Structure theory of $L(\mathbb{R}, \mu)$ and its applications

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Abstract

In this paper, we explore the structure theory of $L(\mathbb{R}, \mu)$ under the hypothesis $L(\mathbb{R}, \mu) \vDash$ "$\mathsf{AD} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$" and give some applications. First we show that "ZFC + there exist ω^2 Woodin cardinals"[1] has the same consistency strength as "$\mathsf{AD} + \omega_1$ is $\mathbb{R}$-supercompact". During this process we show that if $L(\mathbb{R}, \mu) \vDash \mathsf{AD}$ then in fact $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+$. Next we prove important properties of $L(\mathbb{R}, \mu)$ including Σ_1-reflection and the uniqueness of μ in $L(\mathbb{R}, \mu)$. Then we give the computation of full HOD in $L(\mathbb{R}, \mu)$. Finally, we use Σ_1-reflection and $\mathbb{P}_{\max}$ forcing to construct a certain ideal on $\mathcal{P}_{\omega_1}(\mathbb{R})$ (or equivalently on $\mathcal{P}_{\omega_1}(\omega_2)$ in this situation) that has the same consistency strength as "ZFC + there exist ω^2 Woodin cardinals."

1 Introduction

Recall that under $\mathsf{ZF} + \mathsf{DC}$, a measure μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$, the set of countable subsets of $\mathbb{R}$, is:

(1) *fine* iff $\{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid x \in \sigma\} \in \mu$ for each $x \in \mathbb{R}$;

(2) *normal* iff for each regressive $F : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathcal{P}_{\omega_1}(\mathbb{R})$, that is

$$\{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset\} \in \mu$$

then

[1] By this, we mean "ZFC + there is a set W of Woodin cardinals of order type ω^2". We will say "there exist ω^2 Woodin cardinals" for short.

$$\exists x \in \mathbb{R}\ \{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid x \in F(\sigma)\} \in \mu.$$

It's easy to see that if μ is a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, ZF proves that normality of μ (condition (2) above) is equivalent to the following "diagonal intersection" property:

(2') If $\langle A_x \mid x \in \mathbb{R}\rangle$ is an $\mathbb{R}$-indexed sequence of μ-measure one sets, then

$$\triangle_{x\in\mathbb{R}} A_x =_{\mathrm{def}} \{\sigma \mid \sigma \in \bigcap_{x\in\sigma} A_x\} \in \mu.$$

We first prove the following (previously unpublished) theorem, due to Woodin, which determines the exact consistency strength of the theory "$\mathsf{AD} + \omega_1$ is $\mathbb{R}$-supercompact".

Theorem 1.1 (Woodin)**.** *The following are equiconsistent.*

1. *ZFC + there are ω^2 Woodin cardinals.*

2. *There is a filter μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that $L(\mathbb{R}, \mu) \vDash$ "$\mathsf{ZF} + \mathsf{DC} + \mathsf{AD} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$".*

The proof of this theorem will occupy part of section 2. The (1) $\Rightarrow$ (2) direction is proved using the derived model construction. The converse uses a Prikry forcing that forces a model of ZFC with ω^2 Woodin cardinals that realizes $L(\mathbb{R}, \mu)$ as its derived model. This also shows that $L(\mathbb{R}, \mu) \vDash \mathsf{AD}$ if and only if $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+$.

It's worth mentioning that the existence of a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ was first shown by Solovay to follow from $\mathsf{AD}_\mathbb{R}$ (see [8]); so $\mathsf{AD}_\mathbb{R}$ implies "$\mathsf{AD}+\omega_1$ is $\mathbb{R}$-supercompact". It also follows from [8] that the theory "$\mathsf{AD} + \omega_1$ is $\mathbb{R}$-supercompact" doesn't imply $\mathsf{AD}_\mathbb{R}$. Theorem 1.1 determines the exact consistency strength of the former, which is much weaker than that of the latter. It also follows from $\mathsf{AD}_\mathbb{R}$ that games on reals of fixed countable length are determined. This gives a hierarchy of normal fine measures extending the Solovay measure in some sense. A sequel to this paper ([16]) gives a construction (due to Woodin) of this hierarchy from $\mathsf{AD}_\mathbb{R}$, explores their exact consistency strength, and gives some applications of these measures.

Using the proof of Theorem 1.1, we explore the basic structure theory of $L(\mathbb{R}, \mu)$. We also prove in section 2 the following theorem, which is also due to Woodin.

Theorem 1.2 (Woodin)**.** *The following holds in $L(\mathbb{R}, \mu)$ assuming $L(\mathbb{R}, \mu) \vDash$ "$\mathsf{AD}^+ + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$".*

1. *$(L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})[\mu], \mu) \prec_{\Sigma_1} (L(\mathbb{R}, \mu), \mu)$.*

2. *Suppose $L(\mathbb{R}, \mu) \vDash$ "μ_0 and μ_1 are normal fine measures on $\mathcal{P}_{\omega_1}(\mathbb{R})$". Then*

$$L(\mathbb{R},\mu) \vDash \mu_0 = \mu_1.$$

Using Theorems 1.1, 1.2, and their proofs, we give some applications in sections 3 and 4. Section 3 is dedicated to the HOD computation in $L(\mathbb{R},\mu)$. The precise definition of HOD will be given in section 3. Roughly speaking, HOD of $L(\mathbb{R},\mu)$ will be shown to be $L(\mathcal{M}_\infty, \Lambda)$ where $\mathcal{M}_\infty \subseteq$ HOD is a fine-structural premouse that has ω^2 Woodin cardinals cofinal in $o(\mathcal{M}_\infty)$, where $o(\mathcal{M}_\infty)$ is the ordinal height of the transitive structure $\mathcal{M}_\infty$, and agrees with HOD on all bounded subsets of Θ and Λ is a certain strategy that acts on finite stacks of normal trees in $\mathcal{M}_\infty$ based on $\mathcal{M}_\infty|\Theta$. The reader familiar with the HOD analysis in $L(\mathbb{R})$ will not be surprised here. As an application, [16] uses the HOD analysis to prove a "determinacy transfer theorem" which roughly states that the determinacy for real games of length ω^2 with payoff $\undertilde{\Pi}^1_1$ and those with payoff $<$-ω^2-$\undertilde{\Pi}^1_1$ are equivalent.

Finally, in section 4 we prove the following two theorems. The first one uses $\mathbb{P}_{\max}$ forcing over a model of the form $L(\mathbb{R},\mu)$ as above and the second one is an application of the core model induction. Woodin's book [19] or Larson's handbook article [5] are good sources for $\mathbb{P}_{\max}$; for details on the core model induction, see [7].

Theorem 1.3. *Suppose $L(\mathbb{R},\mu) \vDash$ "$\mathsf{AD}^+ + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$" and let $G \subseteq \mathbb{P}_{max}$ be a generic filter over $L(\mathbb{R},\mu)$. Then in $L(\mathbb{R},\mu)[G]$, there is a normal fine ideal $\mathcal{I}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that*

1. *letting $\mathcal{F}$ be the dual filter of $\mathcal{I}$ and $A \subseteq \mathbb{R}$ such that A is OD_x for some $x \in \mathbb{R}$, either $A \in \mathcal{F}$ or $\mathbb{R}\backslash A \in \mathcal{F}$;*

2. *$\mathcal{I}$ is precipitous;*

3. *for all $s \in \mathrm{OR}^\omega$, for all generics $G_0, G_1 \subseteq \mathcal{I}^+$, letting $j_{G_i} : V \to Ult(V, G_i) = M_i$ for $i \in \{0,1\}$ be the generic embeddings, then $j_{G_0} \restriction \mathrm{HOD}_{\{\mathcal{I},s\}} = j_{G_1} \restriction \mathrm{HOD}_{\{\mathcal{I},s\}}$ and $\mathrm{HOD}^{M_0}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} = \mathrm{HOD}^{M_1}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} \in V$.*

The next theorem establishes the equiconsistency of the conclusion of Theorem 1.3 with the existence of ω^2 Woodin cardinals.

Theorem 1.4 (ZFC)**.** *Suppose there is a normal fine ideal $\mathcal{I}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that*

1. *letting $\mathcal{F}$ be the dual filter of $\mathcal{I}$ and $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ such that A is OD_x for some $x \in \mathbb{R}$, either $A \in \mathcal{F}$ or $\mathbb{R}\backslash A \in \mathcal{F}$;*

2. *$\mathcal{I}$ is precipituous;*

3. *for all* $s \in \mathrm{OR}^\omega$, *for all generics* $G_0, G_1 \subseteq \mathcal{I}^+$, *letting* $j_{G_i} : V \to Ult(V, G_i) = M_i$ *for* $i \in \{0,1\}$ *be the generic embeddings, then* $j_{G_0} \restriction \mathrm{HOD}_{\{\mathcal{I},s\}} = j_{G_1} \restriction \mathrm{HOD}_{\{\mathcal{I},s\}}$ *and* $\mathrm{HOD}^{M_0}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} = \mathrm{HOD}^{M_1}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} \in V$.

Then in a generic extension $V[G]$ *of* V, *there is a filter* μ *on* $\mathcal{P}_{\omega_1}(\mathbb{R})$ *such that*

$$L(\mathbb{R}, \mu) \vDash \text{“}\mathsf{AD} + \mu \textit{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R})\text{”}.$$

Basic notions and notations. For a transitive structure M, we let $o(M)$ denote the ordinal height of M. A transitive $\mathcal{M}$ is a *fine-structural premouse* or simply a *premouse* if $\mathcal{M} = (J_\alpha[E], \in, E, F^{\mathcal{M}})$, where E is a fine-extender sequence in the sense of [14] and $F^{\mathcal{M}}$ is the amenable code for the top extender of M, also in the sense of [14]. We write $\mathcal{M}|\gamma$ for the structure $\mathcal{N} = (J_\gamma[E \restriction \gamma], \in, E \restriction \gamma, F^{\mathcal{N}})$ and $\mathcal{M}||\gamma$ for $\mathcal{N} = (J_\gamma[E \restriction \gamma], \in, E \restriction \gamma, \emptyset)$. Note that $\mathcal{M}|\gamma = \mathcal{M}||\gamma$ if $\mathcal{M}|\gamma$ is passive, that is its predicate for the top extender is empty. If $\mathcal{P}, \mathcal{Q}$ are premice, we write $\mathcal{P} \lhd \mathcal{Q}$ if there is some $\gamma \leq o(\mathcal{Q})$ such that $\mathcal{P} = \mathcal{Q}|\gamma$. For some $k \leq \omega$, a k-sound premouse $\mathcal{M}$ is (k, α, β)-iterable if player II (the good player) has a winning strategy in the game $\mathcal{G}_k(\mathcal{M}, \alpha, \beta)$ (see [14], Section 4). We customarily call a k-sound premouse $\mathcal{M}$ that is $(k, 1, \omega_1 + 1)$-iterable (or $(k, \omega_1, \omega_1 + 1)$-iterable) a *mouse*. When the degree of soundness of $\mathcal{M}$ is clear from the context, we will neglect to mention it in our notations.

The structure $L(\mathbb{R}, \mu)$ considered in this paper is a structure of the language $\mathcal{L}^* = \mathcal{L} \cup \{\dot{\mathbb{R}}, \dot{\mu}\}$, where $\mathcal{L}$ is the language of set theory, $\dot{\mu}$ is a unary predicate symbol, and $\dot{\mathbb{R}}$ is a constant symbol, whose intended interpretation is the reals of the model. We sometimes write $L(\mathbb{R})[\mu]$, or $(L(\mathbb{R})[\mu], \mu)$ for the same structure. If μ is a measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and $P(v)$ is a property, we often write $\forall^*_\mu \sigma P(\sigma)$ for $\{\sigma \mid P(\sigma)\} \in \mu$. Also, we also say “$\omega_1$ is $\mathbb{R}$-supercompact” to mean “there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$”.

We use Θ to denote the supremum of α such that there is a surjection from $\mathbb{R}$ onto α. Under ZFC, Θ is simply the successor cardinal of the continuum. Assuming AD^+, which is a techincal strengthening of AD (see [9] or [15] for more on AD^+), a *Solovay sequence* is a sequence $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ such that: (i) θ_0 is the supremum of ordinals α such that there is an OD surjection from $\mathbb{R}$ onto α; (ii) if $\beta \leq \Omega$ is limit, then $\theta_\beta = \sup_{\gamma<\beta} \theta_\gamma$; (iii) if $\beta = \gamma + 1 \leq \Omega$, then letting $B \subseteq \mathbb{R}$ have Wadge rank θ_γ, θ_β is the supremum of α such that there is an $OD(B)$ surjection from $\mathbb{R}$ onto α. Suppose $\mathsf{AD}^+ + \Theta = \theta_0$. We let $\underset{\sim}{\delta}^2_1$ denote the largest Suslin cardinal. The largest pointclass with the scales property, as shown by Woodin, is $\underset{\sim}{\Sigma}^2_1$.

For cardinals $\alpha \leq \beta$, we write $Col(\alpha, < \beta)$ for the Lévy collapse that adds a surjection from α onto every $\kappa \in [\alpha, \beta)$. If $\beta > \alpha$ is inaccessible then after forcing with $Col(\alpha, < \beta)$, β has cardinality α^+; otherwise, β will have cardinality α.

Finally, suppose γ is a limit of Woodin cardinals. We let $Hom_{<\gamma}$ denote the collection of $<\gamma$-homogeneously Suslin sets of reals. See [9] for more on the basic theory of $Hom_{<\gamma}$.

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2 The equiconsistency and structure theory of $L(\mathbb{R}, \mu)$

We first present a variation of the derived model construction in [9] in the context where we want to construct a model of the form $L(\mathbb{R}, \mu)$. See [9] for facts about AD^+ and the derived model construction.

Lemma 2.1. *Suppose there is a measurable cardinal. Then there is a forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, $L(\mathbb{R}, \mathcal{C}) \vDash$ "$\mathcal{C}$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$" where $\mathcal{C}$ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.*

Proof. Let κ be a measurable cardinal and U be a normal measure on κ. Let $j : V \to M$ be the ultrapower map by U. Let $\mathbb{P}_0$ be $Col(\omega, <\kappa)$. Let $G \subseteq \mathbb{P}_0$ be V-generic. For $\alpha < \kappa$, we write $G \restriction \alpha$ for $G \cap Col(\omega, <\alpha)$. $Col(\omega, <j(\kappa)) = j(\mathbb{P}_0)$ is isomorphic to $\mathbb{P}_0 * \mathbb{Q}$ for some $\mathbb{Q}$ and whenever $H \subseteq \mathbb{Q}$ is $V[G]$-generic, then j can be lifted to an elementary embedding $j^+ : V[G] \to M[G][H]$ defined by $j^+(\tau_G) = j(\tau)_{G*H}$. Let $\mathbb{R}^{**} = \cup_{\alpha<\kappa}\mathbb{R}^{V[G|\alpha]}$ be the symmetric reals. Note that since κ is inaccessible, $\mathbb{R}^{**} = \mathbb{R}^{V[G]}$. We define a filter $\mathcal{F}^*$ on $\mathcal{P}_{\omega_1}(\mathbb{R}^{**})$ as follows.

$$A \in \mathcal{F}^* \Leftrightarrow \forall H \subseteq \mathbb{Q}(H \text{ is } V[G]\text{-generic} \Rightarrow \mathbb{R}^{V[G]} \in j^+(A)).$$

It's clear from the definition that $\mathcal{F}^* \in V[G]$.

We first claim that $\mathcal{F}^*$ is a normal fine filter. Fineness is easy; so we just verify normality. To see normality, suppose F is regressive. Then $A := \{\sigma \mid F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset\} \in \mathcal{F}$. Then $j^+(F)(\mathbb{R}^*) \subseteq \mathbb{R}^{**} \wedge j^+(F)(\mathbb{R}^*) \neq \emptyset$. Fix some $x \in \mathbb{R}^{**}$ such that $x \in j^+(F)(\mathbb{R}^{**})$. Then $\forall^*_{\mathcal{F}}\sigma\ x \in F(\sigma)$.

We now claim that $L(\mathbb{R}^{**}, \mathcal{F}^*) \vDash \mathcal{F}^*$ is a measure on $\mathcal{P}_{\omega_1}(\mathbb{R}^{**})$. Suppose $A \in L(\mathbb{R}^{**}, \mathcal{F}^*)$ is defined in $V[G]$ by a formula φ from a real $x \in \mathbb{R}^{**}$ (without loss of generality, we suppress parameters $\{U, s\}$, where $s \in \mathrm{OR}^{<\omega}$ that go into the definition of A); so $\sigma \in A \Leftrightarrow V[G] \vDash \varphi[\sigma, x]$. Let $\alpha < \kappa$ be such that $x \in V[G \restriction \alpha]$ and we let U^* be the canonical extension of U in $V[G \restriction \alpha]$. Then either

$$\forall^*_{U^*}\beta V[G \restriction \alpha] \vDash \emptyset \Vdash_{Col(\omega,<\beta)} \emptyset \Vdash_{Col(\omega,<\kappa)} \varphi[\dot{\mathbb{R}}_\beta, x]$$

or

$$\forall^*_{U^*}\beta V[G\restriction\alpha]\models\emptyset\Vdash_{Col(\omega,<\beta)}\emptyset\Vdash_{Col(\omega,<\kappa)}\neg\varphi[\dot{\mathbb{R}}_\beta,x].$$

In the above, $\dot{\mathbb{R}}_\beta$ is the canonical $Col(\omega,<\beta)$-name for the symmetric reals in $V^{Col(\omega,<\beta)}$. This easily implies either $A\in\mathcal{F}^*$ or $\neg A\in\mathcal{F}^*$.

