TAME FAILURES OF THE UNIQUE BRANCH HYPOTHESIS AND MODELS OF $\mathsf{AD}_{\mathbb{R}} + \Theta$ IS REGULAR

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Abstract

In this paper, we show that the failure of the unique branch hypothesis (UBH) for tame trees (see Definition 0.1) implies that in some homogenous generic extension of V there is a transitive model M containing $Ord \cup \mathbb{R}$ such that $M \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. The results of this paper significantly extend earlier works from [7] and [11] for tame trees.

We establish, using the core model induction, a lower bound for certain failures of the *Unique Branch Hypothesis*, (UBH), which is the statement that every iteration tree that acts on V has at most one cofinal well-founded branch. This paper is a continuation of [7], but it is self-contained.

For the rest of this paper, all trees considered are *nonoverlapping*, that is whenever E and F are extenders such that E is used before F along a branch of the tree, then $\mathrm{lh}(E) \leq \mathrm{crit}(F)$. Suppose there is a proper class of strong cardinals. We say κ reflects the set of strong cardinals (or κ is a strong reflecting strongs) if for every λ there is an embedding $j : V \to M$ witnessing that κ is λ-strong and for any cardinal $\mu \in [\kappa, \lambda)$, $V \vDash$ "μ is strong" iff $M \vDash$ "μ is strong". Now we recall the definition of *tame trees* from [7].

Definition 0.1 (Tame iteration tree). *An iteration tree $\mathcal{T}$ on V is tame if for all $\alpha < \beta < lh(\mathcal{T})$ such that $\alpha = pred_T(\beta + 1)$, $\mathcal{M}^{\mathcal{T}}_\alpha \vDash$ "$\exists \kappa < \lambda < \mathrm{cp}(E^{\mathcal{T}}_\beta)$ such that λ is a strong cardinal and κ is strong reflecting strongs".*

UBH was first introduced by Martin and Steel in [1]. Towards showing UBH, Neeman, in [3], showed that a certain weakening of UBH called cUBH holds provided there are no non-bland mice[1]. However, in [14], Woodin showed that in the presence of supercompact cardinals UBH can fail for tame trees. Woodin constructs alternating chains whose branches are well-founded. Extenders of such trees can be demanded to reflect the set of strong cardinals which reflect strong cardinals. Hence critical points of the branch embeddings can be demanded to be above the first strong cardinal which reflects strong cardinals. It is still an important open problem whether UBH holds

[1]We will not use this terminology.

for trees that use extenders that are $2^{\aleph_0}$-closed in the models that they are chosen from.[2] A positive resolution of this problem will lead to the resolution of *the inner model problem* for superstrong cardinals and beyond. It is worth remarking that the aforementioned form of UBH for tame trees will also lead to the resolution of the inner model problem for superstrong cardinals and beyond. Our work can be viewed as an attempt to prove UBH for tame trees by showing that its failure is strong consistency-wise.

We recall some material presented in [5] and [7]. Recall Θ is the supremum of ordinals α such that there is a surjection from $\mathbb{R}$ onto α. Working under $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, we say that $(\theta_\alpha : \alpha \leq \Omega)$ is the *Solovay sequence* if: (a) θ_0 is the supremum of ordinals α such that there is an OD surjection from $\mathbb{R}$ onto α, (b) for $\alpha < \Omega$ (and $\theta_\alpha < \Theta$), $\theta_{\alpha+1}$ is the supremum of ordinals α such that for some $A \subseteq \mathbb{R}$ of Wadge rank θ_α, there is an OD_A surjection from $\mathbb{R}$ onto α, (c) for $\beta \leq \Omega$ limit, $\theta_\beta = \sup_{\alpha<\beta}\theta_\alpha$, and (d) $\theta_\Omega = \Theta$. For a set $A \subseteq \mathbb{R}$, we let θ_A be the supremum of α such that there is an OD_A surjection from $\mathbb{R}$ onto α. We may also define the Solovay sequence $(\theta^\Gamma_\alpha : \alpha \leq \Omega)$ of a pointclass Γ with sufficient closure. We list some important determinacy theories in increasing consistency strength: (1) AD^+, (2) $\mathsf{AD}^+ + \Theta > \theta_0$, (3) $\mathsf{AD}_{\mathbb{R}}$, (4) $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$, (5) $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

Con((5)) implies, among others, the consistency of $\mathsf{MM(c)}$, a significant fragment of *Martin's Maximum* (MM) and was conjectured by Woodin to be equiconsistent with a supercompact cardinal. The first author, in [4], shows that (5) is consistent relative to the existence of a Woodin cardinal which is a limit of Woodin cardinals, which is significantly weaker than a supercompact cardinal.

The following is the main theorem of the paper, which improves significantly the lower-bounds obtained by [7] and [11]. [11] obtains (1) as a lower-bound and the main result of [7] obtains (2) as a lower-bound for failures of UBH for tame trees; Theorem 0.2 obtains (5) as a lower-bound under the same hypothesis.

Theorem 0.2 (Main Theorem)**.** *Suppose there is a proper class of strong cardinals and* UBH *fails for tame trees. Then in a set generic extension of V, there is a transitive inner model M such that $Ord \cup \mathbb{R} \subseteq M$ and $M \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular".*

We remark that there are papers in the literature that obtain "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" as a lower bound for certain theories. For instance, in [12], the second author constructed an inner model of "$\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular" from the Proper Forcing Axiom, and in [6], the first author constructed an inner model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" from certain failures of covering. However, the methods developed in this paper are different from those methods developed in the two aforementioned papers in a rather significant way. In the aforementioned papers, the authors work under hypothesis that imply the failure of lower part covering. More precisely, in the aforementioned papers, equivalents of Theorem 3.3 are proved while having the luxury of knowing that $|\mathcal{P}^+| < \omega_2$ in $M[m]$. Here we do not know that $|\mathcal{P}^+| < \omega_2$, yet our large cardinal assumption still allows us to get an (ω_1, ω_1)-iteration strategy with the desired properties. We anticipate that the construction of such a strategy will be useful in other similar contexts as well.

[2]The first extender used in the trees Woodin constructs is not $2^{\aleph_0}$-closed.

1. PRELIMINARIES

1.1. STACKING MICE

We recall the notions used in [7]. Fix some uncountable cardinal λ and assume ZF. Notice that any function $f : H_\lambda \to H_\lambda$ can be naturally coded by a subset of $\wp(\cup_{\kappa<\lambda}\wp(\kappa))$. We then let $Code^*_\lambda : H_\lambda^{H_\lambda} \to \wp(\cup_{\kappa<\lambda}\wp(\kappa))$ be one such coding. If $\lambda = \omega_1$ then we just write $Code^*$. Because for $\alpha \leq \lambda$, any (α, λ)-iteration strategy[3] for a hybrid premouse[4] of size $< \lambda$ is in $H_\lambda^{H_\lambda}$, we have that any such strategy is in the domain of $Code^*_\lambda$.

Suppose $\Lambda \in dom(Code^*_\lambda)$ is a strategy with hull condensation and $\mu \leq \lambda$. Recall that we say F is (μ, Λ)-mouse operator if for some $X \in H_\lambda$ and formula ϕ in the language of Λ-mice, whenever Y is such that $X \in Y$, $F(Y)$ is the minimal μ-iterable Λ-mouse satisfying $\phi[Y]$.

We then let $Code_\lambda$ be $Code^*_\lambda$ restricted to $F \in dom(Code^*_\lambda)$ that are defined by the following recursion.

1. for some $\alpha \leq \lambda$, F is an (α, λ)-iteration strategy with hull condensation[5],
2. for some $\alpha \leq \lambda$ and for some (α, λ)-iteration strategy $\Lambda \in dom(Code^*_\lambda)$ with hull condensation, F is a (λ, Λ)-mouse operator,
3. for some $\alpha \leq \lambda$, for some (α, λ)-iteration strategy $\Lambda \in dom(Code^*_\lambda)$ with hull condensation, for some (λ, Λ)-mouse operator $G \in dom(Code^*_\lambda)$ and for some $\beta \leq \lambda$, F is a (β, Λ)-iteration strategy with hull condensation for some G-mouse $\mathcal{M} \in H_\lambda$.

When $\lambda = \omega_1$ then we just write $Code$ instead of $Code_{\omega_1}$. Given an $F \in dom(Code_\lambda)$ we let $\mathcal{M}_F$ be, in the case F is an iteration strategy, the structure that F iterates and, in the case F is a mouse operator, the base of the cone on which F is defined.

Let $\mathcal{P} \in H_\lambda$ be a hybrid premouse and for some $\alpha \leq \lambda$, let Σ be (α, λ)-iteration strategy with hull condensation for $\mathcal{P}$. Suppose now that $\Gamma \subseteq \wp(\cup_{\kappa<\lambda}\wp(\kappa))$ is such that $Code_\lambda(\Sigma) \in \Gamma$. Given a Σ-premouse $\mathcal{M}$, we say $\mathcal{M}$ is Γ-*iterable* if $|\mathcal{M}| < \lambda$ and $\mathcal{M}$ has a λ-iteration strategy (or (α, λ)-iteration strategy for some $\alpha \leq \lambda$) Λ such that $Code_\lambda(\Lambda) \in \Gamma$[6]. We let $Mice^{\Gamma,\Sigma}$ be the set of Σ-premice that are Γ-iterable.

Definition 1.1. *Given a Σ-premouse $\mathcal{M} \in H_\lambda$, we say $\mathcal{M}$ is countably α-iterable if whenever $\pi : \mathcal{N} \to \mathcal{M}$ is a countable submodel of $\mathcal{M}$, $\mathcal{N}$, as a Σ^π-mouse, is α-iterable. When $\alpha = \omega_1 + 1$ then we just say that $\mathcal{M}$ is countably iterable. We say $\mathcal{M}$ is countably Γ-iterable if whenever π and $\mathcal{N}$ are as above, $\mathcal{N}$ is Γ-iterable.*

[3]This is an iteration strategy for stacks of less than α normal trees, each of which has length less than λ. Typically these are fine-structural n-maximal iteration trees (as defined in [2]), where n is the degree of soundness of the premouse we iterate. We will suppress this parameter thoughout our paper.

[4]For more on hybrid mice, see [4] or [9].

[5]In this case as well as in cases below $\alpha = 0$ is allowed.

[6]Recall that iteration strategy for a Σ-mouse must respect Σ. In particular, all Λ-iterates of $\mathcal{M}$ are Σ-premice.

Suppose $\mathcal{M}$ is a Σ-premouse. We then let $o(\mathcal{M}) = Ord \cap \mathcal{M}$. We also let $\mathcal{M}||\xi$ be $\mathcal{M}$ cutoff at ξ, i.e., we keep the predicate indexed at ξ. We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. We say ξ is a *cutpoint* of $\mathcal{M}$ if there is no extender E on $\mathcal{M}$ such that $\xi \in (\mathrm{cp}(E), lh(E)]$. We say ξ is a *strong cutpoint* if there is no E on $\mathcal{M}$ such that $\xi \in [\mathrm{cp}(E), lh(E)]$. We say $\eta < o(\mathcal{M})$ is *overlapped* in $\mathcal{M}$ if η isn't a cutpoint of $\mathcal{M}$. Given $\eta < o(\mathcal{M})$ we let

$$\mathcal{O}_\eta^{\mathcal{M}} = \cup\{\mathcal{N} \lhd \mathcal{M} : \rho(\mathcal{N}) \leq \eta \text{ and } \eta \text{ is not overlapped in } \mathcal{N}\}.$$

Given a self-wellordered[7] $a \in H_\lambda$ we define the stacks over a by

Definition 1.2.
1. *$Lp^\Sigma(a) = \cup\{\mathcal{N} : \mathcal{N}$ is a countably iterable sound Σ-mouse over a such that $\rho(\mathcal{N}) = a\}$,*
2. *$\mathcal{K}^{\lambda,\Gamma,\Sigma}(a) = \cup\{\mathcal{N} : \mathcal{N}$ is a countably Γ-iterable sound Σ-mouse over a such that $\rho(\mathcal{N}) = a\}$,*
3. *$\mathcal{W}^{\lambda,\Gamma,\Sigma}(a) = \cup\{\mathcal{N} : \mathcal{N}$ is a Γ-iterable sound Σ-mouse over a such that $\rho(\mathcal{N}) = a\}$.*

Remark 1.3. *In the definition above, when we say "Σ-mouse", we really mean "g-organized Σ-mouse" in the sense of [9]. We will suppress the term "g-organized" in this paper as all Σ-mice considered here will be g-organized Σ-mice. The reason for considering "g-organized Σ-mice" is because one can perform S-constructions on g-organized Σ-mice, but not on Σ-mice as defined in [8].*

When $\Gamma = \wp(\cup_{\kappa<\lambda}\wp(\kappa))$ then we omit it from our notation. We can define the sequences $\langle Lp_\xi^\Sigma(a) : \xi < \eta\rangle$, $\langle \mathcal{K}_\xi^{\lambda,\Gamma,\Sigma}(a) : \xi < \nu\rangle$, and $\langle \mathcal{W}_\xi^{\lambda,\Gamma,\Sigma}(a) : \xi < \mu\rangle$ as usual. For Lp operator the definition is as follows:

1. $Lp_0^\Sigma(a) = Lp^\Sigma(a)$,
2. for $\xi < \eta$, if $Lp_\xi^\Sigma(a) \in H_\lambda$ then $Lp_{\xi+1}^\Sigma = Lp_+^\Sigma(Lp_\xi^\Sigma(a))$[8],
3. for limit $\xi < \eta$, $Lp_\xi^\Sigma = \bigcup_{\alpha<\xi} Lp_\alpha^\Sigma(a)$,
4. η is least such that for all $\xi < \eta$, $Lp_\xi^\Sigma(a)$ is defined.

The other stacks are defined similarly.

1.2. (Γ, Σ)-SUITABLE PREMICE

Again fix an uncountable cardinal λ and assume ZF. We also fix $\Sigma \in dom(Code_\lambda)$ such that Σ is a (α, λ)-iteration strategy with hull condensation and $\Gamma \subseteq \wp(\cup_{\kappa<\lambda}\wp(\kappa))$ such that $Code_\lambda(\Sigma) \in \Gamma$. We now import some material from Subsection 1.3 of [5]. The most important notion we need from that subsection is that of (Γ, Σ)-suitable premouse which is defined as follows:

[7]I.e., self well-ordered, a set a is called self well-ordered if $trc(a \cup \{a\})$ is well-ordered in $L_1(a)$.

[8]$Lp_+^\Sigma(Lp_\xi^\Sigma(a))$ is the stack of sound, countably iterable Σ-mice $\mathcal{N}$ projecting to $\leq o(Lp_\xi^\Sigma(a))$ and extends $Lp_\xi^\Sigma(a)$. Similar definitions can be made for $\mathcal{W}_+^\Sigma(\mathcal{W}_\xi^\Sigma(a))$ and $\mathcal{K}_+^\Sigma(\mathcal{K}_\xi^\Sigma(a))$.

Definition 1.4 ((Γ, Σ)-suitable premouse)**.** *A Σ-premouse $\mathcal{P}$ is (Γ, Σ)-suitable if there is a unique cardinal δ such that*

1. *$\mathcal{P} \vDash$ "δ is the unique Woodin cardinal",*

2. *$o(\mathcal{P}) = \sup_{n<\omega}(\delta^{+n})^{\mathcal{P}}$,*

3. *for every $\eta \neq \delta$, if η is a strong cutpoint of $\mathcal{P}$ then $\mathcal{W}_{+}^{\lambda,\Gamma,\Sigma}(\mathcal{P}|\eta) = \mathcal{P}|((\eta)^{+})^{\mathcal{P}}$.*

4. *for any $\eta < o(\mathcal{P})$, if $\eta \neq \delta$, then $C_{\Gamma}(\mathcal{N}|\eta) \vDash$ "η is not Woodin".*

If $\Gamma = \wp(\cup_{\alpha<\lambda}\wp(\alpha))$ then we use λ instead of Γ. In particular, we use λ-suitable to mean Γ-suitable. We will do the same with all the other notions, such as fullness preservation and short tree iterability, defined in this section. Also, if Γ is fixed throughout or is clear from the context, then we simply say $\mathcal{P}$ is Σ-suitable. We let $\mathcal{P}^{-}$ be the structure that Σ iterates.

Suppose $\mathcal{P}$ is (Γ, Σ)-suitable. Then we let $\delta^{\mathcal{P}}$ be the δ of Definition 1.4. We then proceed as in Section 1.3 of [5] to define (1) nice iteration tree, (2) (Γ, Σ)-short tree, (3) (Γ, Σ)-maximal tree, (4) (Γ, Σ)-correctly guided finite stack and (5) the last model of a (Γ, Σ)-correctly guided finite stack by using $\mathcal{W}^{\lambda,\Gamma,\Sigma}$ operator instead of $\mathcal{W}^{\Gamma}$ operator.

1.3. A BRIEF INTRODUCTION TO HOD MICE

In this paper, a hod premouse $\mathcal{P}$ is one defined as in [4][9]. The reader is advised to consult [4] for basic results and notations concerning hod premice and mice. Let us mention some basic first-order properties of a hod premouse $\mathcal{P}$. There are an ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle(\mathcal{P}(\alpha), \Sigma_{\alpha}^{\mathcal{P}}) \mid \alpha < \lambda^{\mathcal{P}}\rangle$ and $\langle\delta_{\alpha}^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ such that

1. $\langle\delta_{\alpha}^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \vDash \delta_{\alpha}^{\mathcal{P}}$ is Woodin;

2. $\mathcal{P}(0) = Lp_{\omega}(\mathcal{P}|\delta_0)^{\mathcal{P}}$; for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1) = (Lp_{\omega}^{\Sigma_{\alpha}^{\mathcal{P}}}(\mathcal{P}|\delta_{\alpha}))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp_{\omega}^{\oplus_{\beta<\alpha}\Sigma_{\beta}^{\mathcal{P}}}(\mathcal{P}|\delta_{\alpha}))^{\mathcal{P}}$;

3. $\mathcal{P} \vDash \Sigma_{\alpha}^{\mathcal{P}}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$[10]-strategy for $\mathcal{P}(\alpha)$ with hull condensation;

4. if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma_{\beta}^{\mathcal{P}}$ extends $\Sigma_{\alpha}^{\mathcal{P}}$.

