Non-tame mice from tame failures of the unique branch hypothesis*†

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Abstract

In this paper, we show that the failure of the unique branch hypothesis (UBH) for tame trees (see Definition 5.1) implies that in some homogenous generic extension of V there is a transitive model M containing $Ord \cup \mathbb{R}$ such that $M \vDash \mathsf{AD}^+ + \Theta > \theta_0$. In particular, this implies the existence (in V) of a non-tame mouse. The results of this paper significantly extend Steel's earlier results from [16] for tame trees.

In this paper, we establish, using the core model induction, a lower bound for certain failures of the *Unique Branch Hypothesis*, (UBH), which is the statement that every iteration

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tree that acts on V has at most one cofinal well-founded branch. For the rest of this paper, all trees considered are nonoverlapping, that is whenever E and F are extenders such that E is used before F along a branch of the tree, then $\mathrm{lh}(E) \leq \mathrm{crit}(F)$. The following is our main theorem. Tame trees[1] are defined in Definition 5.1: roughly speaking, these are the trees in which the critical point of any branch embedding is above a strong cardinal which reflects strong cardinals.

Theorem 0.1 (Main Theorem)**.** *Suppose there is a proper class of strong cardinals and UBH fails for tame trees. Then in a set generic extension of V, there is a transitive inner model M such that $Ord \cup \mathbb{R} \subseteq M$ and $M \vDash \mathsf{AD}^+ + \theta_0 < \Theta$. In particular, there is a non-tame mouse.*

UBH was first introduced by Martin and Steel in [3]. Towards showing UBH, Neeman, in [5], showed that a certain weakening of UBH called cUBH holds provided there are no non-bland mice[2]. However, in [17], Woodin showed that in the presence of supercompact cardinals UBH can fail for tame trees[3]. It is, however, still an important open problem whether UBH holds for trees that use extenders that are $2^{\aleph_0}$-closed in the models that they are chosen from. A positive resolution of this problem will lead to the resolution of *the inner model problem* for superstrong cardinals and beyond. It is worth remarking that the aforementioned form of UBH for tame trees will also lead to the resolution of the inner model problem for superstrong cardinals and beyond. Our work can be viewed as an attempt to prove UBH for tame trees by showing that its failure is strong consistency-wise.

In this direction, in [16], Steel showed that the failure of UBH for (nonoverlapping) trees implies that there is an inner model with infinitely many Woodin cardinals. If in addition UBH fails for some tree $\mathcal{T}$ such that $\delta(\mathcal{T})$ is in the image of two branch embeddings witnessing the failure of UBH for $\mathcal{T}$ then Steel obtained an inner model with a strong cardinal which is a limit of Woodin cardinals. For tame trees (which, as mentioned in the footnote, include a class of examples constructed by Woodin in [17]), the Main Theorem considerably strengthens the aforementioned result of Steel and because the proof presented here is via the core model induction, we expect that it will yield much more: we believe that our proof, coupled with arguments from [8], will give the existence of a transitive inner model M such that $Ord \cup \mathbb{R} \subseteq M$ and $M \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". However, we still do not know if an

[1]The term "tame trees" is our ad-hod terminology and has nothing to do with the well-established term "tame" used to define a certain first-order property of premice.

[2]We will not use this terminology.

[3]Woodin constructs alternating chains whose branches are well-founded. Extenders of such trees can be demanded to reflect the set of strong cardinals which reflect strong cardinals. Hence critical points of the branch embeddings can certainly be demanded to be above the first strong cardinal which reflects strong cardinals.

arbitrary failure of UBH implies the existence of a non-tame mouse. Various arguments presented in this paper resemble the arguments given in [7] and [11], and some familiarity with those articles will be useful.

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1 Preliminaries

In this paper, we will need to make use of the material presented in Section 1 of [7], most of which, especially Section 1.1, carries over to the hybrid context by just changing the word "mouse" with "hybrid mouse". Because of this, we will only introduce a few main notions and will use Section 1 of [7] as our main background material. In particular, we assume that the reader has already translated the material of Section 1.1 of [7] into the language of hybrid mice.

1.1 Stacking mice

Following the notation of Section 1.3 of [7], we fix some uncountable cardinal λ and assume ZF. Notice that any function $f : H_\lambda \to H_\lambda$ can be naturally coded by a subset of $\mathcal{P}(\cup_{\kappa<\lambda}\mathcal{P}(\kappa))$. We then let $Code^*_\lambda : H_\lambda^{H_\lambda} \to \mathcal{P}(\cup_{\kappa<\lambda}\mathcal{P}(\kappa))$ be one such coding. If $\lambda = \omega_1$ then we just write $Code^*$. Because for $\alpha \leq \lambda$, any (α, λ)-iteration strategy[4] for a hybrid premouse of size $< \lambda$ is in $H_\lambda^{H_\lambda}$, we have that any such strategy is in the domain of $Code^*_\lambda$.

Suppose $\Lambda \in dom(Code^*_\lambda)$ is a strategy with hull condensation and $\mu \leq \lambda$. Recall that we say F is (μ, Λ)-mouse operator if for some $X \in H_\lambda$ and formula ϕ in the language of Λ-mice, whenever Y is such that $X \in Y$, $F(Y)$ is the minimal μ-iterable Λ-mouse satisfying $\phi[Y]$.

We then let $Code_\lambda$ be $Code^*_\lambda$ restricted to $F \in dom(Code^*_\lambda)$ which are defined by the following recursion.

1. for some $\alpha \leq \lambda$, F is a (α, λ)-iteration strategy with hull condensation[5],

[4]This is an iteration strategy for stacks of less than α normal trees, each of which has length less than λ. Typically these are fine-structural n-maximal iteration trees (as defined in [4]), where n is the degree of soundness of the premouse we iterate. We will suppress this parameter thoughout our paper.

[5]In this case as well as in cases below $\alpha = 0$ is allowed.

2. for some $\alpha \leq \lambda$ and for some (α, λ)-iteration strategy $\Lambda \in dom(Code^*_\lambda)$ with hull condensation, F is a (λ, Λ)-mouse operator,

3. for some $\alpha \leq \lambda$, for some (α, λ)-iteration strategy $\Lambda \in dom(Code^*_\lambda)$ with hull condensation, for some (λ, Λ)-mouse operator $G \in dom(Code^*_\lambda)$ and for some $\beta \leq \lambda$, F is a (β, Λ)-iteration strategy with hull condensation for some G-mouse $\mathcal{M} \in H_\lambda$.

When $\lambda = \omega_1$ then we just write $Code$ instead of $Code_{\omega_1}$. Given an $F \in dom(Code_\lambda)$ we let $\mathcal{M}_F$ be, in the case F is an iteration strategy, the structure that F iterates and, in the case F is a mouse operator, the base of the cone on which F is defined.

Let $\mathcal{P} \in H_\lambda$ be a hybrid premouse and for some $\alpha \leq \lambda$, let Σ be (α, λ)-iteration strategy with hull condensation for $\mathcal{P}$. Suppose now that $\Gamma \subseteq \mathcal{P}(\cup_{\kappa<\lambda}\mathcal{P}(\kappa))$ is such that $Code_\lambda(\Sigma) \in \Gamma$. Given a Σ-premouse $\mathcal{M}$, we say $\mathcal{M}$ is Γ-*iterable* if $|\mathcal{M}| < \lambda$ and $\mathcal{M}$ has a λ-iteration strategy (or (α, λ)-iteration strategy for some $\alpha \leq \lambda$) Λ such that $Code_\lambda(\Lambda) \in \Gamma$[6]. We let $Mice^{\Gamma,\Sigma}$ be the set of Σ-premice that are Γ-iterable.

Definition 1.1. *Given a Σ-premouse $\mathcal{M} \in H_\lambda$, we say $\mathcal{M}$ is countably α-iterable if whenever $\pi : \mathcal{N} \to \mathcal{M}$ is a countable submodel of $\mathcal{M}$, $\mathcal{N}$, as a Σ^π-mouse, is α-iterable. When $\alpha = \omega_1+1$ then we just say that $\mathcal{M}$ is countably iterable. We say $\mathcal{M}$ is countably Γ-iterable if whenever π and $\mathcal{N}$ are as above, $\mathcal{N}$ is Γ-iterable.*

Suppose $\mathcal{M}$ is a Σ-premouse. We then let $o(\mathcal{M}) = Ord \cap \mathcal{M}$. We also let $\mathcal{M}||\xi$ be $\mathcal{M}$ cutoff at ξ, i.e., we keep the predicate indexed at ξ. We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. We say ξ is a *cutpoint* of $\mathcal{M}$ if there is no extender E on $\mathcal{M}$ such that $\xi \in (\mathrm{cp}(E), lh(E)]$. We say ξ is a *strong cutpoint* if there is no E on $\mathcal{M}$ such that $\xi \in [\mathrm{cp}(E), lh(E)]$. We say $\eta < o(\mathcal{M})$ is *overlapped* in $\mathcal{M}$ if η isn't a cutpoint of $\mathcal{M}$. Given $\eta < o(\mathcal{M})$ we let

$$\mathcal{O}^{\mathcal{M}}_\eta = \cup\{\mathcal{N} \triangleleft \mathcal{M} : \rho(\mathcal{N}) = \eta \text{ and } \eta \text{ is not overlapped in } \mathcal{N}\}.$$

Given a self-wellordered[7] $a \in H_\lambda$ we define the stacks over a by

Definition 1.2. *1. $Lp^\Sigma(a) = \cup\{\mathcal{N} : \mathcal{N}$ is a countably iterable sound Σ-mouse over a such that $\rho(\mathcal{N}) = a\}$,*

2. $\mathcal{K}^{\lambda,\Gamma,\Sigma}(a) = \cup\{\mathcal{N} : \mathcal{N}$ is a countably Γ-iterable sound Σ-mouse over a such that $\rho(\mathcal{N}) = a\}$,

[6]Recall that iteration strategy for a Σ-mouse must respect Σ. In particular, all Λ-iterates of $\mathcal{M}$ are Σ-premice.

[7]I.e., self well-ordered, a set a is called self well-ordered if $trc(a \cup \{a\})$ is well-ordered in $L_1(a)$.

3. $\mathcal{W}^{\lambda,\Gamma,\Sigma}(a) = \cup\{\mathcal{N} : \mathcal{N}$ *is a* Γ*-iterable sound* Σ*-mouse over* a *such that* $\rho(\mathcal{N}) = a\}$.

When $\Gamma = \mathcal{P}(\cup_{\kappa<\lambda}\mathcal{P}(\kappa))$ then we omit it from our notation. We can define the sequences $\langle Lp^{\Sigma}_{\xi}(a) : \xi < \eta\rangle$, $\langle \mathcal{K}^{\lambda,\Gamma,\Sigma}_{\xi}(a) : \xi < \nu\rangle$, and $\langle \mathcal{W}^{\lambda,\Gamma,\Sigma}_{\xi}(a) : \xi < \mu\rangle$ as usual. For Lp operator the definition is as follows:

1. $Lp^{\Sigma}_0(a) = Lp^{\Sigma}(a)$,
2. for $\xi < \eta$, if $Lp^{\Sigma}_{\xi}(a) \in H_{\lambda}$ then $Lp^{\Sigma}_{\xi+1} = Lp^{\Sigma}(Lp^{\Sigma}_{\xi}(a))$,
3. for limit $\xi < \eta$, $Lp^{\Sigma}_{\xi} = \cup_{\alpha<\xi} Lp^{\Sigma}_{\alpha}(a)$,
4. η is least such that for all $\xi < \eta$, $Lp^{\Sigma}_{\xi}(a)$ is defined.

