prec \mathcal{P}|\eta$ for some inaccessible η proved in [ST21].

We adopt some notations from [ST21]. First, the pair $(\mathcal{P}, \Psi)$ is called *an iterable pair* in [ST21]. Given a strong limit cardinal κ and $F \subseteq Ord$, set

$$W_\kappa^\Psi = (H_\kappa, \mathcal{P}|\kappa, \Psi_{\mathcal{P}|\kappa} \restriction H_\kappa, \in).$$

Given a structure Q in a language extending the language of set theory with a transitive universe, and an $X \prec Q$, we let M_X be the transitive collapse of X and $\pi_X : M_X \to Q$ be the inverse of the transitive collapse. In general, the preimages of objects in X will be denoted by using X as a subscript, e.g. $\pi_X^{-1}(\mathcal{P}) = \mathcal{P}_X$. Also, if $\eta < \delta$, we write $\Psi_{\eta,\delta}$ for the fragment of Ψ that acts on the window (η, δ). Suppose in addition $Q = (R, ...\mathcal{P}, \Psi_{\eta,\delta}, ...)$. We will then write $X \prec (Q|\Psi_{\eta,\delta})$ to mean that $X \prec Q$ and the strategy of $\mathcal{P}_X$ that we are interested in is $\Psi_{\eta,\delta}^{\pi_X}$. We set $\Lambda_X = \Psi_{\eta,\delta}^{\pi_X}$. If g is a generic over V, we write $\Psi_{\eta,\delta}^g$ for the canonical interpretation of $\Psi_{\eta,\delta}$ in $V[g]$ (if exists) and $\Lambda_X^g = (\Psi_{\eta,\delta}^g)^{\pi_X}$.

By results of [Ste22], Ψ has all the properties required to run the constructions in [ST21]. In particular, results of [ST21, Sections 2, 3, 4] can be applied to $(\mathcal{P}, \Psi)$. We summarize some key facts that we use in the constructions in Section 4. We need develop some terminology to state these facts. In the following, we will write V for the universe of $\mathcal{P}$ and the notions below will hold in V.

Suppose ν is a Woodin cardinal. We let EA_ν be the ω-generator version of the extender algebra associated with ν (see e.g. [Ste10] for a detailed discussion of Woodin's extender algebras). We say the triple (M, δ, Φ) *Suslin, co-Suslin captures*[1] the set of reals B if there is a pair $(T, S) \in M$ such that $M \vDash$ "(T, S) are δ-complementing" and

1. M is a countable transitive model of some fragment of ZFC,

[1] This notion is probably due to Steel, see [Ste08].

2. Φ is an ω_1-strategy for M,
3. $M \vDash$ "δ is a Woodin cardinal",
4. for $x \in \mathbb{R}$, $x \in B$ if and only if there is an iteration $\mathcal{T}$ of M according to Φ with last model N such that x is generic over N for $\mathsf{EA}^N_{\pi^{\mathcal{T}}(\delta)}$ and $x \in p[\pi^{\mathcal{T}}(T)]$.

Suppose M is a countable transitive model of set theory and Φ is a strategy of M. Let (η, g) be such that g is M-generic for a poset in $M|\eta$. Let Φ' be the fragment of Φ that acts on iterations that are above η. Then Φ' can be viewed as an iteration strategy of $M[g]$. This is because if $\mathcal{T}$ is an iteration of $M[g]$ above η, there is an iteration $\mathcal{U}$ of M that is above η and such that

1. $lh(\mathcal{T}) = lh(\mathcal{U})$,
2. $\mathcal{T}$ and $\mathcal{U}$ have the same tree structure,
3. for each $\alpha < lh(\mathcal{T})$, $M^{\mathcal{T}}_\alpha = M^{\mathcal{U}}_\alpha[g]$,
4. for each $\alpha < lh(\mathcal{T})$, $E^{\mathcal{T}}_\alpha$ is the extension of $E^{\mathcal{U}}_\alpha$ onto $M^{\mathcal{U}}_\alpha[g]$.

Let Φ'' be the strategy of $M[g]$ with the above properties. We then say that Φ'' is induced by Φ'. We will often confuse Φ'' with Φ'. The following lemma is the key fact that we need for 4; it is proved in [ST21, Lemma 4.4]. Say $u = (\eta, \delta, \lambda)$ is a *good triple* if it is increasing, δ is a Woodin cardinal, and λ is an inaccessible cardinal.

Lemma 2.3. *Suppose $u = (\eta, \delta, \lambda)$ is a good triple and g is V-generic for a poset in V_η. Let $A \in \Gamma^\infty_g$. Then, in $V[g]$, there is a club of countable $X \prec (W_\lambda[g]|\Psi^g_{\eta,\delta})$ such that $(M_X, \delta_X, \Lambda^g_X)$ Suslin, co-Suslin captures A.*[2] *For each such X, let $X' = X \cap W_\lambda \prec W_\lambda$, and $(M_{X'}, \Lambda_{X'})$ be the transitive collapse of X' and its strategy. Then A is projective in $\Lambda_{X'}$. Moreover, these facts remain true in any further generic extension by a poset in $V_\eta[g]$.*

3. Partial Tower Sealing implies Sealing

In this section, we prove Theorem 1.6 and give some applications of 1.6 and Partial Tower Sealing.

Suppose δ is a Woodin limit of Woodin cardinals and $G \subseteq \mathbb{Q}_{<\delta}$ is V-generic, we let $\mathbb{R}^*_G = \mathbb{R}^{V[G]}$ and Hom^*_G be the set of $A \subseteq \mathbb{R}^*_G$ such that for any γ, there is a homogeneity system $\bar{\mu} = \langle \mu_s : s \in \omega^{<\omega}\rangle$ with the properties:

- each measure μ_s is γ-complete in $V[G]$.
- $\vec{\mu} \in V[x]$ for some $x \in \mathbb{R}^*_G$.
- $A = \{z : (\mu_{z\restriction n} : n < \omega) \text{ is well-founded}\}$.

In the last item above, we write $A = S_{\bar{\mu}}$. If the measures in $\bar{\mu}$ are all κ-complete for some uncountable cardinal κ and g is $< \kappa$-generic, recall we can canonically extend A to $A_g = (S_{\bar{\mu}})_g$, where

$$(S_{\bar{\mu}})_g = \{z \in V[G, g] : (\mu_{z\restriction n} : n < \omega) \text{ is well-founded in } V[G, g]\}.$$

Also, it is clear that for δ, G as above, there is a γ (sufficiently large) such that any $A \in Hom^*_G$ is witnessed by a homogeneity system $\bar{\mu}$ where each measure $\mu \in \bar{\mu}$ is γ-complete.

Remark 3.1. Let δ, G, A be as above. Suppose further that there is a proper class of Woodin cardinals. We can in fact choose γ large enough so that letting $g \in V(\mathbb{R}^*_G)$ be $< \delta$-generic such that $\bar{\mu} \in V[g]$ consists of γ-complete measures, then $(S_{\bar{\mu}})_g$ is in fact universally Baire in $V[g]$ and that $A = (S_{\bar{\mu}})_G$ is universally Baire in $V[G]$. This follows from results of Woodin and Martin-Steel, [Ste09]. See item (I) of Section 2.

Proof of Theorem 1.6. We fix a Woodin cardinal δ as in the hypothesis of the theorem. We need to verify clause (2) of Definition 1.2. Let $\mathbb{P} \in V_\delta$ and $g \subseteq \mathbb{P}$ be V-generic. We show that there is an elementary embedding $j : L(\Gamma^\infty, \mathbb{R}) \to L(\Gamma^\infty_g, \mathbb{R}_g)$. Even though this only proves a special case of clause (2), but it will be evident that the proof can be generalized to prove the full statement. To simplify the notation, we write $L(\Gamma^\infty)$ for $L(\Gamma^\infty, \mathbb{R})$ etc.

[2]To conform with the above setup, we tacitly assume Λ^g_X to be the iteration strategy acting on trees above η_X.

Let $G \subseteq \mathbb{Q}_{<\delta}$ be V-generic and j_G be the associated embedding. By Partial Tower Sealing, we have an elementary map $l : L(\Gamma^\infty_G) \to L(j_G(\Gamma^\infty))$ such that $l \restriction \Gamma^\infty_G = \mathrm{id}$. We can then define a map

$$i : L(\Gamma^\infty) \to L(\Gamma^\infty_G)$$

as the unique map determined by: $i(A) = A_G$ for each $A \in \Gamma^\infty$ and $i(\alpha) = \beta$ iff $l(\beta) = j_G(\alpha)$ for any indiscernible α of $L(\Gamma^\infty)$. It is easy to see that i can be canonically extended to all of $L(\Gamma^\infty)$ because every $z \in L(\Gamma^\infty)$ has the form $\tau[x, A, s]$ for some term τ, a real x, $A \in \Gamma^\infty$, and s a finite set of indiscernibles. Therefore, we can simply define

$$i(z) = \tau^{L(\Gamma^\infty_G)}[x, A_G, i(s)].$$

Claim 3.2. *i is well-defined and elementary.*

Proof. Suppose φ is a formula, $x \in \mathbb{R}$, $A \in \Gamma_\infty$, and s is a finite sequence of indiscernibles. Then

$$\begin{aligned} L(\Gamma^\infty) \vDash \varphi[A, x, s] &\Leftrightarrow L(j_G(\Gamma^\infty)) \vDash \varphi[j_G(A), x, j_G(s)] \\ &\Leftrightarrow L(\Gamma^\infty_G) \vDash \varphi[A_G, x, i(s)]. \end{aligned}$$

The first equivalence follows from elementarity of j_G and the second equivalence follows from the elementarity of l and the fact that $l(i(s)) = j_G(s)$ and $l(A_G) = j_G(A) = A_G$. It is easy to see that the equivalences above prove the claim. □

Let $G' \subseteq \mathbb{Q}_{<\delta}$ be $V[g]$-generic and $j_{G'} : V[g] \to M' \subseteq V[g][G']$ be the associated embedding. By Partial Tower Sealing, we have an elementary map $l' : L(\Gamma^\infty_{g*G'}) \to L(j_{G'}(\Gamma^\infty_g))$ such that $l' \restriction \Gamma^\infty_{g*G'} = \mathrm{id}$. As before, we can define the elementary map $i' : L(\Gamma^\infty_g) \to L(\Gamma^\infty_{g*G'})$ similar to how i was defined.

Now, we can find G, G' such that the following hold:

(i) $G \subseteq \mathbb{Q}_{<\delta}$ is V-generic such that $g \in V[G]$.

(ii) $G' \subseteq \mathbb{Q}_{<\delta}$ is $V[g]$-generic.

(iii) $\mathbb{R}^{V[G]} = \mathbb{R}^{V[g][G']}$.

(iv) $\Gamma^\infty_G = \Gamma^\infty_{g*G'}$.

By standard facts concerning $\mathbb{Q}_{<\delta}$, we can find G, G' satisfying (i)-(iii), see [Ste09]. We give a little more details here. By the usual factoring property of $\mathrm{Coll}(\omega, <\delta)$ and the fact that g is $<\delta$-generic, there is a V-generic $H \subseteq \mathrm{Coll}(\omega, <\delta)$ such that $g \in V[H]$. By [Ste09, Lemma 6.6], there are generics $G \subset \mathbb{Q}_{<\delta}$ and $G' \subset \mathbb{Q}^{V[g]}_{<\delta}$ such that G is V-generic, G' is $V[g]$-generic such that $\mathbb{R}^{V[G]} = \mathbb{R}^{V[g][G']} = \mathbb{R}^{V[H]}$. These generics G, G' clearly satisfy (i) - (iii).

We show that (iv) is satisfied as well. In fact, we show

$$\Gamma^\infty_G = \Gamma^\infty_{g*G'} = Hom^*_{g*G'} = Hom^*_G. \tag{3.1}$$

First, note that $Hom^*_{g*G'} = Hom^*_G$. This is because $\mathbb{R}^*_G = \mathbb{R}^*_{g*G'}$, so any homogeneity system $\bar{\mu}$ witnessing $A \in Hom^*_G$ is in $V[x]$ for some $x \in \mathbb{R}^*_G = \mathbb{R}^*_{g*G'}$; therefore, $\bar{\mu}$ witnesses $A \in Hom^*_{g*G'}$. The converse is proved the same way.

But then note that by our hypothesis:

$$Hom^*_G = \Gamma^\infty_G$$

and

$$Hom^*_{g*G'} = \Gamma^\infty_{g*G'}.$$

So the equalities in (3.1) hold.

Let G, G' satisfy (i)-(iv) above. So we have elementary embeddings $i_0 : L(\Gamma^\infty) \to L(\Gamma^\infty_G) = L(\Gamma^\infty_{g*G'})$ and $i_1 : L(\Gamma^\infty_g) \to L(\Gamma^\infty_G) = L(\Gamma^\infty_{g*G'})$, where $i_0 = j_G \restriction L(\Gamma^\infty)$ and $i_1 = j_{G'} \restriction L(\Gamma^\infty_g)$. Let $j : L(\Gamma^\infty) \to L(\Gamma^\infty_g)$ be defined by: $j(A) = A_g$ for each $A \in \Gamma^\infty$ and $j(\alpha) = \beta$ iff $i_1(\beta) = i_0(\alpha)$. Just like in the proof of Claim 3.2, we have that k is elementary. This completes the proof of the theorem.

□

Corollary 3.3. *Suppose there is a proper class of Woodin cardinals and δ is a Woodin limit of Woodin cardinals. Then whenever g is $<$-δ generic, whenever $G \subseteq \mathbb{Q}_{<\delta}$ is $V[g]$-generic, then $\Gamma^\infty_{g*G} = Hom^*_{g*G}$. Therefore, if Partial Tower Sealing holds at δ, then Sealing holds at δ.*

Proof. Let g, G be as in the statement of the corollary. As in the proof of Theorem 1.6, we can find $G' \subseteq \mathbb{Q}_{<\delta}$ such that

(a) $G' \subseteq \mathbb{Q}_{<\delta}$ is V-generic such that $g \in V[G']$.

(b) $G' \subseteq \mathbb{Q}_{<\delta}$ is $V[g]$-generic.

(c) $\mathbb{R}^{V[G']} = \mathbb{R}^{V[g][G]}$.

We need to verify that

(d) $\Gamma^\infty_{G'} = \Gamma^\infty_{g*G}$.