We will write $\delta^{\mathcal{P}}$ for $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}} = \oplus_{\beta<\lambda^{\mathcal{P}}}\Sigma_{\beta}^{\mathcal{P}}$. Note that $\mathcal{P}(0)$ is a pure extender model. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ if there is $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We say then that $\mathcal{P}$ is a *hod initial segment* of $\mathcal{Q}$. $(\mathcal{P}, \Sigma)$ is a *hod pair* if $\mathcal{P}$ is a hod premouse and Σ is a strategy for $\mathcal{P}$ (acting on countable stacks of countable normal trees) such that $\Sigma^{\mathcal{P}} \subseteq \Sigma$ and this fact is preserved under Σ-iterations. Typically, we will construct hod pairs $(\mathcal{P}, \Sigma)$ such that Σ has hull condensation, branch condensation, and is Γ-fullness preserving for some pointclass Γ.

[9]By a similar remark, by "hod premice" we mean "reorganized hod premice" in the sense of [4] or "g-organized hod premice" in the sense of [9]. Again, the reason has to do with S-constructions.

[10]This just means $\Sigma_{\alpha}^{\mathcal{P}}$ acts on all stacks of ω-maximal, normal trees in $\mathcal{P}$.

Suppose $(\mathcal{Q},\Sigma)$ is a hod pair. $\mathcal{P}$ is a $(\mathcal{Q},\Sigma)$-hod premouse if there are ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle(\mathcal{P}(\alpha),\Sigma^{\mathcal{P}}_\alpha)\mid\alpha<\lambda^{\mathcal{P}}\rangle$ and $\langle\delta^{\mathcal{P}}_\alpha\mid\alpha\leq\lambda^{\mathcal{P}}\rangle$ such that

1. $\langle\delta^{\mathcal{P}}_\alpha\mid\alpha\leq\lambda^{\mathcal{P}}\rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P}\vDash\delta^{\mathcal{P}}_\alpha$ is Woodin;
2. $\mathcal{P}(0)=Lp^{\Sigma}_\omega(\mathcal{P}|\delta_0)^{\mathcal{P}}$ (so $\mathcal{P}(0)$ is a Σ-premouse built over $\mathcal{Q}$); for $\alpha<\lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1)=(Lp^{\Sigma\oplus\Sigma^{\mathcal{P}}_\alpha}_\omega(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$; for limit $\alpha\leq\lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha)=(Lp^{\oplus_{\beta<\alpha}\Sigma\oplus\Sigma^{\mathcal{P}}_\beta}_\omega(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$;
3. $\mathcal{P}\vDash\Sigma^{\mathcal{P}}_\alpha$ is a $(\omega,o(\mathcal{P}),o(\mathcal{P}))$strategy for $\mathcal{P}(\alpha)$ with hull condensation;
4. if $\alpha<\beta<\lambda^{\mathcal{P}}$ then $\Sigma^{\mathcal{P}}_\beta$ extends $\Sigma^{\mathcal{P}}_\alpha$.

Inside $\mathcal{P}$, the strategies $\Sigma^{\mathcal{P}}_\alpha$ act on stacks above $\mathcal{Q}$ and every Σ^{P}_α iterate is a Σ-premouse. Again, we write $\delta^{\mathcal{P}}$ for $\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}}$ and $\Sigma^{\mathcal{P}}=\oplus_{\beta<\lambda^{\mathcal{P}}}\Sigma^{\mathcal{P}}_\beta$. $(\mathcal{P},\Lambda)$ is a $(\mathcal{Q},\Sigma)$-hod pair if $\mathcal{P}$ is a $(\mathcal{Q},\Sigma)$-hod premouse and Λ is a strategy for $\mathcal{P}$ such that $\Sigma^{P}\subseteq\Lambda$ and this fact is preserved under Λ-iterations. The reader should consult [4] for the definition of $B(\mathcal{Q},\Sigma)$, and $I(\mathcal{Q},\Sigma)$. In a core model induction, we don't quite have at the moment $(\mathcal{Q},\Sigma)$ is constructed an AD^+-model M such that $(\mathcal{Q},\Sigma)\in M$ but we do know that every $(\mathcal{R},\Lambda)\in B(\mathcal{Q},\Sigma)$ belongs to such a model. We then can show (using our hypothesis) that $(\mathcal{Q},\Sigma)$ belongs to an AD^+-model.

[4] constructs under AD^+ (under Strong Mouse Capturing (SMC)) hod pairs that are fullness preserving, positional, commuting, and have branch condensation. Such hod pairs are particularly important for our computation as they are points in the direct limit system giving rise to HOD of AD^+ models. For hod pairs $(\mathcal{M}_\Sigma,\Sigma)$, if Σ is a strategy with branch condensation and $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}_\Sigma$ with last model $\mathcal{N}$, $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ is independent of $\vec{\mathcal{T}}$. Therefore, later on we will omit the subscript $\vec{\mathcal{T}}$ from $\Sigma_{N,\vec{\mathcal{T}}}$ whenever Σ is a strategy with branch condensation and $\mathcal{M}_\Sigma$ is a hod mouse.

1.4. HOD UNDER AD^+

Using techniques above and the theory of hod mice developed in [4], [4] and [13] compute HOD (up to Θ) in AD^+ models of $V=L(\wp(\mathbb{R}))+\mathsf{SMC}$[11] $+\Theta=\theta_{\alpha+1}$ for some α below "$\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular".

These papers show the existence of an $\mathcal{M}_\infty$ such that:

1. $\mathcal{M}_\infty\in$ HOD.
2. $\mathcal{M}|\Theta$ is a hod premouse.
3. $\mathcal{M}_\infty|\Theta=(V^{\mathrm{HOD}_\Sigma}_\Theta,\vec{E}^{\mathcal{M}_\infty|\Theta},S^{\mathcal{M}_\infty},\in)$, where $S^{\mathcal{M}_\infty|\Theta}$ is the predicate for strategies of hod initial segments of $\mathcal{M}_\infty|\Theta$.

We call $\mathcal{M}_\infty$ the *hod limit*.

[4] also computes HOD (up to Θ) in models of $V=L(\wp(\mathbb{R}))+\mathsf{SMC}+\mathsf{AD}_{\mathbb{R}}$ below "$\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular" by exhibiting a hod premouse $\mathcal{M}_\infty$ satisfying (1)-(3) as above. Here $\mathcal{M}_\infty=\bigcup_{(\mathcal{P},\Sigma)}\mathcal{M}_\infty(\mathcal{Q},\Lambda)$,

[11] SMC stands for the Strong Mouse Capturing, which says that for any hod pair $(\mathcal{P},\Sigma)$ such that Σ has branch condensation and is fullness preserving, then $\mathsf{MC}(\Sigma)$ holds.

where $(\mathcal{Q}, \Lambda)$ is a hod pair with branch condensation and is fullness preserving and $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ is the direct limit of all (non-dropping) Λ-iterates of $\mathcal{Q}$.

What's important for us are the notions discussed in those papers to compute HOD in the successor cases. Let $(\mathcal{P}^-, \Sigma)$ be as above and suppose also that the direct limit $\mathcal{M}_\infty(\mathcal{P}^-, \Sigma)$ agrees with HOD up to θ_α. Let

$$\begin{aligned}\mathbb{B}(\mathcal{P}^-, \Sigma) = \{B \subseteq \wp(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \quad &\mid B \text{ is } OD, \text{ for any } (\mathcal{Q}, \Lambda) \text{ iterate of } (\mathcal{P}^-, \Sigma),\\ &\text{and for any } (x, y) \in B_{(\mathcal{Q},\Lambda)}, x \text{ codes } \mathcal{Q}\}.\end{aligned}$$

In the above definition, we identify Λ with the set of reals $\text{Code}(\Lambda)$. We also write "$\mathcal{P}$ is Σ-suitable" for "$(\mathcal{P}, \Sigma)$ is a suitable pair". For such a $\mathcal{P}$, we let $\delta^{\mathcal{P}}$ be the Woodin cardinal of $\mathcal{P}$ (above $\mathcal{P}^-$). If $(\mathcal{P}^-, \Sigma) = (\emptyset, \emptyset)$, then each $B \in \mathbb{B}(\emptyset, \emptyset)$ can be canonically identified with an OD set of reals and hence $\mathbb{B}(\emptyset, \emptyset)$ can be canonically identified with the collection of OD sets of reals. Suppose $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$ and $\kappa < o(\mathcal{P})$. Let $\tau^{\mathcal{P}}_{B,\kappa}$ be the canonical term in $\mathcal{P}$ that captures B at κ i.e. for any $g \subseteq Col(\omega, \kappa)$ generic over $\mathcal{P}$

$$B_{(\mathcal{P}^-,\Sigma)} \cap \mathcal{P}[g] = (\tau^{\mathcal{P}}_{B,\kappa})_g.$$

Let $\delta = \delta^{\mathcal{P}}$. For each $m < \omega$, let

$$\gamma^{\mathcal{P},\Sigma}_{B,m} = \sup(Hull^{\mathcal{P}}_1(\mathcal{P}^- \cup \tau^{\mathcal{P}}_{B,(\delta^{+m})^{\mathcal{P}}}) \cap \delta),$$

$$H^{\mathcal{P},\Sigma}_{B,m} = Hull^{\mathcal{P}}_1(\gamma^{\mathcal{P},\Sigma}_{B,m} \cup \{\tau^{\mathcal{P}}_{B,(\delta^{+m})^{\mathcal{P}}}\}),$$

$$\gamma^{\mathcal{P},\Sigma}_{B} = \sup\nolimits_{m<\omega} \gamma^{\mathcal{P},\Sigma}_{B,m},$$

and

$$H^{\mathcal{P},\Sigma}_{B} = \bigcup_{m<\omega} H^{\mathcal{P},\Sigma}_{B,m}.$$

Similar definitions can be given for $\gamma^{\mathcal{P},\Sigma}_{\vec{B},m}, H^{\mathcal{P},\Sigma}_{\vec{B},m}, \gamma^{\mathcal{P},\Sigma}_{\vec{B}}, H^{\mathcal{P},\Sigma}_{\vec{B}}$ for any finite sequence $\vec{B} \in \mathbb{B}(\mathcal{P}^-, \Sigma)$. One just needs to include relevant terms for each element of $\vec{B}$ in each relevant hull. The usual notions of B-iterability, strong B-iterability, and the corresponding weak iteration games $\mathcal{WG}(\mathcal{P}, \Sigma)$, $\mathcal{WG}(\mathcal{P}, \Sigma, B)$ are defined in [4, Section 3.1]. [4] and [13] show that if $(\mathcal{P}^-, \Sigma)$ is a hod pair such that

(i) Σ is fullness preserving, commuting, positional, and has branch condensation,

(ii) $\delta^{\mathcal{M}_\infty(\mathcal{P}^-,\Sigma)} = \theta_\alpha$ for some α,

(iii) $\mathcal{M}_\infty(\mathcal{P}^-, \Sigma)|\theta_\alpha = \text{HOD}|\theta_\alpha$,

then we can compute $\text{HOD}|\theta_{\alpha+1}$ as follows.

Let

$$\mathcal{F} = \{(\mathcal{P}, \Sigma, \vec{B}) \quad | \quad \vec{B} \in \mathbb{B}(\mathcal{P}^-, \Sigma)^{<\omega}, (\mathcal{P}^-, \Sigma) \text{ satisfies } (i) - (iii), \mathcal{P} \text{ is } \Sigma\text{-suitable} \\ \text{and } (\mathcal{P}, \Sigma) \text{ is strongly } \vec{B}\text{-iterable}\}.$$

The ordering on $\mathcal{F}$ is defined as follows:

$$(\mathcal{P}, \Sigma, \vec{B}) \preccurlyeq (\mathcal{Q}, \Lambda, \vec{C}) \quad \Leftrightarrow \quad \vec{B} \subseteq \vec{C}, \exists r (r \text{ is a run of } \mathcal{WG}(\mathcal{P}, \Sigma, \vec{B}) \text{ with the last model } \mathcal{P}^* \\ \text{such that } (\mathcal{P}^*)^- = \mathcal{Q}^-,\ \Sigma_{(\mathcal{P}^*)^-} = \Lambda, \mathcal{P}^* = \mathcal{Q}|(\eta^{+\omega})^{\mathcal{Q}} \\ \text{where } \mathcal{Q} \vDash \eta > o(\mathcal{Q}^-) \text{ is Woodin}).$$

Suppose $(P, \Sigma, \vec{B}) \preccurlyeq (Q, \Lambda, \vec{C})$ then there is a unique map $\pi_{\vec{B}}^{(\mathcal{P},\Sigma),(\mathcal{Q},\Delta)} : H_{\vec{B}}^{\mathcal{P},\Sigma} \to H_{\vec{B}}^{\mathcal{Q},\Lambda}$ given by strong $\vec{B}$-iterability. $(\mathcal{F}, \preccurlyeq)$ is then directed. Let

$$\mathcal{M}_{\infty,\alpha} = \text{direct limit of } (\mathcal{F}, \preccurlyeq) \text{ under maps } \pi_{\vec{B}}^{(\mathcal{P},\Sigma),(\mathcal{Q},\Delta)}.$$

Then $\mathcal{M}_{\infty,\alpha} \in \mathrm{HOD}$ and $\mathcal{M}_{\infty,\alpha}|\theta_{\alpha+1} = \mathrm{HOD}|\theta_{\alpha+1}$. Also for each $(\mathcal{P}, \Sigma, \vec{B}) \in \mathcal{F}$, let

$$\pi_{\vec{B}}^{(\mathcal{P},\Sigma),\infty} : H_{\vec{B}}^{\mathcal{P},\Sigma} \to \mathcal{M}_{\infty,\alpha}$$

be the natural map, and let for each such $\vec{B}$

$$H_{\vec{B}}^{\mathcal{M}_{\infty,\alpha}} = \bigcup\nolimits_{(\mathcal{P},\Sigma,\vec{B}) \in \mathcal{F}} \pi_{\vec{B}}^{(\mathcal{P},\Sigma),\infty}[H_{\vec{B}}^{\mathcal{P},\Sigma}],$$

and

$$\gamma_{\vec{B}}^{\mathcal{M}_{\infty,\alpha}} = \bigcup\nolimits_{(\mathcal{P},\Sigma,\vec{B}) \in \mathcal{F}} \pi_{\vec{B}}^{(\mathcal{P},\Sigma),\infty}[\gamma_{\vec{B}}^{\mathcal{P},\Sigma}],$$

Now suppose $f : \Theta \to \Theta$ (f could be taken from a parent ZFC universe) is such that for each α such that $\theta_\alpha < \Theta$, $f \restriction (\theta_\alpha + 1) \in \mathrm{HOD}$ and $f(\theta_\alpha) < \theta_{\alpha+1}$. We call such an f *appropriate*. Fix an appropriate f and an α and let $\mathcal{F}, \mathcal{M}_{\infty,\alpha}$ be as above for α. Let $(\mathcal{P}, \Sigma, B) \in \mathcal{F}$ be such that $f \restriction (\theta_\alpha + 1) \cup \{f \restriction (\theta_\alpha + 1)\} \in \mathrm{rng}(\pi_B^{(\mathcal{P},\Sigma),\infty} \restriction H_B^{\mathcal{P},\Sigma})$. In particular, $\gamma_B^{\mathcal{M}_\infty} > f(\theta_\alpha)$. We call such a triple $(\mathcal{P}, \Sigma, B)$ *f-suitable*. We then say that a Σ-suitable $\mathcal{P}$ is *(strongly)* (f, Σ)*-iterable* if letting B_f be the OD-least B in $\mathbb{B}(\mathcal{P}^-, \Sigma)$ such that $(\mathcal{P}, \Sigma, B_f)$ is f-suitable, then $(\mathcal{P}, \Sigma)$ is (strongly) B-iterable. Whenever $(\mathcal{P}, \Sigma, B_f) \in \mathcal{F}$ is f-iterable, we also write $\gamma_f^{\mathcal{P},\Sigma}$ for $\gamma_{B_f}^{\mathcal{P},\Sigma}$ or simply $\gamma_f^{\mathcal{P}}$ if Σ is clear from the context.

2. THE MAXIMAL MODEL AND A FRAMEWORK FOR THE CORE MODEL INDUCTION

The core model induction is a method for constructing models of determinacy while working under various hypothesis. During the induction one climbs up through the Solovay hierarchy. This is a hierarchy of axioms that extend AD^+ and roughly describes how complicated the Solovay sequence is. One first defines, under a certain smallness assumption, for instance "there are no models M

such that $\mathbb{R}\cup Ord \subset M$ such that $M \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular", a so-called *maximal model of* AD^+, $\mathfrak{M}$. We show $\mathfrak{M} \vDash \mathsf{AD}^+$. We also show that $\mathfrak{M}$ cannot satisfy "$\Theta = \theta_{\alpha+1}$" for some α and "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is singular" as in each case, we can construct a hod pair $(\mathcal{P}, \Sigma)$ that generates $\wp(\mathbb{R}) \cap \mathfrak{M}$, but by maximality of $\mathfrak{M}$, $(\mathcal{P}, \Sigma) \in \mathfrak{M}$. Contradiction. This shows that there must indeed be such models M satisfying "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular".

Throughout the paper we work under the smallness assumption

($\dagger$): "there are no models M such that $\mathbb{R} \cup Ord \subset M$ such that $M \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular".[12]

In this section we first recall the notion of the maximal model and some correctness results from [7]; the second part of the section sets up the framework for our core model induction.

We start by introducing universally Baire iteration strategies and mouse operators. We assume ZFC. Throughout this paper we fix a canonical method for coding sets in HC by reals. Given a real x which is a code of a set in HC, we let M_x be the structure coded by x and let $\pi_x : M_x \to N_x$ be the transitive collapse of M_x. We let WF be the set of reals which code sets in HC.