The other stacks are defined similarly.

1.2 (Γ, Σ)-suitable premice

Again we fix an uncountable cardinal λ such that a large fragment of ZF holds in V_{λ}. We also fix $\Sigma \in dom(Code_{\lambda})$ such that Σ is a (α, λ)-iteration strategy with hull condensation and $\Gamma \subseteq \mathcal{P}(\cup_{\kappa<\lambda}\mathcal{P}(\kappa))$ such that $Code_{\lambda}(\Sigma) \in \Gamma$. We now start outlining how to import the material from Subsection 1.3 of [7]. The most important notion we need from that subsection is that of (Γ, Σ)-suitable premouse which is defined as follows:

Definition 1.3 ((Γ, Σ)-suitable premouse)**.** *A* Σ*-premouse* $\mathcal{P}$ *is* (Γ, Σ)*-suitable if there is a unique cardinal* δ *such that*

1. $\mathcal{P} \vDash$ "δ *is the unique Woodin cardinal*",
2. $o(\mathcal{P}) = \sup_{n<\omega}(\delta^{+n})^{\mathcal{P}}$,
3. *for every* $\eta \neq \delta$, $\mathcal{W}^{\lambda,\Gamma,\Sigma}(\mathcal{P}|\eta) \vDash$ "η *isn't Woodin*".
4. *for any* $\eta < o(\mathcal{P})$, $\mathcal{O}^{\mathcal{P}}_{\eta} = \mathcal{W}^{\lambda,\Gamma,\Sigma}(\mathcal{P}|\eta)$.

If $\Gamma = \mathcal{P}(\cup_{\alpha<\lambda}\mathcal{P}(a))$ then we use λ instead of Γ. In particular, we use λ-suitable to mean Γ-suitable. We will do the same with all the other notions, such fullness preservation and short tree iterability, defined in this section

Suppose $\mathcal{P}$ is Γ-suitable. Then we let $\delta^{\mathcal{P}}$ be the δ of Definition 1.3. We then proceed as in Section 1.3 of [7] to define (1) nice iteration tree, (2) (Γ, Σ)-short tree, (3) (Γ, Σ)-maximal tree, (4) (Γ, Σ)-correctly guided finite stack and (5) the last model of a (Γ, Σ)-correctly guided finite stack by using $\mathcal{W}^{\lambda,\Gamma,\Sigma}$ operator instead of $\mathcal{W}^{\Gamma}$ operator. Next, we let

Definition 1.4 ($S(\Gamma,\Sigma)$ and $F(\Gamma,\Sigma)$)**.** $S(\Gamma,\Sigma) = \{\mathcal{Q} : \mathcal{Q}$ *is* (Γ,Σ)*-suitable*$\}$*. Also, we let* $F(\Gamma,\Sigma)$ *be the set of functions* f *such that* $dom(f) = S(\Gamma,\Sigma)$ *and for each* $\mathcal{P} \in S(\Gamma,\Sigma)$*,* $f(\mathcal{P}) \subseteq \mathcal{P}$ *and* $f(\mathcal{P})$ *is amenable to* $\mathcal{P}$*, i.e., for every* $X \in \mathcal{P}$*,* $X \cap f(\mathcal{P}) \in \mathcal{P}$*.*

Given $\mathcal{P} \in S(\Gamma,\Sigma)$ and $f \in F(\Gamma,\Sigma)$ we let $f_n(\mathcal{P}) = f(\mathcal{P}) \cap \mathcal{P}|((\delta^{\mathcal{P}})^{+n})^{\mathcal{P}}$. Then $f(\mathcal{P}) = \cup_{n<\omega} f_n(\mathcal{P})$. We also let

$$\gamma_f^{\mathcal{P}} = sup(\delta^{\mathcal{P}} \cap Hull_1^{\mathcal{P}}(\{f_n(\mathcal{P}) : n < \omega\})).$$

Notice that

$$\gamma_f^{\mathcal{P}} = \delta^{\mathcal{P}} \cap Hull_1^{\mathcal{P}}(\gamma_f^{\mathcal{P}} \cup \{f_n(\mathcal{P}) : n < \omega\}).$$

We then let

$$H_f^{\mathcal{P}} = Hull_1^{\mathcal{P}}(\gamma_f^{\mathcal{P}} \cup \{f_n(\mathcal{P}) : n < \omega\}).$$

If $\mathcal{P} \in S(\Gamma,\Sigma)$, $f \in F(\Gamma,\Sigma)$ and $i : \mathcal{P} \to \mathcal{Q}$ is an embedding then we let $i(f(\mathcal{P})) = \cup_{n<\omega} i(f_n(\mathcal{P}))$.

The following are the next block of definitions that routinely generalize into our context: (1) (f,Σ)-iterability, (2) $\vec{b} = \langle b_k : k < m\rangle$ witness (f,Σ)-iterability for $\vec{\mathcal{T}} = \langle \mathcal{T}_k, \mathcal{P}_k : k < m\rangle$, and (3) strong (f,Σ)-iterability. These definitions generalize by using $S(\Gamma,\Sigma)$ and $f \in F(\Gamma,\Sigma)$ instead of $S(\Gamma)$ and $F(\Gamma)$.

If $\mathcal{P}$ is strongly (f,Σ)-iterable and $\vec{\mathcal{T}}$ is a (Γ,Σ)-correctly guided finite stack on $\mathcal{P}$ with last model $\mathcal{R}$ then we let

$$\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f} : H_f^{\mathcal{P}} \to H_f^{\mathcal{R}}$$

be the embedding given by any $\vec{b}$ which witnesses the (f,Σ)-iterability of $\vec{\mathcal{T}}$, i.e., fixing $\vec{b}$ which witnesses f-iterability for $\vec{\mathcal{T}}$,

$$\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f} = \pi_{\vec{\mathcal{T}},\vec{b}} \restriction H_f^{\mathcal{P}}.$$

Clearly, $\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f}$ is independent of $\vec{\mathcal{T}}$ and $\vec{b}$. Here we keep Σ in our notation for $\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f}$ because it depends on a (Γ,Σ)-correct iterations. It is conceivable that $\mathcal{R}$ might also be a (Γ,Λ)-correct iterate of $\mathcal{P}$ for another Λ, in which case $\pi^{\Sigma}_{\mathcal{P},\mathcal{R},f}$ might be different from $\pi^{\Lambda}_{\mathcal{P},\mathcal{R},f}$. However, the point is that these embeddings agree on $H_f^{\mathcal{P}}$. Also, we do not carry Γ in our notation as it is usually understood from the context.

Given a finite sequence of functions $\vec{f} = \langle f_i : i < n\rangle \in F(\Gamma,\Sigma)$, we let $\oplus_{i<n} f_i \in F(\Gamma,\Sigma)$ be the function given by $(\oplus_{i<n} f_i)(\mathcal{P}) = \langle f_i(\mathcal{P}) : i < n\rangle$. We set $\oplus\vec{f} = \oplus_{i<n} f_i$.

We then let

$$\mathcal{I}_{\Gamma,F,\Sigma} = \{(\mathcal{P}, \vec{f}) : \mathcal{P} \in S(\Gamma,\Sigma),\ \vec{f} \in F^{<\omega} \text{ and } \mathcal{P} \text{ is strongly } \oplus\vec{f}\text{-iterable}\}.$$

Definition 1.5. *Given $F \subseteq F(\Gamma,\Sigma)$, we say F is closed if for any $\vec{f} \subseteq F^{<\omega}$ there is $\mathcal{P}$ such that $(\mathcal{P}, \oplus\vec{f}) \in \mathcal{I}_{\Gamma,F,\Sigma}$ and for any $\vec{g} \subseteq F^{<\omega}$, there is a (Γ,Σ)-correct iterate $\mathcal{Q}$ of $\mathcal{P}$ such that $(\mathcal{Q}, \vec{f} \cup \vec{g}) \in \mathcal{I}_{\Gamma,F,\Sigma}$.*

Fix now a closed $F \subseteq F(\Gamma,\Sigma)$. Let

$$\mathcal{F}_{\Gamma,F,\Sigma} = \{H^{\mathcal{P}}_f : (\mathcal{P}, f) \in \mathcal{I}_{\Gamma,F,\Sigma}\}.$$

We then define $\preceq_{\Gamma,F,\Sigma}$ on $\mathcal{I}_{\Gamma,F,\Sigma}$ by letting $(\mathcal{P}, \vec{f}) \preceq_{\Gamma,F,\Sigma} (\mathcal{Q}, \vec{g})$ iff $\mathcal{Q}$ is a (Γ,Σ)-correct iterate of $\mathcal{P}$ and $\vec{f} \subseteq \vec{g}$. Given $(\mathcal{P}, \vec{f}) \preceq_{\Gamma,F,\Sigma} (\mathcal{Q}, \vec{g})$, we have that

$$\pi^{\Sigma}_{\mathcal{P},\mathcal{Q},\vec{f}} : H^{\mathcal{P}}_{\oplus\vec{f}} \to H^{\mathcal{Q}}_{\oplus\vec{f}}.$$

Notice that if F is closed then $\preceq_{\Gamma,F,\Sigma}$ is directed. Let then

$$\mathcal{M}_{\infty,\Gamma,F,\Sigma}$$

be the direct limit of $(\mathcal{F}_{\Gamma,F,\Sigma}, \preceq_{\Gamma,F,\Sigma})$ under $\pi^{\Sigma}_{\mathcal{P},\mathcal{Q},\vec{f}}$'s. Given $(\mathcal{P}, \vec{f}) \in \mathcal{I}_{\Gamma,F,\Sigma}$, we let $\pi^{\Sigma}_{\mathcal{P},\vec{f},\infty} : H^{\mathcal{P}}_{\oplus\vec{f}} \to \mathcal{M}_{\infty,\Gamma,F,\Sigma}$ be the direct limit embedding. Using the proof of Lemma 1.19 of [7], we get that

Lemma 1.6. *$\mathcal{M}_{\infty,\Gamma,F,\Sigma}$ is wellfounded.*

Let F be as above and $G \subseteq F$. The following list is then the next block of definitions that carry over to our context with no significant changes (see Section 1.4 of [7]): (1) semi (F,G,Σ)-quasi iteration, (2) the embeddings of the (F,G,Σ)-quasi iteration (in this context, we will have Σ in the superscripts), (3) (F,G,Σ)-quasi iterations, (4) the last model of (F,G)-quasi iterations, (5) $\vec{f}$-guided strategies, (6) a Σ-quasi-self-justifying-system (Σ-qsjs) and (7) (ω,Γ,Σ)-suitable premice.

1.3 HOD_Σ under AD^+

It turns out that for certain iteration strategies Σ, $V_\Theta^{\mathrm{HOD}_\Sigma}$ of many models of determinacy can be obtained as $\mathcal{M}_{\infty,\Gamma,F,\Sigma}$ for some Γ and F. For the rest of this section we assume AD^+. Suppose Σ is an iteration strategy of some hod mouse $\mathcal{Q}$ and suppose Σ is $\mathcal{P}(\mathcal{P}(\omega))$-fullness preserving (see [8]) and has branch condensation (i.e., $\lambda = \omega_1$ from the notation of Subsections 1.1 and 1.2). Assume further that $V = L(\mathcal{P}(\mathbb{R})) + MC(\Sigma)$[8] $+ \Theta = \theta_\Sigma$ and that

[8] $MC(\Sigma)$ stands for the Mouse Capturing relative to Σ which says that for $x, y \in \mathbb{R}$, x is $OD(\Sigma, y)$ iff x is in some Σ-mouse over y.