Next, note $\mathcal{P}_{\omega_1}(\mathbb{R}^{**})$ has size ω_1 in $V[G]$, so we can use the iterated club shooting construction to turn $\mathcal{F}^*$ into the club filter. We let $\mathbb{P}_1$ be the forcing defined in 17.2 of [1]. By 17.2 of [1], $\mathbb{P}_1$ does not add any ω-sequence of ordinals. In particular, it does not add reals. Letting $H\subseteq\mathbb{P}_1$ be $V[G]$-generic, in $V[G][H]$, we still have $L(\mathbb{R}^{**},\mathcal{F}^*)\models$ "$\mathcal{F}^*$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})^{**}$" and furthermore, $\mathcal{F}^*\cap L(\mathbb{R}^{**},\mathcal{F}^*)$ is the restriction of the club filter on $L(\mathbb{R}^{**},\mathcal{F}^*)$. Our desirable $\mathbb{P}$ is $\mathbb{P}_0*\mathbb{P}_1$. □

Suppose there exist ω^2 many Woodin cardinals. Let γ be the sup of the first ω^2 Woodin cardinals and for each $i<\omega$, let η_i be the sup of the first ωi Woodin cardinals. Suppose $G\subseteq Col(\omega,<\gamma)$ is V-generic and for each i, let $\mathbb{R}^*=\cup_{\alpha<\gamma}\mathbb{R}^{V[G\restriction\alpha]}$ and $\sigma_i=\mathbb{R}^{V[G\restriction Col(\omega,<\eta_i)]}$. We define a filter $\mathcal{F}^*$ as follows: for each $A\subseteq\mathcal{P}_{\omega_1}(\mathbb{R}^*)$ in $V[G]$

$$A\in\mathcal{F}^*\Leftrightarrow\exists n\forall m\geq n(\sigma_m\in A).$$

We call $\mathcal{F}^*$ defined above the **tail filter**.

Lemma 2.2. *Let $\gamma,\eta_i,\mathbb{R}^*,\mathcal{F}^*$ be as above. Then*

$$L(\mathbb{R}^*,\mathcal{F}^*)\models\text{“}\mathcal{F}^*\text{ is a normal fine measure on }\mathcal{P}_{\omega_1}(\mathbb{R}^*)\text{”}.$$

Proof. Suppose not. So this statement is forced by the empty condition in $Col(\omega,<\gamma)$ by the homogeneity of $Col(\omega,<\gamma)$. By Lemma 2.1 applied to the first measurable cardinal κ and the fact that the forcing $\mathbb{P}$ used there is of size less than the first Woodin cardinal, by working over $V[g]$, where $g\subseteq\mathbb{P}$ is V-generic, we may assume that in V, the club filter $\mathcal{F}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ has the property that $L(\mathbb{R},\mathcal{F})\models\mathcal{F}$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Let $\lambda>>\gamma$ be regular and let

$$\begin{aligned}S=\{X\prec V_\lambda\quad|\ &X\text{ is countable},\gamma\in X,\exists\eta\in X\cap\gamma\text{ such that}\\&\text{for all successor Woodin cardinals }\lambda\in X\cap(\eta,\gamma),\text{ if }D\subseteq\mathbb{Q}_{<\lambda},\\&D\in X,\text{ and }D\text{ is predense then }X\text{ captures }D\}.\end{aligned}$$

By Lemma 3.1.14 of [4], S is stationary and furthermore, letting $H\subseteq\mathcal{P}(\mathcal{P}_{\omega_1}(V_\lambda))/\mathcal{I}_{NS}$ [2] be generic such that $S\in H$, then for some $\xi<\gamma$, for all $\xi<\delta<\gamma$ and δ is Woodin, $H\cap\mathbb{Q}_{<\delta}$

[2]In this section, $\mathcal{I}_{NS}$ is the nonstationary ideal on $\mathcal{P}_{\omega_1}(V_\lambda)$.

is V-generic. We may as well assume ξ is less than the first Woodin cardinal and hence for all $\delta < \gamma$, δ is Woodin, $H \cap \mathbb{Q}_{<\delta}$ is V-generic.

Let $j : V \to (M, E)$ be the induced generic embedding given by H. Of course, (M, E) may not be wellfounded but wellfounded at least up to λ because $j''\lambda \in M$. For each $\alpha < \omega^2$, let $j_\alpha : V \to M_\alpha$ be the induced embedding by $H \cap \mathbb{Q}_{<\delta_\alpha}$, let M^* be the direct limit of the M_α's and $j^* : V \to M^*$ be the direct limit map. Note that j_α, j^* factor into j.

Let $\mathbb{R}^* = \mathbb{R}^{M^*}$ (the $\mathbb{R}^*$ from before is behind us now) and for each $i < \omega$, $\sigma_i = \mathbb{R}^{M_i^*}$ where $M_i^* = lim_n M_{\omega i+n}$. Let $G \subseteq Col(\omega, <\gamma)$ be such that $\cup_{\alpha<\eta_i} \mathbb{R}^{V[G\restriction\alpha]} = \sigma_i$ for all i; so $\mathbb{R}^*$ is the symmetric reals associated to G. Let $\mathcal{F}^*$ be the tail filter defined in $V[G]$. We claim that if $A \in j^*(\mathcal{F})$ then $A \in \mathcal{F}^*$. To see this, let $\pi \in M^*$ witness that A is a club. Let $\alpha < \omega^2$ be such that M_α contains the preimage of π. Then it is clear that $\forall m$ such that $\omega m \geq \alpha$ and $\pi''\sigma_m \subseteq \sigma_m$. This shows $j^*(\mathcal{F}) \subseteq \mathcal{F}^*$ and hence $L_\lambda(\mathbb{R}^*, j^*(\mathcal{F})) = L_\lambda(\mathbb{R}^*, \mathcal{F}^*) \vDash$ "$\mathcal{F}^*$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R}^*)$". Since λ can be chosen arbitrarily large, we're done. □

Next, we prove a "reflection phenomenon" analoguous to that in Lemma 6.4 of [9].

Lemma 2.3. *Let $\gamma, G, \mathbb{R}^*, \mathcal{F}^*$ be defined as above. Suppose $x \in \mathbb{R}^{V[H\restriction\alpha]}$ for some $\alpha < \gamma$, and suppose ψ is a formula in the language of set theory with an additional predicate symbol. Let HC^* be the set of heritarily countable sets (in $V[G]$) coded by $\mathbb{R}^*$. Suppose*

$$\exists B \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}^*, \mathcal{F}^*)((HC^*, \in, B) \vDash \psi[x])$$

then

$$\exists B \in Hom_{<\gamma}^{V[G\restriction\alpha]}((HC^{V[G\restriction\alpha]}, \in, B) \vDash \psi[x]).$$

Proof. Such a B in the statement of the lemma is called a ψ-witness. To see that Lemma 2.3 holds, pick the least γ_0 such that some $OD(x)^{L(\mathbb{R}^*,\mathcal{F}^*)}$ ψ-witness B is in $L_{\gamma_0}(\mathbb{R}^*, \mathcal{F}^*)$ and by minimizing the sequence of ordinals in the definition of B, we may assume B is definable (over $L_{\gamma_0}(\mathbb{R}^*, \mathcal{F}^*)$) from x without ordinal parameters. We may as well assume $x \in V$. We want to produce an absolute definition of B as in the proof of Lemma 6.4 in [9]. We do this as follows. First let φ be such that

$$u \in B \Leftrightarrow L_{\gamma_0}(\mathbb{R}^*, \mathcal{F}^*) \vDash \varphi[u, x],$$

and

$$\overline{\psi}(v) = \text{“}v \text{ is a } \psi\text{-witness”}.$$

Let $\mathcal{C}$ denote the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and $\theta(u,v)$ be the natural formula defining B:

$$\begin{aligned}\theta(u,v) \quad = \quad & \text{“}L(\mathbb{R},\mathcal{C}) \vDash \mathcal{C} \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R}) \text{ and } L(\mathbb{R},\mathcal{C}) \vDash \exists B\overline{\psi}[B] \\ & \text{and if } \gamma_0 \text{ is the least } \gamma \text{ such that } L_\gamma(\mathbb{R},\mathcal{C}) \vDash \exists B\overline{\psi}[B] \\ & \text{then } L_{\gamma_0}(\mathbb{R},\mathcal{C}) \vDash \varphi[u,v]\text{”}.\end{aligned}$$

We apply the tree production lemma (see [9]) to the definition $\theta(u,v)$ with parameter $x \in \mathbb{R}^V$. It's clear that stationary correctness holds. To verify generic absolutenss, let $\delta < \gamma$ be a Woodin cardinal; let g be $< \delta$ generic over V and h be $< \delta^+$ generic over $V[g]$. We want to show that if $y \in \mathbb{R}^{V[g]}$

$$V[g] \vDash \theta[y,x] \Leftrightarrow V[g][h] \vDash \theta[y,x].$$

There are $G_0, G_1 \subseteq Col(\omega, <\gamma)$ such that G_0 is generic over $V[g]$ and G_1 is generic over $V[g][h]$ with the property that $\mathbb{R}^*_{G_0} = \mathbb{R}^*_{G_1}$ and furthermore, if $\eta < \gamma$ is a limit of Woodin cardinals above δ, then $\mathbb{R}^*_{G_0} \restriction \eta = \mathbb{R}^*_{G_1} \restriction \eta$ [3]. Such G_0 and G_1 exist since h is generic over $V[g]$ and $\delta < \gamma$. But this means letting $\mathcal{F}_i$ be the tail filter defined from G_i respectively then $L(\mathbb{R}^*_{G_0}, \mathcal{F}_0) = L(\mathbb{R}^*_{G_1}, \mathcal{F}_1)$. The proof of Lemma 2.2 implies that $L(\mathbb{R},\mathcal{C})^{V[g]}$ is embeddable into $L(\mathbb{R}^*_{G_0}, \mathcal{F}_0)$ and $L(\mathbb{R},\mathcal{C})^{V[g][h]}$ is embeddable into $L(\mathbb{R}^*_{G_1}, \mathcal{F}_1)$. This proves generic absoluteness. This gives us that $B \cap \mathbb{R}^V \in Hom^V_{<\gamma}$ and $B \cap \mathbb{R}^V$ is a ψ-witness. Hence we're done. □

Lemma 2.4. *Let $\gamma, \mathbb{R}^*, \mathcal{F}^*$ be defined as above. Then $L(\mathbb{R}^*, \mathcal{F}^*) \vDash \mathsf{AD}^+$.*

Proof. Suppose not. Then any failure of AD^+ in $L(\mathbb{R}^*, \mathcal{F}^*)$ can be expressed in the form

$$(HC^*, \in, B) \vDash \psi[x]$$

for some $x \in \mathbb{R}^*$, some $B \in L(\mathbb{R}^*, \mathcal{F}^*) \cap \mathcal{P}(\mathbb{R})$, and some formula ψ. Using Lemma 2.3, we can get a ψ-witness B in $L(\mathbb{R}^*, \mathcal{F}^*)$ such that $B = C^*$, where $C \in Hom^{V[g]}_{<\gamma}$ for some $<$-γ generic g such that $x \in V[g]$ and C^* is the canonical blowup of C in the sense of [9]. The lemma then follows verbatim from the proof of Theorem 6.1 from Lemma 6.4 in [9]. □

Now assume $L(\mathbb{R},\mu) \vDash$ “$\mathsf{AD} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$”. We prove that in a generic extension of $L(\mathbb{R},\mu)$, there is a class model N such that

1. $N \vDash \mathsf{ZFC}$ + there are ω^2 Woodin cardinals;
2. letting λ be the sup of the Woodin cardinals of N, $\mathbb{R}$ can be realized as the symmetric reals over N via $Col(\omega, <\lambda)$;

[3] $\mathbb{R}^*_{G_0}$ is the symmetric reals defined by G_0 and similarly for $\mathbb{R}^*_{G_1}$. $\mathbb{R}^*_{G_0} \restriction \eta = \mathbb{R}^{V[g][G_0 \cap Col(\omega,<\eta)]}$ and $\mathbb{R}^*_{G_1} \restriction \eta = \mathbb{R}^{V[g][h][G_1 \cap Col(\omega,<\eta)]}$.

3. letting $\mathcal{F}$ be the tail filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ in $N[G]$ where $G \subseteq Col(\omega, < \lambda)$ is a generic over N such that $\mathbb{R}$ is the symmetric reals induced by G, $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F})$ and $\mu \cap L(\mathbb{R}, \mu) = \mathcal{F} \cap L(\mathbb{R}, \mathcal{F})$.

The proof is given in Lemma 2.6. First we introduce some notions. Assume AD^+. Let T be a tree on $\omega \times \mathrm{OR}$ whose projection is a universal Σ^2_1 set. For any real x, by a Σ^2_1 *degree* d_x, we mean the equivalence class of all y such that $L[T, y] = L[T, x]$. Woodin has shown that the notion of Σ^2_1 degrees does not depend on the choice of T. In fact, we can define d_x to be the equivalence class of all y such that $\mathrm{HOD}_y = \mathrm{HOD}_x$. If d_1, d_2 are Σ^2_1 degrees, we say $d_1 \leq d_2$ if for any $x \in d_1$ and $y \in d_2$, $x \in L[T, y]$. $d_1 < d_2$ iff $d_1 \leq d_2$ and $d_1 \neq d_2$. For any reals x, y, we say $d_x = d_y$ or $x \equiv y$ iff $d_x \leq d_y$ and $d_y \leq d_x$. Just like with Turing cones, we define Σ^2_1 cones to be sets of the form $C_d = \{e \mid d \leq e\}$ for some Σ^2_1 degree d.

Theorem 2.5 (Woodin, see [3])**.** *Assume AD^+. Let R, S be sets of ordinals. Then for a (Turing or Σ^2_1) cone of x, $\mathrm{HOD}_R^{L[R,S,x]} \models \omega_2^{L[R,S,x]}$ is a Woodin cardinal.*

Lemma 2.6. *There is a forcing notion $\mathbb{P}$ in $L(\mathbb{R}, \mu)$ and there is an N in $L(\mathbb{R}, \mu)^{\mathbb{P}}$ satisfying (1)-(3) above.*

Proof. First, by arguments from [17], in $L(\mathbb{R}, \mu)$,

$$\Theta = \theta_0 + L(\mathcal{P}(\mathbb{R})) \models \Theta = \theta_0 + \mathsf{MC}.^{4}$$

Hence Σ^2_1 is the largest Suslin pointclass in $L(\mathbb{R}, \mu)$ and by Theorem 17.1 of [11], every set of reals in $L(\mathbb{R}, \mu)$ is contained in an $\mathbb{R}$-mouse[5]. Working in $L(\mathbb{R}, \mu)$, fix a tree T for a universal Σ^2_1 set as before (we may take T to be OD in $L(\mathcal{P}(\mathbb{R}))$). Let

$$\mathbb{D} = \{\langle d_i \mid i < \omega\rangle \mid \forall i(d_i \text{ is a } \Sigma^2_1 \text{ degree and } d_i < d_{i+1})\}.$$

Next, we define a measure ν on $\mathbb{D}$. We say

$$\begin{aligned} A \in \nu \quad \text{iff} \quad & \text{for any } \infty\text{-Borel code} S \text{ for } A, \\ & \forall^*_\mu \sigma\ L[T, S](\sigma) \models \text{“}\mathsf{AD}^+ + \sigma = \mathbb{R} + \exists(\emptyset, U) \in \mathbb{P}_{\Sigma^2_1}\ (\emptyset, U) \Vdash \dot{G} \in \mathcal{A}_S\text{”}. \end{aligned}$$

In the definition of ν, $\mathbb{P}_{\Sigma^2_1}$ is the usual Prikry forcing using the Σ^2_1 degrees (see, e.g., Section 6.2 of [3]) and the cone measure in $L[T, S](\sigma)$, $\dot{G}$ is the name for the corresponding Prikry sequence, $\mathcal{A}_S$ is the set of reals coded by S. Note that:

---

[4] MC is the statement that whenever $x, y \in \mathbb{R}$ are such that x is OD_y, then there is a sound mouse $\mathcal{M}$ over y such that $\rho(\mathcal{M}) = \omega$ and $x \in \mathcal{M}$.

[5] $\mathcal{M}$ is an $\mathbb{R}$-mouse if $\mathcal{M}$ is a premouse over $\mathbb{R}$ in the sense of [12], $\mathcal{M}$ is ω-sound, $\rho(\mathcal{M}) = \mathbb{R}$, and the transitive collapse of every countable substructure of $\mathcal{M}$ is $(\omega, 1, \omega_1 + 1)$-iterable.

(a) for all set of ordinals S, $\forall^*_\mu \sigma\ L[T,S](\sigma) \vDash$ "$\mathsf{AD}^+ + \sigma = \mathbb{R}$";

(b) whether $A \in \nu$ does not depend on the choice of S;

(c) for $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$, let $A^* = \{d \in \mathbb{D} \mid \cup d \in A\}$[6], then $A \in \mu \Leftrightarrow A^* \in \nu$.