Definition 2.1 (uB operators)**.** *Suppose $\Lambda \in dom(Code)$ and $\lambda \geq \omega_1$ is a cardinal. We say Λ is λ-uB if there are $<\lambda$-complementing trees*[13] *(T, S) witnessing that $Code(\Lambda)$ is $<\lambda$-uB in the following stronger sense: for all $x \in WF$ and $n, m \in x$,*

$$(x, n, m) \in p[T] \iff \pi_x(m) \in \Lambda(\pi_x(n)).$$

If g is a $<\lambda$-generic then we let Λ^g be the canonical interpretation of Λ onto $V[g]$, i.e., given $a, b \in HC^{V[g]}$, $\Lambda^g(a) = b$ if and only if whenever $x \in WF^{V[g]}$ is such that $a \in N_x$ and $n \in x$ is such that $\pi_x(n) = a$ then $b = \pi_x[\{m : (x, n, m) \in (p[T])^{V[g]}\}]$.

If Λ is λ-uB for all λ then we say Λ is uB.

Suppose now λ is an uncountable cardinal, g is a $<\lambda$-generic, $a \in (H_\lambda)^V[g]$ and $\Sigma \in dom(Code)$ is λ-uB. Then we define $Lp^{\Sigma,g}(a)$, $\mathcal{W}^{\lambda,\Sigma,g}(a)$ and $\mathcal{K}^{\lambda,\Sigma,g}(a)$ in $V[g]$ according to Definition 1.2. The following connects the three stacks defined above.

Proposition 2.2. *For every $a \in H^V_\lambda$, $\mathcal{W}^{\lambda,\Sigma}(a) \trianglelefteq \mathcal{K}^{\lambda,\Sigma}(a) \trianglelefteq Lp^{\Sigma}(a)$. Moreover, for any $\eta < \lambda$ and V-generic $g \subseteq Coll(\omega, \eta)$ or $g \subseteq Coll(\omega, <\eta)$, $\mathcal{W}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{W}^{\lambda,\Sigma}(a)$, $\mathcal{K}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{K}^{\lambda,\Sigma}(a)$ and $Lp^{\Sigma,g}(a) \trianglelefteq Lp^{\Sigma}(a)$.*

Definition 2.3 (Hod pair below λ)**.** *Suppose now that $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma \in dom(Code)$ is λ^+-uB. We say $(\mathcal{P}, \Sigma)$ is a hod pair below λ if Σ has branch condensation and whenever $g \subseteq Coll(\omega, \lambda)$ is V-generic, in $V[g]$, Σ^g is ω_1-fullness preserving.*

Note that if $\kappa \leq \lambda$ and $(\mathcal{P}, \Sigma)$ is a hod pair below λ then $(\mathcal{P}, \Sigma)$ is a hod pair below κ. We are now in a position to introduce the maximal model of AD^+.

[12] Another way of stating our smallness assumption is the statement: "there are no hod mice $\mathcal{P}$ such that $\delta^{\mathcal{P}}$ is an inaccessible limit of Woodin cardinals in $\mathcal{P}$."

[13] This means that the trees project to complements in all $<\lambda$-generic extensions.

Definition 2.4 (Maximal model of $\mathsf{AD}^+ + \Theta$)**.** *Suppose $\mu < \lambda$ is a cardinal and $g \subseteq Coll(\omega, <\mu)$*[14] *is V-generic. Suppose in $V[g]$, $(\mathcal{P}, \Sigma)$ is a hod pair below λ. Then we let $\mathcal{S}^{\lambda,\Sigma}_{\mu,g} = L(\mathcal{K}^{\lambda,\Sigma,g}(\mathbb{R}^{V[g]}))$. We also let*

$$\mathfrak{M}^{\lambda}_{\mu,g} = L(\bigcup_{(\mathcal{P},\Sigma)} \mathcal{S}^{\lambda,\Sigma}_{\mu,g} \cap \wp(\mathbb{R})) \text{ and } \Omega^{\lambda}_{\mu,g} = \bigcup_{(\mathcal{P},\Sigma)} \mathcal{S}^{\lambda,\Sigma}_{\mu,g} \cap \wp(\mathbb{R}),$$

where the union is over all such hod pairs $(\mathcal{P}, \Sigma)$.

Thus far strategy mice have been discussed only in situations when the underlying set was self-wellordered. However, $\mathcal{S}^{\lambda,\Sigma}_{\mu,g}$ is a Σ-mouse over the set of reals[15]. Such hybrid mice were defined in Section 2.10 of [4] and a more detailed treatment is given in [9]. We say that $\mathcal{S}^{\lambda,\Sigma}_{\mu,g}$ is the λ-Σ-*maximal model of AD^+ at μ*, $\mathfrak{M}^{\lambda}_{\mu,g}$ is the *λ-maximal model of AD^+ at μ*, and $\Omega^{\lambda}_{\mu,g}$ is the *λ-maximal point class of AD^+ at μ*. Our goal is to show that (under $(\dagger)$) $\mathfrak{M}^{\lambda}_{\mu,g}$ is a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular".

The next lemma connects various degrees of iterability. Below, if $\xi \in Ord$ and N is a transitive model of ZFC then we let $N_\xi = V^N_\xi$.

For the purposes of the next lemma, suppose $\mu < \lambda$ are such that μ is a strong cardinal and λ is inaccessible. Let $j : V \to M$ be an embedding witnessing that μ is λ^+-strong and let $g \subseteq Coll(\omega, <\mu)$ and $h \subseteq Coll(\omega, <j(\mu))$ be two generics such that $g = h \cap Coll(\omega, <\mu)$. Let $j^+ : V[g] \to M[h]$ be the lift of j. Let $W = V[g]$. The following lemma comes from Lemma 2.5 of [7].

Lemma 2.5. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair below μ and $a \in V_\lambda[g]$ is self-wellordered. Then*

$$\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{W}^{\lambda,\Sigma,h\cap Coll(\omega,<\lambda)}(a) = \mathcal{K}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\mu,\Sigma,g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}.$$

The following is an easy corollary of Lemma 2.5.

Corollary 2.6. *Suppose $\mu < \kappa < \lambda$ and $j : V \to M$ are such that μ and κ are strong cardinals, λ is inaccessible, j witness that μ is λ-strong and $M \vDash$ "κ is strong cardinal". Let $(\mathcal{P}, \Sigma)$ be a hod pair below μ which is λ-uB. Let $g \subseteq Coll(\omega, <\kappa)$ and $h \subseteq Coll(\omega, <j(\mu))$ be generic such that $g = h \cap Coll(\omega, <\kappa)$. Let $j^+ : V[g \cap Coll(\omega, <\mu)] \to M[h]$ be the lift of j. Then whenever $a \in V_\lambda[g]$,*

$$\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\kappa,\Sigma,g}(a) = \mathcal{W}^{\lambda,\Sigma,h\cap Coll(\omega,<\lambda)}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}.$$

The proof of the above is given in Section 2 of [7], so we omit it here. Now we develop some basic notions in order to state Theorem 2.9 which we will use as a black box. Our core model induction is a typical one: we have two uncountable cardinals $\kappa < \lambda$, *the core model induction operators* (cmi operators) defined on bounded subsets of κ can be extended to act on bounded subsets of λ, and

[14] In this paper, μ is typically an inaccessible cardinal.

[15] Actually, we need that $\mathcal{S}^{\lambda,\Sigma}_{\mu,g}$ is a Θ-g-organized Σ-mouse over $\mathbb{R}^{V[g]}$ as defined in [9]; this is a slight modification from the hierarchy of g-organized Σ-mice. This modification is needed (only for Σ-mice over $\mathbb{R}$) so that the scales analysis works out.

for any such cmi operator F acting on bounded subsets of λ, the minimal F-closed mouse with one Woodin cardinal exists and is λ-iterable.

The mouse operators that are constructed during core model induction have two additional properties: they *transfer and relativize well*. More precisely, fix $\Sigma \in dom(Code)$ which is λ-uB. Given a Σ-mouse operator $F \in dom(Code_\lambda)$, we say

1. (Relativizes well) F relativizes well if there is a formula $\phi(u, v, w)$ such that whenever $X, Y \in dom(F)$ and N are such that $X \in L_1(Y)$ and N is a transitive rudimentarily closed set such that $Y, F(Y) \in N$ then $F(X) \in N$ and $F(X)$ is the unique U such that $N \vDash \phi[U, X, F(Y)]$.

2. (Transfers well) F transfers well if whenever $X, Y \in dom(F)$ are such that X is generic over $L_1(Y)$ then $F(L_1(Y)[X])$ is obtained from $F(Y)$ via S-constructions (see Section 2.11 of [4]) and in particular, $F(L_1(Y))[X] = F(L_1(Y)[X])$.

We are now in a position to introduce the core model induction operators that we will need in this paper.

Definition 2.7 (Core model induction operator). *Suppose $|\mathbb{R}| = \kappa$, $(\mathcal{P}, \Sigma)$ is a hod pair below κ^+. We say $F \in dom(Code)$ is a Σ core model induction operator or just Σ-cmi operator if one of the following holds:*

1. *For some $\alpha \in Ord$, letting $M = \mathcal{S}_\omega^{\kappa^+,\Sigma}||\alpha$, $\Gamma = \Sigma_1^M$, supple $M \vDash \mathsf{AD}^+ + \mathsf{MC}(\Sigma)$ and one of the following holds:*
   (a) *F is a Σ-mouse operator which transfers and relativizes well.*
   (b) *For some self-wellordered $b \in HC$ and some Σ-premouse $\mathcal{Q} \in HC^V$ over b, F is an (ω_1, ω_1)-iteration strategy (above $o(\mathcal{P})$) for a (Σ, Γ)-suitable $\mathcal{Q}$ which is Γ-fullness preserving, has branch condensation and is guided by some $\vec{A} = (A_i : i < \omega)$ such that $\vec{A} \in OD^M_{b,\Sigma,x}$ for some $x \in b$. Moreover, α ends either a weak or a strong gap in the sense of [9].*
   (c) *For some $H \in dom(Code)$, H satisfies a or b above and for some $n < \omega$, F is $x \to \mathcal{M}_n^{\#,H}(x)$ operator or for some $b \in HC$, F is the ω_1-iteration strategy of $\mathcal{M}_n^{\#,H}(b)$.*

2. *For some $\alpha \in Ord$, $a \in HC$ and $\mathcal{M} \trianglelefteq \mathcal{W}^{\kappa^+,\Sigma}(a)$ such that $\rho(\mathcal{M}) = a$ letting Λ be $\mathcal{M}$'s unique strategy, the above conditions hold for F with $L^\Lambda_{\kappa^+}(\mathbb{R})$ used instead of $\mathcal{S}_\omega^{\kappa^+,\Sigma}$ and Λ used instead of Σ.*

When $\Sigma = \emptyset$ then we omit it from our notation. Often times, when doing core model induction, we have two uncountable cardinals $\kappa < \lambda$ and we need to show that cmi operators in $V^{Coll(\omega,<\kappa)}$ can be extended to act on $V_\lambda^{Coll(\omega,<\kappa)}$. This is a weaker notion than being λ-uB. We also need to know that for any cmi operator $F \in V^{Coll(\omega,<\kappa)}$, $\mathcal{M}_1^{\#,F}$-exists. We make these statements more precise.

Definition 2.8 (Lifting cmi operators)**.** *Suppose $\kappa < \lambda$ are two cardinals such that κ is an inaccessible cardinal and suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ.*

1. *Lift$(\kappa, \lambda, \Sigma)$ is the statement that for every generic $g \subseteq Coll(\omega, < \kappa)$, in $V[g]$, for every every Σ^g-cmi operator F there is an operator $F^* \in dom(Code_\lambda)$ such that $F = F^* \restriction HC$. In this case we say F is λ-extendable. Such an F^* is necessarily unique as can be easily shown by a Skolem hull argument*[16]. *If Lift$(\kappa, \lambda, \Sigma)$ holds, $g \subseteq Coll(\omega, < \kappa)$ is generic, and F is a Σ^g-cmi operator then we let F^λ be its extended version.*

2. *We let Proj$(\kappa, \lambda, \Sigma)$*[17] *be the conjunction of the following statements: Lift$(\kappa, \lambda, \Sigma)$ and for every generic $g \subseteq Coll(\omega, < \kappa)$, in $V[g]$,*
   - *(a) for every Σ^g-cmi operator F, $\mathcal{M}_1^{\#, F}$ exists and is λ-iterable.*
   - *(b) for every $a \in H_{\omega_1}$, $\mathcal{K}^{\omega_1, \Sigma, g}(a) = \mathcal{W}^{\lambda, \Sigma, g}(a)$*

The following is the core model induction theorem that we will use.

Theorem 2.9. *Suppose $\kappa < \lambda$ are two uncountable cardinals and suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ such that Proj$(\kappa, \lambda, \Sigma)$ holds. Then for every generic $g \subseteq Coll(\omega, < \kappa)$, $\mathcal{S}_{\kappa, g}^{\lambda, \Sigma} \vDash \mathsf{AD}^+ + \theta_\Sigma = \Theta$.*

We will not prove the theorem here as the proof of the theorem is very much like the proof of the core model induction theorems in [5] (see Theorem 2.4 and Theorem 2.6), [8] (see Chapter 7) and [10]. To prove the theorem we have to use the scales analysis for $\mathcal{S}_{\kappa, g}^{\lambda, \Sigma}$ (see [9]). For a relevant discussion on how Theorem 2.9 is proved, see [7].

We end this section with the following useful fact on lifting strategies. Among other things it can be used to show clause (b) of Proj$(\kappa, \lambda, \Sigma)$. The following is Lemma 3.5 of [7].

Lemma 2.10 (Lifting cmi operators through strongness embeddings)**.** *Suppose $\kappa < \lambda$ are such that κ is a λ-strong cardinal. Then whenever $(\mathcal{P}, \Sigma)$ is a hod pair below κ, Lift$(\kappa, \lambda, \Sigma)$ and clause (b) of Proj$(\kappa, \lambda, \Sigma)$ hold.*

3. A CORE MODEL INDUCTION

Recall that we say μ reflects the set of strong cardinals (or μ is *strong reflecting strongs*) if μ is a strong cardinal and for every $\lambda > \mu$, there is an embedding $j : V \to M$ witnessing that μ is λ-strong and such that for any cardinal $\kappa \in [\mu, \lambda)$, $V \vDash$ "κ is strong" iff $M \vDash$ "κ is strong". We fix $\mu < \kappa < \lambda$ such that λ is an inaccessible cardinal, μ and κ are strong such that μ is strong reflecting strongs and κ is strong.

[16] Suppose $H_0, H_1 \in dom(Code_\lambda^{V[g]})$ are two extensions of F. Working in $V[g]$, let $\pi : N \to H_{\lambda^+}[g]$ be elementary such that N is countable and $H_0, H_1 \in rng(\pi)$. Let $(\bar{H}_0, \bar{H}_1) = \pi^{-1}(H_0, H_1)$. Then it follows from the definition of being a Σ-cmi operator that $\bar{H}_0 = H_0 \restriction N$ and $\bar{H}_1 = H \restriction N$. However, since $H_0 \restriction N = F \restriction N = H_1 \restriction N$, we get that $N \vDash \bar{H}_0 = \bar{H}_1$, contradiction!

[17] Proj stands for projective determinacy. The meaning is taken from clause (a).

Suppose $n \subseteq Coll(\omega, < j(\mu))$ is V-generic. Let $m = n \cap Coll(\omega, < \kappa)$ and $g = m \cap Coll(\omega, < \mu)$. We also let $j^+ : V[g] \to M[n]$ be the lift of j. Suppose also $Proj(\kappa, \lambda, \Psi)$ holds for all hod pairs $(\mathcal{R}, \Psi)$ below κ. We first prove (under the assumption (†)) that:

Theorem 3.1. *$\mathfrak{M}^\lambda_{\mu,g} \neq \mathcal{S}^{\lambda,\Psi}_{\mu,g}$ for some hod pair $(\mathcal{S}, \Psi)$ below κ, where $\mathcal{S} \in V_\mu[g]$, $\Psi \cap V[g] \in V[g]$.*

We first restate the main theorem (Theorem 4.1) of [7] in our context. The proof of this theorem is an easy generalization of that of Theorem 4.1 of [7] combined with Theorem 2.9, so we omit it.