$\mathcal{P}$ is below "θ is measurable", i.e., below measurable limit of Woodins. We let $\Gamma = \mathcal{P}(\mathcal{P}(\omega))$ and for the duration of this subsection, we drop Γ from our notation. Thus, a Σ-suitable premouse is a (Γ, Σ)-suitable premouse and etc.

Suppose $\mathcal{P}$ is Σ-suitable and $A \subseteq \mathbb{R}$ is OD_Σ. We say $\mathcal{P}$ *weakly term captures* A if letting $\delta = \delta^{\mathcal{P}}$, for each $n < \omega$ there is a term relation $\tau \in \mathcal{P}^{Coll(\omega, (\delta^{+n})^{\mathcal{P}})}$ such that for comeager many $\mathcal{P}$-generics, $g \subseteq Coll(\omega, (\delta^{+n})^{\mathcal{P}})$, $\tau_g = \mathcal{P}[g] \cap A$. We say $\mathcal{P}$ term captures A if the equality holds for all generics. The following lemma is essentially due to Woodin and the proof for mice can be found in [9].

Lemma 1.7. *Suppose $\mathcal{P}$ is Σ-suitable and $A \subseteq \mathbb{R}$ is OD_Σ. Then $\mathcal{P}$ weakly term captures A. Moreover, there is a Σ-suitable $\mathcal{Q}$ which term captures A.*

Given a Σ-suitable $\mathcal{P}$ and an OD_Σ set of reals A, we let $\tau^{\mathcal{P}}_{A,n}$ be the standard name for a set of reals in $\mathcal{P}^{Coll(\omega, (\delta^{+n})^{\mathcal{P}})}$ witnessing the fact that $\mathcal{P}$ weakly captures A. We then define $f_A \in F(\Gamma, \Sigma)$ by letting

$$f_A(\mathcal{P}) = \langle \tau^{\mathcal{P}}_{A,n} : n < \omega \rangle.$$

Let $F_{\Sigma,od} = \{f_A : A \subseteq \mathbb{R} \wedge A \in OD_\Sigma\}$.

All the notions we have defined above using $f \in F(\Gamma, \Sigma)$ can be redefined for OD_Σ sets $A \subseteq \mathbb{R}$ using f_A as the relevant function. To save some ink, in what follows, we will say A-iterable instead of f_A-iterable and similarly for other notions. Also, we will use A in our subscripts instead of f_A.

The following lemma is one of the most fundamental lemmas used to compute HOD and it is originally due to Woodin. Again, the proof can be found in [9].

Theorem 1.8. *For each $f \in F_{\Sigma,od}$, there is $\mathcal{P} \in S(\Gamma, \Sigma)$ which is $(F_{\Sigma,od}, f)$-quasi iterable.*

Let $\mathcal{M}_\infty = \mathcal{M}_{\infty, F_{od}, \Sigma}$.

Theorem 1.9 (Woodin, [9])**.** *$\delta^{\mathcal{M}_\infty} = \Theta$, $\mathcal{M}_\infty \in \mathrm{HOD}_\Sigma$ and*

$$\mathcal{M}_\infty | \Theta = (V_\Theta^{\mathrm{HOD}_\Sigma}, \vec{E}^{\mathcal{M}_\infty | \Theta}, S^{\mathcal{M}_\infty}, \in)$$

where $S^{\mathcal{M}_\infty}$ is the predicate of $\mathcal{M}_\infty$ describing Σ.

Finally, if $a \in H_{\omega_1}$ is self-wellordered then we could define $\mathcal{M}_\infty(a)$ by working with Σ-suitable premice over a. Everything we have said about Σ-suitable premice can also be said about Σ-suitable premice over a and in particular, the equivalent of Theorem 1.9 can be proven using $\mathrm{HOD}_{(\Sigma, a) \cup \{a\}}$ instead of HOD_Σ and $\mathcal{M}_\infty(a)$ instead of $\mathcal{M}_\infty$.

2 The maximal model

The core model induction is a method for constructing models of determinacy while working under various hypothesis. During the induction one climbs up through the Solovay hierarchy. This is a hierarchy of axioms which extends AD^+ and roughly describes how complicated the Solovay sequence is. To pass the successor stages of the Solovay hierarchy, (i.e. the stages where the length of the sequence is a successor) one defines a large enough model, called the *maximal model*, and shows that it satisfies AD^+. The next step is to then construct a hod pair beyond the maximal model. In this section our goal is to introduce the maximal model and prove some correctness results such as Lemma 2.5. For more on the Solovay hierarchy see [6].

We start by introducing universally Baire iteration strategies and mouse operators. We assume ZFC. Throughout this paper we fix a canonical method for sets in HC by reals. Given a real x which is a code of a set in HC, we let M_x be the structure coded by x and let $\pi_x : M_x \to N_x$ be the transitive collapse of M_x. We let WF be the set of reals which code sets in HC.

Definition 2.1 (uB operators)**.** *Suppose $\Lambda \in dom(Code)$ and $\lambda \geq \omega_1$ is a cardinal. We say Λ is λ-uB if there are $< \lambda$-complementing trees*[9] *(T, S) witnessing that $Code(\Lambda)$ is $< \lambda$-uB in the following stronger sense: for all $x \in WF$ and $n, m \in x$,*

$$(x, n, m) \in p[T] \iff \pi_x(m) \in \Lambda(\pi_x(n)).$$

If g is $a < \lambda$-generic then we let Λ^g be the canonical interpretation of Λ onto $V[g]$, i.e., given $a, b \in HC^{V[g]}$, $\Lambda^g(a) = b$ if and only if whenever $x \in WF^{V[g]}$ is such that $a \in N_x$ and $n \in x$ is such that $\pi_x(n) = a$ then $b = \pi_x[\{m : (x, n, m) \in (p[T])^{V[g]}\}]$.

If Λ is λ-uB for all λ then we say Λ is uB.

Suppose now λ is an uncountable cardinal, g is a $< \lambda$-generic, $a \in (H_\lambda)^V[g]$ and $\Sigma \in dom(Code)$ is λ-uB. Then we define $Lp^{\Sigma,g}(a)$, $\mathcal{W}^{\lambda,\Sigma,g}(a)$ and $\mathcal{K}^{\lambda,\Sigma,g}(a)$ in $V[g]$ according to Definition 1.2. The following connects the three stacks defined above.

Proposition 2.2. *For every $a \in H^V_\lambda$, $\mathcal{W}^{\lambda,\Sigma}(a) \trianglelefteq \mathcal{K}^{\lambda,\Sigma}(a) \trianglelefteq Lp^{\Sigma}(a)$. Moreover, for any $\eta < \lambda$ and V-generic $g \subseteq Coll(\omega, \eta)$ or $g \subseteq Coll(\omega, < \eta)$, $\mathcal{W}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{W}^{\lambda,\Sigma}(a)$, $\mathcal{K}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{K}^{\lambda,\Sigma}(a)$ and $Lp^{\Sigma,g}(a) \trianglelefteq Lp^{\Sigma}(a)$.*

We are now in a position to introduce the maximal model of AD^+.

[9] This means that the trees project to complements in all $< \lambda$-generic extensions.

Definition 2.3 (Maximal model of AD^+)**.** *Suppose $\Sigma \in Code$ is λ-uB and $\mu < \lambda$ is a cardinal. Let $g \subseteq Coll(\omega, < \mu)$[10] be generic. Then we let $\mathcal{S}^{\lambda,\Sigma}_{\mu,g} = L(\mathcal{K}^{\lambda,\Sigma,g}(\mathbb{R}^{V[g]}))$.*

Thus far strategy mice have been discussed only in situations when the underlying set was self-wellordered. However, $\mathcal{S}^{\lambda,\Sigma}_{\mu,g}$ is a Σ-mouse over the set of reals. Such hybrid mice were defined in Section 2.10 of [8]. We say that $\mathcal{S}^{\lambda,\Sigma}_{\mu,g}$ is the λ-Σ-*maximal model at* μ. Next we define hod pairs below a cardinal.

Definition 2.4 (Hod pair below λ)**.** *Suppose now that $(\mathcal{P}, \Sigma)$ is a hod pair*[11] *such that $\Sigma \in dom(Code)$ is λ^+-uB. We say $(\mathcal{P}, \Sigma)$ is a hod pair below λ if Σ has branch condensation and whenever $g \subseteq Coll(\omega, \lambda)$ is V-generic, in $V[g]$, Σ^g is ω_1-fullness preserving.*

The next lemma connects various degrees of iterability. Below, if $\xi \in Ord$ and N is a transitive model of ZFC then we let $N_\xi = V^N_\xi$.

For the purposes of the next lemma, suppose $\mu < \lambda$ are such that μ is a strong cardinal and λ is inaccessible. Let $j : V \to M$ be an embedding witnessing that μ is λ^+-strong and let $g \subseteq Coll(\omega, < \mu)$ and $h \subseteq Coll(\omega, < j(\mu))$ be two generics such that $g = h \cap Coll(\omega, < \mu)$. Let $j^+ : V[g] \to M[h]$ be the lift of j. Let $W = V[g]$.

Lemma 2.5. *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair below μ and $a \in V_\lambda[g]$ is self-wellordered. Then*

$$\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{W}^{\lambda,\Sigma,h\cap Coll(\omega,<\lambda)}(a) = \mathcal{K}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\mu,\Sigma,g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}.$$

Proof. We first show that $\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\mu,\Sigma,g}(a)$. Work in W. Clearly $\mathcal{W}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{K}^{\mu,\Sigma,g}(a)$. Let then $\mathcal{M} \trianglelefteq \mathcal{K}^{\mu,\Sigma,g}(a)$ be such that $\rho(\mathcal{M}) = a$. We want to see that $\mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a)$. To see this, notice that by a standard absoluteness argument, there is $\sigma : \mathcal{M} \to j^+(\mathcal{M})$ such that $\sigma \in M[h]$, $\sigma(\mathcal{P}) = \mathcal{P}$ and $M[h] \vDash j(\Sigma^g)^\sigma = j(\Sigma^g)$ (this follows from the fact that Σ has branch condensation). Hence, in $M[h]$, $\mathcal{M}$ is $\omega_1 + 1$-iterable $j(\Sigma^g)$-mouse. Let in $M[h]$, $\Lambda \in M[h]$ be the unique $\omega_1 + 1$-iteration strategy of $\mathcal{M}$ (as a $j(\Sigma^g)$-mouse). It follows from the homogeneity of the collapse and the uniqueness of Λ that $\Lambda \restriction H^W_\lambda \in W$. Hence, $\mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a)$.

To see that $\mathcal{W}^{\lambda,g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}$, first suppose $\mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a)$. Then, in $M[h]$, $j(\mathcal{M}) \trianglelefteq \mathcal{W}^{j(\lambda),j(\Sigma),h}(j^+(a))$. Since, in $M[h]$, $\mathcal{M}$ is embeddable into $j^+(\mathcal{M})$ via σ with the above properties, we get that in $M[h]$, $\mathcal{M} \trianglelefteq \mathcal{W}^{j(\lambda),j(\Sigma),h}(a)$. Next, suppose $\mathcal{M} \trianglelefteq (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}$ is such that $\rho(\mathcal{M}) = a$. It follows from the homogeneity of the collapse and the uniqueness of the strategy of $\mathcal{M}$ that $\mathcal{M} \in V[g]$ and that $\mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a)$.

We thus have that

[10] In this paper, μ is typically an inaccessible cardinal.

[11] Hod pairs are in the sense of [8]. They all satisfy that there is no measurable limit of Woodins.

(1) $\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\mu,\Sigma,g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}$.