We verify (b). Let S_0, S_1 be ∞-Borel codes for A. Let $T^\infty = \prod_\sigma T/\mu$ and $S_i^\infty = \prod_\sigma S_i/\mu$ be the ultraproducts by μ.

Claim. $L[T^\infty, S_0^\infty](\mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = L[T^\infty, S_1^\infty](\mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = L(\mathbb{R},\mu) \cap \mathcal{P}(\mathbb{R})$.

Proof. To see this, first observe that by MC in $L(\mathcal{P}(\mathbb{R}))$, $\mathcal{P}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})$ by [11, Theorem 17.1][7]; the second observation is by Los, $Lp(\mathbb{R}) = \prod_\sigma Lp(\sigma)/\mu$; the final observation is $\forall^*_\mu \sigma\ L[T,S_0](\sigma) \cap \mathcal{P}(\sigma) = L[T,S_1](\sigma) \cap \mathcal{P}(\sigma) = OD(\sigma) \cap \mathcal{P}(\sigma) = Lp(\sigma) \cap \mathcal{P}(\sigma)$.

To see the final observation, note that for $i \in \{0,1\}$, $L[T^\infty, S_i^\infty](\mathbb{R}) \cap \mathcal{P}(\mathbb{R}) \subseteq L(\mathbb{R},\mu) \cap \mathcal{P}(\mathbb{R}) = Lp(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$, so by Los, $\forall^*_\mu \sigma\ L[T,S_i](\sigma) \cap \mathcal{P}(\sigma) \subseteq Lp(\sigma) \cap \mathcal{P}(\sigma)$. To see the converse, it suffices to prove the following claim, whose proof is based on an unpublished note of J.R. Steel.

Subclaim. In $L(\mathcal{P}(\mathbb{R}))$, there is a real z such that whenever a is countable, transitive and $z \in a$, then $\mathcal{P}(a) \cap L[T,a] = \mathcal{P}(a) \cap OD(a)$[8].

Proof. First we prove that: on a cone of reals z, $\mathbb{R} \cap L[T,z] = \mathbb{R} \cap OD(z)$. To prove this, first let

$$A(z,n,m) \Leftrightarrow \exists y \in \mathbb{R}\ (y \in OD(z) \backslash L[T,z]) \wedge \text{letting } y_z \text{ be the } OD(z)\text{-least such } y, \text{ then } y_z(n) = m.$$

Now it is a basic AD^+ fact that since $\Theta = \theta_0$, there is a real z_0 such that for all z Turing above z_0, $A \cap L[T,z] \in L[T,z]$ (in other words, the (boldface) envelope of Σ^2_1 is $\mathcal{P}(\mathbb{R})$). For any such z, $OD(z) \cap \mathbb{R} = L[T,z] \cap \mathbb{R}$. To see this, if not, then $y_z(n) = m$ if and only if $A(z,n,m)$. So y_z is computable from $A \cap L[T,z]$ so y_z is in $L[T,z]$. This contradicts the definition of y_z.

Take z_0 to be the base of the cone in the above argument. For any countable transitive a such that $z_0 \in a$, a set $b \subseteq a$ is $OD(a)$ just in case for comeager many enumerations g of a (in order type ω), b is $OD(g)$. This and the above argument give the subclaim. □

The subclaim and MC give

$$\forall^*_\mu \sigma\ Lp(\sigma) \cap \mathcal{P}(\sigma) = OD(\sigma) \cap \mathcal{P}(\sigma) \subseteq L[T,S_i](\sigma) \cap \mathcal{P}(\sigma).$$

[6] Say $d = \langle d_i \mid i < \omega \rangle$; then $\cup d = \{x \in \mathbb{R} \mid \exists n\ x \in L[T,d_n]\}$.

[7] In [12], $Lp(\mathbb{R})$ is denoted $K(\mathbb{R})$ and is the stack of all $\mathbb{R}$-mice.

[8] We remind the reader that T is OD; so $OD(a) = OD(T,a)$.

This completes the proof of the third observation. The three observations give us the claim. □

The claim gives us that the $\mathbb{P}_{\Sigma^2_1}$ forcing relations in these models are the same, in particular, $L[T^\infty, S^\infty_0](\mathbb{R}) \vDash \exists(\emptyset, U) \in \mathbb{P}_{\Sigma^2_1}\ (\emptyset, U) \Vdash \dot{G} \in \mathcal{A}_{S^\infty_0}$ if and only if $L[T^\infty, S^\infty_1](\mathbb{R}) \vDash \exists(\emptyset, U) \in \mathbb{P}_{\Sigma^2_1}\ (\emptyset, U) \Vdash \dot{G} \in \mathcal{A}_{S^\infty_1}$. This gives us (b). (a) follows from the claim and Los theorem.

To see (c), suppose $A \in \mu$. Let S be an ∞-Borel code for A^*. By (a),

$$\forall^*_\mu \sigma\ (\sigma \in A \wedge L[T, S^*](\sigma) \vDash \text{“}\mathsf{AD}^+ + \sigma = \mathbb{R}\text{”}).$$

For any such σ, if d is the sequence of degrees corresponding to a $\mathbb{P}_{\Sigma^2_1}$-generic over $L[T, S^*](\sigma)$, then clearly $\cup d = \sigma \in A$ since d is cofinal in the Σ^2_1 degrees of $L[T, S^*](\sigma)$. This means $d \in A^*$. This gives $A^* \in \nu$. The converse is proved using the proof of the forward direction applied to $\mathcal{P}_{\omega_1}(\mathbb{R})\backslash A$. This finishes the proof of (c).

Let $\mathbb{P}$ be the usual Prikry forcing using ν (cf. [3]). First let $\nu_1 = \nu$, for $n > 0$, let ν_n be the product measure induced by ν_0 on $\mathbb{D}^{n+1}$; that is $\nu_n(Z) = 1 \Leftrightarrow \forall^*_\nu d_0 \cdots \forall^*_\nu d_n \langle d_0, \cdots, d_n\rangle \in Z$. Conditions in $\mathbb{P}$ are pairs (p, U) where for some $n \in \omega$, $p = \langle \vec{d^i} \mid i \leq n \wedge \vec{d^i} \in \mathbb{D} \wedge \vec{d^i} \in \vec{d^{i+1}}(0)^9\rangle$ and U is such that for all $n < \omega$, $U(n) \subseteq \mathbb{D}^{n+1}$ and $\nu_n(U(n)) = 1$. $(p, U) \leq_{\mathbb{P}} (q, W)$ if p end extends q, say $p = q^\frown r$ for some $r \in \mathbb{D}^n$, and for all k and all $s \in U(k)$, $r^\frown s \in W(n + k)$. $\mathbb{P}$ has the usual Prikry property, that is given any condition (p, U), a term τ, a formula $\varphi(x)$, we can find a $(p, U') \leq_{\mathbb{P}} (p, U)$ such that (p, U') decides the value of $\varphi[\tau]$; furthermore, (p, U') is ordinal definable from p, U, τ (see [13] or Section 6 of [3] for a proof). Let G be $\mathbb{P}$ generic. We identify G with the union of the stems of conditions in G, i.e., G is identified with $\langle \vec{d^i} \mid i < \omega \wedge \exists U(\langle d^j \mid j \leq i\rangle, U) \in G\rangle$. We need some notations before proceeding. We write V for $L(\mathbb{R}, \mu)$ (and use them interchangably); for any $g \in \mathbb{D}$, let $\omega^g_1 = \sup_i \omega_1^{L[T^\infty, g(i)]}$ and $\delta(g \restriction i) = \omega_2^{L[T^\infty, g\restriction i]}$ (note that $\delta(g \restriction i)$ doesn't depend on the representatives for the degrees in g). To produce a model with ω^2 Woodin cardinals, we use Theorem 2.5.

For any countable transitive a which admits a well-ordering rudimentary in a and for any real x coding a, let

$$Q^x_a = \mathrm{HOD}^{L[T^\infty, x]}_{T^\infty, a} \restriction (\delta(x) + 1).$$

The expression on the right hand side above stands for $V_{\delta(x)+1} \cap \mathrm{HOD}^{L[T^\infty, x]}_{T^\infty, a}$. Note that Q^x_a only depends on the degree of x; hence for a cone of Σ^2_1-degree e, $Q^e_a = Q^x_a$ for all $x \in a$. Let a be a base of the cone in the subclaim above. We now let

$$Q^0_0 = Q_a^{\vec{d^0}(0)},$$

[9]We abuse notation here to mean $\vec{d^i} \in L[T, \vec{d^{i+1}}(0)]$ and is countable there.

and

$$\delta_0^0 = \delta(\vec{d^0}(0)).$$

For $i < \omega$, let

$$Q_{i+1}^0 = Q_{Q_i^0}^{\vec{d^0}(i+1)},$$

and

$$\delta_{i+1}^0 = \delta(\vec{d^0}(i+1)).$$

This finishes the first block. Let $Q_\omega^0 = \cup_i Q_i^0$. In general, we let

$$Q_0^{j+1} = Q_{Q_\omega^j}^{\vec{d^{j+1}}(0)},$$

and

$$\delta_0^{j+1} = \delta(\vec{d^{j+1}}(0)).$$

For $i < \omega$, let

$$Q_{i+1}^{j+1} = Q_{Q_i^{j+1}}^{\vec{d^{j+1}}(i+1)},$$

and

$$\delta_{i+1}^{j+1} = \delta(\vec{d^{j+1}}(i+1)).$$

In $V[G]$, let

$$N =_{\text{def}} L[T^\infty, \langle Q_j^i \mid i,j < \omega\rangle]$$

Note that N can be defined in $\text{HOD}_{\{G\}}^{(V[G],V)}$. We claim that

$$N \vDash \delta_j^i \text{ is a Woodin cardinal for all } i,j < \omega.$$

The claim follows from the following observations.

(a) For all $i,j < \omega$, $Q_0^{j+1} \cap \mathcal{P}(\delta_i^j) = Q_i^j \cap \mathcal{P}(\delta_i^j) = Q_{i+1}^j \cap \mathcal{P}(\delta_i^j)$.

(b) For $i,j < \omega$, $N \cap \mathcal{P}(\delta_j^i) = Q_j^i \cap \mathcal{P}(\delta_j^i)$.

The second equality of (a) follows from basic facts about Prikry forcing (see Section 6.2 of [3]). Also from [3], we get $L[T^\infty, Q_\omega^i] \cap \mathcal{P}(\delta_j^i) = Q_j^i \cap \mathcal{P}(\delta_j^i)$ for all $i,j < \omega$.

For the first equality, it's enough to prove: $(\dagger) \equiv$ "for any n, for a cone of d, $\mathcal{P}(Q_\omega^n) \cap L[T^\infty, Q_\omega^n] = \mathcal{P}(Q_\omega^n) \cap L[T^\infty, d]$". $(\dagger)$ easily implies the first equality of (a). To see $(\dagger)$, suppose not. Note that $\mathcal{P}(Q_\omega^n) \cap L[T^\infty, Q_\omega^n] = \mathcal{P}(Q_\omega^n) \cap L[T, Q_\omega^n]$ and $L[T^\infty, d] \cap \mathcal{P}(Q_\omega^n) = L[T,d] \cap \mathcal{P}(Q_\omega^n)$ by Los theorem. Working in $L(\mathcal{P}(\mathbb{R}))$, for a cone of d, let b_d be the least $b \subseteq Q_\omega^n$ in $L[T,d] \backslash L[T, Q_\omega^n]$ (the minimality of b_d is in terms of the canonical well-ordering of

$L[T,d])$. Since Q^n_ω is countable, there is a b and a cone of d such that $b = b_d$, so b is $OD_{Q^n_\omega}$. This means $b \in L[T, Q^n_\omega]$ (by the subclaim and the choice of Q^0_0). Contradiction.

Now to see (b), we use the Prikry property of $\mathbb{P}$. Let $A \subseteq \delta^i_j$ be in N. Then A is ordinal definable in $V[G]$ from $\{T^\infty, \langle Q^i_j \mid i,j<\omega\rangle\}$. Let $\dot{Q}$ be the canonical forcing term for $\langle Q^i_j \mid i,j<\omega\rangle$ and $\varphi(v,\hat{t},\dot{Q})$ be a formula in the forcing language with only v free and $t \in \mathrm{OR}^{<\omega} \cup \{T^\infty\}$ such that φ defines A over $V[G]$ from t and $\langle Q^i_j \mid i,j<\omega\rangle$. Let $(p,U) \in G$ with $\mathrm{dom}(p) > i$. By the fact that δ^i_j is countable, the Prikry property gives a condition $(p,Y) \leq (p,U)$ such that (p,Y) decides $\varphi(\hat{\eta},\hat{t},\dot{Q})$ for all $\eta < \delta^i_j$. By density, we may fix such a $(p,Y) \in G$. Letting $n+1 = \mathrm{dom}(p)$, we claim

$$\eta \in A \Leftrightarrow \exists r \in \mathbb{D}^{\mathrm{dom}(p)+1} \exists X \ (r,X) \Vdash \varphi(\hat{\eta},\hat{t},\dot{Q}) \wedge \forall i \leq n \forall j < \omega \ Q^i_j = (Q^i_j)^r,$$

where in the above $(Q^i_j)^r$ is the model Q^i_j defined relative to the sequence of degrees given by r (over the set a specified above). If the equivalence holds, then A is OD from T^∞ and $\langle Q^i_j \mid j<\omega \wedge i \leq n\rangle$. By the proof of (a), we get that $A \in Q^i_j$, which is what we want to prove. We've already shown the $\Rightarrow$ direction. To see the converse, suppose (r,X) is as on the right hand clause but $\eta \notin A$, then we have $(p,Y) \Vdash \neg\varphi(\hat{\eta},\hat{t},\dot{Q})$. Letting $Z(n) = X(n) \cap Y(n)$, we have $(r,Z) \leq (r,X)$ and $(p,Z) \leq (p,Y)$. Let $H \subseteq \mathbb{P}$ be V-generic with $(p,Z) \in H$ and $p^\frown\langle e_i \mid i > n\rangle$ be the Prikry sequence determined by H. It's easy to see that $r^\frown\langle e_i \mid i < \omega\rangle$ is a Prikry sequence giving rise to a generic I such that

$$(r,Z) \in I \wedge V[H] = V[I].$$

But then since $(Q^i_j)^r = (Q^i_j)^p$ for all $j<\omega$, $i \leq n$, $\dot{Q}^H = \dot{Q}^I$; and so both $\varphi(\dot{\eta},\dot{t},\dot{Q})$ and its negation hold in $V[H]$. Contradiction.[10]

Letting $\lambda = \sup_{i,j}\delta^i_j$ and $\gamma_i = \sup_{j<\omega}\delta^i_j$, by the construction of N, there is a $H \subseteq Col(\omega,<\lambda)$ generic over N such that $\mathbb{R}^*_H = \mathbb{R}^V$. To see this, it suffices to see that every $x \in \mathbb{R}^V$ is N-generic for some poset in V^N_λ. Pick n such that $x \in L[T^\infty, y]$ for some (any) $y \in \vec{d^n}(0)$. In $L[T^\infty,y]$, x is generic over $\mathrm{HOD}_{T^\infty,\langle Q^i_j \ \mid \ i<n \wedge j<\omega\rangle}$ for the Vopenka poset $\mathbb{B}$ (this gives also $\mathbb{B} \in N$). A theorem of Becker and Woodin states that on a cone of x, $L[T^\infty,x]$ satisfies $2^\alpha = \alpha^+$ for all $\alpha < \omega^V_1$. Since we can work in that cone from the beginning (i.e. can demand $\vec{d^0}(0)$ is in the cone), in $L[T,y]$, $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2 = \delta^n_0$. Hence in $L[T,y]$, $|\mathbb{B}| = \delta^n_0 < \lambda$. Furthermore, since $Q^n_0 = V^N_{\delta^n_0+1}$, x is $\mathbb{B}$-generic over M. We're done.

Recall G is the sequence $\langle \vec{d^i} \mid i<\omega\rangle$. For each $i<\omega$, let $\sigma_i = \cup_{\alpha<\gamma_i}\mathbb{R}^{H|\alpha} = \cup\vec{d^i}$ [11]. In $N[H]$, let $\mathcal{F}$ be the tail filter defined by the sequence $\langle \sigma_i \mid i<\omega\rangle$. It remains to see that $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F})$ and $\mu \cap L(\mathbb{R},\mu) = \mathcal{F} \cap L(\mathbb{R},\mathcal{F})$. For this it's enough to show $\mu \subseteq \mathcal{F}$.

[10] We note that there is a canonical name $\dot{N}$ for N and the proof above gives a condition of the form $(0,U)$ forcing that $\dot{N}$ has ω^2 Woodin cardinals.

[11] $\bigcup\vec{d^i}$ is union of all reals in a degree in $\vec{d^i}$.