Theorem 3.2. *Suppose $(\mathcal{R}, \Psi)$ is a hod pair below κ such that $Proj(\kappa, \lambda, \Psi)$ holds. Suppose $(\mathcal{R}, \Psi) \in V_\mu[g]$. Let $\mathcal{P} = (\mathcal{M}_\infty)^{\mathcal{S}^{\lambda,\Psi}_{\mu,g}}$. Then in $M[m]$, $\mathcal{P}$ has an (ω_1, ω_1)-iteration strategy Σ such that Σ is extendable to a $(j(\mu), j(\mu))$-strategy that is $j(\mu)$-fullness preserving. Moreover, there is a stack $\vec{\mathcal{T}} \in HC^{V[m]}$ on $\mathcal{P}$ according to Σ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$ exists and in $V[m]$, $(\mathcal{Q}, \Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$ is a hod pair below ω_1 (so in particular, $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation). Finally, in $V[m]$ (or equivalently in $M[m]$), $\mathcal{S}^{\lambda,\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}}_{\kappa,m} \vDash \mathsf{AD}^+ + \theta_\Psi < \Theta$.*

Proof of Theorem 3.1. This basically follows from Theorem 3.2. We outline the argument. Suppose not; then $\mathfrak{M}^\lambda_{\mu,g} = \mathcal{S}^{\lambda,\Psi}_{\mu,g}$ for some hod pair $(\mathcal{R}, \Psi) \in V[g]$ below κ, where $\mathcal{R} \in V_\mu[g]$ and $\Psi \cap V[g] \in V[g]$. Fix such a $(\mathcal{R}, \Psi)$. Applying Theorem 3.2 to $(\mathcal{R}, \Psi)$ and using elementarity of j^+ and the fact that κ is strong in M, we get that there is a hod pair $(\mathcal{Q}, \Sigma)$ below a strong cardinal $\kappa^* < \mu$ such that $(\mathcal{Q}, \Sigma) \in V_{\kappa^*}[g \cap Coll(\omega, < \kappa^*)]$ and $(\mathcal{Q}, \Sigma)$ is also a hod pair below μ such that $\Sigma \notin \mathfrak{M}^\lambda_{\mu,g}$ and $\mathcal{S}^{\lambda,\Sigma}_{\mu,g} \vDash \mathsf{AD}^+$.[18] This contradicts the definition of $\mathfrak{M}^\lambda_{\mu,g}$. □

The above theorem shows that the Solovay sequence of $\Omega^\lambda_{\mu,g}$ has limit length. We then prove:

Theorem 3.3. *Suppose whenever $(\mathcal{R}, \Psi)$ is a hod pair below κ then $Proj(\kappa, \lambda, \Psi)$ holds; so the Solovay sequence of $\Omega^\lambda_{\mu,g}$ has limit length. Let $\mathcal{P} = (\mathcal{M}_\infty)^{\Omega^\lambda_{\mu,g}}$ and $\mathcal{P}^+ = \mathcal{W}^{\lambda,\Sigma^-,g}_\omega(\mathcal{P})$, where Σ^- is the join of the strategies $\Sigma_{\mathcal{P}(\alpha)}$ of $\mathcal{P}(\alpha)$ for all $\alpha < \lambda^{\mathcal{P}}$*[19]*. Let Θ be the height of the Wadge hierarchy of $\Omega^\lambda_{\mu,g}$ (so $\Theta = o(\mathcal{P})$). If $\mathcal{P}^+ \vDash \Theta$ is singular, then there is an initial segment Γ of $\Omega^\lambda_{\mu,g}$ such that "$L(\Gamma) \vDash AD_{\mathbb{R}} + \Theta$ is regular".*

Proving Theorem 3.3 is the main task of our paper. The rest of the section is devoted to this task. We follow arguments in [6]. Many of the main ideas of our proof come from [6]; however, in this situation, we don't know a priori that $|\mathcal{P}^+|^V < \mu^+$ (unlike in the situation of [6]) and this affects many of the key arguments given there. We now outline the proof of the theorem, making use of results from [6] as much as possible.

Lemma 3.4. *Suppose $\mathcal{P} \trianglelefteq \mathcal{M} \trianglelefteq \mathcal{P}^+$. Then $\rho_\omega(\mathcal{M}) \geq \Theta$.*

[18] To see that Σ is fullness preserving with respect to mice in $\mathcal{S}^{\lambda,\Psi}_{\mu,g}$, using the fact that κ^* is strong, we get that Σ is λ-fullness preserving in $V[g \cap Coll(\omega, < \kappa^*)]$. Suppose $\mathcal{M}$ is a λ-iterable sound (g-organized) Ψ-mouse over $a \in HC^{V[g]}$ and $\rho_\omega(\mathcal{M}) = a$, then by S-construction, $\mathcal{M}$ is (fine structurally) equivalent to a (g-organized), sound Ψ-mouse $\mathcal{M}^*$ over some $a^* \in V_\mu[g \cap Coll(\omega, < \kappa^*)]$ and $\rho_\omega(\mathcal{M}^*) = a^*$. This observation guarantees Σ is $\mathcal{S}^{\lambda,\Psi}_{\mu,g}$-fullness preserving.

[19] $\Sigma_{\mathcal{P}(\alpha)}$ is simply the tail of a hod pair $(\mathcal{Q}, \Lambda) \in \Omega^\lambda_{\mu,g}$ where $\mathcal{M}_\infty(\mathcal{Q}, \Lambda) = \mathcal{P}(\alpha)$. Using j^+, we can extend Λ to a unique strategy, called Λ also, acting on stacks in $j^+(\Omega^\lambda_{\mu,g})$; so $(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)})$ is indeed a hod pair in $j^+(\Omega^\lambda_{\mu,g})$.

Proof sketch. Fix such an $\mathcal{M}$. Note that $|\mathcal{M}|^V = \mu$ since the assumption of Theorem 3.3 implies $\Theta < \mu^+$. The methods of [6], in particular Lemma 11.8, applied to $\mathcal{M}$ show that in fact $\rho_\omega(\mathcal{M}) \geq \Theta$. $\square$

We assume throughout this section that $|\mathcal{P}^+|^V \geq \mu^+$. Otherwise, [6] applies and gives Theorem 3.3. By replacing j by the ultrapower embedding via the $(\mu, j(\mu))$-extender derived from j, we may assume $j[\mathcal{P}^+]$ is cofinal in $j(\mathcal{P}^+)$.

Lemma 3.5. *$\mathcal{P}^+ \vDash$ cof$(\lambda^\mathcal{P})$ is measurable.*

Proof. Suppose not. Recall we set $\Sigma^- = \oplus_{\alpha<\lambda^{\mathcal{P}^+}} \Sigma_{\mathcal{P}(\alpha)}$. Let $\Omega = \Omega^\lambda_{\mu,g}$. We have Σ^- acts on $\mathcal{P}^+$. More precisely, whenever $\vec{\mathcal{T}}$ (based on $\mathcal{P}$) is according to Σ^- and $\pi^{\vec{\mathcal{T}}}$ exists, then letting $\mathcal{Q} = \mathrm{Ult}(\mathcal{P}^+, E)$, where E is the $(\mathrm{crt}(\pi^{\vec{\mathcal{T}}}), \sup\pi^{\vec{\mathcal{T}}}[\mathcal{P}])$-extender derived from $\pi^{\vec{\mathcal{T}}}$, then we can define $\sigma : \mathcal{Q} \to j(\mathcal{P}^+)$ as follows: for any $f \in \mathcal{P}^+$, any $a \in (\mathcal{Q}|\delta^\mathcal{Q})^{<\omega}$,

$$\sigma(i_E(f)(a)) = j(f)(\pi^{\Sigma^-_{\vec{\mathcal{T}},\mathcal{Q}}}_{\mathcal{Q}|\delta^\mathcal{Q},\infty}(a)).$$

Using the fact that $\pi^{\vec{\mathcal{T}}}$ is continuous at $\delta^{\mathcal{P}^+}$ and $j \restriction \mathcal{P} = \pi^{\Sigma^-}_{\mathcal{P},\infty}$, we get that σ is elementary, $\sigma \circ i_E = j \restriction \mathcal{P}^+$, and $\sigma \restriction \delta^\mathcal{Q} = \pi^{\Sigma^-_{\vec{\mathcal{T}},\mathcal{Q}}}_{\mathcal{Q},\infty} \restriction \delta^\mathcal{Q}$. In particular, this implies that $\mathcal{Q}$ is well-founded.

It follows from Theorem 3.1 that $(\mathcal{P}^+, \Sigma^-) \in j^+(\Omega^\lambda_{\mu,g})$. But then letting $\mathcal{M}_\infty$ be the direct limit of all iterates of $(\mathcal{P}^+, \Sigma^-)$ in $j^+(\Omega^\lambda_{\mu,g})$, there is an embedding $\tau : \mathcal{M}_\infty \to j(\mathcal{P}^+)$ with critical point $\delta^{\mathcal{M}_\infty}$. This implies that $\mathcal{M}_\infty$ is a hod initial segment of $j(\mathcal{P}^+)$ and $\mathcal{M}_\infty \vDash$ "$\delta^{\mathcal{M}_\infty}$ is an inaccessible limit of Woodin cardinals". This contradicts our smallness assumption (†). $\square$

Definition 3.6 (Nice strategies)**.** *Suppose $\pi_{\mathcal{P}^+,\mathcal{R}} : \mathcal{P}^+ \to \mathcal{R}$, $\sigma : \mathcal{R} \to j(\mathcal{P}^+)$ are Σ_1-elementary. Suppose $j \restriction \mathcal{P}^+ = \sigma \circ \pi_{\mathcal{P}^+,\mathcal{R}}$. Let $\alpha < \lambda^\mathcal{R}$. We say that an iteration strategy Σ_α for $\mathcal{R}(\alpha)$ is* ***nice*** *if and only if*

(i) Σ_α is a $j^+(\Omega^\lambda_{\mu,g})$-fullness preserving strategy for $\mathcal{R}(\alpha)$ with branch condensation.

(ii) $\pi^{\Sigma_\alpha}_{\mathcal{R}(\alpha),\infty} = \sigma' \restriction \mathcal{R}(\alpha)$ for some Σ_1 elementary map $\sigma' : \mathcal{R} \to j(\mathcal{P}^+)$ such that $j \restriction \mathcal{P}^+ = \sigma' \circ \pi_{\mathcal{P}^+,\mathcal{R}}$ (so Σ_α acts on all of $\mathcal{R}$).

(iii) If $\pi_{\mathcal{P}^+,\mathcal{R}} \in M$, then $\Sigma_\alpha \restriction M \in M$.

Now, we construct a partial strategy Σ of $\mathcal{P}^+$ in $V[n]$ with the following properties (using the terminology of [6]):

(i) Σ extends Σ^-.

(ii) Whenever $\vec{\mathcal{T}} \in M_{j(\mu)} \cup M_\kappa[m]$ is a stack on $\mathcal{P}^+$, we say that $\vec{\mathcal{T}}$ is according to Σ if:

  (a) for all $\mathcal{R}$ a terminal node ([6, Definition 2.1]) of $\vec{\mathcal{T}}$, there is a map $\sigma_\mathcal{R} : \mathcal{R} \to j(\mathcal{P}^+)$ such that

$$j \restriction \mathcal{P}^+ = \sigma_\mathcal{R} \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{P}^+,\mathcal{R}}.$$

Furthermore, if $\mathcal{Q}$ and $\mathcal{R}$ are two terminal nodes and $\pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{R}}$ exists then $\sigma_{\mathcal{Q}} = \sigma_{\mathcal{R}} \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{R}}$.

(b) For all terminal nodes $\mathcal{R}$, for all successor $\alpha < \lambda^{\mathcal{R}}$, letting $\sigma_{\mathcal{R}}$ be as above, there is a unique $\{j(f) : f \in \mathcal{P}^+ \wedge j(f)$ is appropriate$\}$-guided[20], nice strategy $\Sigma_{\mathcal{R}(\alpha)}$ for $\mathcal{R}(\alpha)$. Furthermore, letting $\alpha = \beta + 1$, then $\Sigma_{\mathcal{R}(\alpha)}$ extends $\Sigma_{\mathcal{R}(\beta)}$.

(c) Letting $\mathcal{R}$ be as in (b), then whenever $\vec{\mathcal{U}}$ on $\mathcal{R}$ is according to (the tail of) Σ and $\vec{\mathcal{U}}$ is based on $\mathcal{R}(\alpha)$ for some $\alpha < \lambda^{\mathcal{R}}$, then $\vec{\mathcal{U}}$ is according to $\Sigma_{\mathcal{R}(\alpha)}$.

In the above, we define Σ by inductively defining $\Sigma_{\mathcal{R}}(\alpha)$ for each $\alpha < \lambda^{\mathcal{R}}$, where $\mathcal{R}$ is a terminal node on a stack $\vec{\mathcal{T}}$ as above. First note that $\Sigma_{\mathcal{P}(\alpha)}$ is nice for each $\alpha < \lambda^{\mathcal{P}^+}$ (with clause (ii) in Definition 3.6 being witnessed by j). Now suppose $\vec{\mathcal{T}}, \mathcal{R}$ are as above. It is enough to define $\Sigma_{\mathcal{R}(\alpha)}$ for $\alpha = \beta + 1$, where by induction, we have that $\Sigma_{\mathcal{R}(\beta)}$ is nice and the supremum of the generators of $\vec{\mathcal{T}}_{\mathcal{P}^+,\mathcal{R}}$ is $\leq \delta^{\mathcal{R}}_{\beta}$.[21] We prove a series of lemmas that eventually leads to the construction of $\Sigma_{\mathcal{R}(\alpha)}$.

Lemma 3.7. *Let $\vec{\mathcal{T}}, \mathcal{R}, \sigma_{\mathcal{R}}, \alpha, \beta$ be as above. Then $\mathcal{R}(\alpha)$ is full in $j^+(\Omega^{\lambda}_{\mu,g})$.*

Proof. Suppose not. As before we set $\Omega = \Omega^{\lambda}_{\mu,g}$ and we have already assumed that $\Sigma_{\mathcal{R}(\beta)}$ is nice, so in particular, $\mathcal{R}(\beta)$ is $j^+(\Omega)$-full. Let $\xi < j(\mu)$ be M-inaccessible and $(\mathcal{P}^*, \Psi) \in M_\xi$ be a Σ^--hod pair in $j^+(\Omega)$ witnessing $\mathcal{R}(\alpha)$ is not full and $\lambda^{\mathcal{P}^*}$ is limit of countable cofinality (in $\mathcal{P}^*$). More precisely, there is a cutpoint ξ in $\mathcal{R}(\alpha)\backslash\mathcal{R}(\beta)$ such that in $\Gamma(\mathcal{P}^*, \Psi)$, there is a $\Sigma_{\mathcal{R}(\beta)}$-mouse $\mathcal{M}$ such that $\mathcal{M} \lhd Lp^{\Sigma_{\mathcal{R}(\beta)}}(\mathcal{R}(\alpha)|\xi)\backslash\mathcal{R}(\alpha)$. The existence of such a pair $(\mathcal{P}^*, \Psi)$ follows from the fact that $j(\mu)$ is strong in M and by boolean comparison. Note that no levels of $\mathcal{P}^*$ extending $\mathcal{P}$ projects to or below Θ. This is similar to the proof of Lemma 3.4. We assume that Ψ has branch condensation and is $j^+(\Omega)$-fullness preserving.

Let $\sigma = \pi_{\mathcal{P}^+,\mathcal{R}}$ and $\sigma^+ : \mathcal{P}^* \to \mathcal{R}^*$ be the ultrapower map of $\mathcal{P}^*$ by the $(\mathrm{crt}(\sigma), \delta^{\mathcal{R}})$-extender derived from σ. Let $\sigma^+_{\mathcal{R}} : \mathcal{R}^* \to j(\mathcal{P}^*)$ be defined as: for $g \in \mathcal{P}^*$, $a \in (\delta^{\mathcal{R}})^{<\omega}$,

$$\sigma^+_{\mathcal{R}}(\sigma^+(g)(a)) = j(g)(\sigma_{\mathcal{R}}(a)).$$

We have then that $\sigma^+_{\mathcal{R}}$ is elementary and $j \restriction \mathcal{P}^* = \sigma^+_{\mathcal{R}} \circ \sigma^+$.

In V, let $\dot{\mathcal{T}}, \dot{\mathcal{R}}, \dot{\mathcal{R}}^*, \dot{\mathcal{S}}, \dot{\Sigma}, \dot{\sigma}, \dot{\sigma^+}, \dot{\sigma_{\mathcal{R}}}, \dot{\sigma^+_{\mathcal{R}}} \in V$ be canonical $Coll(\omega, <\kappa)$ names for $\vec{\mathcal{T}}, \mathcal{R}, \mathcal{R}^*, \mathcal{R}(\alpha)$, $\Sigma_\beta, \sigma, \sigma^+, \sigma_{\mathcal{R}}, \sigma^+_{\mathcal{R}}$ respectively. Let γ be a sufficiently large regular cardinal in $V[g]$ such that $V_\gamma[g]$ contains all relevant objects and let $\mu + 1 \subset X \prec V_\gamma[g]$ be of size μ and contain all relevant objects. Let $\pi : N \to X$ be the uncollapse map. Let $\bar{m} \in V[g]$ be $Coll(\omega, \pi^{-1}(\kappa))$-generic over N. For any $a \in X$, let $\bar{a} = \pi^{-1}(a)$.

Let $\mathcal{M} = \mathcal{M}^{\sharp,\Psi}_{\omega}$ and let Π be $\mathcal{M}$'s $j(\mu)$-strategy in $M[n]$ (the $\mathcal{M}$ before is behind us now). We assume also that $(\mathcal{M}, \Pi \restriction V[g]) \in X$. Let $\mathcal{N}$ be an iterate (below the first Woodin cardinal of $\mathcal{M}$)

[20]Recall that this means $\Sigma_{\mathcal{R}(\alpha)}$ witnesses $\mathcal{R}(\alpha)$ is strongly $(j(f), \Sigma_{\mathcal{R}(\beta)})$-iterable for all $f \in \mathcal{P}^+$ such that $j(f)$ is appropriate. Furthermore, for any correctly guided, non-dropping $\mathcal{T}$ according to $\Sigma_{\mathcal{R}(\alpha)}$, $\delta(\mathcal{T}) = \sup_{f \in \mathcal{P}^+}(\gamma^{\mathcal{M}^{\mathcal{T}}_b}_{j(f)})$.