As in the proof of (3.1), $Hom^*_{g*G} = Hom^*_{G'}$. We need to verify:

$$Hom^*_{G'} = \Gamma^\infty_{G'} \tag{3.2}$$

and

$$Hom^*_{g*G} = \Gamma^\infty_{g*G}. \tag{3.3}$$

We just prove 3.2 as the proof of 3.3 is the same. Let $A \in Hom^*_{G'}$. Then there is some $g \in V(\mathbb{R}^*_{G'})$ such that g is $< \delta$-generic and some $\bar{\mu} \in V[g]$ such that $A = (S_{\bar{\mu}})_{G'}$. Let $B = (S_{\bar{\mu}})_g$. We may assume all measures in $\bar{\mu}$ are γ-complete for a sufficiently large γ so that B is in fact universally Baire in $V[g]$ and that A is universally Baire in $V[G']$ (see Remark 3.1). So $A \in \Gamma^\infty_{G'}$. Conversely, suppose $A \in \Gamma^\infty_{G'}$. By work of Martin-Steel and Woodin [Ste09] and the fact that there is a proper class of Woodin cardinals, A is κ-homogeneously Suslin for some sufficiently large $\kappa > \delta$. Let $\bar{\mu}$ witness A is κ-homogenously Suslin. Since $\bar{\mu}$ is countable and $\delta = \omega_1^{V[G']}$, $\bar{\mu} \in V[g]$ for some $< \delta$ generic $g \in V(\mathbb{R}^*_{G'})$. Let $B = (S_{\bar{\mu}})_g$ be the κ-homogeneously Suslin set in $V[g]$ witnessed by $\bar{\mu}$. So $A = (S_{\bar{\mu}})_{G'}$ is the canonical extension of B. This means $A \in Hom^*_{G'}$ as desired.[3]

□

Assume AD^+, we say that a pointclass Γ such that $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma)$ is OD-full if whenever $A \in \Gamma$ and $x, y \in \mathbb{R}$ are such that $y \in OD(A, x)$, then $y \in OD(A, x)$ in $L(\Gamma)$. We write Θ for the supremum of ordinals α for which there is a surjection of $\mathbb{R}$ onto α; we write $(\theta_\alpha : \alpha \leq \gamma)$ for the Solovay sequence. These notations can be relativized to pointclasses like Γ and we write $\Theta^\Gamma, \theta^\Gamma_\alpha$ for such objects. For Σ an (ω_1, ω_1)-iteration strategy for a countable mouse or a hod mouse $\mathcal{P}$, for $a \in \mathrm{HC}$, we let $\mathrm{Lp}^\Sigma(a)$ be the stack of all sound Σ-mice $\mathcal{M}$ over a such that $\rho_\omega(\mathcal{M}) = \omega$.

Theorem 3.4. *Suppose Partial Tower Sealing holds at a Woodin cardinal δ. Let $G \subseteq \mathbb{Q}_{<\delta}$ be V-generic and $j_G : V \to M \subseteq V[G]$ be the associated embedding, then Γ^G_∞ is OD-full in $j_G(\Gamma_\infty)$. In particular, the following hold.*

1. *Suppose $\Sigma \in \Gamma^\infty_G$ is an iteration strategy, then for any $a \in HC^{V[G]}$, $Lp^\Sigma(a) \cap L(\Gamma^\infty_G) = Lp^\Sigma(a) \cap L(j_G(\Gamma^\infty))$.*

2. *$\Theta^{\Gamma^\infty_G} = \theta^{j(\Gamma^\infty)}_\alpha$ for some limit ordinal α.*

Proof. Fix G, j_G and let $l : L(\Gamma^G_\infty) \to L(j_G(\Gamma^\infty))$ be given by Partial Tower Sealing. Let $a \in HC^{V[G]}$ and let $A \in \Gamma^\infty_G$. Suppose $y \in OD(A, a) \cap HC^{V[G]}$ in $L(\Gamma^\infty_G)$, then by elementarity and the fact that $l(A) = A$ and $l \restriction \mathbb{R}^{V[G]} = \mathrm{id}$, we have that $y \in OD(A, a)$ in $L(j_G(\Gamma^\infty))$. Conversely, if $y \in OD(A, a) \cap HC^{V[G]}$ in $L(j(\Gamma^\infty))$, then letting T be the tree projecting to the universal $\undertilde{\Sigma}^2_1(A)$-set in $L(j_G(\Gamma^\infty))$; the existence of T follows from the fact that

$$L(j_G(\Gamma^\infty)) \vDash \text{“ there is no largest Suslin cardinal”}$$

which follows from (1.1). So $y \in L[T, a]$. Now by our choice of T and the fact that $\undertilde{\Sigma}^2_1(A)$ is the same in $L(j_G(\Gamma^\infty))$ and in $L(\Gamma^\infty_G)$, $T \in L(\Gamma^\infty_G)$. We have then that $l(T) = T$. Since $L(\Gamma^\infty_G) \vDash y \in L[T, a]$, we see that $y \in OD(A, a)$ in $L(\Gamma^\infty_G)$.

[3] We may assume κ is large enough that $\bar{\mu}$ witnesses $A \in Hom^*_{G'}$.

To see (1), note that by elementarity and the fact that $l(\Sigma) = \Sigma$,

$$l(\mathrm{Lp}^\Sigma(a)^{L(\Gamma^\infty_G)}) = (\mathrm{Lp}^\Sigma(a))^{L(j_G(\Gamma^\infty))}.$$

Clearly $\mathrm{Lp}^\Sigma(a)^{L(\Gamma^\infty_G)} \trianglelefteq \mathrm{Lp}^\Sigma(a))^{L(j_G(\Gamma^\infty))}$. Now suppose $\mathcal{M} \lhd (\mathrm{Lp}^\Sigma(a))^{L(j_G(\Gamma^\infty))}$ is the least that is not in $(\mathrm{Lp}^\Sigma(a))^{L(\Gamma^\infty_G)}$, then since $\mathcal{M} \in OD(\Sigma, a)$ in $L(j_G(\Gamma^\infty))$, $\mathcal{M} \in OD(\Sigma, a)$ in $L(\Gamma^\infty_G)$. Since $L(j_G(\Gamma^\infty)) \vDash \mathcal{M}$ is a Σ-mouse over a, using l, we see that

$$L(\Gamma^\infty_G) \vDash \mathcal{M} \text{ is a } \Sigma\text{-mouse over } a.$$

This means $\mathcal{M} \lhd (\mathrm{Lp}^\Sigma(a))^{L(\Gamma^\infty_G)}$ as desired.

For (2), the fact that $\Theta^{\Gamma^\infty_G} = \theta_\alpha^{j_G(\Gamma^\infty)}$ follows from OD-fullness of Γ^∞_G; in fact, for each β such that $\theta_\beta < \theta_\alpha$ in $j(\Gamma^\infty)$, $\theta_\beta^{\Gamma^\infty_G} = \theta_\beta^{j(\Gamma^\infty)}$. α is limit because $L(\Gamma^\infty_G) \vDash$ "every set of reals is Suslin" as mentioned above. □

4. Derived model representation of Γ^∞ and Sealing

In this section, we summarize the construction in [ST21] that realizes Γ_∞ as a derived model via a direct limit construction. We assume the hypothesis of Theorem 1.3 and write V for the universe of $\mathcal{P}$. We fix a generic $g \subseteq \mathrm{Coll}(\omega, \kappa^+)$ and write ι for κ^+. We note that by our assumption and standard theory of hod mice [Ste22], the hypotheses required to apply Theorem 0.4 of [ST21] are satisfied in $V[g]$.

We say $u = (\eta, \delta, \delta', \lambda)$ is a *good quadruple* if (η, δ, λ) and (η, δ', λ) are good triples with $\delta < \delta'$ (see Section 2). Suppose $u = (\eta, \delta, \delta', \lambda)$ is a good quadruple and h is a $V[g]$-generic such that $g * h$ is generic for a poset in V_η. Working in $V[g*h]$, let $D(h, \eta, \delta, \lambda)$ be the club of countable

$$X \prec ((W_\lambda[g*h], u)|\Psi^g_{\eta,\delta})$$

such that $H^V_\iota \cup \{g\} \subseteq X$.

Suppose $A \in \Gamma^\infty_{g*h}$. Then for a club of $X \in D(h, \eta, \delta, \lambda)$, A is Suslin, co-Suslin captured by $(M_X, \delta_X, \Lambda^{g*h}_X)$ and A is projective in $\Lambda_{X'}$ where $X' = X \cap W_\lambda$ (see Lemma 2.3). Given such an X, we say X *captures* A.

Let $k \subseteq \mathrm{Col}(\omega, \Gamma^\infty_{g*h})$ be generic, and let $(A_i : i < \omega), (w_i : i < \omega)$ be generic enumerations of Γ^∞_{g*h} and $\mathbb{R}_{g*h}$ respectively in $V[g*h*k]$. Let $(X_i : i < \omega) \in V[g*h*k]$ be such that for each i

1. $X_i \in D(h, \eta, \delta, \lambda)$, and
2. X_i captures A_i.

In particular, A_i is projective in $\Lambda_{X_i'}$, where $X_i' = X_i \cap W_\lambda$. We set $M^0_n = M_{X_n'}$, $\pi^0_n = \pi_{X_0}$, $\kappa_0 = \kappa_{X_0}$, $\nu_0 = \delta_{X_0}$, $\nu_0' = \delta'_{X_0}$, $\eta_0 = \eta_{X_0}$, $\delta_0 = \delta$, $\mathcal{P}_0 = \mathcal{P}$.

Next we inductively define sequences $(M^i_n : i, n < \omega)$, $(\pi^i_n : i, n < \omega)$, $(\Lambda_i : i \leq \omega)$, $(\tau^{i,i+1}_n : i, n < \omega)$, $(\nu_n : i < \omega)$, $(\nu_n' : i < \omega)$, $(\eta_n : n < \omega)$, $(\kappa_i : i < \omega)$, $(\theta_i : i < \omega)$, $(\mathcal{T}_i, E_i : i < \omega)$, $(M_i' : i < \omega)$, $(\mathcal{U}_i, F_i : i < \omega)$, $(\mathcal{P}_i : i \leq n)$, $(\mathcal{P}_i' : i < \omega)$, and $(\sigma_i : i < \omega)$ satisfying the following conditions (see Figure 5.1 of [ST21]).

(a) For all $i, n < \omega$, $\pi^i_n : M^i_n \to \mathcal{P}_i$ and $rng(\pi^i_n) \subseteq rng(\pi^i_{n+1})$.

(b) $\tau^{i,i+1}_n : M^i_n \to M^{i+1}_n$. Let $\tau_n : M^0_n \to M^n_n$ be the composition of $\tau^{j,j+1}_n$'s for $j < n$.

(c) For all $i, n < \omega$, $\kappa_n = \tau_n(\kappa_0)$, $\eta_n = \tau_n(\eta_0)$, $\nu_n = \tau_n(\nu_0)$ and $\nu_n' = \tau_n(\nu_0')$.

(d) For all $n < \omega$, $\mathcal{T}_n$ is an iteration of $M^n_n|\nu_n'$ above ν_n that makes w_n generic and M_n' is its last model.

(e) $\theta_n = \pi^{\mathcal{T}_n}(\nu_n')$ and $E_n \in \vec{E}^{M_n'}$ is such that $lh(E_n) > \theta_n$ and $\mathrm{cp}(E_n) = \kappa_n$.

(f) for all m, n, $M^{n+1}_m = Ult(M^n_m, E_n)$ and $\tau^{n,n+1}_m = \pi^{M^n_m}_{E_n}$.

(g) $\mathcal{U}_n = \pi^n_n \mathcal{T}_n$, $\mathcal{P}_n'$ is the last model of $\mathcal{U}_n$, $\sigma_n : M_n' \to \mathcal{P}_n'$ is the copy map and $F_n = \sigma_n(E_n)$.[4]

(h) $\mathcal{P}_{n+1} = Ult(\mathcal{P}_n, F_n)$ and $\psi^{n+1}_m : M^{n+1}_m \to \mathcal{P}_{n+1}$ is given by $\pi^{n+1}_m(\pi^{M^n_m}_{E_n}(f)(a)) = \pi^{\mathcal{P}_n}_{F_n}(\pi^n_m(f))(\sigma_n(a))$.

[4]So $\oplus_{i \leq n}\mathcal{T}_i$ and $\oplus_{i \leq n}\mathcal{U}_i$ are sealed iterations based on κ.

(i) $\Lambda_n = (\pi^n_n\text{-pullback of } (\Psi^{g*h}_\lambda)_{\mathcal{P}_n|\psi_n(\nu_n)})_{\eta_n,\nu_n} = (\sigma_n\text{-pullback of } (\Psi^{g*h}_\lambda)_{\mathcal{P}'_n|\sigma_n(\nu_n)})_{\eta_n,\nu_n}$ (see [ST21, Corollary 3.6]).

Let M^ω_n be the direct limit of $(M^m_n : m < \omega)$ under the maps $\tau^{m,m+1}_n$. Letting $\mathcal{P}_\omega$ be the direct limit of $(\mathcal{P}_n : n < \omega)$ and the compositions of $\pi^{\mathcal{P}_n}_{F_n}$, we have natural maps $\pi^\omega_n : M^\omega_n \to \mathcal{P}_\omega$. Notice that

(1) for each $n < \omega$, $\kappa_n < \omega_1^{V[g*h]}$ and $sup_n\kappa_n = \omega_1^{V[g*h]}$.

It follows that if $\tau^m_n : M^m_n \to M^\omega_n$ is the direct limit embedding then

(2) $\tau^m_n(\kappa_n) = \omega_1^{V[g*h]}$.

Next, notice that

(3) for each m, n, p, letting $\iota_n = \tau_n(\iota_{X_0}) = \tau_n(\iota)$, $M^n_m|\iota_n = M^n_p|\iota_n$ and $\iota_n = (\kappa_n^+)^{M^n_m}$.
(4) for each m, n, p, $\pi^n_m \restriction (M^n_m|\iota_n) = \pi^n_p \restriction (M^n_p|\iota_n)$
(5) for each m, $n > 1$ and $p > n$, $M^n_m|\theta_{n-1} = M^p_m|\theta_{n-1}$.
(6) for each m, $n > 1$ and p with $p > n$, $\pi^n_m \restriction (M^n_m|\theta_{n-1}) = \pi^p_m \restriction (M^p_m|\theta_{n-1})$.

Because of condition (d) above we can find $G \subseteq Coll(\omega, <\omega_1^{V[g*h]})$ generic over M^ω_n (for each $n < \omega$) such that $\mathbb{R}^{M^\omega_n[G]} = \mathbb{R}_{g*h}$ and $G \in V[g*h*k]$. By constructions, $\omega_1^{V[g*h]}$ is a limit of Woodin cardinals in M^ω_n. [ST21] shows that

Lemma 4.1. *For each $n < \omega$, $DM(G)^{M^\omega_n[G]} = L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$.*

Lemma 4.1 implies that clause 1 of Sealing and of Partial Tower Sealing holds. [ST21] uses Lemma 4.1 to also verify clause 2 of Sealing holds.