Let $A \in \mu$. Then $A^* = \{d \in \mathbb{D} \mid \cup d \in A\} \in \nu$. Since $\mathbb{P}$ is the Prikry forcing relative to ν, $\exists n \forall m \geq n\ \vec{d^m} \in A^*$; this means $\exists n \forall m \geq n\ \sigma_m \in A$. This implies $A \in \mathcal{F}$. On the other hand, if $A \notin \mu$ then $\nu(A^*) = 0$. This implies $\neg A \in \mathcal{F}$. So $\mu \subseteq \mathcal{F}$. □

Proof of Theorem 1.1. The (1) ⇒ (2) direction follows from Lemmas 2.4 and 2.2. The (2) ⇒ (1) direction follows from Lemma 2.6 □

Proof of Theorem 1.2. Let $N, \lambda, G, \mathcal{F}$ be defined as in the paragraph after the proof of Lemma 2.4. In $N[G]$, let $D = L(\Gamma, \mathbb{R})$ [12] where $\Gamma = \{A \subseteq \mathbb{R} \mid A \in N(\mathbb{R}) \wedge L(A, \mathbb{R}) \vDash \mathsf{AD}^+\}$. Woodin has shown that $D \vDash \mathsf{AD}^+$ and $\Gamma = \mathcal{P}(\mathbb{R})^D$ (see [20]). Letting T^∞ be defined as in the proof of Lemma 2.6, we already know $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \mu) = \mathcal{P}(\mathbb{R}) \cap L(T^\infty, \mathbb{R}) \subseteq \Gamma$ and $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F})$ and $\mu \cap L(\mathbb{R}, \mu) = \mathcal{F} \cap L(\mathbb{R}, \mathcal{F})$. Also, by the proof of Lemma 2.2 and the ⇒ direction of Theorem 1.1, $\mathcal{F} \cap L(\mathbb{R}, \mathcal{F}) = \mathcal{C} \cap L(\mathbb{R}, \mathcal{F})$ where $\mathcal{C}$ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ in $N[G]$.

Suppose $(L(\mathbb{R}, \mu), \mu) \vDash \phi$ where ϕ is a Σ_1 statement. Then since Θ is regular in $L(\mathbb{R}, \mu)$, there is a $\kappa < \Theta$ such that $(L_\kappa(\mathbb{R}, \mu), \mu \cap L_\kappa(\mathbb{R}, \mu)) \vDash \phi$. There is a set $B \subseteq \mathbb{R}$ in $L(\mathbb{R}, \mu)$ such that B codes the structure $(L_\kappa(\mathbb{R}, \mu), \mu \cap L_\kappa(\mathbb{R}, \mu))$ and hence there is a φ such that

$$(L(\mathbb{R}, \mu), \mu) \vDash \phi \Leftrightarrow (HC, \in, B) \vDash \varphi.$$

Such a B is called a φ-witness as before. We let γ_0 be the least such that $L_{\gamma_0}(\mathbb{R}, \mu)$ ordinal defines a φ-witness. By minimizing the ordinal parameters, we assume then that the φ-witness B is definable over $L_{\gamma_0}(\mathbb{R}, \mu)$ by (Φ, x) for some $x \in \mathbb{R}$, that is

$$y \in B \Leftrightarrow L_{\gamma_0}(\mathbb{R}, \mu) \vDash \Phi[y, x].$$

By the construction of N and the proof of Lemma 2.3, there is $\alpha < \lambda$ and a $B \in Hom_{<\lambda}^{N[G \restriction \alpha]}$ such that

$$(HC^{N[G \restriction \alpha]}, \in, B) \vDash \varphi.$$

But $(HC^{N[G \restriction \alpha]}, \in, B) \prec (HC, \in, B^*)$ where $B^* \in (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R}, \mu)}$ is the canonical blowup of B by Lemma 6.3 of [9].[13] This gives us a $\kappa < \underset{\sim}{\delta}^2_1$ such that $(L_\kappa(\mathbb{R}, \mu), \mu \cap L_\kappa(\mathbb{R}, \mu)) \vDash \phi$.[14] Since ϕ is Σ_1, we have $(L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}, \mu), \mu \cap L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}, \mu)) \vDash \phi$.

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[12] D is called the "new derived model" of N at λ.

[13] To see this, first note that $B^* \in L(\mathbb{R}, \mu)$. By Theorem 4.3 of [9], B has a $Hom_{<\lambda}^{N[G \restriction \alpha]}$-scale and so does $\neg B$. This fact is projective in B so the structure $(HC, \in, B^*)$ sees that $B^*, \neg B^*$ both have a scale. Hence $B^* \in (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R}, \mu)}$.

[14] The proof of Lemma 2.3, in particular, the definition of the formula $\theta(u, v)$ there, tells us that B codes a structure of the form $(L_\kappa(\mathbb{R}, \nu), \nu)$ where ν comes from the club filter in $N[G \restriction \alpha]$ and $\kappa < \underset{\sim}{\delta}^2_1$ so in fact $L_\kappa(\mathbb{R}, \nu) = L_\kappa(\mathbb{R}, \mu)$ and $\mu \cap L(\mathbb{R}, \mu) = \nu \cap L(\mathbb{R}, \nu)$.

This finishes the proof of (1) in Theorem 1.2. (2) of Theorem 1.2 is also a corollary of the proof of Lemma 2.6. One first modifies the definition of $\mathbb{P}$ in Lemma 2.6 by redefining the set U in the condition (p, U) to be: $U(2n) \in \nu_0$ and $U(2n+1) \in \nu_1$ for all n where ν_i is defined from μ_i in the exact way that ν is defined from μ in the proof of Lemma 2.6. Everything else in the proof of the lemma stays the same. This implies $L(\mathbb{R}, \mu_0) = L(\mathbb{R}, \mu_1) = L(\mathbb{R}, \mathcal{F})$ and $\mu_0 = \mu_1 = \mathcal{F}$. To see this, just note that since we already know

$$L(\mathbb{R}, \mathcal{F}) \vDash \mathsf{AD}^+ + \mathcal{F} \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R}),$$

it suffices to show if $A \in \mathcal{F}$ then $A \in \mu_0$ and $A \in \mu_1$. Suppose there is an $A \in \mathcal{F}$ such that $A \in \mu_0$ and $A \notin \mu_1$ (the cases $A \in \mu_1 \backslash \mu_0$ and $A \notin \mu_0 \cap \mu_1$ are handled similarly). Let

$$A^* = \{d \in \mathbb{D} \mid \cup d \in A\}.$$

Then $A^* \in \nu_0 \backslash \nu_1$. For any condition (p, U), just shrink U to U^* by setting $U^*(2n) = U(2n) \cap A^*$ and $U^*(2n+1) = U(2n+1) \cap \neg A^*$. Then $(p, U^*) \Vdash A \notin \mathcal{F}$. Contradiction. This finishes the proof of Theorem 1.2. □

3 The HOD analysis

Throughout this section, we assume $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+$. The following theorem is due to Woodin.

Theorem 3.1. *Suppose $L(\mathbb{R}, \mu) \vDash AD^+ + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then in $L(\mathbb{R}, \mu)$, there is a set $A \subseteq \Theta$ such that* HOD $= L[A]$.

Proof. Working in $L(\mathbb{R}, \mu)$, let $N = L(\mathcal{P}(\mathbb{R}))$. Note that $\Theta^N = \Theta$ and $N \vDash \mathsf{AD}^+ + \Theta = \theta_0$ (see [17]). By general AD^+ theory,

1. $\mathrm{HOD}^N = L[B]$ for some $B \subseteq \Theta$ in HOD^N;

2. $\mathrm{HOD}^N[x] = \mathrm{HOD}^N_x$ for any $x \in \mathbb{R}$.

Let $\delta = \underset{\sim}{\delta}^2_1$. Since $\mu \cap L_\delta(\mathbb{R})[\mu]$ is the club filter, $N|\delta = L_\delta(\mathbb{R})[\mu]$ and hence HOD^N and HOD agree up to δ by Σ_1-reflection. Again, by general AD^+ theory, δ is strong to Θ via embeddings given by measures (see [3]) and these measures are unique (and hence OD) in N, hence HOD^N and HOD agree up to Θ. The same conclusion holds for HOD^N_x and HOD_x.[15] This is key to our proof.

Let $j : \mathrm{HOD} \rightarrow M$ be the ultrapower embedding given by μ using all functions in $L(\mathbb{R}, \mu)$. j is definable from μ. By Theorem 1.2, μ is unique hence j is OD. Similarly, μ also induces an embedding $j_x : HOD_x \rightarrow M_x$ for all $x \in \mathbb{R}$. Note that $\mathrm{HOD}^N[x] = \mathrm{HOD}^N[G_x]$ for a

[15] This has a consequence that Mouse Capturing holds in $L(\mathbb{R}, \mu)$ since Mouse Capturing holds in N

generic G_x for the Vopenka algebra whose elements are OD ∞-Borel codes. By (2) and the fact that $\mathrm{HOD}[x]^N|\Theta = \mathrm{HOD}[x]|\Theta$, j's restriction on bounded subsets of Θ can compute j_x's restriction on bounded subsets of Θ.

Claim: $L(\mathbb{R}, \mu) = L[\mathrm{HOD}^N, j \restriction \Theta](\mathbb{R}) = L[A](\mathbb{R})$ for some $A \subseteq \mathrm{OR}$ in HOD.[16]

Proof. The second equality is clear since $\mathrm{HOD}^N = L[B]$ for some $B \subseteq \Theta$ so now we prove the first equality. First it's easy to see that

$$L[\mathrm{HOD}^N, j \restriction \Theta](\mathbb{R}) = L[\mathrm{HOD}^N_x, j \restriction \Theta](\mathbb{R}) = L[\mathrm{HOD}^N[x], j \restriction \Theta](\mathbb{R}) \quad (*)$$

Let $X \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$. Note that $X \in N$. To see whether X is in μ, let S be an ∞-Borel code for X. S is a bounded subset of Θ. First suppose S is OD in N. So $X \in \mu$ if and only if whenever $g \subseteq Col(\omega, \mathbb{R})$ is generic over $L[\mathrm{HOD}^N, j \restriction \Theta](\mathbb{R})$, in $L[\mathrm{HOD}^N, j \restriction \Theta](\mathbb{R})[g]$, $\mathbb{R}$ is in the set with code $j(S)$. The case where S is OD^N_x for some $x \in \mathbb{R}$ can be handled by using $(*)$. This means $L[HOD^N, j \restriction \Theta](\mathbb{R})$ can compute μ by consulting the homogeneous forcing $Col(\omega, \mathbb{R})$; this gives us the first equality. □

Pick a large γ and consider the elementary substructure Z of $L_\gamma[\mathrm{HOD}^N, j \restriction \Theta]$ consisting of elements definable in $L(\mathbb{R}, \mu)$ from $\{\mathrm{HOD}^N, j\}$, reals, and ordinals less than Θ. Hence Z is OD and has size at most Θ. Let j^* be the transitive collapse of j. Note that

$$\mathrm{HOD}^N = L[B]$$

for some $B \subseteq \Theta$ and since $\Theta \subseteq Z$, B collapses to itself. Hence there is a set $A \subseteq \Theta$ in HOD such that $L[\mathrm{HOD}^N, j^*] \subseteq L[A]$ and it's easy to see that $L(\mathbb{R}, \mu) = L[\mathrm{HOD}^N, j^*](\mathbb{R}) = L[A](\mathbb{R})$. Now since

$$V_\Theta^{L[\mathrm{HOD}^N, j^*]} = V_\Theta^{\mathrm{HOD}^N} = V_\Theta^{\mathrm{HOD}},$$

there is a Θ-c.c. forcing $\mathbb{P}$ ($\mathbb{P}$ is a variation of the Vopenka algebra) such that we have $L[A] \subseteq \mathrm{HOD} \subseteq L[A](\mathbb{R}) = L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \mu)$ is the symmetric part of $L[A][g]$ where $g \subseteq \mathbb{P} \in L[A]$ is generic over HOD (such a g exists). This implies HOD$= L[A]$ hence completes our proof of the theorem. □

We further assume μ comes from the club filter in V, $\mathcal{M}^\sharp_{\omega^2}$ exists and has unique $(\omega, \omega_1, \omega_1 + 1)$ iteration strategy in all generic extensions of V.[17] We'll show how to get rid of these assumptions later on. We first show how to iterate $\mathcal{M}_{\omega^2}$ to realize μ as the tail filter.

[16]By $j \restriction \Theta$ we mean the set of (a, γ) such that $a \in V_\Theta^{\mathrm{HOD}}$ and $\gamma \in j(a)$

[17]In fact, it's enough to assume $\mathcal{M}^\sharp_{\omega^2}$ to be iterable in $V^{Col(\omega, \mathcal{P}(\mathbb{R}))}$.

Lemma 3.2. *There is an iterate $\mathcal{N}$ of $\mathcal{M}_{\omega^2}$ such that letting λ be the limit of $\mathcal{N}$'s Woodin cardinals, $\mathbb{R}$ can be realized as the symmetric reals over $\mathcal{N}$ at λ and letting $\mathcal{F}$ be the tail filter over $\mathcal{N}$ at λ, $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F})$.*

Proof. Let δ_i be the sup of the first ωi Woodin cardinals of $\mathcal{M}_{\omega^2}$ and $\gamma = \sup_i \delta_i$. Let $\xi \geq \omega_1$ be such that $H(\xi) \vDash \mathsf{ZFC}^-$. In $V^{Col(\omega,H(\xi))}$, let $\langle X_i \mid i < \omega\rangle$ be an increasing and cofinal chain of countable (in V) elementary substructures of $H(\xi)$ and $\sigma_i = \mathbb{R} \cap X_i$. To construct the $\mathcal{N}$ as in the statement of the lemma, we do an $\mathbb{R}$-genericity iteration (in $V^{Col(\omega,H(\xi))}$) as follows. Let $\mathcal{P}_0 = \mathcal{M}^{\sharp}_{\omega^2}$ and assume $\mathcal{P}_0 \in X_0$. For $i > 0$, let $\mathcal{P}_i$ be the result of iterating $\mathcal{P}_{i-1}$ in X_{i-1} in the window between the $\omega(i-1)^{th}$ and ωi^{th} Woodin cardinals of $\mathcal{P}_{i-1}$ to make σ_{i-1} generic. We can make sure that each finite stage of the iteration is in X_{i-1}. Let $\mathcal{P}_\omega$ be obtained from the direct limit of the $\mathcal{P}_i$'s and iterating the top extender out of the universe. Let λ be the limit of Woodin cardinals in $\mathcal{P}_\omega$. It's clear that there is a $G \subseteq Col(\omega, <\lambda)$ generic over $\mathcal{P}_\omega$ such that $\mathbb{R} =_{def} \mathbb{R}^V$ is the symmetric reals over $\mathcal{P}_\omega$ and $L(\mathbb{R},\mu)$ is in $\mathcal{P}_\omega[G]$. Let $\mathcal{F}$ be the tail filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ defined over $\mathcal{P}_\omega[G]$. By section 2, $L(\mathbb{R},\mathcal{F}) \vDash \mathcal{F}$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

We want to show $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F})$. To show this, it's enough to see that if $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ is in $L(\mathbb{R},\mu)$ and A is a club (i.e. $A \in \mu$) then $A \in \mathcal{F}$. Let $\pi : \mathbb{R}^{<\omega} \to \mathbb{R} \in V$ witness that A is a club. By the choice of the X_i's, there is an n such that for all $m \geq n$, $\pi \in X_m$ and hence $\pi''\sigma_m^{<\omega} \subseteq \sigma_m$. This shows $A \in \mathcal{F}$. This in turns implies $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F})$ and $\mathcal{F} \cap L(\mathbb{R},\mathcal{F}) = \mu \cap L(\mathbb{R},\mu)$. □

We fix some notation. For a nondropping iterate $\mathcal{P}$ of $\mathcal{M}_{\omega^2}$, let $\gamma_i^{\mathcal{P}}$ be the supremum of the first $\omega(i+1)$ Woodin cardinals of $\mathcal{P}$ and $\lambda^{\mathcal{P}} = \sup_{i<\omega}\gamma_i$. From this point on to the end of the section, we assume the reader has in hands a copy of [13]. Our construction follows closely that paper. There's no point in rewritting every detail there.

Let $\mathcal{M}^+_\infty$ be the direct limit of all nondropping iterates (via countable stacks of countable normal trees) $\mathcal{P}$ of $\mathcal{M}_{\omega^2}$ below the first Woodin cardinal and $\mathcal{H}^+$ be the corresponding direct limit system. By definition, $\mathcal{H}^+$ is countably directed and hence $\mathcal{M}^+_\infty$ is well-founded. We'll define a direct limit system $\mathcal{H}$ in $L(\mathbb{R},\mu)$ that approximates $\mathcal{H}^+$. Working in $L(\mathbb{R},\mu)$, we say $\mathcal{P}$ is *suitable* if

1. $\mathcal{P}$ has only one Woodin cardinal $\delta^{\mathcal{P}}$;

2. it is full (with respect to mice), that is for all $\xi < o(\mathcal{P})$ such that ξ is a cutpoint of $\mathcal{P}$, $Lp(\mathcal{P}|\xi) \lhd \mathcal{P}$ and for all $\xi \neq \delta^{\mathcal{P}}$, $Lp(\mathcal{P}|\xi) \vDash \xi$ is not Woodin and $Lp(\mathcal{P}|\xi) \in \mathcal{P}$;

3. $\mathcal{P} = Lp_\omega(\mathcal{P}|\delta^{\mathcal{P}})$.