[21]If $\alpha < \lambda^{\mathcal{R}}$ is limit and $\mathrm{cof}^{\mathcal{R}}(\alpha)$ is not measurable in $\mathcal{R}$ then we set $\Sigma_{\mathcal{R}(\alpha)}$ to be $\oplus_{\beta<\alpha}\Sigma_{\mathcal{R}(\beta)}$. If $\mathrm{cof}^{\mathcal{R}}(\alpha) = \kappa$ is measurable in $\mathcal{R}$, then we let $\mathcal{S} = \mathrm{Ult}(\mathcal{R}, E)$ where E is the total extender on κ with the least index in $\mathcal{R}$ and we define $\Sigma_{\mathcal{R}(\alpha)}$ by inductively defining $\Sigma_{\mathcal{S}(\beta)}$ for $\beta < \lambda^{\mathcal{S}}$.

such that $H =_{\mathrm{def}} H^M_\xi$ is generically generic[22] over $\mathcal{N}$ for the extender algebra $\mathbb{B}^{\mathcal{N}}_\delta$, where δ is the first Woodin cardinal of $\mathcal{N}$. Then in $\mathcal{N}[H][m]$, the following hold:

$$D(\mathcal{N}[H][m]) \vDash \text{"in } L(\Gamma(\mathcal{P}^*, \Psi), \mathbb{R}),\ \mathcal{R}(\alpha) \text{ is not full"},$$

where $D(\mathcal{N}[H][m])$ is the derived model of $\mathcal{N}[H][m]$. A similar fact holds of $\bar{\mathcal{N}}$ inside $N[\bar{m}]$. In fact, letting $\mathcal{M} = \bar{\mathcal{N}}[\bar{H}]$, then inside $\mathcal{M}$, letting λ be the sup of $\mathcal{M}$'s Woodin cardinals:

$$\emptyset \Vdash_{Col(\omega,<\bar{\kappa})} \Vdash_{Col(\omega,<\lambda)} \text{ in the derived model, } L(\Gamma(\bar{\mathcal{P}^*}, \bar{\Psi})) \text{ witnesses that } \dot{\mathcal{S}} \text{ is not full.} \tag{3.1}$$

Note that

$$\bar{\Psi} = \Psi^\pi \restriction N \text{ and } \bar{\Pi} = \Pi^\pi \restriction N, \tag{3.2}$$

and

$$\pi \restriction \pi^{-1}(j(\mathcal{P}^*)) \circ (\dot{\bar{\sigma}}^+_{\mathcal{R}})_{\bar{m}} =_{\mathrm{def}} \tau \in M[g], \tag{3.3}$$

and

$$\Psi^\pi = j(\Psi)^{j\circ\pi} = j(\Psi)^{\tau\circ(\dot{\bar{\sigma}}^+)_{\bar{m}}} \text{ is Wadge reducible to } \Lambda =_{\mathrm{def}} j(\Psi)^\tau. \tag{3.4}$$

Combining 3.1, 3.2, and 3.4, letting $\mathcal{W} = \dot{\bar{\mathcal{R}}}^*{}_{\bar{m}}$ and $\mathcal{S} = \dot{\bar{\mathcal{S}}}$, we get in $\Omega^\lambda_{\kappa,m}$,

$$\text{in } L(\Gamma(\mathcal{W}, \Lambda), \mathbb{R}), \mathcal{S} \text{ is not full.} \tag{3.5}$$

This means that if we perform an $\mathbb{R}^{V[m]}$-genericity iteration via Λ, then letting $\mathcal{W}^*$ be the iterate, inside $D(\mathcal{W}^*)$, we have

$$\mathcal{S} \text{ is not full.} \tag{3.6}$$

This contradicts results in [4] on internal fullness of hod mice. $\square$

Definition 3.8. *For $f \subseteq \delta^{\mathcal{P}}$ and $f \in \mathcal{P}^+$. We say an $\mathcal{M} \lhd \mathcal{P}^+$ is **f-nice** if $\rho_\omega(\mathcal{M}) = \Theta$, $f \in \mathcal{M}$, $\mathcal{M} \vDash \Theta$ is the largest cardinal, and $j \restriction \mathcal{M}$ is cofinal in $j(\mathcal{M})$.*

Fix an appropriate $f \in \mathcal{P}^+$. Let $\mathcal{M} \lhd \mathcal{P}^+$ be f-nice; note that the set of f-nice $\mathcal{M}$'s is unbounded in $\mathcal{P}^+$. We construct a strategy Σ_f witnessing $\mathcal{R}(\alpha)$ is strongly $(j(f), \Sigma_{\mathcal{R}(\beta)})$-iterable. First, we construct a realizable strategy for $\mathcal{R}(\alpha)$. Let $\tau_{\mathcal{M}} = j \restriction \mathcal{M}$. Note that $\tau_{\mathcal{M}} \in M$ and by f-niceness of $\mathcal{M}$, $\pi_{\mathcal{P}^+,\mathcal{R}} \restriction \mathcal{M}$ is cofinal in $\pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M})$. By absoluteness, $\Sigma_{\mathcal{R}(\beta)} \in M[n]$, and the fact that $\pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M})$ is countable in $M[n]$, there is in $M[n]$ an elementary $\sigma_{\mathcal{M}} : \pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M}) \to j(\mathcal{M})$ such that

- $\sigma_{\mathcal{M}} \circ \pi_{\mathcal{P}^+,\mathcal{R}} \restriction \mathcal{M} = \tau_{\mathcal{M}}$.
- $\sigma_{\mathcal{M}} \restriction \mathcal{R}(\beta) = \pi^{\Sigma_{\mathcal{R}(\beta)}}_{\mathcal{R}(\beta),\infty}$.

[22] See [4, Section 2.10] for the definition.

Let $\Sigma' = j(\mathcal{P})^{\sigma_{\mathcal{M}}}_{\mathcal{R}(\alpha)}$ be the $\Sigma_{\mathcal{M}}$-pullback of $\mathcal{R}(\alpha)$. By constructions in [6, Section 11], whenever $\mathcal{S}$ is a nondropping Σ'-iterate of $\mathcal{R}(\alpha)$, then there is an embedding $\sigma_{\mathcal{S}} : \mathcal{S} \to \sigma_{\mathcal{M}}(\mathcal{R}(\alpha))$ such that $\sigma_{\mathcal{S}} \circ \pi^{\Sigma'}_{\mathcal{R}(\alpha),\mathcal{S}} = \sigma_{\mathcal{M}} \restriction \mathcal{R}(\alpha)$.

Remark 3.9. *The above construction, though stated as a definition for a strategy of $\mathcal{R}(\alpha)$ in $V[m]$ as part of defining a partial strategy Σ for $\mathcal{P}^+$, indeed gives an inductive definition of a strategy $\Lambda_{\mathcal{M}} \in M[n]$ for $\mathcal{M}$ for stacks in $M_{j(\mu)}[n]$; the reason is because $\tau_{\mathcal{M}} \in M$. Furthermore by construction, given any $\Lambda_{\mathcal{M}}$-iterate $\mathcal{S}$ of $\mathcal{M}$, there is some $\sigma : \mathcal{S} \to j(\mathcal{M})$ such that $\sigma \circ \pi_{\mathcal{M},\mathcal{S}} = j \restriction \mathcal{M}$.*

Lemma 3.10. *All nondropping Σ'-iterates of $\mathcal{R}(\alpha)$ are $j^+(\Omega^\lambda_{\mu,g})$-full. Furthermore, Σ' has branch condensation and is positional and commuting.*

Proof sketch. The proof is almost the same as that of Lemma 3.7. We only outline the main changes. Let $\mathcal{S}$ be a non dropping Σ'-iterate of $\mathcal{R}(\alpha)$ and suppose $\mathcal{S}$ is not full. Let $(\mathcal{P}^*, \Psi)$ be as in the proof of Lemma 3.7 witnessing this. Let E be the $(\mathrm{crt}(\pi_{\mathcal{P}^+,\mathcal{R}}), \delta^{\mathcal{R}})$-extender derived from $\pi_{\mathcal{P}^+,\mathcal{R}}$. Let $\mathcal{Q} = \mathrm{Ult}(\mathcal{P}^*, E)$, $\mathcal{N}' = i^{\mathcal{Q}}_E(\mathcal{M})$, and $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, E)$.

Claim 3.11. *$\mathcal{N}' = \mathcal{N} = \pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M})$ and $i^{\mathcal{Q}}_E \restriction \mathcal{M} = i^{\mathcal{M}}_E = \pi_{\mathcal{P}^+,\mathcal{R}} \restriction \mathcal{M}$.*

Proof. We just prove $\mathcal{N} = \pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M})$ and $i^{\mathcal{M}}_E = \pi_{\mathcal{P}^+,\mathcal{R}} \restriction \mathcal{M}$. By definition of E and the choice of $\mathcal{M}$, there is a factor map $l : \mathcal{N} \to \pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M})$ such that $\mathrm{crt}(l) \geq \delta^{\mathcal{R}}$ and l is cofinal in $\pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M})$. Note that both $\mathcal{N}$ and $\pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M})$ both satisfy $\delta^{\mathcal{R}}$ is the largest cardinal. This means that l is the identity. Similarly, $\mathcal{N}' = \pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M})$ and $i^{\mathcal{Q}}_E = \pi_{\mathcal{P}^+,\mathcal{R}} \restriction \mathcal{M}$. □

Now as in the proof of Lemma 3.7, $\pi =_{\mathrm{def}} \pi_{\mathcal{P}^+,\mathcal{R}}$ lifts to $\pi^+ : \mathcal{P}^* \to \mathcal{Q}$ and there is a map $\sigma_{\mathcal{Q}} : \mathcal{Q} \to j(\mathcal{P}^*)$ extending $\sigma_{\mathcal{M}}$ (this uses the claim) such that $\sigma_{\mathcal{Q}} \circ \pi^+ = j \restriction \mathcal{P}^*$.

Now, $\tau =_{\mathrm{def}} \pi^{\Sigma'}_{\mathcal{R}(\alpha),\mathcal{S}}$ can be extended to $\tau^+ : \mathcal{Q} \to \mathcal{S}^+$ (τ^+ is simply the ultrapower map by the $(\mathrm{crt}(\tau), \delta^{\mathcal{S}})$-extender derived from τ) and there is a map $\sigma_{\mathcal{S}^+} : \mathcal{S}^+ \to j(\mathcal{P}^*)$ such that $\sigma_{\mathcal{Q}} = \sigma_{\mathcal{S}^+} \circ \tau^+$. The rest of the proof is just like that of Lemma 3.7.

That Σ' has branch condensation, is positional and commuting follows from [4, Lemma 3.26] and the fact that $\mathrm{cof}(\delta^{\mathcal{P}})$ is measurable in $\mathcal{P}^+$ since Σ' can be taken to be the pullback of some hod pair $(\mathcal{R}, \Lambda)$ in $j^+(\Omega^\lambda_{\mu,g})$ such that $\lambda^{\mathcal{R}}$ is limit and Λ has branch condensation and is fullness preserving. □

Now we reset our notations. Let $f \subseteq \delta^{\mathcal{P}}$ and $f \in \mathcal{P}^+$; let $\mathcal{M}$ be f-nice, and $\tau_{\mathcal{M}} = j \restriction \mathcal{M}$. Again, note that $\tau_{\mathcal{M}} \in M$. We now define the notion of $(f, \mathcal{M})$-condensation. Suppose in $M[n]$, $\mathcal{S}$ is a hod premouse such that $\mathcal{S}$ is $\tau_{\mathcal{M}}$*-realizable*, that is, there are maps $\pi : \mathcal{M} \to \mathcal{S}$ and $\tau_{\mathcal{S}} : \mathcal{S} \to j(\mathcal{M})$ in $M[n]$ such that $\tau_{\mathcal{M}} = \tau_{\mathcal{S}} \circ \pi$. Letting $\Sigma_{\tau_{\mathcal{S}}} = j(\mathcal{P})^{\tau_{\mathcal{S}}}$, we define the set $A_{f,\mathcal{M},\tau_{\mathcal{S}}}$ as follows: for any Σ_1 formula ϕ, for any $s \in (\mathcal{S}|\delta^{\mathcal{S}})^{<\omega}$,

$$(\phi, s) \in A_{f,\mathcal{M},\tau_{\mathcal{S}}} \Leftrightarrow j(\mathcal{M}) \vDash \phi[\pi^{\Sigma_{\tau_{\mathcal{S}}}}_{\mathcal{S}(\gamma),\infty}(s), \tau_{\mathcal{M}}(f)],$$

where γ is such that $s \in \mathcal{S}(\gamma)$. We also let

$$T_{f,\mathcal{M}} = \{(\phi, s) \mid \phi \text{ is a } \Sigma_1 \text{ formula, } s \in (\mathcal{P})^{<\omega}, \text{ and } \mathcal{M} \vDash \phi[s, f]\}.$$

In the following definition, we reuse the notions just defined.

Definition 3.12. *We say $\tau_{\mathcal{S}}$* ***has*** *$(f, \mathcal{M})$**-condensation** (in $M[n]$) if whenever $\mathcal{W}$ is $\tau_{\mathcal{S}}$-realizable as witnessed by $(\pi^*, \tau_{\mathcal{W}})$, then $\pi^*(\pi(T_{f,\mathcal{M}})) = A_{f,\mathcal{M},\tau_{\mathcal{W}}}$.*

The following theorem and its proof is from [6], but here we apply it to $\mathcal{M}$.

Theorem 3.13 ($(f, \mathcal{M})$-condensation lemma)**.** *$\tau_{\mathcal{M}}$ has $(f, \mathcal{M})$-condensation.*

Proof. Working in $M[n]$, let $\mu \leq \nu < j(\mu)$ be such that ν is M-inaccessible. Let $\mathcal{R}_\nu$ be the direct limit of all hod pairs $(\mathcal{W}, \Sigma)$ such that $\mathcal{W} \in M[n \cap Coll(\omega, <\nu)]$, Σ is $j^+(\Omega^\lambda_{\mu,g})$-fullness preserving, positional, commuting, and has branch condensation. Let $Y_\nu = \bigcup_{\alpha < \lambda^{\mathcal{R}_\nu}} \pi^{\Sigma_{\mathcal{R}_\nu(\alpha)}}_{\mathcal{R}_\nu(\alpha),\infty}[\mathcal{R}_\nu(\alpha)]$. Let $X \subset j(\mathcal{M})$ be countable in $M[n]$[23]. Let $\mathcal{R}^*_\nu$ be the transitive collapse of $H_1^{j(\mathcal{M})}(X \cup Y_\nu)$ and σ_ν be the uncollapse map. We say that ν is X-*good* if $\sigma_\nu \restriction \delta^{\mathcal{R}^*_\nu} = \bigcup_{\alpha < \lambda^{\mathcal{R}_\nu}} \pi^{\Sigma_{\mathcal{R}_\nu(\alpha)}}_{\mathcal{R}_\nu(\alpha),\infty}$. The proof of [6, Lemma 11.9] shows that there are cofinally many $\nu < j(\mu)$ that are X-good for any such X. When $X = \tau_{\mathcal{M}}[\mathcal{M}]$, and ν is X-good, we say ν is a *good point*.

For a good point ν, we can define an iteration strategy Λ_ν (for stacks in $M_{j(\mu)}[n]$) for $\mathcal{R}^*_\nu$ the same way $\Lambda_{\mathcal{M}}$ was defined in Remark 3.9, but using σ_ν instead of j. Λ_ν has the following properties:

- Whenever $\mathcal{S}$ is a nondropping Λ_ν-iterate of $\mathcal{R}^*_\nu$, $\mathcal{S}|\delta^{\mathcal{S}}$ is $j^+(\Omega^\lambda_{\mu,g})$-full[24]. Furthermore, for each $\alpha < \lambda^{\mathcal{S}}$, $(\Lambda_\nu)_{\mathcal{S}(\alpha)}$ has branch condensation.
- Letting $\mathcal{S}$ be as above, there is a map $\sigma : \mathcal{S} \to j(\mathcal{M})$ such that $\sigma \circ \pi_{\mathcal{R}^*_\nu,\mathcal{S}} = \sigma_\nu$.

Let $\mathcal{M}_\nu$ be the direct limit of all Λ_ν-iterates in $j^+(\Omega^\lambda_{\mu,g})$ and m_ν be the σ_ν-realization map given by the construction of Λ_ν.

As in the proof of [6, Lemma 11.15], it suffices to show:

$$\text{there is a } \tau_{\mathcal{M}}[\mathcal{M}]\text{-good } \nu \text{ such that } \sigma_\nu \text{ has } (f, \mathcal{M})\text{-condensation.} \tag{3.7}$$

The proof of this now is just that of [6, Lemma 11.15] using remarks in Lemma 3.10 and the fact that $\tau_{\mathcal{M}} \in M$ (this replaces the hypothesis $|\mathcal{P}^+|^V < \mu^+$ used in [6, Lemma 11.9]). We outline the proof here for the reader's convenience.

Suppose 3.7 fails. We can then find a sequence $(\mathcal{Q}_i, \pi_i, \tau_i, k_i, \psi_i, \nu_i : i < \omega) \in M[n]$ such that

1. $\nu_0 = \mu$, $\mathcal{R}_0 = \mathcal{M}$, and $(\nu_i : i < \omega)$ is an increasing sequence of good points,
2. for $i < \omega$, $\mathcal{Q}_i$ is σ_{ν_i}-realizable as witness by (π_i, τ_i) and $k_i : \mathcal{Q}_i \to \mathcal{R}^*_{\nu_{i+1}} =_{\text{def}} \mathcal{R}_{i+1}$ is given by $k_i = \sigma^{-1}_{\nu_i} \circ \tau_i$

[23] $\tau_{\mathcal{M}}[\mathcal{M}]$ is an example of such an X.
[24] $\mathcal{S}$ is not full at the top, so we can't demand more than this.

3. for $i < \omega$, $\sigma_{\nu_i}[\mathcal{R}_i] \subseteq rng(\sigma_{\nu_{i+1}})$, $\psi_i = \sigma^{-1}_{\nu_{i+1}} \circ \sigma_{\nu_i}$ and for $i < m$, letting $\psi_{i,m} = \sigma^{-1}_{\nu_m} \circ \sigma_{\nu_i}$ and $f_i = \psi_{0,i}(f)$, $\pi_i(T_{f_i,\mathcal{R}_i}) \neq A_{f_i,\mathcal{Q}_i,\tau_i}$ (i.e., $(\mathcal{Q}_i, \pi_i, \tau_i)$ witnesses that σ_{ν_i} doesn't have $(f_i, \mathcal{R}_i)$-condensation).

Let now ν be a good point such that $\sup_{i<\omega} \nu_i < \nu < j(\mu)$ and letting $X = \cup_{i<\omega}(\tau_i[\mathcal{Q}_i] \cup \sigma_{\nu_i}[\mathcal{R}_{\nu_i}])$, $X \subseteq rng(\sigma_\nu)$. Let $(\mathcal{S}^*, \Phi^*) \in j^+(\Omega^\lambda_{\mu,g})$ be a hod pair below $j(\mu)$ such that $\mathcal{M}_\nu \triangleleft \mathcal{M}_\infty(\mathcal{S}^*, \Phi^*)$[25] and $\lambda^{\mathcal{S}}$ is limit with cofinality ω in $\mathcal{S}^*$. Let $B = m^{-1}_\nu(j(f))$. Let now $\sigma_i = m^{-1}_\nu \circ \sigma_{\nu_i}$ and $\tau^*_i = m^{-1}_\nu \circ \tau_i$. Notice now that we can define the notion of $(f_i, \mathcal{R}_i)$-condensation also for the embeddings σ_i. We leave it to the reader to fill in the definition. Now notice that we have that

4. for $i < \omega$, $\mathcal{Q}_i$ is σ_i-realizable as witness by (π_i, τ^*_i) and $k_i : \mathcal{Q}_i \to \mathcal{R}_{i+1}$ is given by $k_i = \sigma^{-1}_i \circ \tau^*_i$

5. $(\mathcal{Q}_i, \pi_i, \tau^*_i)$ witnesses that σ_i doesn't have $(f_i, \mathcal{R}_i)$-condensation.