In the next section, we will use the above constructions to verify clause 2 of Partial Tower Sealing holds. We say that the sequence $(X_i : i < \omega)$ is cofinal in Γ^∞_{g*h} as witnessed by $(A_i : i \in \omega)$ and $(w_i : i < \omega)$. We also say that $(M^n_0, \Lambda_n, \theta_n, \tau_{n,m} : n < m < \omega)$ is a Γ^∞_{g*h}-genericity iteration induced by $(X_i : i < \omega)$ where $\tau_{n,m} : M^n_0 \to M^m_0$ is the composition of $\tau^{i,i+1}_0$ for $i \in [n, m)$.

5. Partial Tower Sealing

In this section, we use the results of the previous section to prove Theorem 1.3. We work in the universe of $\mathcal{P}$, which we call V. Let κ be the least strong cardinal. Let $g \subseteq \mathrm{Coll}(\omega, \kappa^+)$ be V-generic and let $\delta > \kappa$ be Woodin. We prove Partial Tower Sealing holds in $V[g]$ at δ.

Work in $V[g]$, let $G \subseteq \mathbb{Q}_{<\delta}$ be $V[g]$-generic and let $j_G : V[g] \to M \subseteq V[g, G]$ be the associated embedding. We want to find an embedding $j : L(\Gamma^\infty_{g*G}) \to L(j_G(\Gamma^\infty_g))$ such that $j \restriction \Gamma^\infty_{g*G}$ is the identity and furthermore, j is an order-preserving bijection on the class of indiscernibles of the models.

We note that the main result of [ST21] already shows Sealing holds in $V[g]$ at δ, therefore, there is an elementary embedding $i : L(\Gamma^\infty_g) \to L(\Gamma^\infty_{g*G})$ such that $i(A) = A_G$ for all $A \in \Gamma^\infty_g$. Furthermore, $(\Gamma^\infty_g)^\sharp, (\Gamma^\infty_{g*G})^\sharp$ exist and i is the order-preserving bijection on the class of indiscernibles of the models.

j_G induces an elementary embedding $k : L(\Gamma^\infty_g) \to L(j_G(\Gamma^\infty_g))$ such that $j_G(A) = A_G$ for all $A \in \Gamma^\infty_g$. If $\Gamma^\infty_{g*G} = j_G(\Gamma^\infty_g)$ then we simply let j be the identity. In general, let $W = L(\Gamma^\infty_{g*G})$, $W' = L(j_G(\Gamma^\infty_g))$, τ a term, $A \in \Gamma^\infty_{g*G}$, $x \in \mathbb{R}_{g*G}$, and s a finite sequence of indiscernibles for both W, W' such that $j_G(s) = s$ and $i(s) = s$, then we define

$$j(\tau^W(A, x, s)) = \tau^{W'}(A, x, s).$$

Since $(\Gamma^\infty_{g*G})^\sharp, (j_G(\Gamma^\infty_g))^\sharp$ exist, everything in W has the form $\tau^W(A, x, s)$ for some A, x, s (and similarly for W'), j is defined on all of W. We need to check that j is elementary.

Let $(\xi_i : i < \omega)$ be the first ω indiscernibles for both W, W' with the properties described above; we may assume that $s = (\xi_i : i < lh(s))$. Let $u = (\eta, \delta', \delta'', \lambda)$ be a good quadruple such that $\sup_{i<\omega} \xi_i < \eta$. Let $k \subseteq \mathrm{Col}(\omega, \Gamma^\infty_{g*G})$ be $V[g*G]$-generic and $k' \subseteq Coll(\omega, j_G(\Gamma^\infty_g))$ be M-generic. We may assume $k' \in V[g*G*k]$

We have that Γ^∞_{g*G} is the Wadge closure of strategies of the countable substructures of $V[g]_\lambda$. More precisely, given $A \in \Gamma^\infty_{g*G}$, there is an $X \prec (W_\lambda|\Psi^{g*G}_{\eta,\delta'})$ such that A is Wadge reducible to Λ_X and $\Lambda_X \in \Gamma^\infty_{g*G}$. It follows that to show that j is elementary it is enough to show that given a formula ϕ, $m \in \omega$, u_m being the first m common indiscernbiles of W, W' that are fixed points of all relevant embeddings, $X \prec ((V[g]_\lambda, u)|\Psi^{g*G}_{\eta,\delta'})$ and a real $x \in \mathbb{R}_{g*G}$,

$$\begin{tikzcd}
& & M_0^2 \arrow[r] & M_0^3 \arrow[r] & \cdots \arrow[r] & M_0^\omega \\
M_0^0 = N_0^0 \arrow[r] & M_0^1 = N_0^1 \arrow[ur] \arrow[dr] & & & & \\
& & N_0^2 \arrow[r] & N_0^3 \arrow[r] & \cdots \arrow[r] & N_0^\omega
\end{tikzcd}$$

Figure 1: Two genericity iterations: the derived mode of M_0^ω is $L(\Gamma^\infty_{g*G})$ and the derived model of N_0^ω is $L(j_G(\Gamma^\infty_g))$.

$$W \vDash \phi[u_m, \Lambda_X, x] \Rightarrow W' \vDash \phi[u_m, \Lambda_X, x].$$ [5]

Fix then a tuple (ϕ, n, X, x) as above.

Working inside $V[g*G*k]$, let $(Y_i : i < \omega)$ be a cofinal sequence in Γ^∞_{g*G} as witnessed by some $\vec{A}$ and $\vec{w}$ such that $A_0 = \emptyset$, $w_0 = x$ and $Y_0' = X$. Using k', we also construct $(Z_i : i < \omega)$, a cofinal sequence in $j_G(\Gamma^\infty_g)$ as witnessed by some $\vec{B}$ and $\vec{v}$ such that $B_0 = \emptyset$, $v_0 = x$ and $Z_0' = X$. Here the Z_i's are elementary substructures of $V^M_{j_G(\lambda)}$.

Let $(M_0^n, \Lambda_n, \theta_n, \tau_{n,l} : n < l < \omega)$ be a Γ^∞_{g*G}-genericity iteration induced by $(Y_i : i < \omega)$ and $(N_0^n, \Phi_n, \nu_n, \sigma_{n,l} : n < l < \omega)$ be a $j_G(\Gamma^\infty_g)$-genericity iteration induced by $(Z_i : i < \omega)$. It is not hard to see that we can make sure that $M_0^1 = N_0^1$ by simply selecting the same extender E_0 after $\mathcal{T}_0$; by our assumptions, $M_0^0 = N_0^0$ and $w_0 = v_0$. Note that this makes sense because if $X \prec V[g]_\lambda$, then $X \prec V^M_{j_G(\lambda)}$ since $V[g]_\lambda \prec V^M_{j_G(\lambda)}$. Furthermore, by elemenarity of j_G and the fact that $\omega_1^{V[G]} > \iota > \kappa$, $M = j_G(V)[g]$ and $j(V) \vDash$ "κ is strong". See Figure 1.

$L(\Gamma^{g*G}_\infty)$ is realized as the derived model of M_0^ω and $L(j_G(\Gamma^g_\infty))$ is realized as the derived model of N_0^ω, and the two iterations agree on the first extender used (namely E_0) and therefore $M_0^1 = N_0^1$.

Let $\zeta = \eta_X$ and $\Gamma = (\Psi_{\eta,\delta'})_X$. Let M_0^ω be the direct limit along $(M_0^n : n < \omega)$ and N_0^ω the direct limit along $(N_0^n : n < \omega)$. For $n < \omega$, let κ_n be the least strong cardinal of M_0^n and κ_n' be the least strong cardinal of N_0^n. Let s_m^n be the first m (cardinal) indiscernibles of $L[M_0^n|\kappa_n]$ and t_m^n be the first m (cardinal) indiscernibles of $L[N_0^n|\kappa_n']$. Notice that $(M_0^n|\kappa_n)^\# \in M_0^n$ and $(N_0^n|\kappa_n')^\# \in N_0^n$. It follows that $\tau_{n,l}(s_m^n) = s_m^l$ and $\sigma_{n,l}(t_m^n) = t_m^l$ for $n < l \leq \omega$. We may and do modify the s_m^n, t_m^n's so that $s_m^\omega = u_m$ and $t_m^\omega = u_m$ for each m.

We then have the following sequence of implications. Below we let Γ^* be the name for the generic extension of Γ in the relevant model and $\dot{DM}$ be the name for the derived model. The third implication below uses the fact that $M_0^1 = N_0^1$.

$$\begin{aligned}
W \vDash \phi[u_m, \Lambda_X, x] &\Rightarrow M_0^\omega[x] \vDash \emptyset \Vdash_{Coll(\omega,<\kappa_\omega)} \dot{DM} \vDash \phi[s_m^\omega, \Gamma^*, x] \\
&\Rightarrow M_0^1[x] = N_0^1[x] \vDash \emptyset \Vdash_{Coll(\omega,<\kappa_1)} \dot{DM} \vDash \phi[s_m^1, \Gamma^*, x] \\
&\Rightarrow N_0^\omega[x] \vDash \emptyset \Vdash_{Coll(\omega,<\kappa_\omega')} \dot{DM} \vDash \phi[t_m^\omega, \Gamma^*, x] \\
&\Rightarrow W' \vDash \phi[u_m, \Lambda_X, x].
\end{aligned}$$

The converse has the same proof. We therefore have proved the equivalence and the theorem.

6. Failure of Tower Sealing

We show in this section that in general, Tower Sealing fails in hod mice. We recall some terminology from [ST24]. The reader can consult [ST24, Definition 2.6] for the definition of an *excellent* hybrid premouse $\mathcal{P}$. Let $\mathcal{P}$ be such a premouse and $\mathcal{P}_0 \lhd \mathcal{P}$ be the unique lsa type hod premouse that is an initial segment of $\mathcal{P}$ from [ST24, Definition 2.6]. In particular, there is a Woodin cardinal δ_0 of $\mathcal{P}$ such that $\mathcal{P}_0 = (\mathcal{P}|\delta_0)^\sharp \lhd \mathcal{P}$ and

$$\mathcal{P}_0 \vDash \text{“}\exists\kappa < \delta_0\ \kappa \text{ is } < \delta_0\text{-strong and a limit of Woodin cardinals.”}$$

Let $\Lambda = S^{\mathcal{P}}$ be the short-tree strategy predicate for $\mathcal{P}_0$ defined in $\mathcal{P}$, then $\mathcal{P}$ is a Λ-premouse with a proper class of Woodin cardinals.[6] As shown in [ST24], Λ has canonical interpretations in all generic

[5] The $\Leftarrow$ is similar as will be evident by the following proof.

[6] In [ST24], we demand that $\mathcal{P}$'s class of measurable limit of Woodin cardinals is stationary.

extensions of $\mathcal{P}$; furthermore, let $g \subseteq Coll(\omega, \mathcal{P}_0)$ be $\mathcal{P}$-generic, then $\mathcal{P}[g] \vDash \mathsf{Sealing}$. In fact, Section 5 shows that

$$\mathcal{P}[g] \vDash \forall\delta \text{ if } \delta \text{ is Woodin, then Partial Tower Sealing holds at } \delta.$$

The next theorem shows that Tower Sealing at any cardinal δ cannot hold in $\mathcal{P}[g]$.

Theorem 6.1. *1. Suppose $(\mathcal{P}, \Sigma)$ is excellent. Let $\mathcal{P}_0, \Lambda$ be the associated lsa type hod premouse derived from $\mathcal{P}$ and Λ be the short-tree strategy of $\mathcal{P}_0$ defined in $\mathcal{P}$. Let $g \subseteq Coll(\omega, \mathcal{P}_0)$ be $\mathcal{P}$-generic. Then*

$$\mathcal{P}[g] \vDash \text{“}\forall\delta \ \textit{if } \delta \textit{ is Woodin, then Tower Sealing fails at } \delta.\text{”}$$

2. Suppose $(\mathcal{P}, \Psi)$ is the minimal lbr hod pair such that $\mathcal{P} \vDash$ “there is a strong cardinal and a proper class of Woodin cardinals”. Let κ be the least strong cardinal of $\mathcal{P}$ and $g \subset Coll(\omega, \kappa^+)$ be $\mathcal{P}$-generic. Then

$$\mathcal{P}[g] \vDash \text{“}\forall\delta \ \textit{if } \delta \textit{ is Woodin, then Tower Sealing fails at } \delta.\text{”}$$

To prove part (1), let δ be a Woodin cardinal of $\mathcal{P} > \delta_0$, equivalently, δ is a Woodin cardinal in $\mathcal{P}[g]$. Working in $\mathcal{P}[g]$, let $G \subseteq \mathbb{Q}_{<\delta}$ be $\mathcal{P}[g]$-generic and $j_G : \mathcal{P}[g] \to M \subseteq \mathcal{P}[g][G]$ be the associated embedding. We show $j_G(\Gamma^g_\infty) \neq \Gamma^{g*G}_\infty$. Suppose not. Letting $\Gamma = \Gamma^\infty_{g*G} = j_G(\Gamma^\infty_g)$ and Λ_G be the canonical interpretation of Λ in $\mathcal{P}[g]$, then

$$\mathrm{Lp}^{\Lambda_G,\Gamma}(\mathcal{P}_0) = \mathrm{Lp}^{j_G(\Lambda),\Gamma}(\mathcal{P}_0).$$

$\mathrm{Lp}^{\Lambda_G,\Gamma}(\mathcal{P}_0) \trianglelefteq \mathrm{Lp}^{j_G(\Lambda),\Gamma}(\mathcal{P}_0)$ since $\Lambda_G \subseteq j_G(\Lambda)$. The other direction holds by our assumption that $\Gamma^\infty_{g*G} = j_G(\Gamma^\infty_g)$; if $\mathcal{M} \lhd \mathrm{Lp}^{j_G(\Lambda),\Gamma}(\mathcal{P}_0)$, then letting $\Sigma_\mathcal{M}$ be the unique iteration strategy for $\mathcal{M}$, then $\Sigma_\mathcal{M} \in \Gamma = \Gamma(\mathcal{P}_0, \Lambda_G)$[7], so $\mathcal{M} \lhd \mathrm{Lp}^{\Lambda_G,\Gamma}(\mathcal{P}_0)$.

Now, let $\mathcal{M} \lhd j_G(\mathcal{P})$ be the least such that $\rho_\omega(\mathcal{M}) < \omega_1^{\mathcal{P}[g]}$ and $\Sigma_\mathcal{M}$ be the canonical strategy of $\mathcal{M}$ as a $j_G(\Lambda)$-mouse in M. Then since $\Sigma_\mathcal{M}$ is a total strategy, it must be in Γ. This means $\mathcal{M} \lhd \mathrm{Lp}^{j_G(\Lambda),\Gamma}(\mathcal{P}_0)$. $\Sigma_\mathcal{M}$ is universally Baire in $\mathcal{P}[g][G]$, so in $\mathcal{P}[g][G][h]$ where h is $\mathrm{Col}(\omega, \delta)$-generic over $\mathcal{P}[g][G]$, $\mathcal{M} \lhd \mathrm{Lp}^{\Lambda_{G*h}}(\mathcal{P}_0)$ and hence $\mathcal{M} \in \mathcal{P}[g]$ by homogeneity. However, $\mathcal{M}$ defines a surjection from some $\alpha < \omega_1^{\mathcal{P}[g]}$ onto $\omega_1^{\mathcal{P}[g]}$, $\mathcal{M}$ cannot be in $\mathcal{P}[g]$. This is a contradiction. Therefore, $\Gamma^\infty_{g*G} \neq j_G(\Gamma^\infty_g)$ as claimed.