The following definition comes from Definition 6.21 in [13].

Definition 3.3. *Working in $L(\mathbb{R},\mu)$, we let $\mathcal{O}$ be the collection of all functions f such that f is an ordinal definable function with domain the set of all countable, suitable $\mathcal{P}$, and $\forall\mathcal{P}\in\mathrm{dom}(f)(f(\mathcal{P})\subseteq\delta^{\mathcal{P}})$.*

Definition 3.4. *Suppose $\vec{f}\in\mathcal{O}^{<\omega}$, $\mathcal{P}$ is suitable, and $\mathrm{dom}(\vec{f})=n$. Let*

$$\gamma_{(\mathcal{P},\vec{f})}=\sup\{Hull^{\mathcal{P}}(\vec{f}(0)(\mathcal{P}),\cdots,\vec{f}(n-1)(\mathcal{P}))\cap\delta^{\mathcal{P}}\},$$

and

$$H_{(\mathcal{P},\vec{f})}=Hull^{\mathcal{P}}(\gamma_{(\mathcal{P},\vec{f})}\cup\{\vec{f}(0)(\mathcal{P}),\cdots,\vec{f}(n-1)(\mathcal{P})\}).$$

We refer to reader to Section 6.3 of [13] for the definitions of $\vec{f}$-iterability, strong $\vec{f}$-iterability. The only difference between our situation and the situation in [13] is that our notions of "suitable", "short", "maximal", "short tree iterable" etc. are relative to the pointclass $(\Sigma^2_1)^{L(\mathbb{R},\mu)}$ instead of $(\Sigma^2_1)^{L(\mathbb{R})}$ as in [13].

Now, let $(\mathcal{P},\vec{f})\in\mathcal{H}$ if $\mathcal{P}$ is strongly $\vec{f}$-iterable. The ordering on $\mathcal{H}$ is defined as follows:

$$(\mathcal{P},\vec{f})\leq_{\mathcal{H}}(\mathcal{Q},\vec{g})\Leftrightarrow\vec{f}\subseteq\vec{g}\wedge\mathcal{Q}\text{ is a psuedo-iterate of }\mathcal{P}.^{18}$$

Note that if $(\mathcal{P},\vec{f})\leq_{\mathcal{H}}(\mathcal{Q},\vec{g})$ then there is a natural embedding $\pi_{(\mathcal{P},\vec{f}),(\mathcal{Q},\vec{g})}:H_{\mathcal{P},\vec{f}}\to H_{\mathcal{Q},\vec{g}}$. We need to see that $\mathcal{H}\neq\emptyset$.

Lemma 3.5. *Let $\vec{f}\in\mathcal{O}^{<\omega}$. Then there is a $\mathcal{P}$ such that $(\mathcal{P},\vec{f})\in\mathcal{H}$.*

Proof sketch. For simplicity, assume $\mathrm{dom}(\vec{f})=1$. The proof of this lemma is just like the proof of Theorem 6.29 in [13]. We only highlight the key changes that make that proof work here.

First let $\nu,\mathbb{P}$ be as in the proof of Lemma 2.6. Let a be a countable transitive self-wellordered set and x be a real that codes a. We need to modify the Q^x_a defined in the proof of Lemma 2.6. Fix a coding of relativized premice by reals and write $\mathcal{P}_z$ for the premouse coded by z. Then let

$$\mathcal{F}^x_a=\{\mathcal{P}_z\mid z\leq_T x\text{ and }\mathcal{P}_z\text{ is a suitable premouse over }a\text{ and }\mathcal{P}_z\text{ is short-tree iterable}\}.$$

Let

$$\mathcal{Q}^x_a=Lp(\mathcal{Q}^{x,-}_a),$$

[18] See definition 6.20 of [13] for the definition of psuedo-iterate.

where $\mathcal{Q}_a^{x,-}$ is the direct limit of the simultaneous comparison and $\{y \mid y \leq_T x\}$-genericity iteration of all $\mathcal{P} \in \mathcal{F}_a^x$. The definition of $\mathcal{Q}_a^x$ comes from Section 6.6 of [13]. As in the proof of Lemma 2.6, we have:

1. letting $\langle \vec{d^i} \mid i < \omega \rangle$ be the generic sequence for $\mathbb{P}$ and $\langle \mathcal{Q}_j^i \mid i, j < \omega \rangle$ be the sequence of models associated to $\langle \vec{d^i} \mid i < \omega \rangle$ as defined in the proof of Lemma 2.6, we have that the model $N = L[T^\infty, \mathcal{M}^{\langle \vec{d^i} \rangle_i}] \models$ "there are ω^2 Woodin cardinals", where $\mathcal{M}^{\langle \vec{d^i} \rangle_i} = L[\cup_i \cup_j \mathcal{Q}_j^i]$;

2. letting λ be the sup of the Woodin cardinals of N, there is a $G \subseteq Col(\omega, < \lambda)$, G is N-generic such that letting $\mathbb{R}_G^*$ be the symmetric reals of $N[G]$ and $\mathcal{F}$ be the tail filter defined over $N[G]$, then $L(\mathcal{R}_G^*, \mathcal{F}) = L(\mathbb{R}, \mu)$ and $\mathcal{F} \cap L(\mathbb{R}, \mu) = \mu$.

The second key point is that whenever $\mathcal{P} \in \mathcal{H}^+$, we can then iterate $\mathcal{P}$ to $\mathcal{Q}$ (above any Woodin cardinal of $\mathcal{P}$) so that $\mathbb{R}^V$ can be realized as the symmetric reals for some $G \subseteq Col(\omega, < \delta_{\omega^2}^{\mathcal{Q}})$ and $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F})$ and $\mu \cap L(\mathbb{R}, \mu) = \mathcal{F} \cap L(\mathbb{R}, \mu)$, where $\mathcal{F}$ is the tail filter defined over $\mathcal{Q}[G]$. This is proved in Lemma 3.2.

We leave it to the reader to check that the proof of Theorem 6.29 of [13] goes through for our situation. This completes our sketch. $\square$

Remark: The lemma above obviously shows $\mathcal{H} \neq \emptyset$. Its proof also shows for any $\vec{f} \in \mathcal{O}^{<\omega}$ and any $(\mathcal{P}, \vec{g}) \in \mathcal{H}$, there is a $\vec{g}$-iterate $\mathcal{Q}$ of $\mathcal{P}$ such that $\mathcal{Q}$ is $(\vec{f} \cup \vec{g})$-strongly iterable.

Now we outline the proof that $\mathcal{M}_\infty^+ \subseteq \mathrm{HOD}^{L(\mathbb{R},\mu)}$. We follow the proof in Section 6.7 of [13]. Suppose $\mathcal{P}$ is suitable and $s \in [\mathrm{OR}]^{<\omega}$, let $\mathcal{L}_{\mathcal{P},s}$ be the language of set theory expanded by constant symbols c_x for each $x \in \mathcal{P}|\delta^{\mathcal{P}} \cup \{\mathcal{P}\}$ and d_x for each x in the range of s. Since s is finite, we can fix a coding of the syntax of $\mathcal{L}_{\mathcal{P},s}$ such that it is definable over $\mathcal{P}|\delta^{\mathcal{P}}$ and the map $x \mapsto c_x$ is definable over $\mathcal{P}|\delta^{\mathcal{P}}$. We continue to use $\mathbb{P}$ to denote the Prikry forcing in Lemma 2.6.

Definition 3.6. *Let $\mathcal{P}$ be suitable and $s = \{\alpha_1, \cdots, \alpha_n\}$. We set*

$$T_s(\mathcal{P}) = \{\phi \in \mathcal{L}_{\mathcal{P},s} \mid \exists p \in \mathbb{P}\ (p = (\emptyset, X) \wedge p \Vdash (\mathcal{M}^{\vec{d}_{\dot{G}}}, \alpha_1, \cdots, \alpha_n, x)_{x \in \mathcal{P}|\delta^{\mathcal{P}}} \models \phi\}.$$

In the above definition, $\mathcal{M}^{\vec{d}_{\dot{G}}}$ is the canonical name for the model $\mathcal{M}^{\langle \vec{d^i} \rangle_i}$ defined in Lemma 3.5 where $\langle \vec{d^i} \rangle_i$ is the Prikry sequence given by a generic $G \subseteq \mathbb{P}$. Note that $T_s(\mathcal{P})$ is a complete, consistent theory of $\mathcal{L}_{\mathcal{P},s}$ and if $s \subseteq t$, we can think of $T_s(\mathcal{P})$ as a subtheory of $T_t(\mathcal{P})$ in a natural way (after appropriately identifying the constant symbols of one with those of the other). Furthermore, $T_s \in \mathcal{O}$ for any $s \in [\mathrm{OR}]^{<\omega}$.

Let $\mathcal{N}_\infty$ be the direct limit of $\mathcal{H}$ under maps $\pi_{(\mathcal{P},\vec{f}),(\mathcal{Q},\vec{g})}$ for $(\mathcal{P}, \vec{f}) \leq_{\mathcal{H}} (\mathcal{Q}, \vec{g})$. Let $\pi_{(\mathcal{P},\vec{f}),\infty}$:

$H_{\mathcal{P},\vec{f}} \to \mathcal{N}_\infty$ be the direct limit map. For each $s \in [\mathrm{OR}]^{<\omega}$ and $\mathcal{P}$ which is strongly T_s-iterable, we let

$$T_s^* = \pi_{(\mathcal{P},T_s),\infty}(T_s(\mathcal{P})).$$

Again, $s \subseteq t$ implies $T_s^* \subseteq T_t^*$, so we let

$$T^* = \bigcup\{T_s^* \mid s \in [\mathrm{OR}]^{<\omega}\}.$$

We have that T^* is a complete, consistent, and Skolemized[19] theory of $\mathcal{L}$, where $\mathcal{L} = \bigcup\{\mathcal{L}_{\mathcal{N}_\infty,s} \mid s \in [\mathrm{OR}]^{<\omega}\}$. We note that T^* is definable in $L(\mathbb{R},\mu)$ because the map $s \mapsto T_s^*$ is definable in $L(\mathbb{R},\mu)$.

Let $\mathcal{A}$ be the unique pointwise definable $\mathcal{L}$-structure such that $\mathcal{A} \vDash T^*$. We show $\mathcal{A}$ is wellfounded and let $\mathcal{N}_\infty^+$ be the transitive collapse of $\mathcal{A}$, restricted to the language of premice.

Lemma 3.7. $\mathcal{N}_\infty^+ = \mathcal{M}_\infty^+$

Proof sketch. We sketch the proof which completely mirrors the proof of Lemma 6.51 in [13]. Let Σ be the iteration strategy of $\mathcal{M}_{\omega^2}$ and $\Sigma_\mathcal{P}$ be the tail of Σ for a Σ-iterate $\mathcal{P}$ of $\mathcal{M}_{\omega^2}$. We will also use $\langle \delta_\alpha^\mathcal{P} \mid \alpha < \omega^2\rangle$ to denote the Woodin cardinals of a Σ-iterate $\mathcal{P}$ of $\mathcal{M}_{\omega^2}$. We write $\mathcal{P}^- = \mathcal{P}|((\delta_0^\mathcal{P})^{+\omega})^\mathcal{P}$. Working in $V^{Col(\omega,\mathbb{R})}$, we define sequences $\langle \mathcal{N}_k \mid k < \omega\rangle$, $\langle \mathcal{N}_k^\omega \mid k < \omega\rangle$, $\langle j_{k,l} \mid k \le l \le \omega\rangle$, $\langle i_k \mid k < \omega\rangle$, $\langle G_k \mid k < \omega\rangle$, and $\langle j_{k,l}^\omega \mid k \le l \le \omega\rangle$ such that

(a) $\mathcal{N}_k \in \mathcal{H}^+$ for all k;

(b) for all k, $\mathcal{N}_{k+1}$ is a $\Sigma_{\mathcal{N}_k}$-iterate of $\mathcal{N}_k$ (below the first Woodin cardinal of $\mathcal{N}_k$) and the corresponding iteration map is $j_{k,k+1}$;

(c) the $\mathcal{N}_k$'s are cofinal in $\mathcal{H}^+$;

(d) $i_k : \mathcal{N}_k \to \mathcal{N}_k^\omega$ is an iteration map according to $\Sigma_{\mathcal{N}_k}$ with critical point $> \delta_0^{\mathcal{N}_k}$;

(e) G_k is generic over $\mathcal{N}_k^\omega$ for the symmetric collapse up to the sup of its Woodins and $\mathbb{R}^*_{G_k} = \mathbb{R}^V$;

(f) $\mathcal{N}_k^\omega = \mathcal{M}^{\langle \vec{d}^i\rangle_i}$ for some $\langle \vec{e}^i\rangle_i$ which is $\mathbb{P}$-generic over $L(\mathbb{R},\mu)$ such that $(\mathcal{N}_k^\omega)^-$ is coded by a real in $\vec{e}^0(0)$;

(g) $j_{k,k+1}^\omega : \mathcal{N}_k^\omega \to \mathcal{N}_{k+1}^\omega$ is the iteration map;

(h) for $k < l$, $j_{k,l}^\omega \circ i_k = i_l \circ j_{k,l}$, where $j_{k,l} : \mathcal{N}_k \to \mathcal{N}_l$ and $j_{k,l}^\omega : \mathcal{N}_k^\omega \to \mathcal{N}_l^\omega$ are natural maps;

[19] This is because of the Prikry property of $\mathbb{P}$.

(i) $j_{k,k+1}|\mathcal{N}_k^- = j^\omega_{k,k+1}|(\mathcal{N}^\omega_k)^-$;

(j) the direct limit $\mathcal{N}^\omega_\omega$ of the $\mathcal{N}^\omega_k$ under maps $j^\omega_{k,l}$'s embeds into a $\Sigma_{\mathcal{M}^+_\infty}$-iterate of $\mathcal{M}^+_\infty$;

(k) for each $s \in [\mathrm{OR}]^{<\omega}$, for all sufficiently large k,

$$\mathcal{N}^\omega_k \vDash \phi[x, s] \Leftrightarrow \exists p \in \mathbb{P}\ (p = (\emptyset, X) \wedge p \Vdash (\mathcal{M}^{\vec{d}_{\dot{G}}} \vDash \phi[x, s]),$$

for $x \in \mathcal{N}^\omega_k|\delta_0^{\mathcal{N}^\omega_k}$.

Everything except for (f) is as in the proof of Lemma 6.51 of [13]. To see (f), fix a $k < \omega$. We fix a Prikry sequence $\langle \vec{d^i} \rangle_i$ such that $(\mathcal{N}^\omega_k)^-$ is coded into $\vec{d^0}(0)$ and letting $\sigma_i = \{y \in \mathbb{R}^V \mid y$ is recursive in $\vec{d^i}(j)$ for some $j < \omega\}$, then for each i, σ_i is closed under the iteration strategy $\Sigma_{\mathcal{N}_k}$ (this can be done in V). We then (inductively) for all i, construct a sequence $\langle \vec{e^i} \mid i < \omega \rangle$ such that $\vec{e^i}$ is a Prikry generic subsequence of $\vec{d^i}$ such that $M^{\langle \vec{e^i} \rangle_i}$ is an iterate of $\mathcal{N}_k$ (see Lemma 6.49 of [13]). The sequence $\langle \vec{e^i} \rangle_i$ satisfies (f) for $\mathcal{N}^\omega_k$.

Having constructed the above objects, the proof of Lemma 6.51 in [13] adapts here to give an isomorphism between $\mathcal{A}$ (viewed as a structure for the language of premice) and $\mathcal{M}^+_\infty$. The isomorphism is the unique extension to all of $\mathcal{A}$ of the map σ, where $\sigma(c^{\mathcal{A}}_x) = x$ (for $x \in \mathcal{M}^+_\infty|\delta_0^{\mathcal{M}^+_\infty}$) and $\sigma(d^{\mathcal{A}}_\alpha) = j^\omega_{k,\omega}(\alpha)$ for k large enough such that $j^\omega_{l,l+1}(\alpha) = \alpha$ for all $l \geq k$. This completes our sketch. □

Now we continue with the sketch of the proof that $\mathrm{HOD}^{L(\mathbb{R},\mu)}$ is a strategy mouse in the presence of $\mathcal{M}^\sharp_{\omega^2}$. Let λ_∞ be the supremum of the Woodin cardinals of $\mathcal{M}^+_\infty$. Let $\mathbb{R}^*$ be the symmetric reals given by an M^+_∞ generic $G \subseteq Col(\omega, <\lambda_\infty)$ and $\mathcal{F}^*$ be the corresponding tail filter defined in $\mathcal{M}^+_\infty[G]$. Since $L(\mathbb{R}^*, \mathcal{F}^*) \equiv L(\mathbb{R}, \mu)$, $L(\mathbb{R}^*, \mathcal{G}^*)$ has its own version of $\mathcal{H}$ and $\mathcal{N}^+_\infty$, so we let

$$\mathcal{H}^* = \mathcal{H}^{L(\mathbb{R}^*,\mathcal{F}^*)} \text{ and } (\mathcal{N}^+_\infty)^* = (\mathcal{N}^+_\infty)^{L(\mathbb{R}^*,\mathcal{F}^*)}.$$

Let Λ be the restriction of $\Sigma_{\mathcal{M}^+_\infty}$ to stacks $\vec{\mathcal{T}} \in \mathcal{M}^+_\infty|\lambda_\infty$, where

- $\vec{\mathcal{T}}$ is based on $\mathcal{M}^+_\infty|\delta_0^{\mathcal{M}^+_\infty}$;
- $L(\mathbb{R}^*, \mathcal{F}^*) \vDash \vec{\mathcal{T}}$ is a finite full stack[20].

