Compactness of ω_1

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Elementary embeddings and large cardinals

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We say that κ is *supercompact/strongly compact* if κ is λ-*supercompact/*λ-*strongly compact for all* λ. Clearly, supercompact $\to$ strongly compact $\to$ measurable.

Ultrafilters/measures

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Strong compactness was introduced by Keisler and Tarski (1963/64) and it turns out that under ZFC, the two notions of strong compactness are equivalent. Without the Axiom of Choice, this is not true.

Compactness ultrafilters (cont.)

Let κ, X be as above. Let μ be a fine, κ-complete measure on $\mathcal{P}_\kappa(X)$. Let $(A_x : x \in X)$ be a sequence of sets in μ. Then

$$\triangle_x A_x = \{\sigma : \sigma \in \bigcap_{x \in \sigma} A_x\}.$$

We say that μ is *normal* if and only if for every sequence $(A_x : x \in X)$ as above,

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Open problem: (ZFC) Is strong compactness equiconsistent with supercompactness?

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It is easy to check that μ is countably complete and fine. So ω_1 is $\mathbb{R}$-strongly compact.

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What about measures on $\mathcal{P}_{\omega_1}(X)$ for X "bigger" than $\mathbb{R}$?

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So we get ω_1 is $\mathcal{P}(\mathbb{R})$-strongly compact. To get a normal measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$, we seem to need Θ is measurable. It is known that $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ is not enough.

Classical constructions of models with ω_1 being $\mathbb{R}$-compact

Suppose $V \models$ ZFC+ there is a measurable cardinal. Let κ is a measurable witnessed by μ, $j : V \to M$ be the μ-ultrapower map, and $G \subseteq Col(\omega, < \kappa)$.

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Though, for example, if $V = L[\mu]$, the minimal model of a measurable cardinal, then $L(\mathbb{R}, F)$ fails to satisfy AD.

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Corollary

"AD + ω_1 *is* $\mathbb{R}$*-supercompact" is strictly stronger (consistencywise) than "AD +* ω_1 *is* $\mathbb{R}$*-strongly compact".*

Canonical models of ω_1 is $\mathbb{R}$-supercompact

Under $\mathsf{AD}_{\mathbb{R}}$, Woodin (early 1980's) has shown that the Solovay measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is unique and asked about uniqueness of models of the form $L(\mathbb{R}, \mu) \vDash$ "μ is a supercompact measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ (under AD).

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Assume $\mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$. Recall that working in a minimal model of $\mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$ (so $cof(\Theta) = \omega_1$), we can construct a countably complete, fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$ by "integrating the Solovay measure along a cofinal, continuous function $f : \omega_1 \rightarrow \Theta$".

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The following are equiconsistent.

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To see this, note that from the proof of the above theorem, we get a model $L(\Omega^*, \mathbb{R}) \vDash \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$, where $\Omega^* \subseteq \mathcal{P}(\mathbb{R})$. Fix a countably complete, fine, normal measure μ on $\mathcal{P}_{\omega_1}(\Omega^*)$. Then note that by normality,

$$\forall^*_\mu \sigma \ M_\sigma = L(\Omega^*_\sigma, \mathbb{R}_\sigma) \vDash \mathsf{AD}_{\mathbb{R}} + \mathsf{DC},$$

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Now, $\forall^*_\mu \sigma \ (\Omega^*_\sigma, \mathbb{R}_\sigma)^\sharp$ exists (because ω_1 is measurable); by normality again, the sharp for $L(\Omega^*, \mathbb{R})$ exists. This demonstrates that the theory $\mathsf{ZF} + \mathsf{DC} +$ "ω_1 is $\mathcal{P}(\mathbb{R})$-supercompact" is strictly stronger than $\mathsf{ZF} + \mathsf{DC} +$ "ω_1 is $\mathcal{P}(\mathbb{R})$-strongly compact".

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Here are some determinacy theories in increasing strength: (1) AD, (2) $\mathsf{AD}^+ + \Theta > \theta_0$, (3) $\mathsf{AD}_{\mathbb{R}}$, (4) $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$, (5) $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular, (6) $\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable, (7) $\mathsf{AD}_{\mathbb{R}} + \Theta$ is Mahlo, (8) $\mathsf{AD}^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal (LSA).

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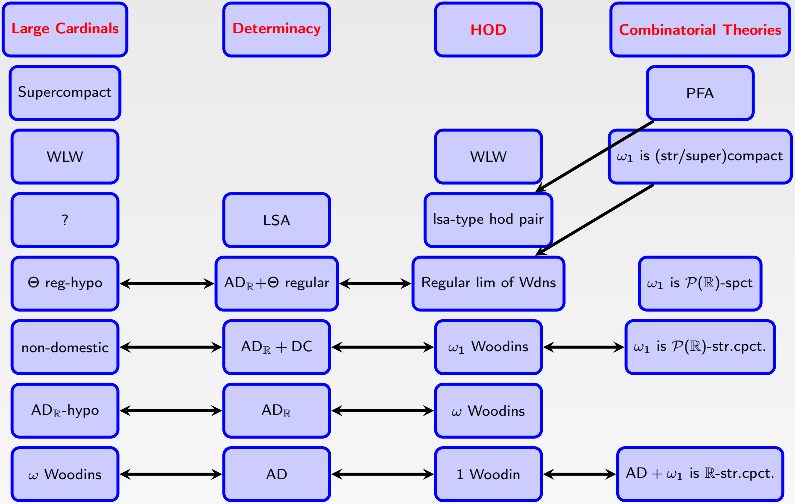
Hierarchies



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