Now we prove part (2). Let κ, g be as in the statement of the theorem. Fix a Woodin cardinal $\delta > \kappa$. Let $G \subseteq \mathbb{Q}_{<\delta}$ be $V[g]$-generic. Let $j_G : V[g] \to M \subseteq V[g][G]$ be the associated generic embedding. We will show that

$$j_G(\Gamma^g_\infty) \neq \Gamma^{g*G}_\infty. \tag{6.1}$$

Suppose (6.1) fails. We write V for the universe of $\mathcal{P}$. Recall that $\iota = \kappa^+$ and hence $\iota^+ = \omega_1^{V[g]}$. In $V[g][G]$, let $\mathcal{M}^G_\infty$ be the direct limit of all countable iterates of $\mathcal{P}|\delta$ via Ψ^{g*G} and let $i_G : \mathcal{P}|\delta \to \mathcal{M}^G_\infty$ be the direct limit embedding. We let $\mathcal{M}^M_\infty$ be the direct limit of all countable iterates of $j_G(\mathcal{P})|\delta$ via $j_G(\Psi^g)$ and $i_M : j_G(\mathcal{P})|\delta \to \mathcal{M}^M_\infty$ be the direct limit embedding. By our assumption, letting $\Gamma = j_G(\Gamma^g_\infty) = \Gamma^{g*G}_\infty$ and $\Theta = \Theta^\Gamma$, then by the general properties of the direct limit construction,

$$i_G(\kappa) = i_M(\kappa) = \Theta$$

and

$$\mathcal{M}^G_\infty|\Theta = \mathcal{M}^M_\infty|\Theta = \mathrm{HOD}^{L(\Gamma)}|\Theta.$$

In the following, we will write j for j_G. Let $(\mathcal{M}, \Phi) \lhd (j(P)|\delta, j(\Psi^g))$ be such that

- $(\iota^+)^V < o(\mathcal{M})$;
- $\rho_\omega(\mathcal{M}) < (\iota^+)^V$ and $\mathcal{M}$ is the minimal such level of M with this property;
- Φ is the canonical strategy of $\mathcal{M}$.

Let $\tau = (\iota^+)^V$. We note that $\mathcal{M}|\tau = \mathcal{P}|\tau$ and τ is a cardinal of $\mathcal{M}$.

Lemma 6.2. $\mathcal{M}^G_\infty|(\Theta)^{+,\mathcal{M}^G_\infty} = \mathcal{M}^M_\infty|(\Theta)^{+,\mathcal{M}^M_\infty}$.

[7]The equality $\Gamma = \Gamma(\mathcal{P}_0, \Lambda_G)$ follows from [ST24], here $\Gamma(\mathcal{P}_0, \Lambda_G)$ is the pointclass generated by $\mathcal{P}_0, \Lambda_G$. $\Gamma(\mathcal{P}_0, \Lambda_G)$ consists of $A \subseteq \mathbb{R}$ such that there is an embedding $i : P_0 \to \mathcal{Q}$ according to Λ_G such that $A <_w \Psi$ where $\Psi = (\Lambda_G)_{\mathcal{Q}|i(\kappa)}$.

Proof. Work in $V[g][G]$, let $\gamma >> \omega_1$ be a regular cardinal and $X \prec H_\gamma^{V[g][G]}$ be countable such that $\omega_1^{V[g]} < X \cap \delta \in \delta$. Let $\pi_X : M_X \to X$ be the uncollapse map and $\delta_X = \mathrm{crt}(\pi_X)$. Let $(\mathcal{Q}, \mathcal{R}) = \pi_X^{-1}(\mathcal{P}|\delta, j(\mathcal{P})|\delta)$. Note that $\mathcal{Q}, \mathcal{R} \vDash$ "κ is strong" and

$$\mathcal{Q}|\iota = \mathcal{R}|\iota = \mathcal{P}|\iota = j(\mathcal{P})|\iota.$$

Furthermore, let $G_X = \pi_X^{-1}(G)$, then $\mathcal{Q} = \mathcal{P}|\delta_X$ and $\mathcal{R} = j_{G_X}(\mathcal{P})|\delta_X$ where j_{G_X} is the generic ultrapower map induced by G_X. Since X is transitive below δ, we in fact get (by condensation, cf. [ST22, Theorem 4.6]) that $(\mathcal{Q}, \Psi_X) = (\mathcal{P}|\delta_X, \Psi_{\mathcal{Q}}^{g*G})$ and $(\mathcal{R}, \Lambda_X) = (j(\mathcal{P})|\delta_X, j(\Psi^g)_{\mathcal{R}})$, where Ψ_X is the π_X^{-1}-pullback of Ψ^{g*G} and Λ_X is the π_X^{-1}-pullback of $j_G(\Psi^g)$. Note also that $\delta_X > \kappa$ is an inaccessible cardinal in $\mathcal{P}$ and $j(\mathcal{P})$.

Let $\Theta_X = \pi_X^{-1}(\Theta)$ and $(\mathcal{Q}_\infty, \mathcal{R}_\infty) = \pi_X^{-1}(\mathcal{M}_\infty^G, \mathcal{M}_\infty^M)$. By elementarity, $\mathcal{Q}_\infty|\Theta_X = \mathcal{R}_\infty|\Theta_X$, $\mathcal{Q}_\infty$ is a Ψ_X-iterate of $\mathcal{Q}$, and $\mathcal{R}_\infty$ is a Λ_X-iterate of $\mathcal{R}$. Note that Ψ_X, Λ_X are fullness preserving in $L(\Gamma)$. $\mathcal{Q}_\infty|\Theta_X = \mathcal{R}_\infty|\Theta_X$, and by fullness preservation of Ψ_X, Λ_X[8] that

$$\mathcal{Q}_\infty|(\Theta_X)^{+,\mathcal{Q}_\infty} = \mathcal{R}_\infty|(\Theta_X)^{+,\mathcal{R}_\infty}.$$

This gives the lemma by elementarity. □

Let $\mathcal{M}_\infty = \mathcal{M}_\infty^G|(\Theta)^{+,\mathcal{M}_\infty^G} = \mathcal{M}_\infty^M|(\Theta)^{+,\mathcal{M}_\infty^M}$. Now let $(\mathcal{Q}, \Sigma) \lhd (\mathcal{Q}^+, \Sigma^+)$ where $(\mathcal{Q}^+, \Sigma^+) \in I(j(\mathcal{P}), j_G(\Psi^g))$[9] and $(\mathcal{R}, \Lambda) \lhd (\mathcal{R}^+, \Lambda^+) \in I(\mathcal{P}, \Psi^{g*G})$ with the following properties:

(i) $\mathcal{M}_\infty(\mathcal{Q}, \Sigma) \lhd \mathcal{M}_\infty$ and $\mathcal{M}_\infty(\mathcal{R}, \Lambda) \lhd \mathcal{M}_\infty$.

(ii) $\mathcal{M}_\infty(\mathcal{Q}, \Sigma) \trianglelefteq \mathcal{M}_\infty(\mathcal{R}, \Lambda)$.

(iii) $\pi_{\mathcal{Q}^+,\infty}^{\Sigma^+} \restriction \iota_\mathcal{Q} = \pi_{\mathcal{R}^+,\infty}^{\Lambda^+} \restriction \iota_\mathcal{R}$, where $\iota_\mathcal{Q} = \pi_{j(\mathcal{P})|\delta,\mathcal{Q}^+}^{j_G(\Psi^g)}(\iota)$ and $\iota_\mathcal{R} = \pi_{\mathcal{P}|\delta,\mathcal{R}^+}^{\Psi^{g*G}}(\iota)$.

(iv) $o(\mathcal{R})$ is a cardinal of $\mathcal{R}^+$ and therefore is a limit of indices of extenders on the sequence of $\mathcal{R}$ with critical point $\kappa_\mathcal{R}$, where $\kappa_\mathcal{R} = \pi_{\mathcal{P}|\delta,\mathcal{R}^+}^{\Psi^{g*G}}(\kappa)$, and $o(\mathcal{M}_\infty(\mathcal{R}, \Lambda))$ is a limit cardinal of $\mathcal{M}_\infty$.

(v) $o(\mathcal{Q})$ is a cardinal of $\mathcal{Q}^+$ and therefore is a limit of indices of extenders on the sequence of $\mathcal{R}$ with critical point $\kappa_\mathcal{Q}$, where $\kappa_\mathcal{Q} = \pi_{j(\mathcal{P})|\delta,\mathcal{Q}^+}^{j_G(\Psi^g)}(\kappa)$, and $o(\mathcal{M}_\infty(\mathcal{Q}, \Sigma))$ is a limit cardinal of $\mathcal{M}_\infty$.

We note that such pairs can easily be constructed by the general properties of the direct limit systems. Item (ii) follows from the assumption that 6.1 fails. Item (iii) follows from the proof of Lemma 6.2; indeed, using the notation as there, we can let $\mathcal{R}^+$ be $\mathrm{Ult}(\mathcal{P}|\delta, E)$ where E is the (long) extender of length Θ_X derived from the iteration map $\Pi_{\mathcal{P}|\delta_X,\infty}^{\Psi_X}$ computed in $M_X[G_X]$ and $\mathcal{Q}^+ = \mathrm{Ult}(j(\mathcal{P})|\delta_X, F)$ where F is the (long) extender of length Θ_X derived from the iteration map $\Pi_{j_{G_X}(\mathcal{P})|\delta_X|\delta_X,\infty}^{\Lambda_X}$ computed in $M_X[G_X]$, then

$$\pi_{\mathcal{Q}^+,\infty}^{\Sigma^+} \restriction \iota_\mathcal{Q} = \pi_{\mathcal{R}^+,\infty}^{\Lambda^+} \restriction \iota_\mathcal{R} = \pi_X \restriction (\Theta_X)^{+,\mathcal{Q}_\infty}.$$

In the following, whenever $(Q, \Psi') \in I(\mathcal{P}|\delta, \Psi^{g*G})$, we write κ_Q, ι_Q etc. for the images of κ, ι etc. under the iteration embedding.

Let $\mathcal{M}_Q = \mathrm{Ult}_0(\mathcal{M}, K)$ where K is the extender derived from $\pi_{j(\mathcal{P})|\delta,\mathcal{Q}^+}^{j_G(\Psi^g)} \restriction \iota$. Then we have, by standard fine-structural computations, that

(vi) $\rho_1(\mathcal{M}_\mathcal{Q}) < \tau_\mathcal{Q}^*$ where $\tau_\mathcal{Q}^* = \pi_K^M(\tau) = \mathrm{sup}\pi_K^M \restriction \tau$.

(vii) $\mathcal{M}_\mathcal{Q} \lhd \mathcal{Q}^+$.

Lemma 6.3. $\pi_{\mathcal{P}|\delta,\mathcal{R}^+}^{\Psi^{g*G}} \restriction \iota = \pi_{j(\mathcal{P})|\delta,\mathcal{Q}^+}^{j_G(\Psi^g)} \restriction \iota$ *and* $\tau_\mathcal{Q}^* = \pi_{\mathcal{P}|\delta,\mathcal{R}^+}^{\Psi^{g*G}}(\tau)$.

Proof. To see the second clause, first note that the first clause implies that K is the extender derived from $\pi_{\mathcal{P}|\delta,\mathcal{R}^+}^{\Psi^{g*G}} \restriction \iota$; furthermore, $\pi_{\mathcal{P}|\delta,\mathcal{R}^+}^{\Psi^{g*G}}$ is continuous at τ because τ is a successor cardinal in $\mathcal{P}$ and hence by (vi), $\tau_\mathcal{Q}^* = \pi_{\mathcal{P}|\delta,\mathcal{R}^+}^{\Psi^{g*G}}(\tau)$.

[8] This is because $\delta_X > \kappa$ is an inaccessible cardinal of $\mathcal{P}$ and $j(\mathcal{P})$. By condensation, $\Psi_X = \Psi_\mathcal{Q}^{g*G}$, and since Ψ^{g*G} is fullness preserving, Ψ_X is as well. A similar argument applies to Λ_X.

[9] For a hod pair $(\mathcal{S}, \Upsilon)$, the set $I(\mathcal{S}, \Upsilon)$ denotes the collection of non-dropping iterates $(\mathcal{S}', \Upsilon')$ of $(\mathcal{S}, \Upsilon)$.

To see the first clause, we use the minimality assumption on our mouse $\mathcal{P}$. We note that

$$\pi^{\Psi^{g*G}}_{\mathcal{P},\mathcal{R}^+} \restriction \kappa = \pi^{j_G(\Psi^g)}_{j(\mathcal{P}),\mathcal{Q}^+} \restriction \kappa.$$

This is because $\Psi^{g*G}_{\mathcal{P}|\kappa} = j(\Psi^g)_{j(\mathcal{P})|\kappa}$. We write σ for this map. Let $T^{\mathcal{P}}_n$ be the theory of the first n-indiscernibles for $\mathcal{P}$. For any $A \subseteq \kappa$, let τ be a term such that $A = \tau^{\mathcal{P}}[T^{\mathcal{P}}_n, s]$ for some $s \in [\kappa]^{<\omega}$. Then $\pi^{\Psi^{g*G}}_{\mathcal{P},\mathcal{R}^+}(A) = \tau^{\mathcal{R}^+}[T^{\mathcal{R}^+}_n, \sigma(s)]$. Similarly, $\pi^{j(\Psi^g)}_{j(\mathcal{P}),\mathcal{Q}^+}(A) = \tau^{\mathcal{Q}^+}[T^{\mathcal{Q}^+}_n, \sigma(s)]$. By the minimality assumption, $T^{\mathcal{R}^+}_n = T^{\mathcal{Q}^+}_n$ for all n. This means

$$\pi^{\Psi^{g*G}}_{\mathcal{P}|\delta,\mathcal{R}^+}(A) = \pi^{j(\Psi^g)}_{j(\mathcal{P})|\delta,\mathcal{Q}^+}(A)$$

as desired.

□

By the choice of $\mathcal{Q}$, (vi), (vii), and Lemma 6.3, we easily get that

(viii) $\mathcal{M}_{\mathcal{Q}} \lhd \mathcal{Q}$.

(ix) $\mathcal{Q}|\tau^*_{\mathcal{Q}} = \mathcal{R}|\tau^*_{\mathcal{Q}}$.

