CONDENSATION FOR MOUSE PAIRS

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Abstract

This is the first of two papers on the fine structure of HOD in models of the Axiom of Determinacy (AD). Let $M \vDash \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\wp(\mathbb{R}))$. [10] shows that under a natural hypothesis on the existence of iteration strategies, the basic fine structure theory for pure extender models goes over to HOD^M. In this paper, we prove a fine condensation theorem, quite similar to Theorem 9.3.2 of Zeman's book [14], except that condensation for iteration strategies has been added to the mix. In the second paper, we shall use this theorem to show that in HOD^M, $\square_\kappa$ holds iff κ is not subcompact.

1. INTRODUCTION

One goal of descriptive inner model theory is to elucidate the structure of HOD (the universe of hereditarily ordinal definable sets) in models M of the Axiom of Determinacy. HOD^M is close to M in various ways; for example, if $M \vDash \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\wp(\mathbb{R}))$[1], then M can be realized as a symmetric forcing extension of HOD^M, so that the first order theory of M is part of the first order theory of its HOD. [2] For this and many other reasons, the study of HOD in models of AD has a long history. We refer the reader to [11] for a survey of this history.

The study of HOD involves ideas from descriptive set theory (for example, games and definable scales) and ideas from inner model theory (mice, comparison, fine structure). One early result showing that inner model theory is relevant is due to the first author, who showed in 1994 ([9]) that if there are ω Woodin cardinals with a measurable above them all, then in $L(\mathbb{R})$, HOD up to θ is a pure extender mouse. Shortly afterward, this result was improved by Hugh Woodin, who reduced its hypothesis to $\mathsf{AD}^{L(\mathbb{R})}$, and identified the full $\mathrm{HOD}^{L(\mathbb{R})}$ as a model of the form $L[M, \Sigma]$, where M is a pure extender premouse, and Σ is a partial iteration strategy for M. $\mathrm{HOD}^{L(\mathbb{R})}$ is thus a new type of mouse, sometimes called a *strategy mouse*, sometimes called a *hod mouse*. See [12] for an account of this work.

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[1] AD^+ is a technical strengthening of AD. It is not known whether $\mathsf{AD} \Rightarrow \mathsf{AD}^+$, though in every model of AD constructed so far, AD^+ also holds. The models of AD that we deal with in this paper satisfy AD^+.

[2] This is a theorem of Woodin from the early 1980s. Cf. [13].

Since the mid-1990s, there has been a great deal of work devoted to extending these results to models of determinacy beyond $L(\mathbb{R})$. Woodin analyzed HOD in models of AD^+ below the minimal model of $\mathsf{AD}_\mathbb{R}$ fine structurally, and Sargsyan pushed the analysis further, first to determinacy models below $\mathsf{AD}_\mathbb{R}$ + "θ is regular" (see [2]), and more recently, to determinacy models below the minimal model of the theory "$\mathsf{AD}^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal" (commonly known as LSA). (See [3].) The hod mice used in this work have the form $M = L[\vec{E}, \Sigma]$, where $\vec{E}$ is a coherent sequence of extenders, and Σ is an iteration strategy for M. The strategy information is fed into the model M slowly, in a way that is dictated in part by the determinacy model whose HOD is being analyzed. One says that the hierarchy of M is *rigidly layered*, or *extender biased*. The object (M, Σ) is called a rigidly layered (extender biased) hod pair.

Putting the strategy information in this way makes comparison easier, but it has serious costs. The definition of "premouse" becomes very complicated, and indeed it is not clear how to extend the definition of rigidly layered hod pairs much past that given in [3]. The definition of "extender biased hod premouse" is not uniform, in that the extent of extender bias depends on the determinacy model whose HOD is being analyzed. Fine structure, and in particular condensation, become more awkward. For example, it is not true in general that the pointwise definable hull of a level of M is a level of M. (The problem is that the hull will not generally be sufficiently extender biased.) Because of this, it is open whether the hod mice of [3] satisfy $\forall\kappa\Box_\kappa$. (The second author did show that $\forall\kappa\Box_{\kappa,2}$ holds in these hod mice; cf. [3].)