Finally notice that

(2) $(\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]} \trianglelefteq \mathcal{W}^{\lambda,\Sigma,h\cap Coll(\omega,<\lambda)}(a) \trianglelefteq \mathcal{W}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{K}^{\lambda,\Sigma,g}(a) \trianglelefteq \mathcal{K}^{\mu,\Sigma,g}(a)$.

(1) and (2) now easily imply the claim. □

The following is an easy corollary of Lemma 2.5.

Corollary 2.6. *Suppose $\mu < \kappa < \lambda$ and $j : V \to M$ are such that μ and κ are strong cardinals, λ is inaccessible, j witness that μ is λ-strong and $M \vDash$ "κ is strong cardinal". Let $(\mathcal{P}, \Sigma)$ be a hod pair below μ which is λ-uB. Let $g \subseteq Coll(\omega, < \kappa)$ and $h \subseteq Coll(\omega, < j(\mu))$ be generic such that $g = h \cap Coll(\omega, < \kappa)$. Let $j^+ : V[g \cap Coll(\omega, < \mu)] \to M[h]$ be the lift of j. Then whenever $a \in V_\lambda[g]$,*

$$\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\kappa,\Sigma,g}(a) = \mathcal{W}^{\lambda,\Sigma,h\cap Coll(\omega,<\lambda)}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}.$$

Proof. Let $k = g \cap Coll(\omega, < \mu)$. Notice that because $j(\Sigma)$ has a unique extension in $M[h]$, we have that $j^+(\Sigma^k) \restriction V_\lambda[g] = \Sigma^g$. Because κ is a strong cardinal in V, it follows from Lemma 2.5 that

(1) $\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\kappa,\Sigma,g}(a)$.

Because κ is a strong cardinal in M, it follows from Lemma 2.5 that

(2) $\mathcal{K}^{\kappa,\Sigma,g}(a) = \mathcal{W}^{j(\lambda),j(\Sigma),g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}$.

Therefore, $\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\kappa,\Sigma,g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}$. □

3 The core model induction

The goal of this section is to develop some basic notions in order to state Theorem 3.3 which we will use as a black box. Our core model induction is a typical one: we have two uncountable cardinals $\kappa < \lambda$, *the core model induction operators* (cmi operators) defined on bounded subsets of κ can be extended to act on bounded subsets of λ, and for any such cmi operator F acting on bounded subsets of λ, the minimal F-closed mouse with one Woodin cardinal exists and is λ-iterable. Having these three conditions is enough to show, by using

the *scales analysis* developed in [13] and [10], that the -λ-*maximal model at* κ indeed satisfies AD^+. The details of the proof of Theorem 3.3 have appeared, in a less general form, in [9] and [11].

The mouse operators that are constructed during core model induction have two additional properties: they *transfer and relativize well*. To make these notions precise, fix $\Sigma \in dom(Code)$ which is λ-uB. Given a Σ-mouse operator $F \in dom(Code_\lambda)$, we say

1. (Relativizes well) F relativizes well if there is a formula $\phi(u, v, w)$ such that whenever $X, Y \in dom(F)$ and N are such that $X \in L_1(Y)$ and N is a transitive rudimentarily closed set such that $Y, F(Y) \in N$ then $F(X) \in N$ and $F(X)$ is the unique U such that $N \vDash \phi[U, X, F(Y)]$.

2. (Transfers well) F transfers well if whenever $X, Y \in dom(F)$ are such that X is generic over $L_1(Y)$ then $F(L_1(Y)[X])$ is obtained from $F(Y)$ via S-constructions (see Section 2.11 of [8]) and in particular, $F(L_1(Y))[X] = F(L_1(Y)[X])$.

We are now in a position to introduce the core model induction operators that we will need in this paper.

Definition 3.1 (Core model induction operator)**.** *Suppose* $|\mathbb{R}| = \kappa$, $(\mathcal{P}, \Sigma)$ *is a hod pair below* κ^+. *We say* $F \in dom(Code)$ *is a* Σ *core model induction operator or just* Σ*-cmi operator if one of the following holds:*

1. *For some* $\alpha \in Ord$, *letting* $M = \mathcal{S}_\omega^{\kappa^+,\Sigma}||\alpha$, $M \vDash \mathsf{AD}^+ + MC(\Sigma)$ *and one of the following holds:*

   (a) F *is a* Σ*-mouse operator which transfers and relativizes well.*

   (b) *For some self-wellordered* $b \in HC$ *and some* Σ*-premouse* $\mathcal{Q} \in HC^V$ *over* b, F *is an* (ω_1, ω_1)*-iteration strategy for* $\mathcal{Q}$ *which is* $(\mathcal{P}(\mathbb{R}))^M$*-fullness preserving, has branch condensation and is guided by some* $\vec{A} = (A_i : i < \omega)$ *such that* $\vec{A} \in OD^M_{b,\Sigma,x}$ *for some* $x \in b$. *Moreover,* α *ends either a weak or a strong gap in the sense of [10].*

   (c) *For some* $H \in dom(Code)$, H *satisfies a or b above and for some* $n < \omega$, F *is* $x \to \mathcal{M}_n^{\#,H}(x)$ *operator or for some* $b \in HC$, F *is the* ω_1*-iteration strategy of* $\mathcal{M}_n^{\#,H}(b)$.

2. *For some* $\alpha \in Ord$, $a \in HC$ *and* $\mathcal{M} \trianglelefteq \mathcal{W}^{\kappa^+,\Sigma}(a)$ *such that* $\rho(\mathcal{M}) = a$ *letting* Λ *be* $\mathcal{M}$*'s unique strategy, the above conditions hold for* F *with* $L^\Lambda_{\kappa^+}(\mathbb{R})$ *used instead of* $\mathcal{S}_\omega^{\kappa^+,\Sigma}$ *and* Λ *used instead of* Σ.

When $\Sigma = \emptyset$ then we omit it from our notation. Often times, when doing core model induction, we have two uncountable cardinals $\kappa < \lambda$ and we need to show that cmi operators in $V^{Coll(\omega,<\kappa)}$ can be extended to act on $V_\lambda^{Coll(\omega,<\kappa)}$. This is a weaker notion than being λ-uB. We also need to know that for any cmi operator $F \in V^{Coll(\omega,<\kappa)}$, $\mathcal{M}_1^{\#,F}$-exists. We make these statements more precise.

Definition 3.2 (Lifting cmi operators)**.** *Suppose $\kappa < \lambda$ are two cardinals such that κ is an inaccessible cardinal and suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ.*

1. *Lift$(\kappa, \lambda, \Sigma)$ is the statement that for every generic $g \subseteq Coll(\omega, < \kappa)$, in $V[g]$, for every every Σ^g-cmi operator F there is an operator $F^* \in dom(Code_\lambda)$ such that $F = F^* \restriction HC$. In this case we say F is λ-extendable. Such an F^* is necessarily unique as can be easily shown by a Skolem hull argument*[12]*. If Lift$(\kappa, \lambda, \Sigma)$ holds, $g \subseteq Coll(\omega, < \kappa)$ is generic, and F is a Σ^g-cmi operator then we let F^λ be its extended version.*

2. *We let Proj$(\kappa, \lambda, \Sigma)$*[13] *be the conjunction of the following statements: Lift$(\kappa, \lambda, \Sigma)$ and for every generic $g \subseteq Coll(\omega, < \kappa)$, in $V[g]$,*

   (a) *for every Σ^g-cmi operator F, $\mathcal{M}_1^{\#,F}$ exists and is λ-iterable.*

   (b) *for every $a \in H_{\omega_1}$, $\mathcal{K}^{\omega_1,\Sigma,g}(a) = \mathcal{W}^{\lambda,\Sigma,g}(a)$*

Recall that under AD, if X is any set then θ_X is the least ordinal which isn't a surjective image of $\mathbb{R}$ via an OD_X function. The following is the core model induction theorem that we will use.

Theorem 3.3. *Suppose $\kappa < \lambda$ are two uncountable cardinals and suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ such that Proj$(\kappa, \lambda, \Sigma)$ holds. Then for every generic $g \subseteq Coll(\omega, < \kappa)$, $\mathcal{S}_{\kappa,g}^{\lambda,\Sigma} \vDash \mathsf{AD}^+ + \theta_\Sigma = \Theta$.*

We will not prove the theorem here as the proof of the theorem is very much like the proof of the core model induction theorems in [7] (see Theorem 2.4 and Theorem 2.6), [9] (see Chapter 7) and [11]. To prove the theorem we have to use the scales analysis for $\mathcal{S}_{\kappa,g}^{\lambda,\Sigma}$ which is unpublished but see [10]. Also, the readers familiar with the scales analysis of $Lp(\mathbb{R})$ as developed by Steel in [13] and [14] should have no problem seeing how the general theory should be developed. However, there is one point worth going over.

[12]Suppose $H_0, H_1 \in dom(Code_\lambda^{V[g]})$ are two extensions of F. Working in $V[g]$, let $\pi : N \to H_{\lambda^+}[g]$ be elementary such that N is countable and $H_0, H_1 \in rng(\pi)$. Let $(\bar{H}_0, \bar{H}_1) = \pi^{-1}(H_0, H_1)$. Then it follows from the definition of being a Σ-cmi operator that $\bar{H}_0 = H_0 \restriction N$ and $\bar{H}_1 = H \restriction N$. However, since $H_0 \restriction N = F \restriction N = H_1 \restriction N$, we get that $N \vDash \bar{H}_0 = \bar{H}_1$, contradiction!

[13]Proj stands for projective determinacy. The meaning is taken from clause (a).

Suppose we are doing core model induction to prove Theorem 3.3. Fix then $g \subseteq Coll(\omega, < \kappa)$. During this core model induction, we climb through the levels of $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$ some of which project to $\mathbb{R}$ but do not satisfy that "$\Theta = \theta_{\Sigma^g}$". It is then the case that the scales analysis of [10] cannot help us in producing the next "new" set. However, such levels can never be problematic for proving that AD^+ holds in $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$. This follows from the following lemma.