The importance of this move is that the badness of $(\mathcal{Q}_i, \mathcal{R}_i, \pi_i, \tau^*_i, k_i, \psi_i, \sigma_i : i < \omega)$ can now be witnessed in the derived model of $\mathcal{S}^*$ as computed by Φ^*. More precisely, letting $\Sigma_i = \oplus_{\alpha<\lambda^{\mathcal{R}_i}} \Lambda_{\nu_i}(\alpha)$ and $\Psi_i = (\tau_i\text{-pullback of } j(\Sigma_i)^h)$,

(1): in $M[n]$, letting $N = D(\mathcal{S}^*, \Phi^*) = L(\Gamma(\mathcal{S}^*, \Phi^*), \mathbb{R})$, in N, there is a formula $\theta(u, v)$ and a finite set of ordinals t such that for every i, $(\phi, s) \in T_{f_i,\mathcal{R}_i}$ if and only if $\theta[\pi^{\Sigma_i}_{\mathcal{R}_i(\alpha),\infty}(s), t]$ where α is the least such that $s \in [\delta^{\mathcal{R}_i}_\alpha]^{<\omega}$. However, in N, for each i, there is a pair $(\phi_i, s_i) \in T_{\mathcal{Q}_i,\pi_i(f_i)}$ such that $\neg\theta[\pi^{\Psi_i}_{\mathcal{Q}_i(\alpha),\infty}(s_i), t]$ where α is the least such that $s \in [\delta^{\mathcal{Q}_i}_\alpha]^{<\omega}$.

Suppose K is a transitive model of AD^+ and $b = ((\mathcal{M}_i, \Sigma_i), \mathcal{N}_i, \gamma_i, l_i, \xi_i, C : i < \omega) \in K$ is such that $(\mathcal{M}_i, \mathcal{N}_i, \gamma_i, l_i, \xi_i, C : i < \omega) \in HC^K$. Suppose $\theta(u, v)$ is a formula and t is a finite sequence of ordinals. We write $K \vDash$ "$(b, \theta(u, v), t)$ is bad" if in K, letting $K^* = L(\{D \subseteq \mathbb{R} : w(D) \leq t(0)\})$ then $b \in K^*$ and in K^*

6. for every $i < \omega$, $\mathcal{M}_i$ is a hod premouse such that $\lambda^{\mathcal{M}_i}$ is limit and Σ_i is an ω_1-iteration strategy for $\mathcal{M}_i|\delta^{\mathcal{M}_i}$ with the property that for every $\alpha < \lambda^{\mathcal{M}_i}$, $(\Sigma_i)_{\mathcal{M}_i(\alpha)}$ has branch condensation and is fullness preserving,

7. for every i, $\xi_i : \mathcal{M}_i \to \mathcal{M}_{i+1}$,

8. for every i, $\mathcal{N}_i$ is a ξ_i-realizable as witnessed by (γ_i, l_i),

9. for every $\alpha < \lambda^{\mathcal{N}_i}$, letting $\Psi_i = (l_i\text{-pullback of } \Sigma_i)$, $(\Psi_i)_{\mathcal{N}_i(\alpha)}$ has branch condensation and is fullness preserving,

10. $C \in M_0 \cap \wp(\delta^{\mathcal{M}_i})$ and letting $C_0 = C$ and $C_{i+1} = \xi_i(C_i)$, for every i,

$$(\phi, s) \in T_{C_i,\mathcal{M}_i} \text{ if and only if } \theta[\pi^{\Sigma_i}_{\mathcal{M}_i(\alpha),\infty}(s), t]$$

[25] The direct limit is taken inside $j^+(\Omega^\lambda_{\mu,g})$.

where α is least such that $s \in [\delta_\alpha^{\mathcal{M}_i}]^{<\omega}$ but for every i, there is $(\phi_i, s_i) \in T_{\gamma_i(C_i), \mathcal{N}_i}$ such that $\neg\theta[\pi^{\Psi_i}_{\mathcal{N}_i(\alpha),\infty}(s), t]$ where α is least such that $s_i \in [\delta_\alpha^{\mathcal{N}_i}]^{<\omega}$.

In $M[n]$, let $(\mathcal{W}^*, \Pi^*)$ be a $(\mathcal{P}^+, \Sigma^-)$-hod pair such that $\mathcal{W}^* \in M_\gamma[g]$ for some M-cardinal $\gamma < j(\mu)$ but greater than ν, Π^* is a $(j(\lambda), j(\lambda))$-strategy that is $j^+(\Omega^\lambda_{\mu,g})$-fullness preserving, $\Pi^* \cap j^+(\Omega^\lambda_{\mu,g}) \in (\Omega^\lambda_{\mu,g})$, and $\Gamma(\mathcal{W}^*, \Pi^*) = \Gamma(\mathcal{S}^*, \Phi^*)$ in $j^+(\Omega^\lambda_{\mu,g})$. Let $b = ((\mathcal{R}_i, \Sigma_i), \mathcal{Q}_i, \pi_i, k_i, \psi_i, f_i : i < \omega)$. We can then rewrite (1) in terms of $(\mathcal{W}^*, \Pi^*)$ and get that

(2): in $M[h]$, letting $N = D(\mathcal{W}^*, \Pi^*) = L(\Gamma(\mathcal{W}^*, \Pi^*), \mathbb{R})$, in N, there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(b, \theta(u, v), t)$ is bad.

Let then $\mathcal{N}^* = \mathcal{N}_\omega^{\#, \Pi^*, \oplus_{i<\omega}\Sigma_i}$. Let $\mathcal{N}$ be an iterate of $\mathcal{N}^*$ via the canonical iteration strategy of $\mathcal{N}^*$ such that H^M_γ is generically generic over the extender algebra of $\mathcal{N}$ at its bottom Woodin cardinal. We can now witness (3) inside $\mathcal{N}[H^M_\gamma][h_\gamma]$ as follows:

(3): $D(\mathcal{N}[H^M_\gamma][h_\gamma]) \vDash$ "letting $N = D(\mathcal{W}^*, \Pi^*) = L(\Gamma(\mathcal{W}^*, \Pi^*), \mathbb{R})$, in N, there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(b, \theta(u, v), t)$ is bad".

We will get a contradiction using (3). Notice that the sequence $a = (\mathcal{R}_i, \psi_i, \Sigma_i, f_i : i < \omega) \in M$. However, the sequence $(\mathcal{Q}_i, \pi_i, k_i : i < \omega)$ may not be in M. Let then $d \in M^{Coll(\omega,<\gamma)}$ be a name for $(\mathcal{Q}_i, \pi_i, k_i : i < \omega)$. Let $\zeta = (j(\mu)^+)^M$, and let $\pi : P[g] \to (H^M_\zeta)[g]$ be such that $P \in V$, $\mathrm{cp}(\pi) > \mu$, $|P|^V = \mu$, and all relevant objects are contained in $rng(\pi)$. Let $\mathcal{M} = \pi^{-1}(\mathcal{N})$, $e = \pi^{-1}(a)$ and $c = \pi^{-1}(d)$. Let for $i < \omega$, $e(i) = (\mathcal{K}_i, \xi_i, \bar{\Sigma}_i, g_i : i < \omega)$ and $(\mathcal{W}, \Pi) = \pi^{-1}(\mathcal{W}^*, \Pi^*)$. Also we let $\bar{\gamma} = \pi^{-1}(\gamma)$. By elementarity, (3) gives that

(4): whenever $\bar{m} \subseteq Coll(\omega, < \pi^{-1}(j(\mu)))$ is $P[g]$-generic then in $P[g][\bar{m}]$, letting $\bar{k} = \bar{m} \cap Coll(\omega, < \bar{\gamma})$ $d = d_{g*\bar{k}}$, for $i < \omega$, $d(i) = (\mathcal{S}_i, \gamma_i, l_i)$ and $g = ((\mathcal{K}_i, \bar{\Sigma}_i), \mathcal{S}_i, \gamma_i, l_i, \xi_i, g_i : i < \omega)$, $D(\mathcal{M}[H^P_{\bar{\gamma}}][g * \bar{k}]) \vDash$ "letting $N = D(\mathcal{W}, \Pi) = L(\Gamma(\mathcal{W}, \Pi), \mathbb{R})$, in N, there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(g, \theta(u, v), t)$ is bad".

Using genericity iterations we can completely internalize (5) to $\mathcal{M}^* = \mathcal{M}[H^{\mathcal{P}}_{\bar{\gamma}}]$ and get that

(5): in $\mathcal{M}^*$, there is a name $d^* \in (\mathcal{M}^*)^{Coll(\omega,<\bar{\gamma})}$ such that whenever $\bar{k} \subseteq Coll(\omega, < \bar{\gamma})$ is $\mathcal{M}^*$-generic then letting $d = d^*_{\bar{k}}$, for $i < \omega$, $d(i) = (\mathcal{S}_i, \gamma_i, l_i)$ and $g = ((\mathcal{K}_i, \bar{\Sigma}_i), \mathcal{S}_i, \gamma_i, l_i, \xi_i, g_i : i < \omega)$, $D(\mathcal{M}^*[\bar{k}]) \vDash$ "letting $N = D(\mathcal{W}, \Pi) = L(\Gamma(\mathcal{W}, \Pi), \mathbb{R})$, in N, there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(g, \theta(u, v), t)$ is bad".

Work now in $M[n]$. Notice that for every i, $\bar{\Sigma}_i = ((\pi\text{-pullback of } \Sigma_i)) \restriction P$ and $\Pi = ((\pi\text{-pullback of } \Pi^*)) \restriction P$. In what follows, we abuse our notation and let for every i, $\bar{\Sigma}_i = (\pi\text{-pullback of } \Sigma_i)$

and $\Pi = (\pi\text{-pullback of } \Pi^*)$ in all $M[n]$. It then follows that in $M[n]$, $\mathcal{M}$ is a $\Pi \oplus (\oplus_{i<\omega}\bar{\Sigma}_i)$-mouse. Let now $C = D(\mathcal{W}, \Pi)$. It is easy to see that (5) gives $(\mathcal{S}_i, \gamma_i, l_i : i < \omega)$ such that if $g = ((\mathcal{K}_i, \bar{\Sigma}_i), \mathcal{S}_i, \gamma_i, l_i, \xi_i, g_i : i < \omega)$

> (6): in C, there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(g, \theta(u, v), t)$ is bad.

Fix then $\theta(u, v)$ and t as in (6). Let E_i be the $(\delta^{\mathcal{K}_i}, \delta^{\mathcal{K}_{i+1}})$-extender derived from ξ_i and F_i be $(\delta^{K_i}, \delta^{\mathcal{S}_i})$-extender derived from γ_i. Let $\mathcal{K}_0^+ = \mathcal{W}$, $\mathcal{S}_i^+ = Ult(\mathcal{K}_i, F_i)$ and $\mathcal{K}_{i+1}^+ = Ult(\mathcal{K}_i^+, E_i)$. Let $p_i = \sigma_{\nu_i} \circ (\pi \restriction \mathcal{K}_i)$. Then we have that p_i, γ_i, ξ_i and l_i extend to $p_i^+ : \mathcal{K}_i^+ \to j(\mathcal{W})$, $\gamma_i^+ : \mathcal{K}_i^+ \to \mathcal{S}_i^+$, $\xi_i^+ : \mathcal{K}_i^+ \to \mathcal{K}_{i+1}^+$ and $l_i^+ : \mathcal{S}_i^+ \to \mathcal{K}_{i+1}^+$ such that $p_i^+ = p_{i+1}^+ \circ \xi_i^+$ and $\xi_i^+ = l_i^+ \circ \gamma_i^+$.

By a standard argument (e.g. see [13, Lemma 4.3]), we can simultaneously iterate $(\mathcal{K}_i^+, \mathcal{S}_i^+ : i < \omega)$ using strategies $\Pi_i = (p_i^+\text{-pullback of } \pi(\Pi))$ and $\Omega_i = (l_i^+ \circ p_i^+\text{-pullback of } \pi(\Pi))$ to make $\mathbb{R}^{M[n]}$-generic. Such genericity iterations have been used by many authors. The details of such genericity iterations are spelled out in Definition 1.35 of [5]. The outcome of this iteration is a sequence of models $(\mathcal{K}_{i,\omega}, \mathcal{S}_{i,\omega} : i < \omega)$ and embeddings $(\xi_{i,\omega}, \gamma_{i,\omega}, l_{i,\omega} : i < \omega)$ with the property that $\xi_{i,\omega} : \mathcal{K}_{i,\omega} \to \mathcal{K}_{i+1,\omega}$, $\gamma_{i,\omega} : \mathcal{K}_{i,\omega} \to \mathcal{S}_{i,\omega}$, $l_{i,\omega} : \mathcal{S}_{i,\omega} \to \mathcal{K}_{i+1,\omega}$ and for every $i < \omega$, $\xi_{i,\omega} = l_{i,\omega} \circ \xi_{i,\omega}$. Moreover, the iterations $\mathcal{K}_i^+$-to-$\mathcal{K}_{i,\omega}$ and $\mathcal{S}_i$-to-$\mathcal{S}_{i,\omega}$ are above respectively $\delta^{\mathcal{K}_i}$ and $\delta^{\mathcal{S}_i}$. Let then $C_i = D(\mathcal{K}_{i,\omega})$ and $D_i = D(\mathcal{S}_{i,\omega})$. One important remark is that for every $i < \omega$, $\mathcal{K}_{i,\omega}$ is a $\bar{\Sigma}_i$-hod premouse and $\mathcal{S}_{i,\omega}$ is a Ψ_i-premouse where $\Psi_i = (l_i\text{-pullback of } \bar{\Sigma}_i)$. Another important remark is that $C_i \subseteq D_i \subseteq C_{i+1}$. The most important remark, however, is that the the construction of the sequences $(\mathcal{K}_{i,\omega}, \mathcal{S}_{i,\omega} : i < \omega)$ and $(\xi_{i,\omega}, \gamma_{i,\omega}, l_{i,\omega} : i < \omega)$ guarantees that the direct limit of $K_{i,\omega}$ under $\xi_{i,\omega}$ is well-founded. Let then n be such that for every $m \geq n$, $\xi_{m,\omega}(t) = t$. It then follows from (6) and the fact that for every $i < \omega$, $C \subseteq C_i$ and $C \subseteq D_i$ that

> (7): for every $i < \omega$, in C_i, for every (ϕ, s) such that ϕ is a formula and $s \in [\delta^{\mathcal{K}_i}]^{<\omega}$, $\mathcal{K}_i \vDash \phi[s, B_i]$ if and only if $\theta[\pi^{\bar{\Sigma}_i}_{\mathcal{K}_i(\alpha),\infty}(s), t]$ where $\alpha < \lambda^{\mathcal{K}_i}$ is least such that $s \in [\delta_\alpha^{\mathcal{K}_i}]^{<\omega}$.

> (8): for every i, in D_i, there is a formula ϕ and $s \in [\delta^{\mathcal{S}_i}]^{<\omega}$ such that $\mathcal{S}_i \vDash \phi[s, \gamma_i(B_i)]$ and $\neg\theta[\pi^{\Psi_i}_{\mathcal{S}_i(\alpha),\infty}(s), t]$ where $\alpha < \lambda^{\mathcal{S}_i}$ is least such that $s \in [\delta_\alpha^{\mathcal{S}_i}]^{<\omega}$.

It follows from elementarity of $\gamma_{i,\omega}$, (7) and the fact that if $i \geq n$ then $\gamma_{i,\omega}(t) = t$ that

> (9): for every $i \geq n$, in D_i, for every (ϕ, s) such that ϕ is a formula and $s \in [\delta^{\mathcal{S}_i}]^{<\omega}$ and $\mathcal{S}_i \vDash \phi[s, \gamma_i(B_i)]$ if and only if $\theta[\pi^{\Psi_i}_{\mathcal{S}_i(\alpha),\infty}(s), t]$ where $\alpha < \lambda^{\mathcal{S}_i}$ is least such that $s \in [\delta_\alpha^{\mathcal{S}_i}]^{<\omega}$.

Clearly (8) and (9) contradict one another. This completes the proof of the theorem. □

Lemmas 3.10 and 3.13 immediately give us the following corollary.

Lemma 3.14. *Let $f, \mathcal{M}, \sigma_{\mathcal{M}}, \mathcal{R}, \Sigma'$ be defined prior to Remark 3.9. Let $\sigma'_{\mathcal{M}} : \pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M}) \to j(\mathcal{M})$ be defined by: $\sigma'_{\mathcal{M}}(\pi_{\mathcal{P}^+,\mathcal{R}}(g)(a)) = \tau_{\mathcal{M}}(g)(\pi^{\Sigma'}_{\mathcal{R}(\alpha),\infty}(a))$ for all $g \in \mathcal{M}$ and $a \in \mathcal{R}(\alpha)^{<\omega}$. Then $\sigma'_{\mathcal{M}} \in M[n]$ is Σ_1-elementary, and*

$$\pi^{\Sigma'}_{\mathcal{R}(\alpha),\infty} \restriction H^{\mathcal{R}(\alpha)}_f = \sigma'_{\mathcal{M}} \restriction H^{\mathcal{R}(\alpha)}_f. \tag{3.8}$$

Furthermore, Σ' is commuting, positional, witnesses $\mathcal{R}(\alpha)$ is strongly $(\Sigma_{\mathcal{R}(\beta)}, j(f))$-iterable and has branch condensation.

Proof. First, Σ_1-elementarity of $\sigma'_{\mathcal{M}}$ follows from Theorem 3.13 and the fact that $\mathcal{M}$ is g-suitable for every $g \subseteq \delta^{\mathcal{P}}$ and $g \in \mathcal{M}$.