We show $L[\mathcal{M}^+_\infty, \Lambda] = \mathrm{HOD}^{L(\mathbb{R},\mu)}$ through a sequence of lemmas. For an ordinal α, put

$$\alpha^* = d^{\mathcal{A}}_\alpha,$$

and for $s = \{\alpha_1, \cdots, \alpha_n\}$ a finite set of ordinals, put

[20]See Definition 6.20 of [13] for the precise definition of finite full stacks.

$$s^* = \{\alpha_1^*, \cdots, \alpha_n^*\}.$$

Lemma 3.8 (Derived model resemblance)**.** *Let $(\mathcal{P}, \vec{f}) \in \mathcal{H}$ and $\overline{\eta} < \gamma_{(\mathcal{P},\vec{f})}$, and $\eta = \pi_{(\mathcal{P},\vec{f}),\infty}(\overline{\eta})$. Let $s \in [\mathrm{OR}]^{<\omega}$, and $\phi(v_0, v_1, v_2)$ be a formula in the language of set theory; then the following are equivalent*

(a) $L(\mathbb{R}^*, \mathcal{F}^*) \vDash \phi[\mathcal{M}_\infty, \eta, s^*]$;

(b) $L(\mathbb{R}, \mu) \vDash$ "there is an $(\mathcal{R}, \vec{f}) \geq_{\mathcal{F}} (\mathcal{P}, \vec{f})$ such that whenever $(\mathcal{Q}, \vec{f}) \geq_{\mathcal{H}} (\mathcal{R}, \vec{f})$, then $\phi(\mathcal{Q}, \pi_{(\mathcal{P},\vec{f}),(\mathcal{Q},\vec{f})}(\overline{\eta}), s)$".

The proof of this lemma is almost exactly like the proof of Lemma 6.54 of [13], so we omit it. The only difference is in Lemma 6.54 of [13], the proof of Lemma 6.51 of [13] is used, here we use that of Lemma 3.7.

Lemma 3.9. *Λ is definable over $L(\mathbb{R}, \mu)$, and hence $L[\mathcal{M}_\infty^+, \Lambda] \subseteq \mathrm{HOD}^{L(\mathbb{R},\mu)}$*

Proof. Suppose $f \in \mathcal{O}$ is definable in $L(\mathbb{R}, \mu)$ by a formula ψ and $s \in [\mathrm{OR}]^{<\omega}$, then we let $f^* \in \mathcal{O}^{L(\mathbb{R}^*,\mathcal{F}^*)}$ be definable in $L(\mathbb{R}^*, \mathcal{F}^*)$ from ψ and s^*.

Sublemma 3.10. *Let $\vec{\mathcal{T}}$ be a finite full stack on $\mathcal{M}_\infty^+|\delta_0^{\mathcal{M}_\infty^+}$ in $L(\mathbb{R}^*, \mathcal{F}^*)$ and let $\vec{b} = \Sigma_{\mathcal{M}_\infty^+}(\vec{\mathcal{T}})$. Then $\vec{b}$ respects f^*, for all $f \in \mathcal{O}$.*

The proof of Sublemma 3.10 is just like that of Claim 6.57 in [13] (with appropriate use of the proof of Lemma 3.7). Sublemma 3.10 implies $\mathcal{M}_\infty$ is strongly f^*-iterable in $L(\mathbb{R}^*, \mathcal{F}^*)$ for all $f \in \mathcal{O}$. Sublemma 3.10 also gives the following.

Sublemma 3.11. *Suppose $\mathcal{Q}$ is a psuedo-iterate[21] of $\mathcal{M}_\infty$ and $\mathcal{T}$ is a maximal tree on $\mathcal{Q}$ in the sense of $L(\mathbb{R}^*, \mathcal{F}^*)$. Let $b = \Lambda(\mathcal{T})$; then for all $\eta < \delta^{\mathcal{Q}}$, the following are equivalent:*

(a) $i_b^{\mathcal{T}}(\eta) = \xi$;

(b) there is some $f \in \mathcal{O}$ such that $\eta < \gamma_{(\mathcal{Q},f^*)}$ and exists some branch choice[22] of $\mathcal{T}$ that respects f^* and $i_c^{\mathcal{T}}(\eta) = \xi$.

Since the $\gamma_{(\mathcal{Q},f^*)}$'s sup up to $\delta^{\mathcal{Q}}$ and i_b is continuous at $\delta^{\mathcal{Q}}$, clause (b) defines Λ over $L(\mathbb{R}, \mu)$. □

We have an iteration map

$$\pi_\infty : \mathcal{N}_\infty \to \mathcal{N}_\infty^*$$

[21]See Definition 6.13 of [13].
[22]See Definition 6.23 of [13].

which is definable over $L(\mathbb{R}^*, \mathcal{F}^*)$ by the equality

$$\pi_\infty = \cup_{f\in\mathcal{O}} \pi^{\mathcal{H}^*}_{(\mathcal{N}_\infty, f^*),\infty}.$$

By Boolean comparison, π_∞ is definable over $L[\mathcal{M}^+_\infty, \Lambda]$. This implies $\mathcal{N}^*_\infty$ is the direct limit of all Λ-iterates of $\mathcal{N}_\infty$ which belong to $\mathcal{M}^+_\infty$ and π_∞ is the canonical map into the direct limit. Lemma 3.8 also gives us the following.

Lemma 3.12. *For all $\eta < \delta_0^{\mathcal{M}^+_\infty}$, $\pi_\infty(\eta) = \eta^*$.*

Finally, we have

Theorem 3.13. *Suppose $\mathcal{M}^\sharp_{\omega^2}$ exists and is $(\omega, \mathrm{OR}, \mathrm{OR})$-iterable. Suppose μ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+ + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then the following models are equal:*

1. $\mathrm{HOD}^{L(\mathbb{R},\mu)}$,
2. $L[\mathcal{M}^+_\infty, \pi_\infty]$,
3. $L[\mathcal{M}^+_\infty, \Lambda]$.

Proof. Since $\pi_\infty \in L[\mathcal{M}^+_\infty, \Lambda]$, $L[\mathcal{M}^+_\infty, \pi_\infty] \subseteq L[\mathcal{M}^+_\infty, \Lambda]$. Lemma 3.9 implies $L[\mathcal{M}^+_\infty, \Lambda] \subseteq \mathrm{HOD}^{L(\mathbb{R},\mu)}$. It remains to show $\mathrm{HOD}^{L(\mathbb{R},\mu)} \subseteq L[\mathcal{M}_\infty, \pi_\infty]$. By Theorem 3.1, in $L(\mathbb{R}, \mu)$, there is some $A \subseteq \Theta$ such that $\mathrm{HOD} = L[A]$. Let ϕ define A. By Lemma 3.8

$$\alpha \in A \Leftrightarrow L[\mathcal{M}^+_\infty, \pi_\infty] \vDash \mathcal{M}^+_\infty \vDash (1 \Vdash L(\mathbb{R}^*, \mathcal{F}^*) \vDash \phi[\alpha^*]).$$

By Lemma 3.12, $\alpha^* = \pi_\infty(\alpha)$ and hence the above equivalence defines A over $L[\mathcal{M}^+_\infty, \pi_\infty]$. This completes the proof of the theorem. □

We now describe how to compute HOD just assuming $V = L(\mathbb{R}, \mu)$ satisfying AD^+. Let $\mathcal{H}$ be as above. The idea is that we use Σ_1 reflection to reflect a "bad" statement φ (like "$\mathcal{N}^+_\infty$ is illfounded" or "$\mathrm{HOD} \neq L(\mathcal{N}^+_\infty, \Lambda)$") to a level $L_\kappa(\mathbb{R}, \mu)$ where $\kappa < \underset{\sim}{\delta}^2_1$ (i.e. we have that $L_\kappa(\mathbb{R}, \mu) \vDash \varphi$). But then since $\mu \cap L_\kappa(\mathbb{R}, \mu)$ comes from the club filter, all we need to compute HOD in $L_\kappa(\mathbb{R}, \mu)$ is to construct a mouse $\mathcal{N}$ related to N just like $M^\sharp_{\omega^2}$ related to $L(\mathbb{R}, \mu)$. Once the mouse $\mathcal{N}$ is constructed, we sucessfully compute HOD of $L_\kappa(\mathbb{R}, \mu)$ and hence show that $L_\kappa(\mathbb{R}, \mu) \vDash \neg\varphi$. This gives us a contradiction.

We now proceed to construct $\mathcal{N}$. To be concrete, we fix a "bad" statement φ (like "HOD is illfounded") and let $N = L_\kappa(\mathbb{R}, \mu)$ be least such that $N \vDash (T)$ where $(T) \equiv$ "$\mathsf{MC} + \mathsf{AD}^+ + \mathsf{DC} + \mathsf{ZF}^- + \Theta = \theta_0 + \varphi$". Let $\Gamma^* = (\Sigma^2_1)^N$, $\Phi = \mathcal{P}(\mathbb{R})^N$ and U be the universal Φ-set. We have that Γ^* is a good pointclass and $Env(\underset{\sim}{\Gamma}^*) = \Phi$ by closure of N.

Let $\vec{B} = \langle B_i \mid i < \omega \rangle$ be a sjs sealing $Env(\underset{\sim}{\Gamma}^*)$ with each $B_i \in N$ and $B_0 = U$. Such a $\vec{B}$ exists (see Section 4.1 of [18]).

Because MC holds and $\Phi \subsetneq \underset{\sim}{\delta}^2_1$, there is a real x such that there is a sound mouse $\mathcal{M}$ over x such that $\rho(\mathcal{M}) = x$ and $\mathcal{M}$ doesn't have an iteration strategy in N. Fix then such an $(x, \mathcal{M})$ and let Σ be the strategy of $\mathcal{M}$. Let $\Gamma \subsetneq \underset{\sim}{\Delta}^2_1$ be a good pointclass such that $Code(\Sigma), \vec{B}, U, U^c \in \underset{\sim}{\delta}_\Gamma$. By Theorem 10.3 in [11], there is a z such that $(\mathcal{N}^*_z, \delta_z, \Sigma_z)$ Suslin captures $Code(\Sigma), \vec{B}, U, U^c$ and $\mathcal{N}^*_z$ is coarse mouse with iteration stratetgy $\Sigma_z \in \underset{\sim}{\delta}^2_1$ and δ_z is the unique Woodin cardinal of $\mathcal{N}^*_z$.

Because $\vec{B}$ is Suslin captured by $\mathcal{N}^*_z$, we have $(\delta_z^+)^{\mathcal{N}^*_z}$-complementing trees $T, S \in \mathcal{N}^*_z$ [23] with the property that for any Σ_z-iterate N^* of $\mathcal{N}^*_z$ such that the iteration map $i : \mathcal{N}^*_z \to N^*$ exists, for any $<$-$i((\delta_z^+)^{\mathcal{N}^*_z})$-generic g over N^*, $p[i(T)] \cap N^*[g] = \vec{B} \cap N^*[g] = \mathbb{R}^{N^*[g]} \backslash p[i(S)]$. Let κ be the least cardinal of $\mathcal{N}^*_z$ which, in $\mathcal{N}^*_z$ is $< \delta_z$-strong.

Claim 1. $\mathcal{N}^*_z \vDash$ "κ is a limit of points η such that $Lp^{\Gamma^*}(\mathcal{N}^*_z|\eta) \vDash$ "η is Woodin".

Proof. The proof is an easy reflection argument. Let $\lambda = \delta_z^+$ and let $\pi : M \to \mathcal{N}^*_z|\lambda$ be an elementary substructure such that

1. $T, S \in ran(\pi)$,

2. if $\mathrm{cp}(\pi) = \eta$ then $V_\eta^{\mathcal{N}^*_z} \subseteq M$, $\pi(\eta) = \delta_z$ and $\eta > \kappa$.

By elementarity, we have that $M \vDash$ "η is Woodin". Letting $\pi^{-1}(\langle T, S \rangle) = \langle \bar{T}, \bar{S} \rangle$, we have that $(\bar{T}, \bar{S})$ Suslin captures $\vec{B}$ over M at η. This implies that M is Φ-full and in particular, $Lp^{\Gamma^*}(\mathcal{N}^*_z|\eta) \in M$. Therefore, $Lp^{\Gamma^*}(\mathcal{N}^*_z|\eta) \vDash$ "η is Woodin". The claim then follows by a standard argument. □

Let now $\langle \eta_i : i < \omega^2 \rangle$ be the first ω^2 points $< \kappa$ such that for every $i < \omega$, $Lp^{\Gamma^*}(\mathcal{N}^*_z|\eta_i) \vDash$ "η_i is Woodin". Let now $\langle \mathcal{N}_i : i < \omega^2 \rangle$ be a sequence constructed according to the following rules:

1. $\mathcal{N}_0 = L[\vec{E}]^{\mathcal{N}^*_z|\eta_0}$,

2. if i is limit, $\mathcal{N}'_i = \cup_{j<i} \mathcal{N}_i$ and $\mathcal{N}_i = (L[\vec{E}][\mathcal{N}'_i])^{\mathcal{N}^*_z|\eta_i}$,

3. $\mathcal{N}_{i+1} = (L[\vec{E}][\mathcal{N}_i])^{\mathcal{N}^*_z|\eta_{i+1}}$.

Let $\mathcal{N}_{\omega^2} = \cup_{i<\omega^2} \mathcal{N}_i$.

Claim 2. For every $i < \omega^2$, $\mathcal{N}_{\omega^2} \vDash$ "η_i is Woodin" and $\mathcal{N}_{\omega^2}|(\eta_i^+)^{\mathcal{N}_\omega} = Lp^{\Gamma^*}(\mathcal{N}_i)$.

[23] This means that whenever g is $< (\delta_z^+)^{\mathcal{N}^*_z}$-generic over $\mathcal{N}^*_z$, then in $\mathcal{N}^*_z[g]$, $p[T]$ and $p[S]$ project to complements.

Proof. It is enough to show that

1. $\mathcal{N}_{i+1} \vDash$ "η_i is Woodin",
2. $\mathcal{N}_i = V_{\eta_i}^{\mathcal{N}_{i+1}}$,
3. $\mathcal{N}_{i+1}|(\eta_i^+)^{\mathcal{N}_{i+1}} = Lp^{\Gamma^*}(\mathcal{N}_i)$,
4. if i is limit, then $\mathcal{N}_i|((\sup_{j<i}\eta_j^+)^{\mathcal{N}_i}) = Lp^{\Gamma^*}(\mathcal{N}_i')$.

To show 1-4, it is enough to show that if $\mathcal{W} \trianglelefteq \mathcal{N}_{i+1}$ is such that $\rho_\omega(W) \leq \eta_i$ or if i is limit and $\mathcal{W} \triangleleft \mathcal{N}_i$ is such that $\rho_\omega(W) \leq \sup_{j<i}\eta_j$ then the fragment of $\mathcal{W}$'s iteration strategy which acts on trees above η_i ($\sup_{j<i}\eta_j$ respectively) is in Γ^*. Suppose first that i is a successor and $\mathcal{W} \trianglelefteq \mathcal{N}_{i+1}$ is such that $\rho_\omega(W) \leq \eta_i$. Let ξ be such that the if $\mathcal{S}$ is the ξth model of the full background construction producing $\mathcal{N}_{i+1}$ then $\mathbb{C}(\mathcal{S})$[24] $= \mathcal{W}$. Let $\pi : \mathcal{W} \to \mathcal{S}$ be the core map. The iteration strategy of $\mathcal{W}$ is the π-pullback of the iteration strategy of $\mathcal{S}$. Let then $\nu < \eta_{i+1}$ be such that $\mathcal{S}$ is the ξth model of the full background construction of $\mathcal{N}_x^*|\nu$. To determine the complexity of the induced strategy of $\mathcal{S}$ it is enough to determine the strategy of $\mathcal{N}_x^*|\nu$ which acts on non-dropping stacks that are completely above η_i. Now, notice that by the choice of η_{i+1}, for any non-dropping tree $\mathcal{T}$ on $\mathcal{N}_x^*|\nu$ which is above η_i and is of limit length, if $b = \Sigma(\mathcal{T})$ then $\mathcal{Q}(b,\mathcal{T})$ exists and $\mathcal{Q}(b,\mathcal{T})$ has no overlaps, and $\mathcal{Q}(b,\mathcal{T}) \trianglelefteq Lp^{\Gamma^*}(\mathcal{M}(\mathcal{T}))$. This observation and the fact that Γ^* is closed under real quantifiers indeed show that the fragment of the iteration strategy of $\mathcal{N}_x^*|\nu$ that acts on non-dropping stack that are above η_i is in Γ^*. Hence, the strategy of $\mathcal{W}$ is in Γ^*.