Lemma 3.4. *Suppose in $V[g]$, $\mathcal{M} \trianglelefteq \mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$ is such that $\rho(\mathcal{M}) = \mathbb{R}$ and $\mathcal{M} \vDash$ "$\Theta \neq \theta_{\Sigma^g}$". Then there is $\mathcal{N} \trianglelefteq \mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$ such that $\mathcal{M} \trianglelefteq \mathcal{N}$, $\mathcal{N} \vDash$ "$\mathsf{AD}^+ + \Theta = \theta_{\Sigma^g}$".*

Proof. Since $\mathcal{M} \vDash$ "$\Theta \neq \theta_{\Sigma^g}$" it follows that $\mathcal{P}(\mathbb{R})^{\mathcal{M}} \cap (Lp^{\Sigma^g}(\mathbb{R}))^{\mathcal{M}} \neq \mathcal{P}(\mathbb{R})^{\mathcal{M}}$. It then follows that there is some $\alpha < o(\mathcal{M})$ such that $\rho(\mathcal{M}|\alpha) = \mathbb{R}$ but $\mathcal{M}|\alpha \not\trianglelefteq (Lp^{\Sigma^g}(\mathbb{R}))^{\mathcal{M}}$. Let $\pi : \mathcal{N} \to \mathcal{M}|\alpha$ be such that $\mathcal{N}$ is countable and its iteration strategy is not in $\mathcal{M}$. Let $\Lambda \in V[g]$ be the λ-iteration strategy of $\mathcal{N}$. Then a core model induction through $L^{\Lambda}(\mathbb{R})$ shows that $L^{\Lambda}(\mathbb{R}) \vDash \mathsf{AD}^+$ (this is where we needed clause 2 of Definition 3.1). However, its not hard to see that $L^{\Lambda}(\mathbb{R}) \vDash$ "$\Theta = \theta_{\Sigma^g}$". It then follows from an unpublished result of the the first author and Steel that $L^{\Lambda}(\mathbb{R}) \vDash \mathcal{P}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap Lp^{\Sigma^g}(\mathbb{R})$ (for the case $\Sigma^g = \emptyset$, see [12]). Let then $\mathcal{K} \trianglelefteq (Lp^{\Sigma^g}(\mathbb{R}))^{L^{\Lambda}(\mathbb{R})}$ be such that $\rho(\mathcal{K}) = \mathbb{R}$, $\mathcal{K} \vDash \Theta = \theta_{\Sigma^g}$ and $\Lambda \restriction HC^{V[g]} \in \mathcal{K}$ (there is such a $\mathcal{K}$ by an easy application of $\Sigma^2_1(Code(\Sigma^g))$ reflection). Since countable submodels of $\mathcal{K}$ are λ-iterable (see clause (b) of $Proj(\kappa, \lambda, \Sigma)$), we have that $\mathcal{K} \trianglelefteq \mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$. Also we cannot have that $\mathcal{K} \triangleleft \mathcal{M}$ as otherwise $\mathcal{N}$ would have a strategy in $\mathcal{M}$. Therefore, $\mathcal{M} \trianglelefteq \mathcal{K}$. □

We can now do core model induction through the levels of $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g}$ as follows. If we have reached a gap satisfying "$\Theta = \theta_{\Sigma^g}$" then we can use the scales analysis of [10] to go beyond. If we have reached a level that satisfies "$\Theta \neq \theta_{\Sigma^g}$" then using Lemma 3.4 we can skip through it and go to the least level beyond it that satisfies "$\Theta = \theta_{\Sigma^g}$". We leave the rest of the details to the reader.

One final remark is that under the hypothesis of Theorem 3.3, whenever $\Lambda \in V[g]$ is an iteration strategy of some Σ-mouse $\mathcal{M}$ over some self-wellordered $a \in HC^{V[g]}$ with the property that $\rho(\mathcal{M}) = a$ then $L^{\Lambda}(\mathbb{R}^{V[g]}) \vDash \mathsf{AD}^+$ (which can be proved by a core model induction argument through $L^{\Lambda}(\mathbb{R}^{V[g]})$). It then follows that $\mathcal{S}^{\lambda,\Sigma}_{\kappa,g} \vDash$ '$\Theta = \theta_{\Sigma^g}$".

We end this section with the following useful fact on lifting strategies. Among other things it can be used to show clause (b) of $\mathrm{Proj}(\kappa, \lambda, \Sigma)$.

Lemma 3.5 (Lifting cmi operators through strongness embeddings)**.** *Suppose $\kappa < \lambda$ are such that κ is a λ-strong cardinal. Then whenever $(\mathcal{P}, \Sigma)$ is a hod pair below κ, $Lift(\kappa, \lambda, \Sigma)$ and clause (b) of $Proj(\kappa, \lambda, \Sigma)$ hold.*

Proof. Fix an embedding $j : V \to M$ witnessing κ is λ-strong. We only show that $\mathrm{Lift}(\kappa, \lambda, \Sigma)$ holds as the proof of clause b of $\mathrm{Proj}(\kappa, \lambda, \Sigma)$ is very similar. Let $g \subseteq Coll(\omega, <$

κ) and $h \subseteq Coll(\omega, < j(\kappa))$ be V-generic such that $g = h \cap Coll(\omega, < \kappa)$. We can then extend j to $j^+ : V[g] \to M[h]$. Working in $V[g]$, fix a Σ^g-cmi operator F. Let $F^\lambda = j^+(F) \restriction H_\lambda[g]$. Fix $X \in HC^{V[g]}$ such that $V[g] \vDash F \in OD_{\{X, \Sigma^g\}}$. It then follows that $M[h] \vDash j^+(F) \in OD_{\{X, j^+(\Sigma^g)\}}$. This in turns implies $F^\lambda \in V[g]$. □

4 A core model induction at a strong cardinal

In this section we present a useful application of Theorem 3.3 which we will later use to prove our main theorem. Recall that we say μ reflects the set of strong cardinals if for every λ there is an embedding $j : V \to M$ witnessing that μ is λ-strong and for any cardinal $\kappa \in [\mu, \lambda)$, $V \vDash$ "κ is strong" iff $M \vDash$ "κ is strong".

Theorem 4.1. *Suppose $\mu < \kappa < \lambda$ are such that λ is an inaccessible cardinal, μ and κ are strong such that μ reflects the set of strong cardinals and whenever $(\mathcal{R}, \Psi)$ is a hod pair below κ such that $\lambda^{\mathcal{R}} = 0$, Proj$(\kappa, \lambda, \Psi)$ holds. Suppose $m \subseteq Coll(\omega, < \kappa)$ is generic. Then in $V[m]$, there is $A \subseteq \mathbb{R}$ such that $L(A, \mathbb{R}) \vDash \theta_0 < \Theta$.*

More specifically let $g = m \cap Coll(\omega, < \mu)$ and $\mathcal{P} = (\mathcal{M}_\infty)^{\mathcal{S}^\lambda_{\mu,g}}$. Then in $V[m]$, $\mathcal{P}$ has an (ω_1, ω_1)-iteration strategy Ψ such that Ψ is λ-fullness preserving. Moreover, there is a stack $\vec{\mathcal{T}} \in HC^{V[m]}$ on $\mathcal{P}$ according to Ψ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$ exists and in $V[m]$, $(\mathcal{Q}, \Psi_{\mathcal{Q}, \vec{\mathcal{T}}})$ is a hod pair below ω_1. Finally, in $V[m]$, Ψ is λ-extendible and $L(\Psi_{\mathcal{Q}, \vec{\mathcal{T}}}, \mathbb{R}) \vDash \mathsf{AD}^+ + \theta_0 < \Theta$.

Clearly it's enough to prove the second part of the theorem which we do in a sequence of lemmas. Fix then $\mu < \kappa < \lambda$ as in Theorem 4.1. Fix a V-generics m, g as in the Theorem and let $j : V \to M$ be an embedding witnessing that μ is λ^+-strong and such that κ is strong in M. Also fix a V-generic $h \subseteq Coll(\omega, < j(\mu))$ such that $h \cap Coll(\omega, < \kappa) = m$. It then follows that j lifts to $j^+ : V[g] \to M[h]$. Notice that it follows from our hypothesis, Theorem 3.3 and Lemma 3.5 that $\mathcal{S}^\lambda_{\mu,g} \vDash \mathsf{AD}^+ + \Theta = \theta_0$.

Let $k = h \cap Coll(\omega, < \lambda)$, $\mathcal{S} = \mathcal{S}^\lambda_{\mu,g}$ and $\Gamma^* = (F_{od})^{\mathcal{S}}$. The following is an immediate corollary of Lemma 2.5.

Corollary 4.2. *For any $a \in HC^{V[g]}$, $(Lp(a))^{\mathcal{S}} = \mathcal{W}^{\lambda, g}(a)$.*

We will use the next lemma along with Lemma 1.29 of [7] to construct an iteration strategy for $\mathcal{P}$.

Lemma 4.3. *$j^+[\Gamma^*]$ is a qsjs for $j^+(S(\Gamma^*))$ as witnessed by $\mathcal{P}$.*

Proof. We first prove the following.

Claim. Suppose $\mathcal{R} \in j(\mathcal{S})$ is such that there are $\pi : \mathcal{P} \to \mathcal{R}$ and $\sigma : \mathcal{R} \to j(\mathcal{P})$ such that $j \restriction \mathcal{P} = \sigma \circ \pi$. Then $\mathcal{R} \in S(j^+(\Gamma^*))$.

Proof. First let $T \in j^+(\mathcal{S})$ be the tree projecting to the universal $(\Sigma^2_1)^{j^+(\mathcal{S})}$ set. We have that $L[T, \mathcal{P}] \vDash \mathcal{P} = H_{(\delta^{\mathcal{P}})^{+\omega}}$. Notice that $T \in V$. It then follows that we can lift $j \restriction \mathcal{P}$, π and σ to

$$j^* : L[T, \mathcal{P}] \to L[j(T), j(\mathcal{P})],\ \pi^* : L[T, \mathcal{P}] \to L[\pi^*(T), \mathcal{R}] \text{ and}$$
$$\sigma^* : L[\pi^*(T), \mathcal{R}] \to L[j(T), j(\mathcal{P})].$$

such that $j^* = \sigma^* \circ \pi^*$. The proof of Lemma 2.21 of [7] now shows that $\mathcal{R} \in j^+(S(\Gamma^*))$. □

To finish the proof, we need to show that for every $A \in \Gamma^*$, in $j^+(\mathcal{S})$,

(1) $\mathcal{P}$ is $(j^+[\Gamma^*], j(A))$-quasi iterable[14].

To see (1), fix $A \in \Gamma^*$ and fix $\mathcal{Q} \in S(\Gamma^*)$ such that in $\mathcal{S}$, $\mathcal{Q}$ is (Γ^*, A)-quasi iterable. Then $j^+(\mathcal{S}) \vDash$ "$\mathcal{Q}$ is $(j^+(\Gamma^*), j(A))$-quasi iterable". Since we have that $j^+(\mathcal{S}) \vDash$ "$\mathcal{P}$ is a $(j^+(\Gamma^*), j(A))$-quasi iterate of $\mathcal{Q}$", we have that $j^+(\mathcal{S}) \vDash$ "$\mathcal{P}$ is a $(j^+(\Gamma^*), j(A))$-quasi iterable". Repeating the argument for every A, we get that

(2) for every $A \in \Gamma^*$, $j^+(\mathcal{S}) \vDash$ "$\mathcal{P}$ is $(j^+(\Gamma^*), j(A))$-quasi iterable".

It follows from (2) that to finish the proof of (1) it's enough to show that

(3) for every $A \in \Gamma^*$, in $j^+(\mathcal{S})$, every $(j^+(\Gamma^*), j^+(A))$-quasi iteration is also a $(j^+[\Gamma^*], j^+(A))$-quasi iteration.

To prove (3), it is enough to show that whenever $\mathcal{Q}$ is a $j^+(\Gamma^*)$-quasi iterate of $\mathcal{P}$ then $\delta^{\mathcal{Q}} = \cup_{B \in j^+[\Gamma^*]} H^{\mathcal{Q}}_{\tau^{\mathcal{Q}}_B}$. Fix then $\mathcal{Q}$ which is a $j^+(\Gamma^*)$-quasi iterate of $\mathcal{P}$. Let $\pi = \cup_{B \in j^+[\Gamma^*]} \pi_{\mathcal{P},\mathcal{Q},B}$. Let $\mathcal{W}$ be the transitive collapse of $\cup_{B \in j^+[\Gamma^*]} H^{\mathcal{Q}}_{\tau^{\mathcal{Q}}_B}$. Let $\sigma : \mathcal{W} \to \mathcal{Q}$ be the uncollapse map and $\tau = \cup_{B \in j^+(\Gamma^*)} \pi_{\mathcal{Q},\infty,B}$. Because $\mathcal{P} = \cup_{B \in j^+[\Gamma^*]} H^{\mathcal{P}}_B$, π is total. It then follows that

$$j \restriction \mathcal{P} = \tau \circ (\sigma^{-1} \circ \pi).$$

[14]Technically we should write $(j^+[\Gamma^*], \{j(A)\})$ but we abuse notation here.