By changing $\mathcal{M}$ if necessary, we can assume that $\rho_1(\mathcal{M}) = \delta^{\mathcal{P}}$ and there is some $h \in \mathcal{M}$ such that $h \subseteq \delta^{\mathcal{P}}$ and f, $\tau^{\mathcal{R}(\alpha)}_{B_f,\kappa}, \gamma^{\mathcal{R}(\alpha)}_f$ are Σ_1 computable in $\pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M})$ from $\pi_{\mathcal{P}^+,\mathcal{R}}(h)$ for all $\kappa \in \{((\delta^{\mathcal{R}}_\alpha)^{+n})^{\mathcal{R}} \mid n < \omega\}$. Then applying Lemma 3.13 to $(h, \mathcal{M})$, we get 3.8. The second clause follows from Lemma 3.10. $\square$

Working in $V[n]$, we fix an enumeration $\langle g_k \mid k < \omega\rangle$ and $\langle f_k = j(g_k) \mid k < \omega\rangle$ of $\{f \mid f \in \mathcal{P}^+ \wedge j(f) \text{ is appropriate}\}$ and $\{j(f) \mid f \in \mathcal{P}^+ \wedge j(f) \text{ is appropriate}\}$ respectively, so that whenever $H^{\mathcal{R}(\alpha)}_{f_k} \subseteq H^{\mathcal{R}(\alpha)}_{f_l}$ then $k \leq l$. Note that for any k, there is some $l \geq k$ such that $H^{\mathcal{R}(\alpha)}_{f_k} \subseteq H^{\mathcal{R}(\alpha)}_{f_l}$.

For each l and f_l-suitable $\mathcal{M}$, fix strategy $\Lambda_l \in M[n]$ for $\mathcal{R}(\alpha)$ extending $\Sigma_{\mathcal{R}}(\beta)$, map $\tau'_{\mathcal{M},l}$ satisfying Lemma 3.14 for f_l. We also demand for $l \leq k$ such that $H^{\mathcal{R}(\alpha)}_{f_l} \subseteq H^{\mathcal{R}(\alpha)}_{f_k}$,

$$\pi^{\Lambda_k} \restriction H^{\mathcal{R}(\alpha)}_{f_l} = \pi^{\Lambda_l} \restriction H^{\mathcal{R}(\alpha)}_{f_l}. \tag{3.9}$$

We plan to construct strategy $\Sigma_{\mathcal{R}(\alpha)}$ by taking "limit" of the Λ_l's as follows. For simplicity, suppose $\mathcal{T} \in M_{j(\mu)}[n]$ is a normal, correctly guided, maximal tree on $\mathcal{R}(\alpha)$[26] using extenders above $\mathcal{R}(\beta)$. Let $\mathcal{M}(\mathcal{T})^+$ be the end model of the tree $\mathcal{T}$.[27] For each n, let $b_n = \Lambda_n(\mathcal{T})$. We let $\Sigma_{\mathcal{R}(\alpha)}(\mathcal{T}) = b$, where

$$\xi \in b \Leftrightarrow \exists k \forall l \geq k (\xi \in b_l). \tag{3.10}$$

Let $\mathcal{H}$ be the transitive collapse of $\bigcup_n H^{\mathcal{M}(\mathcal{T})^+}_{f_n}$ and τ be the uncollapse map. We want to show that

(i) b is cofinal in $\mathcal{T}$;

(ii) $\mathcal{H} = \mathcal{M}(\mathcal{T})^+$ and τ is the identity. This gives that $\Sigma_{\mathcal{R}(\alpha)}$ is fullness preserving;

(iii) $\Sigma_{\mathcal{R}(\alpha)}$ acts on all of $\mathcal{R}$ and is $\sigma'_{\mathcal{R}}$-realizable for some Σ_1 elementary embedding $\sigma'_{\mathcal{R}} : \mathcal{R} \to j(\mathcal{P}^+)$ such that $\sigma'_{\mathcal{R}} \circ \pi_{\mathcal{P}^+,\mathcal{R}} = j \restriction \mathcal{P}^+$;

[26] If $\mathcal{T}$ is short, then there is a unique branch b given by the $\mathcal{Q}$-structure.

[27] Recall that maximal trees always have the last model; regardless of whether there is a cofinal branch.

(iv) $\Sigma_{\mathcal{R}(\alpha)}$ has branch condensation and is guided by $\{j(f) \mid f \in \mathcal{P}^+ \wedge j(f)$ is appropriate$\}$.

Lemma 3.15. *$\mathcal{H} = \mathcal{M}(\mathcal{T})^+$ and τ is the identity.*

Proof. Let $\pi = \bigcup_l \pi^{\Lambda_l}_{\mathcal{M}(\mathcal{T})^+,\infty} \restriction H^{\mathcal{M}(\mathcal{T})^+}_{f_l}$ and $k = \pi \circ \tau$. Note that

$$\mathcal{R}(\alpha) = \bigcup_l H^{\mathcal{R}(\alpha)}_{f_l}.$$

This is because every $x \in \mathcal{R}(\alpha)$ has the form $\pi_{\mathcal{P}^+,\mathcal{R}}(g_l \restriction (\delta^{\mathcal{R}}_\beta + 1))(a)$ for some $l < \omega$ and some $a \in (\mathcal{R}|\delta^{\mathcal{R}}_\beta)^{<\omega}$ and $\mathcal{R}(\beta) \cup \{\pi_{\mathcal{P}^+,\mathcal{R}}(g_k \restriction \delta^{\mathcal{R}}_\beta + 1)\} \subset H^{\mathcal{R}(\alpha)}_{f_k}$ for all k.[28] This means that there is a Σ_1 map $i : \mathcal{R}(\alpha) \to \mathcal{H}$. Furthermore, letting E be the $(\mathrm{crt}(i), \delta^{\mathcal{H}})$-extender derived from i, then E gives the ultrapower map $i^+ : \mathcal{R} \to \mathcal{H}^+ =_{\mathrm{def}} \mathrm{Ult}(\mathcal{R}, E)$ extending i. Letting $k^+(i^+(g)(a)) = \sigma_{\mathcal{R}}(g)(k(a))$, we have that: $k^+ \circ i^+ = \sigma_{\mathcal{R}}$.

Now we can use the proof of Lemma 3.7 to conclude that $\mathcal{H}$ is full. If τ is not the identity, then letting $\gamma = \mathrm{crt}(\tau)$, we have: γ is Woodin in $\mathcal{H}$, $\tau(\gamma) = \delta(\mathcal{T})$ is Woodin in $\mathcal{M}(\mathcal{T})^+$, and $\mathcal{H} \lhd \mathcal{M}(\mathcal{T})^+$. Since $\mathcal{H}$ is full, γ is Woodin in $\mathcal{M}(\mathcal{T})^+$. This contradicts the fact that there are no Woodin cardinals in $\mathcal{M}(\mathcal{T})^+$ between $\delta^{\mathcal{R}}_\beta$ and $\delta(\mathcal{T})$. $\square$

Lemma 3.15 proves (ii); furthermore, it implies that $\sup \gamma^{\mathcal{M}(\mathcal{T})^+}_{f_n} = \delta(\mathcal{T})$. This means that b is cofinal in $\mathcal{T}$, hence proves (i) (see [8, Theorem 5.4.14] for an argument). We also get that $\Sigma_{\mathcal{R}(\alpha)}$ is guided by $\{f_n \mid n < \omega\}$ (and is the unique such strategy). At this point, we don't know that $\Sigma_{\mathcal{R}(\alpha)} \in M[n]$ and has branch condensation. The following is the main technical lemma.

Lemma 3.16 (Notations as above)**.** *The following hold:*

1. *$\Sigma_{\mathcal{R}(\alpha)}$ acts on all of $\mathcal{R}$ and whenever $i : \mathcal{R} \to \mathcal{S}$ is according to $\Sigma_{\mathcal{R}(\alpha)}$, there are embeddings $\sigma'_{\mathcal{R}} : \mathcal{R} \to j(\mathcal{P}^+)$ such that $j \restriction \mathcal{P}^+ = \sigma'_{\mathcal{R}} \circ \pi_{\mathcal{P}^+,\mathcal{R}}$ and $\tau : \mathcal{S} \to j(\mathcal{P}^+)$ such that $\sigma'_{\mathcal{R}} = \tau \circ i$ and $\tau \restriction \mathcal{S}(i(\alpha)) = \pi^{\Phi}_{\mathcal{S}(i(\alpha)),\infty}$, where Φ is the i-tail of $\Sigma_{\mathcal{R}(\alpha)}$.*

2. *$\Sigma_{\mathcal{R}(\alpha)}$ has branch condensation.*

3. *If $\pi_{\mathcal{P}^+,\mathcal{R}} \in M$, then $\Sigma_{\mathcal{R}(\alpha)}$ is in $j^+(\Omega^\lambda_{\mu,g})$.*

Proof. For (1), the map $\sigma'_{\mathcal{R}}$ is defined as follows: for any $x \in \mathcal{R}$, letting $x = \pi_{\mathcal{P}^+,\mathcal{R}}(g_l)(a)$ for some $l < \omega$ and $a \in (\delta^{\mathcal{R}}_\alpha)^{<\omega}$, then letting $k \geq l$ be such that $a \in H^{\mathcal{R}(\alpha)}_{f_k}$

$$\sigma'_{\mathcal{R}}(x) = j(g_l)(\pi^{\Lambda_k}_{\mathcal{R}(\alpha),\infty}(a)) = f_l(\pi^{\Lambda_k}_{\mathcal{R}(\alpha),\infty}(a)).$$

The map is well-defined by line 3.9. Using Theorem 3.13, we can show that $\sigma'_{\mathcal{R}}$ is Σ_1 elementary as follows. Suppose φ is Σ_1 and x, g_l, a are as above (we may increase l and assume $x = \pi^{\mathcal{P}^+,\mathcal{R}}(g_l)(a)$ and $a \in H^{\mathcal{R}(\alpha)}_{f_l}$). Suppose

$$j(\mathcal{P}^+) \vDash \varphi[f_l(\pi^{\Lambda_l}_{\mathcal{R}(\alpha),\infty}(a))] \Leftrightarrow \mathcal{R} \nvDash \varphi[x].$$

[28] $\pi_{\mathcal{P}^+,\mathcal{R}}(g_k \restriction \delta^{\mathcal{R}}_\beta + 1) \subset H^{\mathcal{R}(\alpha)}_{f_k}$ holds by elementarity of $\pi_{\mathcal{P}^+,\mathcal{R}}$ and the fact that the corresponding containment holds in $\mathcal{P}^+$.

Since φ is Σ_1 and $j[\mathcal{P}^+]$ is cofinal in $j(\mathcal{P}^+)$, we can find some $\mathcal{M} \lhd \mathcal{P}^+$ such that the above is equivalent to

$$j(\mathcal{M}) \vDash \varphi[f_l(\pi^{\Lambda_l}_{\mathcal{R}(\alpha),\infty}(a))] \Leftrightarrow \pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{M}) \nvDash \varphi[x].$$

This contradicts Theorem 3.13 applied to $\mathcal{M}, f_l$. Theorem 3.13 also gives that $j \restriction \mathcal{P}^+ = \sigma'_{\mathcal{R}} \circ \pi_{\mathcal{P}^+,\mathcal{R}}$ and

$$\sigma'_{\mathcal{R}} \restriction \mathcal{R}(\alpha) = \pi^{\Sigma_{\mathcal{R}(\alpha)}}_{\mathcal{R}(\alpha),\infty}. \tag{3.11}$$

By Lemma 3.15, $\mathcal{S}(i(\alpha)) = \bigcup_l H^{\mathcal{S}(i(\alpha))}_{f_l}$. Now define $\tau : \mathcal{S} \to j(\mathcal{P}^+)$ as follows. Let $\vec{\mathcal{U}}$ be the stack giving rise to i and $\Lambda = (\Sigma_{\mathcal{S}(i(\beta))})_{\vec{\mathcal{U}}}$. For $x \in \mathcal{S}$, say $x = i(g)(a)$ for some $g \in \mathcal{R}$ and $a \in \mathcal{S}(i(\alpha))^{<\omega}$, and say $g = \pi_{\mathcal{P}^+,\mathcal{R}}(g_l)(b)$ for some $b \in \mathcal{R}(\beta)^{<\omega}$, we let

$$\tau(x) = f_l(\pi^{\Lambda}_{\mathcal{S}(i(\beta)),\infty}(i(b)))(\pi^{\Phi}_{\mathcal{S}(i(\alpha)),\infty}(a)).$$

Using line 3.11, we get that τ is Σ_1 elementary and $\tau \circ i = \sigma'_{\mathcal{R}}$. This proves (1). The following claim proves (2).

Claim 3.17. *$\Sigma_{\mathcal{R}(\alpha)}$ has branch condensation.*

Proof. Suppose not. Then there are a (non dropping) stack $\vec{\mathcal{W}}$ with last model $\mathcal{S} \in M[n]$ and a normal tree $\mathcal{T}$ of limit length based on a window $(\delta^{\mathcal{S}}_{\beta^*}, \delta^{\mathcal{S}}_{\alpha^*})$ such that

1. $\beta^* = \pi^{\vec{\mathcal{W}}}(\beta)$, $\alpha^* = \pi^{\vec{\mathcal{W}}}(\alpha) = \beta^* + 1$.
2. $\vec{\mathcal{W}}$ and $\mathcal{T}$ are according to $\Sigma_{\mathcal{R}(\alpha)}$.
3. generators of $\vec{\mathcal{W}}$ are below $\delta^{\mathcal{S}}_{\beta^*}$.
4. $\Sigma_{\mathcal{S}(\beta^*),\vec{\mathcal{W}}}$ has branch condensation.
5. Let $b = \Sigma_{\mathcal{S},\vec{\mathcal{W}}}(\mathcal{T})$. There are a cofinal branch $c \neq b$, an iteration map $i : \mathcal{R} \to \mathcal{Y}$ according to $\Sigma_{\mathcal{R}(\alpha)}$, and a $\sigma : \mathcal{M}^{\mathcal{T}}_c \to \mathcal{Y}$ such that $\sigma \circ \pi^{\mathcal{T}}_c \circ \pi^{\vec{\mathcal{W}}} = i$.

We proceed to obtain a contradiction. Let $\tau : \mathcal{Y} \to j(\mathcal{P}^+)$ come from the construction of $\Sigma_{\mathcal{R}(\alpha)}$ and $\sigma_{\mathcal{S}} : \mathcal{S} \to j(\mathcal{P}^+)$ be the realization map. By arguments above and the fact that $\mathcal{M}^{\mathcal{T}}_c$ realizes into $j(\mathcal{P}^+)$ via $\tau \circ i$ and $\tau \circ i$ factors into $\sigma_{\mathcal{S}}$, there is a strategy Λ such that $\pi^{\Lambda}_{\mathcal{M}^{\mathcal{T}}_c(\alpha^*),\infty} = \psi \restriction \mathcal{M}^{\mathcal{T}}_c(\alpha^*)$ for some ψ such that $j \restriction \mathcal{P}^+ = \psi \circ \pi^{\mathcal{T}}_c \circ \pi^{\vec{\mathcal{W}}}$ and Λ witnesses that $\mathcal{M}^{\mathcal{T}}_c(\alpha^*)$ is strongly $(\mathcal{S}(\beta^*), f_k)$-iterable for all $k < \omega$. This means $\Lambda_{\mathcal{M}^{\mathcal{T}}_c(\alpha^*)} = \Sigma_{\mathcal{M}^{\mathcal{T}}_b(\alpha^*),\vec{\mathcal{W}}^\frown\mathcal{T}^\frown b}$.

Let $\Psi = \Sigma_{\mathcal{M}^{\mathcal{T}}_b(\alpha^*),\vec{\mathcal{W}}^\frown\mathcal{T}^\frown b}$. Let $\phi : \mathcal{M}^{\mathcal{T}}_b \to j(\mathcal{P}^+)$ be the realization map. Note that

$$\phi \restriction \mathcal{M}^{\mathcal{T}}_b(\alpha^*) = \psi \restriction \mathcal{M}^{\mathcal{T}}_c(\alpha^*) = \pi^{\Lambda}_{\mathcal{M}^{\mathcal{T}}_c(\alpha^*),\infty} = \pi^{\Psi}_{\mathcal{M}^{\mathcal{T}}_b(\alpha^*),\infty}.$$

Now we aim to show $b = c$, which contradicts our assumption.

By assumptions on $\mathcal{W}$, we have

$$\delta^{\mathcal{S}}_{\alpha^*} = \sup(A), \text{ where } A = \{\gamma < \delta^{\mathcal{S}}_{\alpha^*} \mid \exists g \in \mathcal{P}^+ \exists b \in (\delta^{\mathcal{R}}_{\beta})^{<\omega} \exists a \in (\delta^{\mathcal{S}}_{\beta^*})^{<\omega}\ \gamma = \pi^{\mathcal{W}}(\pi_{\mathcal{P}^+,\mathcal{R}}(g)(b))(a)\}.$$

For each $\gamma = \pi^{\mathcal{W}}(\pi_{\mathcal{P}^+,\mathcal{R}}(g)(b))(a) \in A$,

$$\psi \circ \pi_c^{\mathcal{T}}(\gamma) = j(g)(\pi^{\Psi_{\mathcal{M}_b^{\mathcal{T}}(\alpha^*)}}(\pi^{\vec{\mathcal{W}}}(b), a)). \tag{3.12}$$

And

$$\phi \circ \pi_b^{\mathcal{T}}(\gamma) = j(g)(\pi^{\Psi_{\mathcal{M}_b^{\mathcal{T}}(\alpha^*)}}(\pi^{\vec{\mathcal{W}}}(b), a)). \tag{3.13}$$

Since $\phi \restriction \mathcal{M}_b^{\mathcal{T}}(\alpha^*) = \psi \restriction \mathcal{M}_c^{\mathcal{T}}(\alpha^*)$, we get that

$$\pi_c^{\mathcal{T}}(\gamma) = \pi_b^{\mathcal{T}}(\gamma). \tag{3.14}$$

3.12, 3.13, and 3.14 imply $\mathrm{rng}(\pi_b^{\mathcal{T}}) \cap \mathrm{rng}(\pi_c^{\mathcal{T}})$ is cofinal in $\delta(\mathcal{T})$. So $b = c$. Contradiction.