Suppose $i < \omega^2$ is limit and (1)-(4) are satisfied for all $j < i$. We first claim that the induced strategy $\Sigma_{\mathcal{N}_i'}$ from Σ_z is Γ^*-fullness preserving: suppose $k : \mathcal{N}_i' \to \mathcal{P}$ is according to $\Sigma_{\mathcal{N}_i'}$ then $\mathcal{P}$ is Γ^*-i-suitable, that is

- $\langle k(\eta_j) \mid j < i\rangle$ are the only Woodin cardinals of $\mathcal{P}$;
- for any cut point ξ of $\mathcal{P}$, $Lp^{\Gamma^*}(\mathcal{P}|\xi) \triangleleft \mathcal{P}$ and for any $\xi \neq i(\eta_j)$ for any $j < i$, $Lp^{\Gamma^*}(\mathcal{P}|\xi) \vDash$ ξ is not Woodin.

First we see that $\mathcal{N}_i'$ is Γ^*-i-suitable. We show for instance if $\eta < \eta_0$ then $C_{\Gamma^*}(\mathcal{N}_i'|\eta) \vDash$ "η is not Woodin" (the rest of the verification is similar). Otherwise, $\mathcal{N}_i'|\eta$ is the η-th model in the $L[E]$-construction of $\mathcal{N}_z^*$ and $L[T_{\Gamma^*}, \mathcal{N}_i'|\eta] \vDash$ "η is Woodin", where T_{Γ^*} is the tree projecting to the Γ^*-universal set. We also get that $L[T_{\Gamma^*}, \mathcal{N}_i'|\eta] \cap V_\eta = \mathcal{N}_i'|\eta$ and $V_\eta^{\mathcal{N}_z^*}$ is generic over $L[T_{\Gamma^*}, \mathcal{N}_i'|\eta]$ for $\mathbb{B}_\eta$, the η-generic extender algebra at η. $\mathbb{B}_\eta$ is η-cc, so every $f : \eta \to \eta$ in $L[T_{\Gamma^*}, \mathcal{N}_i'|\eta][V_\eta^{\mathcal{N}_z^*}]$ is bounded by a function $g : \eta \to \eta$ in $L[T_{\Gamma^*}, \mathcal{N}_i'|\eta]$. Furthermore, if E

[24] $\mathbb{C}(\mathcal{S})$ denotes the core of $\mathcal{S}$.

witnesses the Woodin property for g in $L[T_{\Gamma^*}, \mathcal{N}_i'|\eta]$ and $\nu(E)$ is a cardinal in $L[T_{\Gamma^*}, \mathcal{N}_i'|\eta]$ then the background extender E^* witnesses the Woodin property for f in $L[T_{\Gamma^*}, \mathcal{N}_i'|\eta][V_\eta^{\mathcal{N}_z^*}]$ (note also $E \restriction \nu(E) = E^* \restriction \nu(E)$). So η is Woodin in $L[T_{\Gamma^*}, V_\eta^{\mathcal{N}_z^*}]$. By the minimality of η_0, $\eta = \eta_0$. Contradiction. The proof works also for any η_j.

Now let k be as in the claim. Let $k^* : \mathcal{N}_z^* \to N^*$ be the map coming from resurrecting the tree giving rise to k. Let $\sigma : \mathcal{P} \to k^*(\mathcal{N}_i')$ be the resurrection map. Since $\mathcal{N}_z^*, N^*$ have absolute definitions of Γ^*, $k^*(\mathcal{N}_i')$ is Γ^*-i-suitable. This and the fact that σ has in its range all the term relations for $\vec{B}$, we get that $\mathcal{P}$ is Γ^*-i-suitable.

The argument in Lemma 3.7 that an iterate of $\mathcal{M}_{\omega^2}$ extends a Prikry generic and the fact that $\Sigma_{\mathcal{N}_i'}$ is Γ^*-fullness preserving show that $\mathcal{W}$ cannot project across $\sup_{j<i}\eta_j$ and that $\mathcal{W} \lhd Lp^{\Gamma^*}(\mathcal{N}_i')$. This completes the proof of the claim. □

Working in $L(\mathbb{R}, \mu)$, we now claim that there is $\mathcal{W} \unlhd Lp(\mathcal{N}_{\omega^2})$ such that $\rho(W) < \eta_{\omega^2}$. To see this suppose not. It follows from MC that $Lp(\mathcal{N}_{\omega^2})$ is Σ_1^2-full. We then have that x is generic over $Lp(\mathcal{N}_{\omega^2})$ at the extender algebra of $\mathcal{N}_{\omega^2}$ at η_0. Because $Lp(\mathcal{N}_{\omega^2})[x]$ is Σ_1^2-full, we have that $\mathcal{M} \in Lp(\mathcal{N}_{\omega^2})[x]$ and $Lp(\mathcal{N}_{\omega^2})[x] \vDash$ "$\mathcal{M}$ is η_{ω^2}-iterable" by fullness of $Lp(\mathcal{N}_{\omega^2})[x]$. Let $\mathcal{S} = (L[\vec{E}][x])^{\mathcal{N}_{\omega^2}[x]|\eta_2}$ where the extenders used have critical point $> \eta_0$. Then working in $\mathcal{N}_{\omega^2}[x]$ we can compare $\mathcal{M}$ with $\mathcal{S}$. Using standard arguments, we get that $\mathcal{S}$ side doesn't move and by universality, $\mathcal{M}$ side has to come short (see [6]). This in fact means that $\mathcal{M} \unlhd \mathcal{S}$. But the same argument used in the proof of Claim 2 shows that every $\mathcal{K} \unlhd \mathcal{S}$ has an iteration strategy in Γ^*, contradiction!

Let $\eta_{\omega^2} = \sup_{i<\omega^2} \eta_i$ and $\mathcal{W} \unlhd Lp(\mathcal{N}_{\omega^2})$ be least such that $\rho_\omega(\mathcal{W}) < \eta_{\omega^2}$.We can show the following.

Lemma 3.14. *$\mathcal{W} = \mathcal{J}_{\xi+1}(\mathcal{N}_{\omega^2})$ where ξ is least such that for some τ, $\mathcal{J}_\xi(\mathcal{N}_{\omega^2}) \vDash$ "$\mathsf{ZF}^- + \tau$ is a limit of Woodin cardinals $+(T)$ holds in my derived model below τ [25]."*

Since the proof of this lemma is almost the same as that of Claim 7.5 in [13], we will not give it here. However, we have a few remarks regarding the proof:

- we typically replace N by a countable transitive $\overline{N}$ elementarily embeddable into N since the strategy of $\mathcal{W}$ is not known to extend to $V^{Col(\omega,\mathbb{R})}$. Having said this, we will confuse our N with its countable copy.
- We can then do an $\mathbb{R}^N$-iteration of $\mathcal{W}$ to "line up" its iterate with a $\mathbb{P}^N$-generic.

Asides from these remarks, everything else can just be transferred straightforwardly from the proof of Lemma 7.5 in [13] to the proof of Lemma 3.14. Now we just let $\mathcal{N}$ be the

[25]Here "derived model" means the model $L(\mathbb{R}^*, \mathcal{F}^*)$ where $\mathbb{R}^*$ is the symmetric reals for the Levy collapse at τ and $\mathcal{F}^*$ is the corresponding tail filter.

pointwise definable hull of $\mathcal{W}|\xi$. Letting $\mathcal{N}$'s unique iteration strategy be Λ, we can show Λ is Φ-fullness preserving and for any $\vec{f} \in (\mathcal{O}^{<\omega})^N$, there is a strongly $\vec{f}$-iterable, N-suitable $\mathcal{P}$ (in fact, $\mathcal{P} = \mathcal{Q}^-$ for some Λ-iterate $\mathcal{Q}$ of $\mathcal{N}$). We leave the rest of the details to the reader.

4 Further applications

We first prove a series of lemmas which imply Theorem 1.3. For each $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$, let

$$M_\sigma = \text{HOD}^{(L(\mathbb{R},\mu),\mu)}_{\sigma\cup\{\sigma\}}.$$

Suppose G is a $\mathbb{P}_{\max}$ generic over $L(\mathbb{R},\mu)$, where

$$L(\mathbb{R},\mu) \vDash \text{"}\mathsf{AD}^+ + \mu \text{ is a normal fine measure on } \mathcal{P}_{\omega_1}(\mathbb{R})\text{"}.$$

Note that $L(\mathbb{R},\mu)[G] \vDash \mathsf{ZFC}$ since $\mathbb{P}_{\max}$ wellorders the reals. In $L(\mathbb{R},\mu)[G]$, let

$$\mathcal{I} = \{A \mid \exists\langle A_x \mid x \in \mathbb{R}\rangle(A \subseteq \nabla_{x\in\mathbb{R}} A_x \wedge \forall x\ (\mu(A_x) = 0 \text{ or } A_x = \neg S))\},$$

where $S = \{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid G \cap \sigma \text{ is } \mathbb{P}_{\max} \restriction \sigma\text{-generic over } M_\sigma\}$. It's clear that in $L(\mathbb{R},\mu)[G]$, $\mathcal{I}$ is a normal fine ideal. Let $\mathcal{F}$ be the dual filter of $\mathcal{I}$.

Lemma 4.1. *Let $\mathcal{I}^- = \{A \mid \exists\langle A_x \mid x \in \mathbb{R}\rangle(A \subseteq \nabla_{x\in\mathbb{R}} A_x \wedge \forall x\ \mu(A_x) = 0)\}$. Let $\mathcal{F}^-$ be the dual filter of $\mathcal{I}^-$. Suppose $A \in \mathcal{F}^-$. Then $\exists B, C$ such that $\mu(B) = 1$ and C is a club in $L(\mathbb{R},\mu)[G]$ such that $B \cap C \subseteq A$.*

Proof. Suppose $1 \Vdash_{\mathbb{P}_{\max}} \tau : \mathbb{R} \to \mu$ witnesses $\{\sigma \mid \forall x \in \sigma\ \sigma \in \tau(x)\} \in \mathcal{F}^-$. For each $x \in \mathbb{R}$. let $D_x = \{p \mid p \parallel \tau(x)\}$. It's easy to see that D_x is dense for each x. Furthermore,

$$\forall^*_\mu \sigma \forall x \in \sigma (D_x \cap \sigma \text{ is dense in } \mathbb{P}_{\max} \restriction \sigma \wedge \{q \in D_x \cap \sigma \mid q \Vdash \sigma \in \tau(x)\} \text{ is dense.})$$

For otherwise, $\exists x, q \forall^*_\mu \sigma\ x \in \sigma \wedge q \in D_x \cap \sigma \wedge q \Vdash \sigma \notin \tau(x)$. This contradicts that $q \Vdash \tau(x) \in \mu$. Let B be the set of σ having the property displayed above. $\mu(B) = 1$.

Let $A \subseteq \mathbb{R}$ code the function $x \mapsto D_x$ and let G be a $\mathbb{P}_{\max}$-generic over $L(\mathbb{R},\mu)$. Hence $D = \{\sigma \mid \forall x \in \sigma\ \sigma \in \tau_G(x)\} \in \mathcal{F}^-$. Let $C = \{\sigma \mid (\sigma, A \cap \sigma, G \cap \sigma) \prec (\mathbb{R}, A, G)\}$. Hence C is a club in $L(\mathbb{R},\mu)[G]$ and $B \cap C \subseteq D$. □

Lemma 4.2. *Let $\mathcal{I}^-, \mathcal{F}^-$ be as in Lemma 4.1. Then $S \notin \mathcal{I}^-$.*

Proof. Suppose not. Then $\neg S \in \mathcal{F}^-$. The following is a Σ_1-statement (with predicate μ) that $L(\mathbb{R},\mu)[G]$ satisfies:

$$\exists B, C(\mu(B) = 1 \wedge C \text{ is a club } \wedge \forall\sigma(\sigma \in B \cap C \Rightarrow \exists D \subseteq \mathbb{P}_{\max}(M_\sigma \vDash \text{"}D \text{ is dense"} \wedge G \cap D = \emptyset))).$$

By part (1) of Theorem 1.2 and the fact that $\mathbb{P}_{\max}$ is a forcing of size $\mathbb{R}$, $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R},\mu)[G]$ satisfies the same statement. Here μ coincides with the club measure and hence $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R},\mu)[G] \vDash$ "$\neg S$ contains a club". Let $\mathcal{C}$ be a club of elementary substructures X_σ containing everything relevant (and a pair of complementing trees for the universal Σ^2_1 set). Then it's easy to see that $\mathcal{C}^* \subseteq S$ where $\mathcal{C}^* = \{\sigma \mid \sigma = \mathbb{R} \cap X_\sigma \wedge X_\sigma \in \mathcal{C}\}$. This is a contradiction. □

The above lemmas say that $\mathcal{I}$ strictly contains $\mathcal{I}^-$, i.e. S adds nontrivial information to $\mathcal{I}^-$. We now proceed to characterize $\mathcal{I}$-positive sets in terms of the $\mathbb{P}_{\max}$ forcing relation over $L(\mathbb{R},\mu)$.

Lemma 4.3. *Suppose $p \in \mathbb{P}_{\max}$ and τ is a $\mathbb{P}_{\max}$ term for a subset of $\mathcal{P}_{\omega_1}(\mathbb{R})$ in generic extensions of $L(\mathbb{R},\mu)$. Then the following is true in $L(\mathbb{R},\mu)$.*

$$p \Vdash_{\mathbb{P}_{\max}} \tau \textit{ is } \mathcal{I}\textit{-positive} \Leftrightarrow \forall^*_\mu \sigma\ \forall^* g \subseteq \mathbb{P}_{\max} \restriction \sigma\ (p \in g \Rightarrow \exists q < g\ q \Vdash_{\mathbb{P}_{\max}} \sigma \in \tau).$$

Proof. Some explanations about the notation in the lemma are in order. "$\forall^* g \subseteq \mathbb{P}_{\max} \restriction \sigma$" means "for comeager many filters g over $\mathbb{P}_{\max} \restriction \sigma$"; "$\exists^* g \subseteq \mathbb{P}_{\max} \restriction \sigma$" means "for nonmeager many filters g over $\mathbb{P}_{\max} \restriction \sigma$". These category quantifiers make sense because σ is countable. Also we only force with $\mathbb{P}_{\max}$ here so we'll write " $\Vdash$ " for " $\Vdash_{\mathbb{P}_{\max}}$ " and "$p < q$" for "$p <_{\mathbb{P}_{\max}} q$". Finally, "$q < g$" means "$\forall r \in g\ q < r$".

Claim. *Suppose in $L(\mathbb{R},\mu)$, $\forall \sigma\ X_\sigma$ is comeager in $\mathbb{P}_{\max} \restriction \sigma$. Then $\forall^*_\mu \sigma \forall G_\sigma$ (G_σ is $\mathbb{P}_{\max} \restriction \sigma$-generic over $M_\sigma \Rightarrow G_\sigma \in X_\sigma$).*

Proof. Suppose $\sigma \mapsto X_\sigma$ is $OD_{\mu,x}$ for some $x \in \mathbb{R}$. Let $A = \{y \in \mathbb{R} \mid y \text{ codes } (\sigma, g) \text{ where } g \in X_\sigma\}$. Hence A is $OD_{\mu,x}$. Let S be an $OD_{\mu,x}$ ∞-Borel code for A and $\mathcal{A}_S$ be the set of reals coded by S. Hence, $\forall^*_\mu \sigma\ S \in M_\sigma$.

For each such σ, let $G_\sigma \in X_\sigma$ be M_σ-generic and H be $M_\sigma[G_\sigma]$-generic for $Col(\omega,\sigma)$. Then

$$M_\sigma[G_\sigma][H] \vDash (\sigma, G_\sigma) \in \mathcal{A}_S.$$

In the above, note that we use $S \in M_\sigma$. Also no $p \in \mathbb{P}_{\max} \restriction \sigma$ can force $(\sigma, \dot{G}) \notin \mathcal{A}_S$. Hence we're done. □

Suppose the conclusion of the lemma is false. There are two directions to take care of.
Case 1. $p \Vdash \tau$ is $\mathcal{I}$-positive but $\forall^*_\mu \sigma \exists^* g\ (p \in q \wedge \forall q < g\ q \Vdash \sigma \notin \tau)$.

Extending p if necessary and using normality, we may assume $\forall^*_\mu \sigma \forall^* g (p \in g \wedge \forall q < g\ q \vDash \sigma \notin \tau)$. Let T be the set of such σ. Let G be a $\mathbb{P}_{\max}$ generic and $p \in G$. By the claim and the fact that $S \in \mathcal{F}$, $\tau_G \cap S \cap T \neq \emptyset$. So let $\sigma \in \tau_G \cap S \cap T$ such that $p \in G \cap \sigma$. Then $G \cap \sigma$ is M_σ-generic and $\forall q < G \cap \sigma\ q \Vdash \sigma \notin \tau$. But $\exists q < G \cap \sigma$ such that $q \in G$ by density. This implies $\sigma \notin \tau_G$. Contradiction.

Case 2. $p \Vdash \tau \in \mathcal{I}$ and $\forall^*_\mu \sigma \forall^* g\ (p \in g \Rightarrow \exists q < g\ q \Vdash \sigma \in \tau)$.