The claim then implies that $\mathcal{W} \in S(j^+(\Gamma^*))$. This finishes the proof of (1). A similar proof gives the following.

(4) whenever $\mathcal{Q}$ is a $j^+[\Gamma^*]$-quasi iterate of $\mathcal{P}$ and $\epsilon : \mathcal{R} \to_{\Sigma_1} \mathcal{Q}$ is such that for every $A \in \Gamma^*$, $\tau^{\mathcal{Q}}_{j^+(A)} \in rng(\epsilon)$ then $\mathcal{R} \in j^+(\mathcal{S}(\Gamma^*))$.

The key point again is that the embedding π defined above is total. This finishes the proof of the lemma. □

We can now use Lemma 1.29 of [7] to get a strategy $\Sigma^* = \Sigma^{j^+[\Gamma^*]}$. In our current situation, there is one important difference with [7]: here Σ^* may not act on all trees that are in $M[h]$ as $j^+[\Gamma^*]$ isn't in $M[h]$. However, it acts on all stacks that are in $V_\lambda[k]$. This is simply because

$$F = \{B \cap \mathbb{R}^{V[k]} : B \in j^+[\Gamma^*]\} \in V[k].$$

Also, $\Sigma = \Sigma^* \restriction V_\lambda[m]$ and $\Psi = \Sigma \restriction V_\kappa[m]$.

Lemma 4.4. *In $V[m]$, Σ is a (λ, λ)-iteration strategy which is λ-fullness preserving and is guided by F.*

Proof. It is enough to show that Σ is λ-fullness preserving as we have already established the remaining clauses. That Σ is λ-fullness preserving follows easily from Corollary 2.6. □

Next, we show that there is a stack $\vec{\mathcal{T}}$ on $\mathcal{P}$ according to Ψ with last model $\mathcal{Q} \in HC^{V[m]}$ such that $\pi^{\vec{\mathcal{T}}}$ exists and $\Psi_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation. We follow the proof of branch condensation that first appeared in [2] and also in Chapter 7 of [9] (see especially the proofs of Lemma 7.9.6 and Lemma 7.9.7 of [9]). Below we summarize what we need in order to carry out the proof.

Recall that if Λ is a (possibly partial) iteration strategy for a λ-suitable premouse $\mathcal{R}$ then we say Λ has *weak-condensation* on its domain if whenever $\mathcal{R}^*$ is a Λ-iterate of $\mathcal{R}$ such that the iteration embedding $i : \mathcal{R} \to \mathcal{R}^*$ exists and $\mathcal{R}^{**}$ is such that there are $\pi : \mathcal{R} \to \mathcal{R}^{**}$ and $\sigma : \mathcal{R}^{**} \to \mathcal{R}^*$ with the property that $i = \sigma \circ \pi$ then $\mathcal{R}^{**}$ is λ-suitable.

Suppose (R, J) is a pair such that R is a transitive set such that for some ν which is a cardinal in R, $R \vDash$ "$V = H_{\nu^+} + J$ is a precipitous ideal on ω_1". We say (R, J) captures Ψ if in $V[m]$,

1. (R, J) is countable and an iterable pair via taking generic ultrapowers by J and its images;

2. $\mathcal{P} \in HC^R$, $\Psi \restriction HC^R \in R$ and letting $\Psi^R = \Psi \restriction HC^R$, $R \vDash$ "no tail of Ψ^R has branch condensation";

3. whenever $\xi < \omega_1$ and $(R_\alpha, J_\alpha, G_\alpha{}^{15}, \pi_{\alpha,\beta} : \alpha < \beta \leq \xi)$ is some iteration of (R, J) of length $\xi + 1$ then $\pi_{0,\xi}(\Psi^R)$ has weak-condensation and fullness preservation on its domain.

The main lemma towards showing that some tail of Ψ has branch condensation is that

Lemma 4.5. *In $V[m]$, there is no (R, J) which captures Ψ.*

We do not give the proof of the lemma as it can be found in [2] and in Chapter 7 of [9]. We then derive a contradiction by showing that

Lemma 4.6. *Suppose no tail of Ψ has branch condensation. Then in $V[m]$, there is a pair (R, J) which captures Ψ.*

Proof. It follows from Theorem 3.3 and Lemma 3.5 that $\mathcal{S}^\lambda_{\kappa,m} \vDash \mathsf{AD}^+ + \theta_0 = \Theta$. Let then $\mathcal{Q} = \mathcal{M}_\infty^{\mathcal{S}^\lambda_{\kappa,m}}$. It easily follows from the fact that $j^+(\mathcal{S}) \vDash$ "$\mathcal{Q}$ is a F_{od}-quasi iterate of $\mathcal{P}$", from (1) and (2) of Lemma 4.3 and Lemma 4.4 that $\mathcal{Q} = \mathcal{M}_\infty(\mathcal{P}, \Psi)$. It then follows that $V \vDash |\mathcal{Q}| < \kappa^+$.

To finish let $\pi : \mathcal{P} \to \mathcal{Q}$ be the iteration map according to Ψ. We also let T be the tree of the universal $(\Sigma^2_1)^{\mathcal{S}^\lambda_{\kappa,m}}$-set, $\nu = ((2^\kappa)^+)^V$ and μ be a κ-complete normal measure on κ. Working in $V[m]$, let $\sigma : R \to (H_{\nu^+})^V[m]$ be such that R is countable and $\{\Psi, \mathcal{Q}, \pi, T, \mu\} \in rng(\sigma)$. Let $n \in \omega$ be such that T_n projects onto $\{(x, \mathcal{M}) : x \in \mathbb{R}^{V[m]} \wedge \mathcal{M} \trianglelefteq \mathcal{W}^{\lambda,m}(x) \wedge \rho(\mathcal{M}) = x\}$. Also let $r \in \omega$ be such that T_r projects to the set of (x, y, z) such that x codes a self-wellordered X, y codes an $\mathcal{M} \triangleleft \mathcal{W}^{\lambda,m}(X)$ such that $\rho(\mathcal{M}) = X$ and z is a tree on $\mathcal{M}$ according to the unique iteration strategy of $\mathcal{M}$.

Let then $\{\bar\Psi, \bar{\mathcal{Q}}, \bar\pi, \bar T, \bar\mu\} = \sigma^{-1}(\{\Psi, \mathcal{Q}, \pi, T, \mu\})$, $\bar R = \sigma^{-1}((H_{\nu^+})^V)$ and $\bar m = \sigma^{-1}(m)$. We then have that $R = \bar R[\bar m]$. Let then $J \in R$ be the precipitous ideal on ω_1 induced by $\bar\mu$. (see Theorem 22.33 [1]).

Suppose now that no tail of Ψ has branch condensation. It then follows by elementarity of σ that $R \vDash$ "no tail of $\sigma^{-1}(\Psi)$ has branch condensation". Since we already know that in $V[m]$, (R, J) is countable and iterable, to finish, it remains to show that the (R, J) captures Ψ.

Let then $\Psi^R = \Psi \restriction HC^R = \sigma^{-1}(\Psi)$, $\mathcal{Q}^R = \sigma^{-1}(\mathcal{Q})$ and $\pi^R = \sigma^{-1}(\pi)$. Notice that by the construction of Ψ we have that whenever $\mathcal{R}$ is a Ψ-iterate of $\mathcal{P}$ via $\vec{\mathcal{T}}$ such that the iteration embedding $\pi^{\vec{\mathcal{T}}}$-exists then $\mathcal{M}_\infty(\mathcal{R}, \Psi_{\mathcal{R},\vec{\mathcal{T}}}) = \mathcal{Q}$ and letting $\pi_{\mathcal{R},\mathcal{Q}}$ be the iteration

[15] $G_\alpha \subseteq (\mathcal{P}(\omega_1)/J_\alpha)^{R_\alpha}$ is a generic over R_α.

map, $\pi = \pi_{\mathcal{R},\mathcal{Q}} \circ \pi^{\vec{\mathcal{T}}}$. We then have that

(1) $R \vDash$"whenever $\mathcal{R}$ is a Ψ^R-iterate of $\mathcal{P}$ via $\vec{\mathcal{T}}$ such that the iteration embedding $\pi^{\vec{\mathcal{T}}}$-exists then $\mathcal{M}_\infty(\mathcal{R}, \Psi^R) = \mathcal{Q}^R$ and letting $\pi_{\mathcal{R},\mathcal{Q}^R}$ be the iteration map, $\pi^R = \pi_{\mathcal{R},\mathcal{Q}^R} \circ \pi^{\vec{\mathcal{T}}}$".

To show that (R, J) captures Ψ, let $(R_\alpha, J_\alpha, G_\alpha, \pi_{\alpha,\beta} : \alpha < \beta \leq \xi)$ be some iteration of (R, J) of length $\xi + 1$. Let $\vec{\mathcal{T}} \in HC^{R_\xi}$ be according to $\pi_{0,\xi}(\Psi^R)$ with last model $\mathcal{R}$ such that $\pi^{\vec{\mathcal{T}}}$-exists. We need to show that $\mathcal{S}^\lambda_{\kappa,m} \vDash$ "$\mathcal{R}$ is Σ^2_1-suitable". By (1), we have that there is $p : \mathcal{R} \to \pi_{0,\xi}(\mathcal{Q})$ such that $\pi_{0,\xi}(\pi^R) = p \circ \pi^{\vec{\mathcal{T}}}$.

It follows from the construction of J that $\pi_{0,\xi} \restriction \bar{R}$ is actually an iteration of $\bar{R}$ via $\bar{\mu}$ and there is $q : \pi_{0,\xi}(\bar{R}) \to (H_{\nu^+})^V$ such that $\sigma \restriction \bar{R} = q \circ (\pi_{0,\xi} \restriction \bar{R})$. We then have that $\pi = (q \restriction (\pi_{0,\xi} \restriction \mathcal{Q}^R)) \circ p \circ \pi^{\vec{\mathcal{T}}}$, implying that, by weak condensation of Ψ, that $\mathcal{S}^\lambda_{\kappa,m} \vDash$ "$\mathcal{R}$ is Σ^2_1-suitable". The proof that $\pi_{0,\xi}(\Psi^R)$ has weak branch condensation is very similar and we omit it.

It remains to show that iterations according to $\pi_{0,\xi}(\Psi^R)$ are correctly guided. We do this only for normal trees as the general case is only notationally more complicated. To show this, we first consider the case of trees that don't have fatal drops. Notice that if $\mathcal{T} \in HC^{V[m]}$ is a correctly guided tree[16] which is according to Ψ and letting $b = \Psi(\mathcal{T})$, $\mathcal{Q}(b, \mathcal{T})$-exists then whenever $x, y \in \mathbb{R}^{V[m]}$ are such that x codes $\mathcal{M}(\mathcal{T})$ and y codes $\mathcal{Q}(b, \mathcal{T})$ then $(x, y) \in p[T_n]$. We then have that

(2) if $\mathcal{T} \in HC^R$ is according to Ψ^R, is correctly guided and letting $b = \Psi^R(\mathcal{T})$, $\mathcal{Q}(b, \mathcal{T})$-exists then whenever $x, y \in \mathbb{R}^R$ are such that x codes $\mathcal{M}(\mathcal{T})$ and y codes $\mathcal{Q}(b, \mathcal{T})$ then $(x, y) \in p[\bar{T}_n]$.