Let $\mathcal{T}$ be the normal tree on $\mathcal{Q}$ according to Σ with last model $\mathcal{M}_\infty(\mathcal{Q},\Sigma)$ and $\mathcal{U}$ the normal tree on $\mathcal{R}$ according to Λ with last model $\mathcal{M}_\infty(\mathcal{R},\Lambda)$.

Lemma 6.4. *Suppose $\alpha < lh(\mathcal{T})$ is on the main branch of $\mathcal{T}$ and $\beta < lh(\mathcal{U})$ is on the main branch of $\mathcal{U}$. Suppose*

- $\mathcal{M}^{\mathcal{T}}_\alpha|\pi^{\mathcal{T}}_{0,\alpha}(\kappa_{\mathcal{Q}}) = \mathcal{M}^{\mathcal{U}}_\beta|\pi^{\mathcal{U}}_{0,\beta}(\kappa_{\mathcal{R}})$,
- *the generators of $[0,\alpha]_T$ are contained in $\pi^{\mathcal{T}}_{0,\alpha}(\kappa_{\mathcal{Q}})$,*
- *the generators of $[0,\beta]_U$ are contained in $\pi^{\mathcal{U}}_{0,\beta}(\kappa_{\mathcal{R}})$.*

Then letting $\kappa^* = \pi^{\mathcal{T}}_{0,\alpha}(\kappa_{\mathcal{Q}}) = \pi^{\mathcal{U}}_{0,\beta}(\kappa_{\mathcal{R}})$, α^*+1 be the successor α on the main branch of $\mathcal{T}$ (if one exists) and β^*+1 the successor of β on the main branch of $\mathcal{U}$ (if one exists), then if $E^{\mathcal{T}}_{\alpha^*}$ has critical point κ^*, then $E^{\mathcal{U}}_{\beta^*}$ also has critical point κ^* and $E^{\mathcal{T}}_{\alpha^*} = E^{\mathcal{U}}_{\beta^*}$.

Proof. We show this by induction on the branches $[0,\alpha]_T, [0,\beta]_U$. We assume the lemma holds for pairs (α',β') where $\alpha' \in [0,\alpha)_T$, $\beta' \in [0,\beta)_U$. We first make a couple of simple observations.

First, suppose α^*+1 exists, so $E^{\mathcal{T}}_{\alpha^*}$ is defined. Suppose $\mathrm{crt}(E^{\mathcal{T}}_{\alpha^*}) = \kappa^*$. Then $E^{\mathcal{U}}_{\beta^*}$ is defined and $\mathrm{crt}(E^{\mathcal{U}}_{\beta^*}) = \kappa^*$. This is easily seen to be true as otherwise, $\kappa_{\mathcal{M}_\infty(\mathcal{R},\Lambda)} < \kappa_{\mathcal{M}_\infty(\mathcal{Q},\Sigma)}$, but $\mathcal{M}_\infty(\mathcal{Q},\Sigma) \unlhd \mathcal{M}_\infty(\mathcal{R},\Lambda)$. This implies $\mathcal{M}_\infty(\mathcal{Q},\Sigma)$ has more than one strong cardinal. Contradiction.

Now observe that if $\kappa^* < \pi^{\Lambda}_{\mathcal{R},\infty}(\kappa_{\mathcal{R}})$, then $E^{\mathcal{U}}_{\beta^*}$ exists and $\mathrm{crt}(E^{\mathcal{U}}_{\beta^*}) = \kappa^*$. This is because if $\mathrm{crt}(E^{\mathcal{U}}_{\beta^*}) > \kappa^*$, then future extenders used along the main branch of $\mathcal{U}$ must have critical points $> \kappa^*$, but this means κ^* is a strong cardinal of $\mathcal{M}_\infty(\mathcal{R},\Lambda)$ since α is on the main branch of $\mathcal{U}$. Since $\pi^{\Lambda}_{\mathcal{R},\infty}(\kappa_{\mathcal{R}})$ is a strong cardinal of $\mathcal{M}_\infty(\mathcal{R},\Lambda) > \kappa^*$, this means $\mathcal{M}_\infty(\mathcal{R},\Lambda)$ has more than one strong cardinal. Contradiction. A similar statement holds for the $\mathcal{T}$-side, namely if $\kappa^* < \pi^{\Sigma}_{\mathcal{Q},\infty}(\kappa_{\mathcal{Q}})$, then $E^{\mathcal{T}}_{\alpha^*}$ exists and $\mathrm{crt}(E^{\mathcal{T}}_{\alpha^*}) = \kappa^*$

The two observations above easily imply most of the conclusions of the lemma except for the last equality. So we assume that $E^{\mathcal{T}}_{\alpha^*}, E^{\mathcal{U}}_{\beta^*}$ both exist and have critical point κ^*. By the initial segment condition, we know that $\mathrm{lh}(E^{\mathcal{T}}_{\alpha^*}) = \mathrm{lh}(E^{\mathcal{U}}_{\beta^*}) =_{def} \xi$. Furthermore, $(\mathcal{M}^{\mathcal{T}}_{\alpha^*}||\xi, \Sigma_{\mathcal{M}^{\mathcal{T}}_{\alpha^*}||\xi}) = (\mathcal{M}^{\mathcal{U}}_{\beta^*}||\xi, \Lambda_{\mathcal{M}^{\mathcal{U}}_{\beta^*}||\xi})$. Letting $\iota^* = \pi^{\mathcal{T}}_{0,\alpha}(\iota_{\mathcal{Q}}) = \pi^{\mathcal{U}}_{0,\beta}(\iota_{\mathcal{R}})$, we have

$$\pi^{\Sigma^+}_{\mathcal{M}^{\mathcal{T},+}_\alpha,\infty} \restriction \iota^* = \pi^{\Lambda,+}_{\mathcal{M}^{\mathcal{U},+}_\beta,\infty} \restriction \iota^*. \tag{6.2}$$

This follows from (iii) and by our induction hypothesis which implies that $\pi^{\mathcal{T}}_{0,\alpha} \restriction \iota_{\mathcal{Q}} = \pi^{\mathcal{U}}_{0,\beta} \restriction \iota_{\mathcal{R}}$. To see (6.2), let $\xi < \iota^*$, so $\xi = \pi^{\mathcal{T}}_{0,\alpha}(f)(a) = \pi^{\mathcal{U}}_{0,\beta}(f)(a)$ for $f \in \mathcal{Q}|\iota_{\mathcal{Q}}$ and $a \in [\kappa^*]^{<\omega}$. So

$$\begin{aligned}
\pi^{\Sigma^+}_{\mathcal{M}^{\mathcal{T},+}_\alpha,\infty}(\xi) &= \pi^{\Sigma^+}_{\mathcal{Q}^+,\infty}(\pi^{\mathcal{T}}_{0,\alpha}(f))(a) \\
&= \pi^{\Lambda^+}_{\mathcal{R}^+,\infty}(\pi^{\mathcal{U}}_{0,\beta}(f))(a) \\
&= \pi^{\Lambda^+}_{\mathcal{M}^{\mathcal{U},+}_\beta,\infty}(\xi).
\end{aligned}$$

In fact, we get that $\pi^{\Sigma^+}_{\mathcal{M}^{\mathcal{T},+}_{\alpha},\infty}$ and $\pi^{\Lambda,+}_{\mathcal{M}^{\mathcal{U},+}_{\beta},\infty}$ agree on all the elements of the H_{ι^*} of the models.

Now we can show the equality of the two extenders by the following calculations: let $a \in [\lambda(E^{\mathcal{T}}_{\alpha^*})]^{<\omega}$ and $A \subseteq [\kappa^*]^{|a|}$,

$$\begin{aligned}(a, A) \in E^{\mathcal{T}}_{\alpha^*} &\Leftrightarrow a \in \pi_{E^{\mathcal{T}}_{\alpha^*}}(A)\\ &\Leftrightarrow a \in \pi^{\Sigma^+}_{\mathcal{M}^{\mathcal{T},+}_{\alpha},\infty}(A)\\ &\Leftrightarrow a \in \pi^{\Lambda^+}_{\mathcal{M}^{\mathcal{U},+}_{\beta},\infty}(A)\\ &\Leftrightarrow (a, A) \in E^{\mathcal{U}}_{\beta^*}.\end{aligned}$$

The second equivalence follows from the following facts:

- $\pi^{\Sigma}_{\alpha,\infty}(A) = \pi^{\Sigma}_{\alpha^*+1,\infty} \circ \pi_{E^{\mathcal{T}*}_{\alpha}}(A)$.
- By the general properties of direct limits, there is a factor map $\sigma : \mathcal{M}_\infty(\mathcal{Q}, \Sigma) \to \mathcal{M}_\infty(\mathcal{Q}^+, \Sigma^+)$ such that $\mathrm{crt}(\sigma) = \kappa_{\mathcal{M}_\infty(\mathcal{Q},\Sigma)}$.
- $\pi^{\Sigma}_{\alpha^*+1,\infty}(a) = a$.

Combining the above facts, we see that $a \in \pi_{E^{\mathcal{T}}_{\alpha^*}}(A)$ is equivalent to $a \in \sigma \circ \pi^{\Sigma}_{\alpha^*+1,\infty} \circ \pi_{E^{\mathcal{T}*}_{\alpha}}(A) = \pi^{\Sigma^+}_{\mathcal{M}^{\mathcal{T},+}_{\alpha},\infty}(A)$. This gives the second equivalence. The third equivalence follows from (6.2) and the remark after. The last equivalence is proved just like the second equivalence. □

Now we have two cases:
Case 1: $\pi^{\mathcal{T}}_{0,\infty}(\kappa_{\mathcal{Q}}) = \pi^{\mathcal{U}}_{0,\infty}(\kappa_{\mathcal{R}})$.

Let this ordinal be γ. Then γ is the strong cardinal of $\mathcal{M}_\infty(\mathcal{R}, \Lambda)$ and of $\mathcal{M}_\infty(\mathcal{Q}, \Sigma)$. Furthermore, $\pi^{\mathcal{M}_{\mathcal{Q}}}_F(\mathcal{M}_{\mathcal{Q}}) \lhd \mathcal{M}_\infty(\mathcal{R}, \Lambda)$ where F is the extender derived from $\pi^{\mathcal{T}}_{0,\infty}$ and $\pi^{\mathcal{M}_{\mathcal{Q}}}_F(\mathcal{M}_{\mathcal{Q}})$ is the 0-ultrapower embedding derived from F on $\mathcal{M}_{\mathcal{Q}}$. Since $\rho_1(\mathcal{M}_{\mathcal{Q}}) < \tau^*_{\mathcal{Q}}$, by elementarity, $\rho_1(\pi^{\mathcal{M}_{\mathcal{Q}}}_F(\mathcal{M}_{\mathcal{Q}})) < \pi^{\mathcal{T}}_{0,\infty}(\tau^*_{\mathcal{Q}})$. On the other hand, since τ is a cardinal of $\mathcal{P}$, $\tau^*_{\mathcal{Q}}$ is a (successor) cardinal of $\mathcal{R}$ and is a continuity point of $\pi^{\mathcal{U}}_{0,\infty}$. This means $\pi^{\mathcal{U}}_{0,\infty}(\tau^*_{\mathcal{Q}}) = \pi^{\mathcal{M}_{\mathcal{Q}}}_F(\tau^*_{\mathcal{Q}})$ (by Lemma 6.4) is a cardinal of $\mathcal{M}_\infty(\mathcal{R}, \infty)$, but $\pi^{\mathcal{M}_{\mathcal{Q}}}_F(\mathcal{M}_{\mathcal{Q}})$ witnesses $\pi^{\mathcal{U}}_{0,\infty}(\tau^*_{\mathcal{Q}})$ is not a cardinal of $\mathcal{M}_\infty(\mathcal{R}, \infty)$. We have a contradiction.
Case 2: $\pi^{\mathcal{T}}_{0,\infty}(\kappa_{\mathcal{Q}}) < \pi^{\mathcal{U}}_{0,\infty}(\kappa_{\mathcal{R}})$.

Let α be the least in $\mathcal{U}$ such that the strong cardinal of $\mathcal{M}^{\mathcal{U}}_{\alpha} = \pi^{\mathcal{T}}_{0,\infty}(\kappa_{\mathcal{Q}})$. It's easy to see such an α exists and in fact α in on the main branch of $\mathcal{U}$ (see the analysis in Lemma 6.4). Now we have that letting E be the extender on the main branch of $\mathcal{U}$ that is applied to $\mathcal{M}^{\mathcal{U}}_{\alpha}$, then $\mathrm{lh}(E) \geq o(\pi^{\mathcal{M}_{\mathcal{Q}}}_F(\mathcal{M}_{\mathcal{Q}}))$. This is because $i_E(\pi^{\mathcal{T}}_{0,\infty}(\kappa_{\mathcal{Q}}))$ is an inaccessible cardinal of $\mathcal{M}_\infty(\mathcal{R}, \Lambda)$ and by our case hypothesis, $\pi^{\mathcal{M}_{\mathcal{Q}}}_F(\mathcal{M}_{\mathcal{Q}}) \lhd \mathcal{M}_\infty(\mathcal{Q}, \Sigma)$, so by the agreement between models, it is easy to see that $\pi^{\mathcal{M}_{\mathcal{Q}}}_F(\mathcal{M}_{\mathcal{Q}}) \lhd \mathcal{M}^{\mathcal{U}}_{\alpha}$. But this leads to a contradiction as in Case 1 because $\pi^{\mathcal{U}}_{0,\alpha}(\tau^*_{\mathcal{Q}})$ is not a cardinal of $\mathcal{M}^{\mathcal{U}}_{\alpha}$.

This completes the proof of Theorem 6.1. One weakness of the above proof is it seems very hard to generalize Lemma 6.2 to obtain the agreements between the two direct limits at a strong cardinal of those limits above Θ. Therefore, one may hope to prove Tower Sealing holds in a generic extension of a hod mouse with two (or more) isolated strong cardinals. We show that this is not the case.

Theorem 6.5 (AD^+). *Suppose $(\mathcal{P}, \Psi)$ is an lbr hod pair such that $\mathcal{P} \vDash$ "there is a proper class of Woodin cardinals and there are finitely many strong cardinals". Let κ be the largest strong cardinal of $\mathcal{P}$, and let $g \subseteq Coll(\omega, \kappa^+)$. Suppose there is no subcompact cardinal in $\mathcal{P}$. Then in $\mathcal{P}[g]$, Tower Sealing fails at every Woodin cardinal.*

Proof. Suppose there are n strong cardinals in $\mathcal{P}$ with κ being the largest one. Let $\delta > \kappa$ be a Woodin cardinal. We show Tower Sealing fails at δ in $\mathcal{P}[g]$. Let $\mathcal{Q} = \mathcal{P}|\delta^{+,\mathcal{P}}$ and fix an $X \prec \mathcal{Q}$ with $|X| = \kappa^+$ and $X \cap \kappa^{++} \in \kappa^{++}$. Let $\pi : M_X \to X$ be the uncollapse map with $\mathrm{crt}(\pi) = \gamma_X$. We may choose X so that γ_X does not index an extender on the $\mathcal{Q}$-sequence and $\mathrm{cof}(\gamma_X) > \kappa$; there is a κ^+-club of such X because there are no subcompact cardinals in $\mathcal{P}$. As in [ST22; Ste22], we coiterate $\mathcal{Q}$ and the phalanx $(\mathcal{Q}, M_X, \gamma_X)$ into a common hod pair construction.