Let T be the set of σ as above. Let G be $\mathbb{P}_{\max}$ generic containing p. Hence $T \in \mathcal{F}$. Let $\sigma \in T \cap S \cap \neg \tau_G$ and $p \in G \cap \sigma$. By density, $\exists q < G \cap \sigma\ q \in G \wedge q \Vdash \sigma \in \tau$. Hence $\sigma \in \tau_G$. Contradiction. □

Now suppose $\dot{f}$ is a $\mathbb{P}_{\max}$ name for a function from an $\mathcal{I}$-positive set into OR and let τ be a name for $\mathrm{dom}(f)$ and for simplicity suppose $\emptyset \Vdash \tau$ is $\mathcal{I}$-positive $\wedge\ \dot{f} : \tau \to \check{\mathrm{OR}}$. Let $F : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathrm{OR} \cup \{\infty\}$ be defined as follows:

$F(\sigma) =$ α_σ where α_σ is the least α such that $\forall^* g \subseteq \mathbb{P}_{\max} \restriction \sigma\ (g \text{ is } M_\sigma\text{-generic} \Rightarrow \exists q < g\ q \Vdash \check{\sigma} \in \tau \wedge \dot{f}(\check{\sigma}) = \check{\alpha})$, if α exists, and ∞ otherwise.

Clearly, $F \in L(\mathbb{R}, \mu)$ and by the fact that τ_G is $\mathcal{I}$-positive and a standard application of Baire category theorem,[26] $\forall^*_\mu \sigma\ F(\sigma) \neq \infty$.

Lemma 4.4. *Suppose $\dot{f}, \tau, F$ are as above. Suppose G is a $\mathbb{P}_{\max}$ generic over $L(\mathbb{R}, \mu)$. Then in $L(\mathbb{R}, \mu)[G]$, $\{\sigma \mid F(\sigma) = f(\sigma)\}$ is $\mathcal{I}$-positive.*

Proof. Suppose not; assume $p \Vdash \tau' = \{\sigma \mid F(\check{\sigma}) = \dot{f}(\check{\sigma})\} \in \mathcal{I}$. Using Lemma 4.3, we get

$$\forall^*_\mu \sigma\ \exists^* g \subseteq \mathbb{P}_{\max} \restriction \sigma\ (p \in g \wedge \forall q < g\ q \Vdash \sigma \notin \tau'). \tag{4.1}$$

Using the Baire category theorem, we get from 4.1

$$\forall^*_\mu \sigma\ \exists p > q_\sigma \in \sigma\ \forall^* q_\sigma \in g \subseteq \mathbb{P}_{\max} \restriction \sigma\ \forall r < g\ r \Vdash \sigma \notin \tau'. \tag{4.2}$$

Now using normality of μ, we "freeze out" the the q_σ's

$$\exists q < p\ \forall^*_\mu \sigma\ \forall^* q \in g \subseteq \mathbb{P}_{\max} \restriction \sigma\ \forall r < g\ r \Vdash \sigma \notin \tau'. \tag{4.3}$$

From 4.2 and 4.3, we get

$$\exists q < p\ \forall^*_\mu \sigma\ \forall^* q \in g \subseteq \mathbb{P}_{\max} \restriction \sigma\ (g \text{ is } M_\sigma\text{-generic } \Rightarrow \forall r < g\ r \Vdash F(\check{\sigma}) \neq \dot{f}(\check{\sigma})). \tag{4.4}$$

[26] More precisely, we use the fact that if $F : A \to \mathrm{OR}$ is a function on a comeager set A then F is constant on some comeager subset of A.

We get a contradiction from 4.4 as follows. Fix a σ in the μ-measure one set from 4.4 such that $F(\sigma) = \alpha_\sigma \neq \infty$. For the chosen σ, fix a g in the set described in 4.4 as well as in the set described in the definition of $F(\sigma)$. Then by 4.4, $\forall r < g$, $r \Vdash F(\sigma) = \alpha_\sigma \neq \dot{f}(\check{\sigma})$ but by the definition of $F(\sigma)$, $\exists r < g$ $r \Vdash \dot{f}(\check{\sigma}) = \alpha_\sigma$. Contradiction. $\square$

Proof of Theorem 1.3. Working in $L(\mathbb{R},\mu)[G]$, let $H \subseteq \mathcal{I}^+$ be generic. We show that (1)-(3) hold. Let $A \subseteq \mathbb{R}$ be OD_x for some $x \in \mathbb{R}$. By countable closure and homogeneity of $\mathbb{P}_{\max}$, $x \in L(\mathbb{R},\mu)$ and hence $A \in L(\mathbb{R},\mu)$. Since $\mathcal{F} \restriction L(\mathbb{R},\mu) = \mu$, we obtain (1) [27]. Lemma 4.4 implies $\forall s \in \mathrm{OR}^\omega$ $j_H \restriction \mathrm{HOD}_s \in V$ and is independent of H. To see this, note that $s \in L(\mathbb{R},\mu)$ as $\mathbb{P}_{\max}$ is countably closed and $L(\mathbb{R},\mu) \vDash \mathsf{DC}$; furthermore, by homogeneity of $\mathbb{P}_{\max}$, $\mathrm{HOD}_s \subseteq \mathrm{HOD}_s^{L(\mathbb{R},\mu)}$ and there is a bijection between OR and HOD_s in $L(\mathbb{R},\mu)$. So Lemma 4.4 applies to functions $f : S \to \mathrm{HOD}_s$ where S is $\mathcal{I}$-positive. This implies $j_H \restriction \mathrm{HOD}_s = j_\mu \restriction \mathrm{HOD}_s$, which also shows (2).

To show $j_H \restriction \mathrm{HOD}_{\mathcal{I}}$ is independent of H, first note that $\mathcal{F}$ is generated by μ and $\mathcal{A} =_{\mathrm{def}} \{T \subseteq \mathcal{P}_{\omega_1}(\mathbb{R}) \mid \exists C(C \text{ is a club and } T \cap C = S \cap C\}$, where S is defined at the beginning of the section in relation to the definition of $\mathcal{I}$. Note that $\mathcal{A}$ is definable in $L(\mathbb{R},\mu)[G]$ (from no parameters). To see this, suppose G_0, G_1 are two $\mathbb{P}_{\max}$ generics (in $L(\mathbb{R},\mu)[G]$) and let S_{G_i} be defined relative to G_i ($i \in \{0,1\}$) the same way S is defined relative to G. Also let $A_{G_i} \subseteq \omega_1$ be the generating set for G_i. Let $p \in G_0 \cap G_1$ and $a_0, a_1 \in \mathcal{P}(\omega_1)^p$ be such that $j_i(a_i) = A_i$ where j_i are unique iteration maps of p. The proof of homogeneity of $\mathbb{P}_{\max}$ gives a bijection π from $\{q \mid q < p\}$ to itself. It's easy to see that

$$C = \{\sigma \mid (\sigma, \mathbb{P}_{\max} \restriction \sigma, \pi \restriction \sigma) \prec (\mathbb{R}, \mathbb{P}_{\max}, \pi)\},$$

is club and $S_{G_0} \cap C = S_{G_1} \cap C$. By homogeneity of $\mathbb{P}_{\max}$, there is a bijection (definable over) $L(\mathbb{R},\mu)$ from OR onto $\mathrm{HOD}_{\mathcal{I}}$. So the ultraproduct $[\sigma \mapsto \mathrm{HOD}_{\mathcal{I}}]_H$ using functions in $L(\mathbb{R},\mu)[G]$ is just $[\sigma \mapsto \mathrm{HOD}_{\mathcal{I}}]_\mu$ using functions in $L(\mathbb{R},\mu)$.

Finally, to see $\mathrm{HOD}^{M_0}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} = \mathrm{HOD}^{M_1}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} \in V$, note that for any generic H, letting $V = L(\mathbb{R},\mu)[G]$, $\mathrm{HOD}^{Ult(V,H)}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}}$ is represented by $\sigma \mapsto \mathrm{HOD}^V_{\sigma \cup \{\sigma\}}$. Let f be such that $\mathrm{dom}(f) = S$ where S is $\mathcal{I}$-positive and $\forall \sigma \in S, f(\sigma) \in \mathrm{HOD}^V_{\sigma \cup \{\sigma\}}$. By normality, shrinking S if necessary, we may assume $\exists x \in \mathbb{R} \forall \sigma \in S, f(\sigma) \in \mathrm{HOD}^V_{\{x,\sigma\}}$ and Lemma 4.4 can be applied to this f. We finished the proof of Theorem 1.3. $\square$

Proof of Theorem 1.4. Let $\mathcal{I}$ be as in the hypothesis of the theorem. Since we're shooting for a model of the form $L(\mathbb{R},\mu)$, we may as well assume there is no model M containing $\mathbb{R} \cup \mathrm{OR}$ such that $M \vDash \mathsf{AD}^+ + \Theta > \theta_0$; the existence of such an M gives a model of $\mathsf{ZFC}+$

[27] The proof of (1) in fact shows more. It shows that if $A \subseteq \mathbb{R}$ is OD_s for some $s \in \mathrm{OR}^\omega$, then $A \in \mathcal{F}$ or $\mathbb{R} \backslash A \in \mathcal{F}$

there are ω^2 Woodin cardinals, which in turns gives a model of the form $L(\mathbb{R}, \mu)$ satisfying the conclusion of the theorem.

By arguments in [18] (see in particular Section 4.6), the existence of a normal fine ideal $\mathcal{I}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that $\mathcal{I}$ is precipituous and for all generics $G_0, G_1 \subseteq \mathcal{I}^+$, $s \in \mathrm{OR}^\omega$, $j_{G_0} \restriction \mathrm{HOD}_s = j_{G_1} \restriction \mathrm{HOD}_s \in V$ and $\mathrm{HOD}^{Ult(V,G_0)}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} = \mathrm{HOD}^{Ult(V,G_1)}_{\mathbb{R}^V \cup \{\mathbb{R}^V\}} \in V$ implies that AD^+ holds in $Lp(\mathbb{R})$. Let $M = Lp(\mathbb{R}) \vDash \mathsf{AD}^+$. Let $\mathcal{F}$ be the hod direct limit system in M, and $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$ in M (see [7] or [18] for the full definition of $\mathcal{F}$). Fix a generic $G \subseteq \mathcal{I}^+$ and let $j = j_G$ be the generic embedding. To prove the theorem, we consider two cases.

Case 1. $\Theta^M < \mathfrak{c}^+$.

We first observe that the argument in Chapter 5 of [18] for getting a strategy with branch condensation from $\mathcal{I}$ being strong and $j_H \restriction \mathrm{HOD}_{\{s,\mathcal{I}\}}$ being independent of V-generic $H \subset \mathcal{I}^+$ for any $s \in \mathrm{OR}^\omega$ can be used in our situation. Here are the two key points. The hypothesis of Case 1 replaces the strength of the ideal, which is used in showing Θ^M is countable in $j(M)$ and $j \restriction \mathcal{M}_\infty \in \mathrm{Ult}(V, G)$ and is countable there. The hypothesis $j_H \restriction \mathrm{HOD}_{\{s,\mathcal{I}\}} \in V$ being independent of V-generic $H \subset \mathcal{I}^+$ for any $s \in \mathrm{OR}^\omega$ is used in getting a strategy with branch condensation (see [2]), and a model N containing $\mathbb{R} \cup \mathrm{OR}$ such that $N \vDash \mathsf{AD}^+ + \Theta > \theta_0$. Working over N, by a similar reasoning as in the first paragraph of this section, we obtain the desired model $L(\mathbb{R}, \mu)$. This finishes the proof of the theorem in Case 1.

Case 2. $\Theta^M \geq \mathfrak{c}^+$

Recall that $\mathcal{F}$ is the dual filter to $\mathcal{I}$. Let $\mu = \mathcal{F} \cap M$. First we observe by (1) that μ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})^M$. Next, we need to see that μ doesn't construct sets of reals beyond M. This is the content of the next claim.

Claim. $L(\mathbb{R}, \mu) \subseteq M$.[28]

Proof. We first prove the following subclaim.

Subclaim. μ is amenable to M in that if $\langle A_x \mid x \in \mathbb{R} \wedge A_x \in \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))^M \rangle \in M$ then $\langle A_x \mid x \in \mathbb{R} \wedge \mu(A_x) = 1 \rangle \in M$.

Proof. Fix a sequence $\mathcal{C} = \langle A_x \mid x \in \mathbb{R} \wedge A_x \in \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))^M \rangle \in M$ and fix an ∞-Borel code S for the sequence. Let T be the tree for a universal $(\Sigma^2_1)^M$ set. We may assume $S \in OD^M$ and is a bounded subset of Θ^M. We also assume S codes T. Let $\mathcal{A}_S$ be the set coded by S over any model containing S. By MC and the definition of T, S in M, it's easy to see that in M,

$$\forall^*_\mu \sigma(\mathcal{P}(\sigma) \cap L(S, \sigma) = \mathcal{P}(\sigma) \cap L(T, \sigma) = \mathcal{P}(\sigma) \cap Lp(\sigma)).$$

[28]We just need from the claim that $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \mu) \subset M$.

Let $S^* = [\sigma \mapsto S]_\mu$ and $T^* = [\sigma \mapsto T]_\mu$ where the ultraproducts are taken with functions in M. Now, S^*, T^* may not be in M but

$$\mathcal{P}(\mathbb{R}) \cap L(S^*, \mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap L(T^*, \mathbb{R}) = \mathcal{P}(\mathbb{R})^M.$$

This implies $\mathcal{C} \in L(S^*, \mathbb{R})$. For each $x \in \mathbb{R}$,

$$\begin{aligned} A_x \in \mu &\Leftrightarrow (\forall^*_\mu \sigma)(\sigma \in A_x \cap \mathcal{P}_{\omega_1}(\sigma)) \\ &\Leftrightarrow (\forall^*_\mu \sigma)(L(S, \sigma) \vDash \emptyset \Vdash_{Col(\omega,\sigma)} \sigma \in (\mathcal{A}_S)_x \\ &\Leftrightarrow L(S^*, \mathbb{R}) \vDash \emptyset \Vdash_{Col(\omega,\mathbb{R})} \mathbb{R} \in (\mathcal{A}^*_S)_x. \end{aligned}$$

The above shows $\mu \restriction \mathcal{C} \in L(S^*, \mathbb{R})$. Since $\mu \restriction \mathcal{C}$ can be coded by a set of reals in $L(S^*, \mathbb{R})$, $\mu \restriction \mathcal{C} \in M$. This finishes the proof of the claim. □

Using the subclaim, we finish the proof of the claim as follows. Suppose α is least such that $\exists A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ $A \in L_{\alpha+1}(\mathbb{R})[\mu] \backslash L_\alpha(\mathbb{R})[\mu]$ and $A \notin M$. By properties of α and condensation of μ, there is a definable over $L_\alpha(\mathbb{R})[\mu]$ surjection of $\mathbb{R}$ onto $L_\alpha(\mathbb{R})[\mu]$. This implies $\alpha < \mathfrak{c}^+$. Also by minimality of α, $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subseteq M$.

Now, if $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subsetneq \mathcal{P}(\mathbb{R})^M$, then the subclaim gives us $\mu \cap L_\alpha(\mathbb{R})[\mu] \in M$ which implies $A \in M$. Contradiction. So we may assume $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] = \mathcal{P}(\mathbb{R})^M$. This means $\Theta^{L_\alpha(\mathbb{R})[\mu]} = \Theta^M \geq \mathfrak{c}^+$. This contradicts the fact that $\alpha < \mathfrak{c}^+$. □

The claim implies $L(\mathbb{R}, \mu) \vDash \mathsf{AD} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. This finishes the proof of the theorem. □

5 Open problems and questions

We list some open problems and questions related to models of the form $L(\mathbb{R}, \mu)$. In Theorem 1.2, we prove the internal uniqueness of μ inside $L(\mathbb{R}, \mu)$. It's natural to ask whether $L(\mathbb{R}, \mu)$ is unique externally.

Question. Suppose μ_0, μ_1 are filters on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that for $i \in \{0, 1\}$, $L(\mathbb{R}, \mu_i) \vDash$ "$\mathsf{AD}^+ + \mu_i$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$". Must $L(\mathbb{R}, \mu_0) = L(\mathbb{R}, \mu_1)$ and $\mu_0 \cap L(\mathbb{R}, \mu_0) = \mu_1 \cap L(\mathbb{R}, \mu_1)$? What is the consistency strength of having distinct models of $\mathsf{AD}^+ + V = L(\mathbb{R}, \mu)$?

In [17], it's shown that $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+$ if and only if $L(\mathbb{R}, \mu) \vDash \Theta > \omega_2$. It's known that the equivalence fails for $L(\mathbb{R})$. However, the following is still open.

Open problem. Suppose $L(\mathbb{R}) \models \Theta$ is strongly inaccessible[29]. Must $L(\mathbb{R}) \models \mathsf{AD}^+$?

A variation of the above that we believe is still open is when we replace the hypothesis "$L(\mathbb{R}) \models \Theta$ is inaccessible" by "$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta$ is inaccessible (or Woodin)". Finally, with regard to constructing $L(\mathbb{R}, \mu)$ in a core model induction, the following is still open (cf. [10]), where NS is the nonstationary ideal on ω_1.

Conjecture. The following are equiconsistent.

1. $\mathsf{ZFC}+$ there are ω^2 Woodin cardinals.
2. NS is saturated and $\mathsf{WRP}^*_{(2)}(\omega_2)$ holds.
3. NS is saturated and $\mathsf{SRP}^*_{(2)}(\omega_2)$ holds.

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