Let now $\mathcal{T} \in HC^{R_\xi}$ be according to $\pi_{0,\xi}(\Psi^R)$ and such that it is correctly guided and if $b = \pi_{0,\xi}(\Psi^R)(\mathcal{T})$ then $\mathcal{Q}(b, \mathcal{T})$-exists. Let $x, y \in \mathbb{R}^{R_\xi}$ be such that x codes $\mathcal{M}(\mathcal{T})$ and y codes $\mathcal{Q}(b, \mathcal{T})$. By (2) we have that $(x, y) \in p[\pi_{0,\xi}(\bar{T}_n)]$. Keeping the above notation, we have that $(x, y) \in p[l \circ \pi_{0,\xi}(\bar{T}_n)] = p[T_n]$ implying that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{W}^{\lambda,m}(\mathcal{M}(\mathcal{T}))$.

Lastly we need to take care of trees with fatal drops. Notice that if $\mathcal{T} \in HC^{V[m]}$ is a tree which has a fatal drop at (α, η) then letting $\mathcal{U}$ be the tail of $\mathcal{T}$ after stage α on $\mathcal{O}^{\mathcal{M}^\mathcal{T}_\alpha}_\eta$ and letting $\mathcal{M} \trianglelefteq \mathcal{O}^{\mathcal{M}^\mathcal{T}_\alpha}_\eta$ be the least such that $\rho(\mathcal{M}) = \eta$ and $\mathcal{U}$ is a tree on $\mathcal{M}$ above η then whenever $x, y, z \in \mathbb{R}^{V[m]}$ are such that x codes $\mathcal{M}^\mathcal{T}_\alpha|\eta$, y codes $\mathcal{M}$ and z codes $\mathcal{U}$ then $(x, y, z) \in p[T_r]$. It then follows that

[16]Recall that correctly guided trees do not have fatal drops, see the paragraph before Definition 1.11 of [7].

(3) if $\mathcal{T} \in HC^R$ is a tree which has a fatal drop at (α, η) then letting $\mathcal{U}$ be the tail of $\mathcal{T}$ after stage α on $\mathcal{O}_\eta^{\mathcal{M}_\alpha^\mathcal{T}}$ and letting $\mathcal{M} \trianglelefteq \mathcal{O}_\eta^{\mathcal{M}_\alpha^\mathcal{T}}$ be the least such that $\rho(\mathcal{M}) = \eta$ and $\mathcal{U}$ is a tree on $\mathcal{M}$ above η then whenever $x, y, z \in \mathbb{R}^R$ are such that x codes $\mathcal{M}_\alpha^\mathcal{T}|\eta$, y codes $\mathcal{M}$ and z codes $\mathcal{U}$ then $(x, y, z) \in p[\bar{T}_r]$.

The rest of the proof is just like the proof of the case when $\mathcal{T}$ doesn't have a fatal drop except we now use (3) instead of (2). □

Using Lemma 4.6 we can fix $\vec{\mathcal{T}} \in HC^{V[m]}$ on $\mathcal{P}$ according to Ψ with last model $\mathcal{Q}$ such that $\pi^{\vec{\mathcal{T}}}$-exists and $\Psi_{\mathcal{Q},\vec{\mathcal{T}}}$ has branch condensation. To finish the proof of Theorem 4.1 we need to show that in $V[m]$, $(\mathcal{Q}, \Psi_{\mathcal{Q},\vec{\mathcal{T}}})$ is a hod pair below ω_1. It would then follow from Lemma 3.3 that in $V[m]$, $L(\Psi_{\mathcal{Q},\vec{\mathcal{T}}}, \mathbb{R}) \vDash \mathsf{AD}^+$. Because in $V[m]$, $\Psi_{\mathcal{Q},\vec{\mathcal{T}}}$ is ω_1-fullness preserving, it follows that $L(\Psi_{\mathcal{Q},\vec{\mathcal{T}}}, \mathbb{R}) \vDash \mathsf{AD}^+ + \Theta > \theta_0$. The following lemma then finishes the proof of Theorem 4.1. Let $\Lambda = \Psi_{\mathcal{Q},\vec{\mathcal{T}}}$.

Lemma 4.7. *$V[m] \vDash (\mathcal{Q}, \Lambda)$ is a hod pair below ω_1.*

Proof. Let $\nu < \mu$ be such that letting $l = m \cap Coll(\omega, \nu)$, $\mathcal{Q} \in HC^{V[l]}$. We claim that

(1) in $V[l]$ there are κ-complementing trees T, S such that in $V[m]$, $(p[T])^{V[m]} = \{(x, n, m) : x \in \mathbb{R}, n, m \in x \text{ and } \pi_x(m) \in \Lambda(\pi_x(n))\}$.

We start our proof of (1) with the following claim.

Claim. The fragment of $\Lambda \restriction HC^{V[l]}$ which acts on normal trees is κ-uB in $V[l]$.

Proof. Given $\eta \in (\nu, \kappa)$ we let $l_\eta = m \cap Coll(\omega, < \eta)$. Also, let $\mathcal{P}_\eta \in V[l]$ be the result of generically comparing all $\mathcal{R} \in HC^{V[l_\eta]}$ such that $V[l_\eta] \vDash$ " $\Vdash_{Coll(\omega,<\kappa)} \mathcal{R}$ is λ-suitable and λ-short tree iterable". Also, let $\mathcal{Q}_\eta$ be the Λ-iterate of $\mathcal{Q}$ obtained by making $H_\eta[l_\eta]$ generically generic for $\mathbb{B}^{\mathcal{Q}_\eta}_{\delta^{\mathcal{Q}_\eta}}$. Let $\pi_\eta : \mathcal{Q} \to \mathcal{P}_\eta$ and $\sigma_\eta : \mathcal{Q} \to \mathcal{Q}_\eta$ be the iteration embeddings. Let in $V[g]$, $U = \{(x, y) \in \mathbb{R}^2 : \mathcal{S} \vDash$ "x codes a and y codes a sound a-mouse projecting to $a\}$ and $Z = \{(x, y, z) \in \mathbb{R}^3 : \mathcal{S} \vDash$ "x codes a, y codes a sound a-mouse $\mathcal{M}$ projecting to a and z codes a normal tree according to the unique strategy of $\mathcal{M}\}$. Then we have that $U, Z \in \Gamma^*$.

Suppose now that $\mathcal{T} \in HC^{V[m]}$ is a stack on $\mathcal{Q}$. We then have that $\mathcal{T}$ is according to Λ if and only if for any $\eta \in (\nu, \kappa)$ such that $\mathcal{T} \in HC^{V[l_\eta]}$

1. $\mathcal{T}$ doesn't have a fatal drop and for any limit $\alpha < lh(\mathcal{T})$ letting b be the branch of $\mathcal{T} \restriction \alpha$ the following holds:

(a) $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ exists if and only if whenever $n \subseteq Coll(\omega, \delta^{\mathcal{Q}_\eta})$ is $\mathcal{Q}_\eta[\mathcal{T}]$-generic and $x \in \mathcal{Q}_\eta[\mathcal{T}][n]$ is a real coding $\mathcal{M}(\mathcal{T} \restriction \alpha)$, there is $y \in \mathcal{Q}_\eta[\mathcal{T}][n]$ such that $(x, y) \in \pi_\eta(\tau_U^{\mathcal{Q}})$ and if $\mathcal{M}$ is the mouse coded by y then $rud(\mathcal{M}) \vDash$ "$\delta(\mathcal{T} \restriction \alpha)$ isn't Woodin".

(b) $\mathcal{Q}(b, \mathcal{T} \restriction \alpha)$ doesn't exist if and only if there is $\sigma : \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{P}_\eta$ such that $\pi_\eta = \sigma \circ \pi_{0,\alpha}^{\mathcal{T}}$.

2. $\mathcal{T}$ has a fatal drop at (α, β) and whenever $n \subseteq Coll(\omega, \delta^{\mathcal{Q}_\eta})$ is $\mathcal{Q}_\eta$-generic, $x \in \mathcal{Q}_\eta[T][n]$ is a real coding $\mathcal{M}_\alpha^{\mathcal{T}}|\beta$ and $y \in \mathcal{Q}_\eta[T][n]$ is a real coding $\mathcal{O}_\beta^{\mathcal{M}_\alpha^{\mathcal{T}}}$, there is $z \in \mathcal{Q}_\eta[T][n]$ such that z codes the part of $\mathcal{T}$ after stage α and $(x, y, z) \in \pi_\eta(\tau_Z^{\mathcal{Q}})$.

It is not hard to see that if we let ϕ be the formula expressed by the the clauses above then club many hulls of $(H_\kappa^{V[l]}, \Lambda \restriction H_\kappa^{V[l]}, \in)$ are generically correct about $\Lambda \restriction HC^{V[l]}$ and hence, about ϕ. More precisely, in $V[l]$, there is a club of $X \in H_\kappa^{V[l]}$ such that letting $\pi : N \to X$ be the transitive collapse of X then whenever $(n, \mathcal{T})$ are such that n is generic over N and $\mathcal{T} \in N[n]$ is a tree on $\mathcal{Q}$ then

$$\mathcal{Q}[n] \vDash \phi[\mathcal{T}] \text{ if and only if } \phi[\mathcal{T}].$$

The claim now follows from Lemma 4.1 of [15]. □

To finish the proof of (1) we first notice that the claim holds for $\Lambda \restriction HC^{V[l]}$. Let then (T, S) be the κ-complementing trees such that in $V[m]$, $p[T] = \{x : x \text{ codes } \mathcal{T} \text{ on } \mathcal{Q} \text{ such that } \phi[\mathcal{T}]\}$ (see Lemma 4.1 of [15]). The proof of the claim then shows that in $V[m]$, $p[T] = \{x : x$ codes a tree $\mathcal{T}$ according to $\Lambda\}$. It is now easy to modify (T, S) so that they satisfy (1). □

5 On the strength of the failure of the UBH for tame trees

In this section, we present the proof of our Main Theorem. For the rest of this section we assume that there is a proper class of strong cardinals. We start by introducing *tame trees*. Recall that we say κ reflects the set of strong cardinals (or κ is a strong reflecting strongs) if for every λ there is an embedding $j : V \to M$ witnessing that κ is λ-strong and for any cardinal $\mu \in [\kappa, \lambda)$, $V \vDash$ "μ is strong" iff $M \vDash$ "μ is strong".

Definition 5.1 (Tame iteration tree)**.** *A normal iteration tree $\mathcal{T}$ on V is tame if for all $\alpha < \beta < lh(\mathcal{T})$ such that $\alpha = pred_T(\beta + 1)$, $\mathcal{M}_\alpha^{\mathcal{T}} \vDash$ "$\exists \kappa < \lambda < \mathrm{cp}(E_\beta^{\mathcal{T}})$ such that λ is a strong cardinal and κ is strong reflecting strongs".*

While our proof will not need the assumption that κ is strong reflecting strongs, we defined tame trees in this particular way because we believe tame failures of UBH give inner models of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular". The full proof of this claim will appear in a future publication.

Towards a contradiction, we assume that there is a tame iteration tree $\mathcal{T}$ on V with two cofinal well-founded branches b and c and the conlusion of the Main Theorem fails. Let $M_b = \mathcal{M}_b^{\mathcal{T}}$, $M_c = \mathcal{M}_c^{\mathcal{T}}$, $M = \mathcal{M}(\mathcal{T})$, $\delta = \delta(\mathcal{T})$, $\delta_b^+ = (\delta^+)^{M_b}$, $\delta_c^+ = (\delta^+)^{M_c}$, $\pi_b = \pi_b^{\mathcal{T}}$ and $\pi_c = \pi_c^{\mathcal{T}}$. Finally, let $\kappa_0 < \kappa_1 < \kappa_2$ be such that:

- κ_0 is the first strong reflecting strongs in V;
- κ_1 is the first strong above κ_0 in V;
- since $\mathcal{T}$ is tame, we have that all the extenders used in $\mathcal{T}$ have critical point $> \kappa_1$; hence we can choose an inaccessible $\kappa_2 > \kappa_1$ and κ_2 is below the critical point of any extender used in $\mathcal{T}$.