More precisely, let $\Sigma = \Psi_{\mathcal{Q}}$ and write M for M_X. Fix a coarse strategy pair $((N^*, \in, w, \mathcal{F}, \Psi^*), \Psi^{**})$, in the sense of [Ste22], that captures Σ, and let $\mathbb{C}$ be the maximal $(w, \mathcal{F})$ construction, with models $M_{\nu,l}$ and induced strategies $\Omega_{\nu,l}$. Let $\delta^* = \delta(w)$. By [ST22, Theorem 3.26], $(*)(M, \Sigma)$ holds, so we can fix $\langle \eta_0, k_0\rangle$ lex least such that $(\mathcal{Q}, \Sigma)$ iterates to $(M_{\eta_0,k_0}, \Omega_{\eta_0,k_0})$, and for all $(\nu, l) <_{\mathrm{lex}} (\eta_0, k_0)$, $(\mathcal{Q}, \Sigma)$

iterates strictly past $(M_{\nu,l},\Omega_{\nu,l})$. Let $\mathcal{U}_{\nu,l}$ be the unique normal tree on $\mathcal{Q}$ witnessing $(\mathcal{Q},\Sigma)$ iterates past $(M_{\mu,l},\Omega_{\nu,l})$.[10]

To make the main points transparent and simplify certain arguments, we assume $n=2$. We write $\kappa_0^{\mathcal{Q}}<\kappa_1^{\mathcal{Q}}$ for the strong cardinals of $\mathcal{Q}$ and for any non-dropping iterate $\mathcal{R}$ of $\mathcal{Q}$, we write $\kappa_0^{\mathcal{R}},\kappa_1^{\mathcal{R}}$ for the strong cardinals of $\mathcal{R}$. Similarly, we denote κ_0^M,κ_1^M for the strong cardinals of M.

We define trees $\mathcal{S}_{\nu,l}$ on $(\mathcal{Q},M,\gamma_X)$ for certain $(\nu,l)\le(\eta_0,k_0)$. Fix $(\nu,l)\le(\eta_0,k_0)$ for now, and assume $\mathcal{S}_{\nu',l'}$ is defined whenever $(\nu',l')<(\nu,l)$. Let $\mathcal{U}=\mathcal{U}_{\nu,l}$, and for $\tau<\mathrm{lh}(\mathcal{U})$, let

$$\Sigma_\tau^{\mathcal{U}}=\Sigma_{\mathcal{U}\restriction(\tau+1)}$$

be the tail strategy for $\mathcal{M}_\tau^{\mathcal{U}}$ induced by Σ. We proceed to define $\mathcal{S}=\mathcal{S}_{\nu,l}$, by comparing the phalanx $(\mathcal{Q},M,\gamma_X)$ (using strategy (Σ,Σ^{π_X})) with $M_{\nu,l}$. As we define $\mathcal{S}$, we lift $\mathcal{S}$ to a padded tree $\mathcal{T}$ on $\mathcal{Q}$, by copying. Let us write

$$\Sigma_\theta^{\mathcal{T}}=\Sigma_{\mathcal{T}\restriction(\theta+1)}$$

for the tail strategy for $\mathcal{M}_\theta^{\mathcal{T}}$ induced by Σ.

We let $\mathcal{Q}=\mathcal{M}_0^{\mathcal{S}}$, $M=\mathcal{M}_1^{\mathcal{S}}$. For $\theta<lh(\mathcal{S})$, we will have copy map π_θ from $\mathcal{M}_\theta^{\mathcal{S}}$ into $\mathcal{M}_\theta^{\mathcal{T}}$. The map π_θ is a nearly elementary.[11] We attach the complete strategy

$$\Lambda_\theta=(\Sigma_\theta^{\mathcal{T}})^{\pi_\theta}$$

to $\mathcal{M}_\theta^{\mathcal{S}}$. We also define a non-decreasing sequence of ordinals $\lambda_\theta=\lambda_\theta^{\mathcal{S}}$ that measure agreement between models of $\mathcal{S}$, and tell us which model we should apply the next extender to.

We start with

$$\mathcal{M}_0^{\mathcal{S}}=\mathcal{Q},\mathcal{M}_1^{\mathcal{S}}=M,\gamma_0=\gamma_X,$$

and

$$\mathcal{M}_0^{\mathcal{T}}=\mathcal{M}_1^{\mathcal{T}}=\mathcal{Q},\pi_0=id,\pi_1=\pi_X,\sigma_0=\pi_X,$$

and

$$\Lambda_0=\Sigma,\ \Lambda_1=\Sigma^{\pi_1}.$$

We say that $0,1$ are distinct roots of $\mathcal{S}$. We say that 0 is unstable, and 1 is stable. As we proceed, we shall declare additional nodes θ of $\mathcal{S}$ to be unstable. We do so because $(\mathcal{M}_\theta^{\mathcal{S}},\Lambda_\theta)=(\mathcal{M}_\gamma^{\mathcal{U}},\Sigma_\gamma^{\mathcal{U}})$[12] for some γ, and when we do so, we shall immediately define $\mathcal{M}_{\theta+1}^{\mathcal{S}}$, as well as σ_θ and γ_θ. Here $\Lambda_{\theta+1}=\Lambda_\theta^{\sigma_\theta}$. In this case, $[0,\theta]_S$ does not drop, and all $\xi\le_S\theta$ are also unstable. We regard $\theta+1$ as a new root of $\mathcal{S}$. This is the only way new roots are constructed.

If θ is unstable, then we define

$$\gamma_\theta=i_{0,\theta}^{\mathcal{S}}(\gamma_0).$$

The construction of $\mathcal{S}$ takes place in rounds in which we either add one stable θ, or one unstable θ and its stable successor $\theta+1$. Thus the current last model is always stable, and all extenders used in $\mathcal{S}$ are taken from stable models. If γ is stable, then $\lambda_\gamma=\lambda(E_\gamma^{\mathcal{S}})$.

For $\theta<lh(\mathcal{S})$, let $\pi_\theta:\mathcal{M}_\theta^{\mathcal{S}}\to\mathcal{M}_\theta^{\mathcal{T}}$ be the copy map. We are maintaining by induction that the last node γ of our current $\mathcal{S}$ is stable, and

Induction hypotheses $(\dagger)_\gamma$. If $\theta<\gamma$ and θ is unstable, then

(1) $0\le_{\mathcal{S}}\theta$ and $[0,\theta]_{\mathcal{S}}$ does not drop (in model or degree), and every $\xi\le_S\theta$ is unstable,

(2) there is a γ such that $(\mathcal{M}_\theta^{\mathcal{S}},\Lambda_\theta)=(\mathcal{M}_\gamma^{\mathcal{U}},\Sigma_\gamma^{\mathcal{U}})$,

(3) $\mathcal{M}_{\theta+1}^{\mathcal{T}}=\mathcal{M}_\theta^{\mathcal{T}}$, and $\pi_{\theta+1}=\pi_\theta\circ\sigma_\theta:\mathcal{M}_{\theta+1}^{\mathcal{S}}\to\mathcal{M}_{\theta+1}^{\mathcal{T}}=\mathcal{M}_\theta^{\mathcal{T}}$.

(4) γ_θ does not index an extender on the $\mathcal{M}_\theta^{\mathcal{S}}$-sequence.

[10] We note that since $k(\mathcal{Q})=0$, $\mathcal{Q}$ is strongly stable in the sense of [Ste22]. The possibility that $(\mathcal{Q},\Sigma)$ iterates to some type 2 pair generated by $(M_{\eta_0,k_0},\Omega_{\eta_0,k_0})$ doesn't occur here.

[11] See [ST22, Section 2.3] for a summary of the types of elementary maps between mouse pairs.

[12] The external strategy agreement does not seem important to require for θ to be declared unstable. We should be able to declare θ unstable when only the models agree.

Setting $\sigma_0 = \pi$, we have $(\dagger)_1$.

For a node γ of $\mathcal{S}$, we write $\mathcal{S}$-pred(γ) for the immediate $\leq_{\mathcal{S}}$-predecessor of $\mathcal{S}$. For γ a node in $\mathcal{S}$, we set

$$\mathrm{st}(\gamma) = \text{the least stable } \theta \text{ such that } \theta \leq_{\mathcal{S}} \gamma,$$

and

$$\mathrm{st}^*(\gamma) = \begin{cases} \mathrm{st}(\gamma) & : \text{ if } \mathrm{st}(\gamma) = \theta + 1 \text{ for some unstable } \theta \\ \text{undefined} & : \text{ otherwise.} \end{cases}$$

The construction of $\mathcal{S}$ ends when we reach a stable θ such that

(I) $M_{\nu,l} \lhd \mathcal{M}^{\mathcal{S}}_{\theta}$, or $M^{\mathcal{S}}_{\theta} = M_{\nu,l}$ and $\mathcal{S}$ drops, or

(II) $\mathcal{M}^{\mathcal{S}}_{\theta} \unlhd M_{\nu,l}$, and $[\mathrm{rt}(\theta), \theta]_{\mathcal{S}}$ does not drop in model or degree.

If case (I) occurs, then we go on to define $\mathcal{S}_{\nu,l+1}$. If case (II) occurs, we stop the construction.

We now describe how to extend $\mathcal{S}$ one more step. First we assume $\mathcal{S}$ has successor length $\gamma + 1$ and let $\mathcal{M}^{\mathcal{S}}_{\gamma}$ be the current last model, so that γ is stable. Suppose $(\dagger)_\gamma$ holds. Suppose (I), (II) above do not hold for γ, so that we have a least disagreement between $\mathcal{M}^{\mathcal{S}}_{\gamma}$ and $M_{\nu,l}$. Suppose the least disagreement involves only an extender E on the sequence of $\mathcal{M}^{\mathcal{S}}_{\gamma}$.[13] Letting $\tau = \mathrm{lh}(E)$, we have

- $M_{\nu,l}|(\tau, 0) = \mathcal{M}^{\mathcal{S}}_{\gamma}|(\tau, -1)$,[14] and
- $(\Omega_{\nu,l})_{(\tau,0)} = (\Lambda_\gamma)_{(\tau,-1)}$.

We now describe how to extend $\mathcal{S}$ one more step. We set $E^{\mathcal{S}}_{\gamma} = E^+$ and $\lambda^{\mathcal{S}}_{\gamma} = \lambda_E$.[15] Let ξ be the least such that $\mathrm{crt}(E) < \lambda^{\mathcal{S}}_{\xi}$. We let $\mathcal{S}$-pred$(\gamma + 1) = \xi$. Let (β, k) be lex least such that either $\rho(\mathcal{M}^{\mathcal{S}}_{\xi}|(\beta, k)) \leq \mathrm{crt}(E)$ or $(\beta, k) = (\hat{o}(\mathcal{M}^{\mathcal{S}}_{\xi}), k(\mathcal{M}^{\mathcal{S}}_{\xi}))$. Set

$$\mathcal{M}^{\mathcal{S}}_{\gamma+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_{\xi}|(\beta, k), E^+),$$

and let $\hat{i}^{\mathcal{S}}_{\xi,\gamma+1}$ be the canonical embedding. Let

$$\mathcal{M}^{\mathcal{T}}_{\gamma+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_{\xi}|(\pi_\xi(\beta), k), \pi_\gamma(E)^+),$$

and let $\pi_{\gamma+1}$ be given by the Shift Lemma. This determines $\Lambda_{\gamma+1}$.

If ξ is stable or $(\beta, k) < (\hat{o}(\mathcal{M}^{\mathcal{S}}_{\xi}), k(\mathcal{M}^{\mathcal{S}}_{\xi}))$, then we declare $\gamma+1$ to be stable. $(\dagger)_{\gamma+1}$ follows vacuously from $(\dagger)_\gamma$.

If ξ is unstable and E^+ is not used in $\mathcal{U}$, then again we declare $\gamma + 1$ stable. Again, $(\dagger)_{\gamma+1}$ follows vacuously from $(\dagger)_\gamma$.

Finally, suppose ξ is unstable and E^+ is used in $\mathcal{U}$, say $E^+ = E^{\mathcal{U}}_{\mu}$. Let τ be such that

$$e^{\mathcal{S}}_{\xi} = e^{\mathcal{U}}_{\tau},$$

where $e^{\mathcal{S}}_{\xi}$ is the sequence of extenders used on the branch $[0, \xi]_S$ and similarly for $e^{\mathcal{U}}_{\tau}$. So in particular,

$$(\mathcal{M}^{\mathcal{S}}_{\xi}, \Lambda_\xi) = (\mathcal{M}^{\mathcal{U}}_{\tau}, \Sigma^{\mathcal{U}}_{\tau}).$$

We have that

$$e^{\mathcal{S}}_{\gamma+1} = e^{\mathcal{S}}_{\xi}{}^\frown\langle E^+\rangle = e^{\mathcal{U}}_{\tau}{}^\frown\langle E^+\rangle = e^{\mathcal{U}}_{\mu+1}.$$

[Ste25] shows that $\tau = U - pred(\mu + 1)$. We then we declare $\gamma + 1$ to be unstable and $\gamma + 2$ stable. We must define the tuple needed for $(\dagger)_{\gamma+2}$. Let $i = i^{\mathcal{S}}_{\xi,\gamma+1}$, and

$$\langle N, G, \sigma, \gamma^*\rangle = \langle \mathcal{M}^{\mathcal{S}}_{\xi}, \mathcal{M}^{\mathcal{S}}_{\xi+1}, \sigma_\xi, \gamma_\xi\rangle.$$

We let

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[13] Later, we will prove that this is the case.

[14] Recall $\mathcal{M}^{\mathcal{S}}_{\gamma}|(\tau, -1)$ is the structure obtained from $\mathcal{M}^{\mathcal{S}}_{\gamma}|\tau$ by removing E. Sometimes, we will write M^- for $M|(o(M), -1)$.