□

Now we show that $\Sigma_{\mathcal{R}(\alpha)} \cap M \in M$ in the case $\pi_{\mathcal{P}^+,\mathcal{R}} \in M$. In this case, $\mathcal{R} \in M$. Note also that the construction of $\Sigma_{\mathcal{R}(\alpha)}$ doesn't depend on the enumeration of the set $\{j(f) \mid f \in \mathcal{P}^+\}$ in $V[n]$; it only depends on the set itself. So in fact, $\Sigma_{\mathcal{R}(\alpha)} \restriction V \in V$.

Claim 3.18. $\Sigma_{\mathcal{R}(\alpha)} \restriction M \in M$.

Proof. Suppose first $\mathcal{R} \in M_\kappa$. As noted before, $\Sigma_{\mathcal{R}(\alpha)} \cap M_\kappa \in M$ (since $M_{\kappa+1} = V_{\kappa+1}$). Using the fact that κ is strong in M, we can lift $\Sigma_{\mathcal{R}(\alpha)} \cap M_\kappa$ to a strategy Λ in $M_j(\mu)$ acting on trees in $M_{j(\mu)}$, and Λ has branch condensation and is $j^+(\Omega^\lambda_{\mu,g})$-fullness preserving. We claim that $\Sigma_{\mathcal{R}(\alpha)} \cap M = \Lambda \in M$. Suppose not. Then by a standard fact, $\mathrm{cof}^V(\delta_\alpha^{\mathcal{R}}) = \omega$.[29] Let $\langle x_n \mid n < \omega \rangle$ be cofinal in $\delta_\alpha^{\mathcal{R}}$ and say, $x_n = \pi_{\mathcal{P}^+,\mathcal{R}}(h_n)(a_n)$ for some $h_n \in \mathcal{P}^+$ and $a_n \in \mathcal{R}(\beta)$.

From our assumption, $((\delta^{\mathcal{P}})^+)^{\mathcal{P}^+} = \mu^+$. Hence, there is some $\mathcal{N} \lhd \mathcal{P}^+$ such that $\rho_\omega(\mathcal{N}) = \delta^{\mathcal{P}}$, $\{h_n \mid n < \omega\} \subseteq \mathcal{N}$ and in $\mathcal{P}^+$, $\mathrm{cof}(o(\mathcal{N})) = \omega$. Let $\mathcal{N}$ be the least that contains some such h_n's and let p be the standard parameter of $\mathcal{N}$ so for each n, h_n is definable in $\mathcal{N}$ from p and some $a \in \mathcal{P}$. Note that we can assume $\pi_{\mathcal{P}^+,\mathcal{R}}[\delta^{\mathcal{P}}] \subseteq \mathcal{R}(\beta)$. This means $\sup(A) = \delta_\alpha^{\mathcal{R}}$, where

$$A = \{g(x) \mid g \in \pi_{\mathcal{P}^+,\mathcal{R}}(\mathcal{N}) \wedge x \in \mathcal{R}(\beta)\}.$$

By the choice of $\mathcal{N}$ and the fact that A can be computed in $\mathcal{R}$, $\delta_\alpha^{\mathcal{R}}$ is singular in $\mathcal{R}$. Contradiction.

Now assume $\mathcal{R} \in M_{j(\mu)} \backslash M_\kappa$. Note that whenever $\mathcal{W} \in V_\kappa[m] = M_\kappa[m]$ is such that there is an embedding $\pi : \mathcal{W} \to \mathcal{R}$ with $\pi(\gamma) = \alpha$ and there is an embedding $\pi^* : \mathcal{P}^+ \to \mathcal{W}$ such that $\pi_{\mathcal{P}^+,\mathcal{R}} = \pi \circ \pi^*$, then $\mathcal{W}(\gamma)$ has a nice strategy as witnessed by some realization map $i : \mathcal{W} \to j(\mathcal{P}^+)$ such that the strategy restricted to $M_\kappa[m]$ has branch condensation. Let $h : M \to N$ witness κ is $j(\mu)^+$-strong in M and we may assume $h = i_F^M$ for some extender F; h can be extended to a map, which we also call h, from $M[m]$ to $N[p]$ for some V-generic $p \subseteq Col(\omega, < h(\kappa))$ such that $p \restriction Col(\omega, < j(\mu)) = n$. Then by absoluteness, in $N[n]$, there is an embedding π from $\mathcal{R}$ into $h(\mathcal{R})$ such that $h(\pi_{\mathcal{P}^+,\mathcal{R}}) = \pi \circ \pi_{\mathcal{P}^+,\mathcal{R}}$. And hence there is a strategy $\Lambda \in M[n]$ of $\mathcal{R}$ with branch

[29]There is a tree $\mathcal{U} \in M$ of limit length with $\Sigma_{\mathcal{R}(\alpha)}(\mathcal{T}) = b \neq c = \Lambda(\mathcal{T})$. Hence $\mathrm{cof}^V(\delta(\mathcal{U})) = \omega$. Since $\delta_\alpha^{\mathcal{R}}$ is mapped cofinally into $\delta(\mathcal{U})$ by either branch embedding , $\mathrm{cof}^V(\delta_\alpha^{\mathcal{R}}) = \omega$

condensation acting on stacks in $M_{j(\mu)}[n] = N_{j(\mu)[n]}$ based on $\mathcal{R}(\alpha)$ such that $\Lambda \restriction M \in M$[30]. By the same argument as above, we get that $\Lambda \restriction M = \Sigma_{\mathcal{R}(\alpha)} \restriction M$.

□

We now prove (3). Let Λ be as in the proof of Claim 3.18; so $\Lambda \in M[n]$, Λ has branch condensation and is $j^+(\Omega^\lambda_{\mu,g})$-fullness preserving. It's enough to prove $\Lambda = \Sigma_{\mathcal{R}(\alpha)}$ so that $\Sigma_{\mathcal{R}(\alpha)} \in M[n]$; (3) then follows from $Proj(j(\kappa), j(\lambda), \Sigma_{\mathcal{R}(\alpha)})$. The equality follows from an argument similar to the proof of Claim 3.17 and the fact that for any $k : \mathcal{R} \to \mathcal{M}$ in $M[n]$ according to Λ and $\Sigma_{\mathcal{R}(\alpha)}$, there is an $i : \mathcal{R} \to \mathcal{S}$ according to $\Lambda \restriction M = \Sigma_{\mathcal{R}(\alpha)} \restriction M$ such that there is some map $\sigma : \mathcal{M} \to \mathcal{S}$ such that $\sigma \in M[n]$ and $\sigma \circ k = i$[31]. This completes the proof of the lemma.

□

We have finished the construction of a partial strategy Σ acting on trees in $M_{j(\mu)} \cup M_\kappa[m]$. During the course of the construction, we also showed that non-dropping iterates $\mathcal{R}$ of Σ are $j^+(\Omega^\lambda_{\mu,g})$-full and there is a Σ_1 elementary embedding $\sigma_{\mathcal{R}} : \mathcal{R} \to j(\mathcal{P}^+)$ such that $\sigma_{\mathcal{R}} \circ \pi_{\mathcal{P}^+,\mathcal{R}} = j \restriction \mathcal{P}^+$, and for each $\alpha < \lambda^{\mathcal{R}}$, $\sigma_{\mathcal{R}} \restriction \mathcal{R}|\delta^{\mathcal{R}}_\alpha$ is the iteration map according to the $\sigma_{\mathcal{R}}$-nice strategy Σ_α constructed above. Next we show branch condensation of Σ.

Lemma 3.19. *Σ has branch condensation.*

Proof. Suppose not. Then as in [4], we can find a "minimal place" where branch condensation fails. More precisely, there are a (non dropping) stack $\vec{\mathcal{W}}$ with last model $\mathcal{R}$ and a normal tree $\mathcal{T}$ of limit length based on a window $(\delta^{\mathcal{R}}_\alpha, \delta^{\mathcal{R}}_{\alpha+1})$ such that

1. $\vec{\mathcal{W}}$ and $\mathcal{T}$ are according to Σ.
2. generators of $\vec{\mathcal{W}}$ are below $\delta^{\mathcal{R}}_\alpha$.
3. $\Sigma_{\mathcal{R}(\alpha),\vec{\mathcal{W}}}$ has branch condensation.
4. Let $b = \Sigma_{\mathcal{R},\vec{\mathcal{W}}}(\mathcal{T})$. There are a cofinal branch $c \neq b$, an iteration map $i : \mathcal{P}^+ \to \mathcal{S}$ according to Σ, and a $\sigma : \mathcal{M}^{\mathcal{T}}_c \to \mathcal{S}$ such that $\sigma \circ \pi^{\mathcal{T}}_c \circ \pi^{\vec{\mathcal{W}}} = i$.

The rest of the proof is just as in the proof of Claim 3.17.

□

Let $\Lambda = \Sigma \restriction M_\kappa[m] \in M[m]$. Using the fact that κ is strong in M, we extend Λ to a strategy $\Lambda^+ \in M[n]$ such that Λ^+ acts on all stacks in $M_{j(\mu)}[n]$, has branch condensation, and is $j(\Omega^\lambda_{\mu,g})$-fullness preserving. Furthermore, since $\Sigma \cap M_\kappa \in M$, we also get that $\Lambda^+ \restriction M \in M$.

[30] From the point of view of $\mathrm{Ult}(V, F)[n]$, the strategy Λ is $\{i^V_F(j(f)) \mid f \in \mathcal{P}^+\}$-guided, acts on trees in $i^V_F(M)[n]$ and doesn't depend on any generic enumeration of the set $\{i^V_F(j(f)) \mid f \in \mathcal{P}^+\}$, so if in addition $\mathcal{W} \in M_\kappa$, then Λ's restriction to $N_{j(\mu)} = M_{j(\mu)} = i^V_F(M)_{j(\mu)}$ is in M even though we first collapsed $\mathcal{R}$ to ω in N to find an embedding from $\mathcal{R}$ into $\pi(\mathcal{R})$.

[31] Suppose $\vec{\mathcal{W}}^\frown\mathcal{T}$ is according to both strategies, $\vec{\mathcal{W}}$ is on $\mathcal{R}(\beta)$ and $\mathcal{T}$ on $\mathcal{M}^{\vec{\mathcal{W}}}$ is such that $b = \Lambda_{\mathcal{M}^{\mathcal{T}}}(\mathcal{T}) \neq c = (\Sigma_{\mathcal{R}(\alpha)})_{\mathcal{M}^{\mathcal{T}}}(\mathcal{T})$. Then there is some iteration $i : \mathcal{R} \to \mathcal{S}$ according $\Lambda \restriction M = \Sigma_{\mathcal{R}(\alpha)} \restriction M$ such that there are iteration maps $\sigma_0 : \mathcal{M}^{\mathcal{T}}_b \to \mathcal{S}$ according to the tail of Λ and $\sigma_1 : \mathcal{M}^{\mathcal{T}}_c \to \mathcal{S}$ according to the tail of $\Sigma_{\mathcal{R}(\alpha)}$. We can then run the proof of Claim 3.17 to get that $b = c$.

Lemma 3.20. *$\Sigma \restriction M \subseteq \Lambda^+ \restriction M \in M$.*

Proof. Suppose $\Sigma \restriction M \nsubseteq \Lambda^+ \restriction M$. Then there are in M a (non dropping) stack $\vec{\mathcal{W}}$ with last model $\mathcal{R}$ and a normal tree $\mathcal{T}$ of limit length based on a window $(\delta_\alpha^{\mathcal{R}}, \delta_{\alpha+1}^{\mathcal{R}})$ such that

1. $\vec{\mathcal{W}}$ and $\mathcal{T}$ are according to both strategies.
2. generators of $\vec{\mathcal{W}}$ are below $\delta_\alpha^{\mathcal{R}}$.
3. Let $b = \Sigma_{\mathcal{R},\vec{\mathcal{W}}}(\mathcal{T})$ and $c = \Lambda^+_{\mathcal{R},\vec{\mathcal{W}}}(\mathcal{T})$. Then $b \neq c$.

This means in V as well as in M, $\mathrm{cof}(\delta_{\alpha+1}^{\mathcal{R}}) = \omega$. The rest of the proof is just as in the proof of Claim 3.18.

□

Let $\Psi = \Lambda^+ \restriction M = \Sigma \restriction M$. By generic comparison using the fact that Λ^+ has branch condensation and is $j^+(\Omega^\lambda_{\mu,g})$-fullness preserving (see [4]), we get:

> For any Λ^+-iterate $\mathcal{R}$ of $\mathcal{P}^+$, letting $i : \mathcal{P}^+ \to \mathcal{R}$ be the iteration map, there is a Ψ-iterate $\mathcal{S}$ of $\mathcal{P}^+$ such that, letting $h : \mathcal{P}^+ \to \mathcal{S}$ be the iteration map, there is a map $\sigma : \mathcal{R} \to \mathcal{S}$ such that $h = \sigma \circ i$. In fact, σ is a Λ^+-iteration map.

Using the above paragraph and the properties of Σ, letting $\mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+)$ be the direct limit of (all countable) Λ^+-iterates of $\mathcal{P}^+$ in $M[n]$, then there is a Σ_1 elementary map $\tau : \mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+) \to \mathcal{P}^+$ such that $\tau \circ \pi^{\Lambda^+}_{\mathcal{P}^+,\infty} = j \restriction \mathcal{P}^+$ and $\mathrm{crt}(\tau) = \delta^{\mathcal{M}_\infty(\mathcal{P}^+,\Lambda^+)}$.

The map τ is defined as follows: for any $x \in \mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+)$, let $\mathcal{R} \in M$ be a Ψ-iterate of $\mathcal{P}^+$ such that there is some $y \in \mathcal{R}$ such that $\pi^{\Psi_{\mathcal{R}}}_{\mathcal{R},\infty}(y) = x$. Now by construction of Ψ, there is a map $\tau_{\mathcal{R}} : \mathcal{R} \to j(\mathcal{P}^+)$ such that $j \restriction \mathcal{P}^+ = \tau_{\mathcal{R}} \circ \pi^{\Psi}_{\mathcal{P}^+,\mathcal{R}}$ and $\tau_{\mathcal{R}} \restriction \delta^{\mathcal{R}}$ agrees with the iteration map by Ψ. We then let $\tau(x) = \tau_{\mathcal{R}}(y)$.

Lemma 3.21. *$(\mathcal{P}^+, \Sigma)$ is a hod pair below κ in $V[m]$ and $(\mathcal{P}^+, \Sigma \restriction HC^{M[n]}) \in j^+(\Omega^\lambda_{\mu,g})$.*

Proof. The second clause follows from the first clause and $Proj(j(\kappa), j(\lambda), \Sigma)$ in M. It suffices to prove the first clause. Let $l = m \cap Coll(\omega, \nu)$ where $\mu < \nu < \kappa$ and $\mathcal{P}^+$ is countable in $V^{Coll(\omega,\nu)}$ and let $a = \{f \mid f \in \mathcal{P}^+ \wedge f \text{ is appropriate}\}$ and $b = \{j(f) \mid f \in \mathcal{P}^+ \wedge f \text{ is appropriate}\}$. Let $\phi(\mathcal{P}^+, a, b, \vec{\mathcal{T}})$ be the formula stating "$\vec{\mathcal{T}}$ is a stack on $\mathcal{P}^+$ according to Σ". Fix $\xi >> j(\mu)$ and let $X \prec V_\xi[l]$ be countable in $V[l]$ and X contains all relevant objects and $o(\mathcal{P}^+) \subset X$. Let $\pi : N \to X$ be the uncollapse map and $(\bar{b}, \bar{\kappa}) = \pi^{-1}(b, \kappa)$. Let $\bar{m} \in V[l]$ be N-generic for a poset in $H^N_{\bar{\kappa}}$. Let $\vec{\mathcal{T}} \in M[\bar{m}]$. Then we claim that

$$M[\bar{m}] \vDash \phi[\mathcal{P}^+, a, \bar{b}, \vec{\mathcal{T}}] \Leftrightarrow V[l] \vDash \phi[\mathcal{P}^+, a, b, \vec{\mathcal{T}}]. \tag{3.15}$$

Suppose $\vec{\mathcal{T}} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha \mid \alpha \leq \eta \rangle$ and suppose $\mathcal{T}_\eta$ is based on $\mathcal{M}_\eta(\gamma + 1)$ for some γ. Suppose by induction, 3.15 holds for $\vec{\mathcal{T}}|\alpha$ for all $\alpha < \eta$. We now show 3.15 holds for $\mathcal{T}_\eta$. But 3.15 holds for $\mathcal{T}_\eta$ because $\pi[\bar{b}] = b$ and so $\mathcal{T}_\eta$ is $\bar{b}$-guided in $M[\bar{m}]$ if and only if $\mathcal{T}_\eta$ is b-guided in $V[l]$. □

The above definition of τ and Lemma 3.21 imply that the direct limit $\mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+)$ is in $j^+(\Omega^\lambda_{\mu,g})$ and $\mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+) \vDash \delta^{\mathcal{M}_\infty(\mathcal{P}^+,\Lambda^+)}$ is regular. By [4], in $M[n]$, $L(\Gamma(\mathcal{P}^+, \Lambda^+), \mathbb{R}) \vDash \mathsf{AD}_\mathbb{R} + \Theta$ is regular. Theorem 3.3 follows from elementarity.

The conclusion of Theorem 3.3 contradicts our smallness assumption (†). So there must be a model of "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular".

4. THE STRENGTH OF FAILURE OF UBH FOR TAME TREES

In this section, we prove Theorem 0.2. The proof of [7, Lemma 5.2] shows that whenever $(\mathcal{R}, \Psi)$ is a hod pair below κ, then $Proj(\kappa, \lambda, \Psi)$. The hypothesis of Theorems 3.2 and 3.3 is satisfied. These theorems in turns imply the conclusion of Theorem 0.2.

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