Suppose $g \subseteq Coll(\omega, < \kappa_1)$ is V-generic. To make the notation as transparent as possible, we will confuse our iteration embeddings that act on V with their extensions that act on $V[g]$. Thus, for instance, $\pi_b : V[g] \to M_b[g]$ and etc. Working in $V[g]$, fix a hod pair $(\mathcal{P}, \Sigma) \in V[g]$ below κ_1 such that $\mathcal{P} \in HC^{V[g]}$ and $\lambda^{\mathcal{P}} = 0$. The next lemma is the key lemma.

Lemma 5.2 (Key Lemma)**.** *For every hod pair $(\mathcal{P}, \Sigma)$ below κ_1 such that $\lambda^{\mathcal{P}} = 0$, Proj$(\kappa_1, \kappa_2, \Sigma)$ holds.*

Given the Key Lemma we can easily get a contradiction by using Theorem 4.1 (applied with κ_0 in place of μ, κ_1 in place of κ and κ_2 in place of λ). It is then enough to show that the Key Lemma holds which is what we will do in the next few lemmas. Towards the proof of the Key Lemma, we fix a hod pair $(\mathcal{P}, \Sigma)$ below κ_1. Since clause (b) of Proj$(\kappa_1, \kappa_2, \Sigma)$ follows from Lemma 2.5, we will only establish clause (a).

We will only verify clause (a) of Proj$(\kappa_1, \kappa_2, \Sigma)$ for Σ-cmi operators defined according to clause 1 of Definition 3.1 as those defined according to clause 2 of Definition 3.1 can be handled in a very similar manner. Lets then fix such a Σ-cmi operator F. Notice that it follows from Lemma 3.5 that for every ξ, both in $M_b[g]$ and in $M_c[g]$, F is ξ-extendable. We then let F_b and F_c be the two Ord-extensions of F in $M_b[g]$ and $M_c[g]$ respectively.

We say F can be *lifted* if for any $x \in H_{\delta^+}^{M_b[g]} \cap H_{\delta^+}^{M_c[g]}$, $F_b(x) = F_c(x)$ and $(Lp^{F_b}(x))^{M_b[g]}$ is compatible with $(Lp^{F_c}(x))^{M_c[g]}$ (i.e., one is an initial segment of the other).

We first present a simple lemma which illustrates some of the key ideas that we will use.

Lemma 5.3. *Suppose $x, \mathcal{M} \in M_b \cap M_c$ are such that $\mathcal{M}$ is a sound x-premouse such that $\rho(\mathcal{M}) = x$. Then $\mathcal{M} \trianglelefteq Lp^{M_b}(x) \iff \mathcal{M} \trianglelefteq Lp^{M_c}(x)$.*

Proof. Suppose $\mathcal{N}$ is a countable hull of $\mathcal{M}$ in V. Then by an absoluteness argument using that HC is in both M_b and M_c, $\mathcal{N}$ is a countable hull of $\mathcal{M}$ both in M_b and M_c. Hence, the claim follows. □

Unfortunately, the lemma doesn't immediately generalize to F-mice since the absoluteness used in the proof isn't in general true. Fixing an $\mathcal{N}$ as in the proof which is a countable submodel of $\mathcal{M} \trianglelefteq (Lp^{F_b}(x))^{M_b}$ it is still true that $\mathcal{N}$ can be realized as a countable hull of $(Lp^{F_b}(x))^{M_b}$ and $(Lp^{F_c}(x))^{M_c}$ in M_b and M_c via certain embeddings $j_b : \mathcal{N} \to \mathcal{M}$ and $j_c : \mathcal{N} \to \mathcal{M}$ in M_b and M_c respectively: however, it is not clear, in the case F is an iteration strategy, that $F_b^{j_b}$ and $F_c^{j_c}$ (i.e., the pullbacks of F_b and F_c) are the same strategies. The following lemma in fact shows that they have to be the same.

Lemma 5.4. *F can be lifted.*

Proof. We already know that F can be extended to F_b and F_c. It remains to show that whenever $x \in M_b \cap M_c$, $F_b(x) = F_c(x)$ and $(Lp^{F_b}(x))^{M_b}$[17] and $(Lp^{F_c}(x))^{M_c}$ are compatible. We show the second clause as the first is only a special case of it. Assume towards a contradiction that $(Lp^{F_b}(x))^{M_b}$ and $(Lp^{F_c}(x))^{M_c}$ are not compatible. Let $\mathcal{S}_b = (Lp^{F_b}(x))^{M_b}$ and $\mathcal{S}_c = (Lp^{F_c}(x))^{M_c}$. Fix an elementary $\sigma : W \to V_\epsilon[g]$ for some very large ϵ[18] such that W is countable in $V[g]$, $(\mathcal{T}, b, c, \mathcal{P}, F, x) \in rng(\sigma)$ and if $(\mathcal{U}, d, e, \mathcal{Q}, G, y) = \sigma^{-1}(\mathcal{T}, b, c, \mathcal{P}, F, x)$ then $\sigma[lh(\mathcal{U})]$ is cofinal in $lh(\mathcal{T})$. Let $\eta = |\mathcal{P}|^{M[g]} = |\mathcal{P}|^{V[g]}$. Note that $\eta < \kappa_1$ by the definition of $\mathcal{P}$. By our choice of κ_1, $\mathrm{cp}(\pi_b) > \eta$ and $\mathrm{cp}(\pi_c) > \eta$. Since $\eta \in rng(\sigma)$, let $\nu = \sigma^{-1}(\eta)$. Also, let $M_d = \mathcal{M}_d^{\mathcal{U}}$, $M_e = \mathcal{M}_e^{\mathcal{U}}$, and $(G_d, G_e) = \sigma^{-1}(F_b, F_c)$. We now have that in W, $(Lp^{G_d}(y))^{M_d}$ isn't compatible with $(Lp^{G_e}(y))^{M_e}$.

Let $\sigma_\xi : \mathcal{M}_\xi^{\mathcal{U}} \to \mathcal{M}_\xi^{\sigma\mathcal{U}}$ be the copy maps. We have that $\sigma_\beta \in \mathcal{M}_0^{\sigma\mathcal{U}} = V[g]$ and there are $m : M_d \to \mathcal{M}_0^{\sigma\mathcal{U}}$ and $n : \mathcal{M}_e \to \mathcal{M}_0^{\sigma\mathcal{U}}$ such that $\sigma_0 = m \circ \pi_{\beta,d}^{\mathcal{U}}$ and $\sigma_0 = n \circ \pi_{\beta,e}^{\mathcal{U}}$. Let $H = \sigma_\beta(G^*) \in \mathcal{M}_\beta^{\sigma\mathcal{U}}$ where $G^* = \sigma^{-1}((\pi_b)^{-1}(F)) = \sigma^{-1}((\pi_c)^{-1}(F)) \in \mathcal{M}_0^{\mathcal{U}}$

Let $\mathcal{R}_d = (Lp^{G_d}(y))^{M_d}$ and $\mathcal{R}_e = (Lp^{G_e}(y))^{M_e}$. Finally, let $\mathcal{W}_d = m(\mathcal{R}_d)$ and $\mathcal{W}_e = n(\mathcal{R}_e)$. Notice that $\sigma_\beta \restriction \mathcal{Q} = m \restriction \mathcal{Q} = n \restriction \mathcal{Q}$ and $m(G_d), n(G_e)$ both extend H. But now, in $V[g] = \mathcal{M}_0^{\sigma\mathcal{U}}$, $\mathcal{R}_d$ and $\mathcal{R}_e$ can be compared as the both are G^+-iterable where G^+ is σ_0-pullback of H. □

[17] This means that whenever $\pi : (\mathcal{N}, \mathcal{P}^*, x^*) \to (\mathcal{M}, \mathcal{P}, x)$ is such that $\mathcal{M} \triangleleft Lp^{F_b}(x)$ and $\mathcal{N}$ is countable transitive, then $\mathcal{N}$ has a unique $\omega_1 + 1$ Λ-strategy where Λ is such that whenever $\mathcal{R}$ is an iterate of $\mathcal{N}$ and $\mathcal{U} \in \mathcal{N}$ is a tree on $\mathcal{P}^*$ according to Λ then $\Lambda(\mathcal{U}) = F(\pi\mathcal{U}) \in \mathcal{R}$.

[18] We will confuse this $V_\epsilon[g]$ with $V[g]$ during the proof.

Next, we show that $M[g] \vDash$ "$\mathcal{M}_1^{\#,F}$ exists and is $< \delta$-iterable". This will complete the proof of the Key Lemma. Suppose not. Without loss of generality, assume $\delta_b^+ \leq \delta_c^+$. By our assumption, in $M[g]$, the F-closed core model K^F derived from a $K^{c,F}$ which is constructed (up to δ) using extenders with critical point $> \kappa_2$ exists and is 1-small[19]. The following claim then gives us a contradiction.

Claim. $Lp^{F_b}(K^F) \vDash \delta$ is Woodin.

Proof. Recall that we assume $(\delta^+)^{M_b} \leq (\delta^+)^{M_c}$ and this along with the proof of Claim 4 of [16] in turns imply that $Lp^{F_b}(K^F) \trianglelefteq Lp^{F_c}(K^F)$ and hence $Lp^{F_b}(K^F) \in M_b \cap M_c$. By the proof of Theorem 4.1 in [16] and Theorem 2.2 of [3], $Lp^{F_b}(K^F) \vDash \delta$ is Woodin. □

The claim and the fact that there is a proper class of strong cardinals (in $M[g]$) imply that $M[g] \vDash$ "$\mathcal{M}_1^{\sharp,F}$ exists and is $< \delta$-iterable." By the agreement between V and M, we have $V[g] \vDash$ "$\mathcal{M}_1^{\sharp,F}$ exists and is $< \kappa_2$-iterable." This finishes the proof of the Main Theorem.

6 On the strength of $\neg UBH$ without strongs

It is possible to prove a similar lower bound for $\neg UBH$ by somewhat strengthening the hypothesis yet dropping the assumption that there are proper class of strong cardinals. In this section, we state the result. Its proof is mostly due to the second author and will appear elsewhere.

Given an iteration tree $\mathcal{T}$ of limit length and $\alpha < lh(\mathcal{T})$, we let $\mathcal{T}_{\geq\alpha}$ be $\mathcal{T}$ starting from α and $\mathcal{T}_{\leq\alpha} = \mathcal{T} \restriction \alpha + 1$. Similarly, we define $\mathcal{T}_{<\alpha}$ and $\mathcal{T}_{>\alpha}$.

Theorem 6.1. *Suppose $\mathcal{T}$ is a normal tree on V with two wellfounded branches b and c such that if $\alpha = \sup(b \cap c)$ then $\delta(\mathcal{T}) \in rng(\pi^{\mathcal{T}}_{\alpha,b}) \cap rng(\pi^{\mathcal{T}}_{\alpha,c})$ and $\mathcal{T}_{\geq\alpha} \in M^{\mathcal{T}}_\alpha$. Then in some homogenous extension of V there is a transitive model M such that $\mathbb{R}, Ord \subseteq M$ and $M \vDash$ "$\mathsf{AD}^+ + \theta_0 < \Theta$". In particular, there is a non-tame mouse.*

The hypothesis of Theorem 6.1 includes, among other trees, alternating chains.

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