[15] This is the notation used in [Ste22], for an extender E on the M-sequence, E^+ is the extender with generators $\lambda_E \cup \{\lambda_E\}$ that represents $i^{\mathrm{Ult}(M,E)}_F \circ i^M_E$, where F is the order zero total measure on λ_E in $\mathrm{Ult}(M, E)$. We also write $\hat{\lambda}(E^+) = \lambda_E$, $\mathrm{lh}(E^+) = \mathrm{lh}(E)$. E^+ is the plus-type extender derived from E.

$$\gamma_{\gamma+1} = i(\gamma^*) = i^{\mathcal{S}}_{0,\gamma+1}(\gamma_0).$$

Now we define $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ and $\sigma_{\gamma+1}$. Note that by our assumption that $\mathrm{cof}(\gamma_X) > \kappa$ and the fact that $\mathrm{crt}(E) \leq \kappa^{\mathcal{M}^{\mathcal{U}}_\tau}$, where $\kappa^{\mathcal{M}^{\mathcal{U}}_\tau}$ is the largest strong cardinal of $\mathcal{M}^{\mathcal{U}}_\tau$,

$$\gamma_{\gamma+1} = \sup i[\gamma^*].$$

Let $\mathcal{M}^{\mathcal{S}}_{\gamma+2} = \mathrm{Ult}(G, E^+)$, $i^{\mathcal{S}}_{\xi+1,\gamma+2}$ be the ultrapower map, and $\sigma_{\gamma+1} : \mathcal{M}^{\mathcal{S}}_{\gamma+2} \to \mathcal{M}^{\mathcal{S}}_{\gamma+1}$ be the copy map and $\pi_{\gamma+1} = \pi_\gamma \circ \sigma_{\gamma+1}$.

If there is a least disagreement between $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ and $M_{\nu,l}$, it has to involve an extender F from the sequence of $M^{\mathcal{S}}_{\gamma+2}$ (by [Ste22, Lemma 5.64]). If no such F exists, we leave $\lambda^{\mathcal{S}}_{\gamma+1}, \lambda^{\mathcal{S}}_{\gamma+2}$ undefined. Otherwise, let

$$\lambda^{\mathcal{S}}_{\gamma+2} = \lambda(F)$$

and

$$\lambda^{\mathcal{S}}_{\gamma+1} = \min(\lambda^{\mathcal{S}}_{\gamma+2}, \gamma_{\gamma+1}).$$

The $\lambda^{\mathcal{S}}_\xi$'s tell us what model should an extender used in $\mathcal{S}$ be applied to.

Claim 6.6. *$(\dagger)_{\gamma+1}$ holds.*

Proof. (1)–(3) are clear. (4) also follows because $\gamma_{\gamma+1} = i^{\mathcal{S}}_{0,\gamma+1}(\gamma_0)$. By the fact that γ_0 does not index an extender on the $\mathcal{M}^{\mathcal{S}}_0$-sequence and elementarity, $\gamma_{\gamma+1}$ does not index an extender on the $\mathcal{M}^{\mathcal{S}}_{\gamma+1}$-sequence. □

If (I) or (II) holds at $\gamma+2$, then the construction of $\mathcal{S}$ is over. Otherwise, we let $E^{\mathcal{S}}_{\gamma+2}$ be the least disagreement between $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ and $M_{\nu,l}$, and we set

$$\lambda^{\mathcal{S}}_{\gamma+1} = \inf(\gamma_{\gamma+1}, \lambda(E^{\mathcal{S}}_{\gamma+2})).$$

This completes the successor step in the construction of $\mathcal{S}$.

Now suppose we are given $\mathcal{S} \restriction \theta$, where θ is a limit ordinal. Let $b = \Sigma(\mathcal{T} \restriction \theta)$.

Case 1. There is a largest $\eta \in b$ such that η is unstable.

Fix η. There are two subcases.

(A) for all $\gamma \in b - (\eta+1)$, $\mathrm{rt}(\gamma) = \eta + 1$. In this case, $b - (\eta+1)$ is a branch of $\mathcal{S}$. Let $\mathcal{S}$ choose this branch,

$$[\eta+1, \theta)_{\mathcal{S}} = b - (\eta+1),$$

and let $\mathcal{M}^{\mathcal{S}}_\theta$ be the direct limit of the $\mathcal{M}^{\mathcal{S}}_\gamma$ for sufficiently large $\gamma \in b - (\eta+1)$. We define the branch embedding $i^{\mathcal{S}}_{\gamma,\theta}$ a usual and $\pi_\theta : \mathcal{M}^{\mathcal{S}}_\theta \to \mathcal{M}^{\mathcal{T}}_\theta$ is given by the fact that the copy maps commute with the branch embeddings. We declare θ to be stable.

(B) for all $\gamma \in b - (\eta+1)$, $\mathrm{rt}(\gamma) = \eta$. Let $\mathcal{S}$ choose

$$[0, \theta)_{\mathcal{S}} = (b - \eta) \cup [0, \eta]_{\mathcal{S}},$$

and let $\mathcal{M}^{\mathcal{S}}_\theta$ be the direct limit of the $\mathcal{M}^{\mathcal{S}}_\gamma$ for sufficiently large $\gamma \in b$. Branch embeddings $i^{\mathcal{S}}_{\gamma,\theta}$ for $\gamma \geq \eta$ are defined as usual. $\pi_\theta : \mathcal{M}^{\mathcal{S}}_\theta \to \mathcal{M}^{\mathcal{T}}_\theta$ is given by the fact that copy maps commute with branch embeddings. We declare θ to be stable.

Since θ is stable, $(\dagger)_\theta$ follows at once from $\forall \gamma < \theta \; (\dagger)_\gamma$.

Case 2. There are boundedly many unstable ordinals in b but no largest one.

We let η be the sup of the unstable ordinals in b. Let $\mathcal{S}$ choose

$$[0, \theta)_{\mathcal{S}} = (b - \eta) \cup [0, \eta]_{\mathcal{S}},$$

and define the corresponding objects as in case 1(B). We declare θ stable, and again $(\dagger)_\theta$ is immediate.

Case 3. There are arbitrarily large unstable ordinals in b. In this case, b is a disjoint union of pairs $\{\gamma, \gamma+1\}$ such that γ is unstable and $\gamma+1$ is stable. We set

$$[0,\theta)_{\mathcal{S}} = \{\xi \in b \mid \xi \text{ is unstable}\},$$

and let $\mathcal{M}^{\mathcal{S}}_\theta$ be the direct limit of the $\mathcal{M}^{\mathcal{S}}_\xi$'s for $\xi \in b$ unstable. There is no dropping in model or degree along $[0,\theta)_{\mathcal{S}}$. We define maps $i^{\mathcal{S}}_{\xi,\theta}, \pi_\theta$ as usual. If $(\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta)$ is not a pair of the form $(\mathcal{M}^{\mathcal{U}}_\tau, \Sigma^{\mathcal{U}}_\tau)$, then we declare θ stable and $(\dagger)_\theta$ is immediate.

Suppose that $(\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta)$ is a pair of $\mathcal{U}$. We declare θ unstable. We set

$$\gamma_\theta = i^{\mathcal{S}}_{0,\theta}(\gamma_0)$$

and

$$\mathcal{M}^{\mathcal{S}}_{\theta+1} = \text{ the direct limit of } i^{\mathcal{S}}_{\gamma+1,\gamma'+1}(\mathcal{M}^{\mathcal{S}}_{\gamma+1}), \text{ for } \gamma <_S \gamma' <_S \theta.$$

We also let

$$\sigma_\theta = \text{ common value of } i^{\mathcal{S}}_{\gamma,\theta}(\sigma_\gamma), \text{ for } \gamma <_S \theta \text{ sufficiently large.}^{16}$$

It is easy then to see that $\Phi_\theta = \langle \mathcal{M}^{\mathcal{S}}_\theta, \mathcal{M}^{\mathcal{S}}_{\theta+1}, \sigma_\theta, \gamma_\theta\rangle$ witnesses $(\dagger)_\theta$ holds.

If (I) holds, then we stop the construction of $\mathcal{S} = \mathcal{S}_{\nu,l}$ and move on to $\mathcal{S}_{\nu,l+1}$. If (II) holds, we stop the construction of $\mathcal{S}$ and do not move on. If neither holds, we let $E^{\mathcal{S}}_{\theta+1}$ be the extender on the $\mathcal{M}^{\mathcal{S}}_{\theta+1}$ sequence that represents its first disagreement with $M_{\nu,l}$, and set

$$\lambda^{\mathcal{S}}_{\theta+1} = \lambda(E^{\mathcal{S}}_{\theta+1}),$$

$$\lambda^{\mathcal{S}}_\theta = \inf(\lambda_{\theta+1}, \gamma_\theta).$$

It then is routine to verify $(\dagger)_{\theta+1}$.

This finishes our construction of $\mathcal{S} = \mathcal{S}_{\nu,l}$ and $\mathcal{T}$. Note that every extender used in $\mathcal{S}$ is taken from a stable node and every stable node, except the last model of $\mathcal{S}$ contributes exactly one extender to $\mathcal{S}$. The last model of $\mathcal{S}$ is stable.

Remark 6.7. It is possible in general that ξ is unstable, $\mathcal{S}$-pred$(\gamma+1) = \xi$, and $\mathrm{crt}(E^{\mathcal{S}}_\gamma) = \lambda_F$ where F is the last extender of $\mathcal{M}^{\mathcal{S}}_\xi|\gamma_\xi$. In this case, $(\beta, k) = (\mathrm{lh}(F), 0)$. The problem then is that $\mathcal{M}^{\mathcal{S}}_{\gamma+1}$ is not an lpm, because its last extender $i_{\xi,\gamma+1}(F)$ has a missing whole initial segment, namely F. This is the *JSZ anomaly.*

In the situations of least-disagreement comparisons, when a JSZ extender occurs, the Schindler-Zeman solution is to just continue comparing anyway. Suppose a JSZ extender, which fails the Jensen ISC, has the form $E^{\mathcal{S}}_{\eta+1} = L \circ E$ is used on the $\mathcal{U}$-side. Say our phalanx is (M, Q, γ_0) where $E^{\mathcal{S}}_\eta = L$ with $\mathrm{crt}(L) = \lambda_E$ and $E = E^M_{\gamma_0}$, then $E \notin M^{\mathcal{S}}_{\eta+1} = \mathrm{Ult}(M|\gamma_0, L)$ and $\mathcal{M}^{\mathcal{S}}_{\eta+2} = \mathrm{Ult}(M, E^{\mathcal{S}}_{\eta+1}) = \mathrm{Ult}(M, L\circ U)$. Note that the Jensen ISC holds everywhere on $\mathcal{U}$, then the main branch of $\mathcal{U}$ uses first E and then L. So L is used on both $\mathcal{S}$ and $\mathcal{U}$; this cannot happen in a comparison by least-extender disagreement. Since our comparison is against a common background construction, the SZ solution does not seem to work here; some care must be taken when JSZ anomalies occur. The JSZ anomaly affects how we lift our problematic phalanx and forces us to modify the rules of $\mathcal{S}_{\nu,l}$ to enable to prove the comparison terminates and other aspects of the comparison. See [Ste25] for how to handle the JSZ anomalies in the context of comparison against a background construction.

The JSZ anomaly does not occur in the comparison we are describing in this paper. The reason is that we chose X so that γ_X does not index an extender on the $\mathcal{Q}$-sequence and this fact propagates to the lifted phalanxes.

Claim 6.8. *For some $(\nu, l) \leq (\eta_0, k_0)$, the construction of $\mathcal{S}_{\nu,l}$ stops for reason (II).*

Proof. This is similar to the proof of [Ste22, Lemma 9.6.2].

□

Fix $(\nu, l) \leq (\eta_0, k_0)$ such that the construction of $\mathcal{S} = \mathcal{S}_{\nu,l}$ terminates at a stable θ such that for some γ, $\mathcal{M}^{\mathcal{S}}_\theta \trianglelefteq \mathcal{M}^{\mathcal{U}_{\nu,l}}_\gamma$. Let $\mathcal{S} = \mathcal{S}_{\nu,l}$, $\mathcal{U} = \mathcal{U}_{\nu,l}$, and let γ be the least such that $\mathcal{M}^{\mathcal{S}}_\theta \trianglelefteq \mathcal{M}^{\mathcal{U}}_\gamma$. We have $\mathrm{lh}(\mathcal{S}) = \theta + 1$, and $[\mathrm{rt}(\theta), \theta]_{\mathcal{S}}$ does not drop in model or degree.

Claim 6.9. *For some unstable ξ, $rt(\theta) = \xi + 1$.*

Proof. Suppose the claim is false. Then

[16] We abuse the notation a bit here when we write $i^{\mathcal{S}}_{\gamma,\theta}(\sigma_\gamma)$ as σ_γ is technically not an element of $\mathcal{M}^{\mathcal{S}}_\gamma$, but the meaning of σ_θ should be clear.

$$
\begin{tikzcd}
\mathcal{M}^{\mathcal{U}}_{\gamma} & & \mathcal{M}^{\mathcal{S}}_{\theta} & \\
S \arrow[u, "i^{\mathcal{U}}_{\tau,\gamma}"] \arrow[r, phantom, "\lhd"] & \mathcal{M}^{\mathcal{U}}_{\tau} & \mathcal{M}^{\mathcal{S}}_{\xi+1} \arrow[u, "i^{\mathcal{S}}_{\xi+1,\theta}"] \arrow[r, "\sigma_\xi"] & \mathcal{M}^{\mathcal{S}}_{\xi} \\
\mathcal{Q} \arrow[ur, "i^{\mathcal{U}}_{0,\tau}"] & & M \arrow[u] \arrow[r, "\sigma_0"] & \mathcal{Q} \arrow[u, "i^{\mathcal{S}}_{0,\xi}"]
\end{tikzcd}
$$

Figure 2: Diagram of the comparison argument: $\mathcal{M}^{\mathcal{S}}_\theta \trianglelefteq \mathcal{M}^{\mathcal{U}}_\gamma$, $\mathcal{M}^{\mathcal{S}}_\xi = \mathcal{M}^{\mathcal{U}}_\tau$, $\mathrm{crt}(i^{\mathcal{S}}_{\xi+1,\theta}) > \gamma_\xi = \mathrm{crt}(\sigma_\xi)$, and the embedding $i^{\mathcal{U}}_{\tau,\gamma}$ acts on $S \lhd \mathcal{M}^{\mathcal{U}}_\tau$ where S the the collapsing structure of γ_ξ.

(i) either $\mathrm{rt}(\theta) = \xi \geq 0$ for ξ unstable,

(ii) or $\mathrm{rt}(\theta) = \xi$ for ξ stable and is a limit of unstable $\xi' < \xi$,

In either case, the embeddings $i^{\mathcal{S}}_{0,\xi}$ and $i^{\mathcal{S}}_{\xi,\theta}$ exist, so $i^{\mathcal{S}}_{0,\theta}$ exists. Let $P = \mathcal{M}^{\mathcal{S}}_\theta$ and $R = \mathcal{M}^{\mathcal{U}}_\gamma$. Let $i = i^{\mathcal{S}}_{0,\theta}$, $j = i^{\mathcal{U}}_{0,\gamma}$, $\sigma : P \to P^* = \mathcal{M}^{\mathcal{T}}_\theta$ be the copy map, and $i^* : M \to P^*$ be the branch embedding of $[0,\theta]_T$ if these maps are defined.

By Dodd-Jensen, $P = R$, i, j, i^* are defined, and $i = j$. Let H be the first extender used along $[0,\theta]_S$ and K be the first extender used along $[0,\gamma]_U$. Since $i = j$, K, H are compatible.

Since we can recover branch extenders from branch embeddings, we have

$$e^{\mathcal{S}}_\theta = e^{\mathcal{U}}_\gamma.$$

Let $\eta \leq_S \theta$ be the least stable. Then $e^{\mathcal{S}}_\eta = e^{\mathcal{S}}_\theta \restriction \delta = e^{\mathcal{U}}_\gamma \restriction \delta$ for some δ.[17] There is a $\tau \leq_U \gamma$ such that $e^{\mathcal{U}}_\tau = e^{\mathcal{U}}_\gamma \restriction \delta = e^{\mathcal{S}}_\eta$. So $\mathcal{M}^{\mathcal{S}}_\eta = \mathcal{M}^{\mathcal{U}}_\tau$. By pullback consistency,[18] we easily get that $\Lambda_\eta = \Sigma^{\mathcal{U}}_\tau$. If η is a limit ordinal, then by the rule of $\mathcal{S}$, we declare η unstable, contradicting our assumption. So let $S - pred(\eta) = \mu$; then μ is unstable by the minimality of η. But then we declare η unstable by the rule of $\mathcal{S}$ at successor stages. Again this is a contradiction. □

Let ξ be as in Claim 6.9, and τ be such that $(\mathcal{M}^{\mathcal{S}}_\xi, \Lambda_\xi) = (\mathcal{M}^{\mathcal{U}}_\tau, \Sigma^{\mathcal{U}}_\tau)$. We have that

- $(\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta) \trianglelefteq (M_{\nu,l}, \Omega_{\nu,l}) \trianglelefteq (\mathcal{M}^{\mathcal{U}}_\gamma, \Sigma^{\mathcal{U}}_\gamma)$ for some $(\nu, l) \leq (\eta_0, k_0)$.
- $[\xi + 1, \theta]_S$ does not drop in model or degree.
- The tuple $(\mathcal{M}^{\mathcal{S}}_\xi, \mathcal{M}^{\mathcal{S}}_{\xi+1}, \sigma_\xi, \gamma_\xi)$ witnesses $(\dagger)_\xi$.

Let $P = \mathcal{M}^{\mathcal{S}}_\theta$ and $R = \mathcal{M}^{\mathcal{U}}_\gamma$. See Figure 2 for the relevant diagram of the comparison.

Claim 6.10. *(i) $\tau \leq \eta < \gamma$ implies $lh(E^{\mathcal{U}}_\eta) \geq \gamma_\xi$ and if $\eta < \tau$ then $\lambda(E^{\mathcal{U}}_\eta) \leq \gamma_\xi$.*

(ii) $P \lhd R$.

Proof. The first clause of part (i) follows from the agreement between $P, M_{\nu,l}, R, \mathcal{M}^{\mathcal{S}}_\xi = \mathcal{M}^{\mathcal{U}}_\tau$, more precisely, these models agree up to γ_ξ, therefore, for any $\tau \leq \eta < \gamma$, $\mathrm{lh}(E^{\mathcal{U}}_\eta) \geq \gamma_\xi$. For the second clause, note that by construction, $e^{\mathcal{U}}_\tau = e^{\mathcal{S}}_\xi$[19] and $\mathrm{lh}(e^{\mathcal{U}}_\tau) = \mathrm{lh}(e^{\mathcal{S}}_\xi) \leq \gamma_\xi$; the fact that $\mathrm{lh}(e^{\mathcal{S}}_\xi) \leq \gamma_\xi$ follows from the rules of lifting phalanx, all extenders used in $e^{\mathcal{S}}_\xi$ have critical point less than γ_ξ and therefore, their length has to be $\leq \gamma_\xi$. If $\eta < \tau$ then $\lambda(E^{\mathcal{U}}_\eta) \leq \mathrm{lh}(E^{\mathcal{U}}_\eta) \leq \mathrm{lh}(e^{\mathcal{U}}_\tau) \leq \gamma_\xi$ as desired.

[17] If $\mathrm{rt}(\theta) = 0$ then $e^{\mathcal{S}}_\eta$ consists of a single extender H and $\delta = 1$. This is a special case and is simpler.

[18] Pullback consistency follows from other properties of mouse pairs specified in [Ste22].

[19] $e^{\mathcal{U}}_\tau$ is the extender sequence used along $[0,\tau]_U$ and similarly $e^{\mathcal{S}}_\xi$ is the extender sequence used along $[0,\xi]_S$.

To see part (ii), suppose $P = R$. First note that the branch $[0, \gamma]_U$ cannot drop because P is a ZFC^--model while if $[0, \gamma]_U$ drops then R is not. So $[0, \gamma]_U$ has no drop. We may assume $\gamma > \tau$ as otherwise, $P = R = \mathcal{M}^{\mathcal{U}}_\tau$ but $\mathcal{M}^{\mathcal{U}}_\tau$ has a collapsing level for γ_ξ while γ_ξ is a cardinal in P. Contradiction. So $\gamma > \tau$. Let $\tau' \in [0, \gamma]_U$ be the least element of this branch $\geq \tau$. It is easy to see that $\mathcal{M}^{\mathcal{U}}_\tau$ and $\mathcal{M}^{\mathcal{U}}_{\tau'}$ agree up to their common γ^+_ξ and the branch embedding $i^{\mathcal{U}}_{\tau',\gamma}$ has critical point $> \gamma_\xi$. But this is also a contradiction because $\mathcal{M}^{\mathcal{U}}_\tau$, hence $\mathcal{M}^{\mathcal{U}}_{\tau'}$ and R, has a collapsing level for γ_ξ while γ_ξ is a cardinal in P. □

Claim 6.11. *$\tau \in [0, \gamma]_U$ and letting $\epsilon + 1 \in [\tau, \gamma]_U$ be the U-successor of τ on the branch $[0, \gamma]_U$, $crt(E^{\mathcal{U}}_\epsilon) > \gamma_\xi$.*

Proof. Let $\alpha <_U \beta + 1 \in [0, \gamma]_U$ be such that $\alpha \leq \tau$, $\beta + 1 > \tau$, and α is the U-predecessor of $\beta + 1$. Suppose $\mathrm{crt}(E^{\mathcal{U}}_\beta) < \gamma_\xi$, then we claim that

Subclaim. $\mathrm{crt}(E^{\mathcal{U}}_\beta)$ must be a strong cardinal in $\mathcal{M}^{\mathcal{U}}_\alpha$ and $i^{\mathcal{U}}_{\alpha,\beta+1}$ exists.

Proof. First note that by an easy induction on $\beta \geq \tau, crt(\mathrm{E}^{\mathcal{U}}_\beta)$ must be a strong cardinal in $\mathcal{M}^{\mathcal{U}}_\tau$. Second, note that since $\alpha \leq \tau$, for $i \in \{0, 1\}$, $\kappa_i^{\mathcal{M}^{\mathcal{U}}_\alpha} \leq \kappa_i^{\mathcal{M}^{\mathcal{U}}_\tau}$. [20]

So let us assume $\kappa = \mathrm{crt}(E^{\mathcal{U}}_\beta) = \kappa_i^{\mathcal{M}^{\mathcal{U}}_\tau}$ for some i and $\kappa_i^{\mathcal{M}^{\mathcal{U}}_\alpha} < \kappa_i^{\mathcal{M}^{\mathcal{U}}_\tau}$. Since $\mathcal{M}^{\mathcal{U}}_{\beta+1} | \gamma_\xi = P | \gamma_\xi$, in P, $\kappa_i^{\mathcal{M}^{\mathcal{U}}_\alpha}$ is strong to $\kappa_i^{\mathcal{M}^{\mathcal{U}}_\tau}$; since $\kappa_i^{\mathcal{M}^{\mathcal{U}}_\tau}$ is strong in P, $\kappa_i^{\mathcal{M}^{\mathcal{U}}_\alpha}$ is strong in P as well. We get that

$$P \vDash \text{" there are } n+1 \text{ strong cardinals."}$$

This is a contradiction. So $\kappa_i^{\mathcal{M}^{\mathcal{U}}_\alpha} = \kappa_i^{\mathcal{M}^{\mathcal{U}}_\tau}$ as desired. Furthermore, this easily gives that $\mathcal{M}^{\mathcal{U}}_\alpha$ and $\mathcal{M}^{\mathcal{U}}_\beta$ agrees up to the successor cardinal of $\kappa^{\mathcal{M}^{\mathcal{U}}_\alpha}$. This gives the second clause and completes the proof of the subclaim.

□

Using the subclaim, let $\kappa = \mathrm{crt}(E^{\mathcal{U}}_\beta) = \kappa_i^{\mathcal{M}^{\mathcal{U}}_\alpha}$ for some i, and $\lambda = i^{\mathcal{U}}_{\alpha,\beta+1}(\kappa)$. Then since $\mathrm{lh}(E^{\mathcal{U}}_\beta) > \gamma_\xi$, $\lambda > \gamma_\xi$ and λ is a strong cardinal in $\mathcal{M}^{\mathcal{U}}_{\beta+1}$. Furthermore, $\lambda < \mathrm{lh}(E^{\mathcal{U}}_\beta) \leq o(P)$ and $\mathcal{M}^{\mathcal{U}}_{\beta+1} || \lambda = P || \lambda$. Since $i^{\mathcal{S}}_{\xi+1,\theta}$ is above γ_ξ, the strong cardinals of P are below γ_ξ. In particular,

$$\mathcal{M}^{\mathcal{U}}_{\beta+1} || \lambda = P || \lambda \vDash \text{" there are } n \text{ strong cardinals."}$$

But then since λ is strong in $\mathcal{M}^{\mathcal{U}}_{\beta+1}$,

$$\mathcal{M}^{\mathcal{U}}_{\beta+1} \vDash \text{" there are } n+1 \text{ strong cardinals".}$$

This is a contradiction.

We have shown that letting $\alpha, \beta + 1$ be as above, then $\mathrm{crt}(E^{\mathcal{U}}_\beta) \geq \gamma_\xi$, in fact it is easy to see that $\mathrm{crt}(E^{\mathcal{U}}_\beta) > \gamma_\xi$. If $\tau \notin [0, \gamma]_U$, then $\alpha < \tau$ and hence $\lambda(E^{\mathcal{U}}_\alpha) \leq \gamma_\xi$, but then $\mathrm{crt}(E^{\mathcal{U}}_\beta) < \lambda(E^{\mathcal{U}}_\alpha) \leq \gamma_\xi$. Contradiction. This shows $\tau \in [0, \gamma]_U$ and completes the proof of the claim.

□

Claim 6.11 and the argument in Claim 6.10 imply that the branch $[\tau, \gamma]_U$ must drop, in fact, letting $S \lhd \mathcal{M}^{\mathcal{U}}_\tau$ be the collapsing structure for γ_ξ, $\mathcal{M}^{\mathcal{U}}_{\epsilon+1} = \mathrm{Ult}(S, E^{\mathcal{U}}_\epsilon)$. In other words, the branch $[\tau, \gamma]_U$ is based on S. Let $i : S \to R$ be the iteration embedding and $j : \mathcal{M}^{\mathcal{S}}_{\xi+1} \to P$ be the iteration embedding $i^{\mathcal{S}}_{\xi+1,\theta}$. We have that by pullback consistency, $(\Sigma^{\mathcal{U}}_\gamma)^i = (\Sigma^{\mathcal{U}}_\tau)_S$ and $\Lambda^j_\theta = \Lambda_{\xi+1}$. Claims 6.10 and 6.11 easily imply that $\Lambda_{\xi+1}$ is projective in $(\Sigma^{\mathcal{U}}_\tau)_S$. Similarly, letting $S^* \lhd \mathcal{Q}$ be the collapsing structure for γ_0, Λ_1 is projective in $(\Sigma^{\mathcal{U}}_0)_{S^*}$.

The above gives us the following: if $j : \mathcal{P}[g] \to M$ is a generic ultrapower induced by a generic $G \subset \mathbb{Q}_\delta$, then letting Ψ be the iteration strategy for the collapsing structure $Q \lhd M$ of $\omega_1^{\mathcal{P}[g]}$, every $A \in \Gamma_\infty^{\mathcal{P}[g][G]}$ is projective in Ψ. This means $\Gamma_\infty^{\mathcal{P}[g][G]} \subsetneq j(\Gamma_\infty^{\mathcal{P}[g]})$. Therefore, Tower Sealing fails in $\mathcal{P}[g]$. □

[20]Note is that it can't happen that $\mathrm{crt}(E^{\mathcal{U}}_\beta) = \kappa_0^{\mathcal{M}^{\mathcal{U}}_\tau}$ and $\kappa_0^{\mathcal{M}^{\mathcal{U}}_\tau} < \kappa_0^{\mathcal{M}^{\mathcal{U}}_\alpha} < \kappa_1^{\mathcal{M}^{\mathcal{U}}_\alpha} < \kappa_1^{\mathcal{M}^{\mathcal{U}}_\tau}$. This is because this means $\lambda(E^{\mathcal{U}}_{\alpha-1}) > \kappa_0^{\mathcal{M}^{\mathcal{U}}_\tau}$, therefore, since $\mathrm{crt}(E^{\mathcal{U}}_\beta) = \kappa_0^{\mathcal{M}^{\mathcal{U}}_\tau}$, the rule of $\mathcal{U}$ implies that $E^{\mathcal{U}}_\beta$ must be applied to model earlier than α. Contradiction.

Remark 6.12. The proof of the above theorem, particularly Claim 6.11 uses the assumption the set of strong cardinals in the model is finite. The proof can be generalized in a straightforward way to models in which the set of strong cardinals is discrete and has order smaller than the least measurable cardinal. It is not clear how to generalize this proof of hod mice with strong cardinals which reflect the class of strong cardinals.

7. Questions

Question 7.1.

- *Can Tower Sealing hold in a generic extension of a hod mouse?*
- *Is Tower Sealing consistent relative to* ZFC+ *"there is a Woodin limit of Woodin cardinals"?*

As mentioned in the previous section, it is plausible that some form of Tower Sealing may be shown to hold in hod mice with strong cardinals which reflect the class of strong cardinals; however, the argument has to be different from what is given in this paper. A natural conjecture is

Conjecture 7.2. *Suppose $(\mathcal{P}, \Psi)$ is a hod pair such that $\mathcal{P} \vDash$ "there is a strong cardinal which reflects the class of strong cardinals and there is a proper class of Woodin cardinals". Let κ be the least strong cardinal which reflects the class of strong cardinals and let $g \subseteq Coll(\omega, \kappa^+)$, then*

$$\mathcal{P}[g] \vDash \text{“}\forall\delta \quad \textit{if } \delta \textit{ is Woodin, then Tower Sealing holds at } \delta.\text{”}$$

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