The more naive notion of hod premouse would abandon extender bias, and simply add the least missing piece of strategy information at essentially every stage. This was originally suggested by Woodin. The first author has recently proved a general comparison theorem that makes it possible to use this approach, at least in the realm of short extenders. The resulting premice are called *least branch premice* (lpm's), and the pairs (M, Σ) are called *least branch hod pairs* (lbr hod pairs). Combining results of [10] and [8], one has

Theorem 1.1 ([10],[8]). *Assume* AD^++ *"there is an* (ω_1, ω_1) *iteration strategy for a pure extender premouse with a long extender on its sequence". Let* $\Gamma \subseteq P(\mathbb{R})$ *be such that* $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}_\mathbb{R}$+ *"there is no* (ω_1, ω_1) *iteration strategy for a pure extender premouse with a long extender on its sequence"; then* $HOD^{L(\Gamma,\mathbb{R})}$ *is a least branch premouse.*

Of course, one would like to remove the iterability hypothesis of 1.1, and prove its conclusion under AD^+ alone. Finding a way to do this is one manifestation of the long standing iterability problem of inner model theory. Although we do not yet know how to do this, the theorem does make it highly likely that in models of $\mathsf{AD}_\mathbb{R}$ that have not reached an iteration strategy for a pure extender premouse with a long extender, HOD is an lpm.

Least branch premice have a fine structure much closer to that of pure extender models than that of rigidly layered hod premice. The paper [10] develops the basics, the solidity and universality of standard parameters, and a coarse form of condensation. The main theorem of this paper, Theorem 3.7, is a stronger condensation theorem. The statement of 3.7 is parallel to that of Theorem 9.3.2 of [14], but it has a strategy-condensation feature that is new even in the pure extender model context.

The proof of 3.7 follows the same outline as the proofs of solidity, universality, and condensation given in [10], but there are a number of additional difficulties to be overcome. These stem from the restricted elementarity we have for the ultrapowers of phalanxes that are taken in the course of the proof.

Theorem 3.7 is one of the main ingredients in the proof of the main theorem of our second paper. We say that (M, Σ) is a *mouse pair* iff M is either a pure extender premouse or a least branch premouse, and Σ is an iteration strategy for M that condenses and normalizes well. See [10, Chapter 5] and section 1 below for a full definition.

Theorem 1.2 (AD^+). *Let (M, Σ) be a mouse pair. Let κ be a cardinal of M such that $M \models$ "κ^+ exists"; then in M, the following are equivalent.*

1. $\square_\kappa$.
2. $\square_{\kappa,<\kappa}$.
3. *κ is not subcompact.*
4. *The set of $\nu < \kappa^+$ such that $M|\nu$ is extender-active is non-stationary in κ^+.*

The special case of this theorem in which M is a pure extender model is a landmark result of Schimmerling and Zeman. (See [4].) Our proof follows the Schimmerling-Zeman proof quite closely.

Theorem 1.2 has applications to consistency strength lower bound questions that we discuss in the second paper. But our work was also motivated by the desire to put the fine structure theory of [10] to the test, so to speak. Determining the pattern of $\square$ is a good way to go one level deeper into the world of projecta, standard parameters, restricted elementarity, and condensation theorems. We found when we did so that the definition of *hod premouse* given in [10] was wrong, in that strategy information was being added in a way that would not in general be preserved by Σ_1 hulls. The correct method for strategy insertion comes from [7], and we describe it further below. [10] has been revised so that it now uses this method.

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2. LEAST-BRANCH HOD PREMICE

We adopt for the most part the fine structure and notation from [10, Chapter 5] concerning least-branch hod premice (lpm's) and lbr hod pairs. We summarize some main points below. The reader can see [10, Chapter 5] for more details.

Least branch premice (lpm). The language for lpm's is $\mathcal{L}_0$ with symbols $\in, \dot{E}, \dot{F}, \dot{\Sigma}, \dot{B}, \dot{\gamma}$. An lpm M is of the form (N, k) where N is an $\mathcal{L}_0$ amenable structure that is k-sound. We write $k = k(M)$. We often identify M with N and suppress k. $o(M)$ denotes the ordinal height of M, and $\hat{o}(M)$ denotes the α such that $o(M) = \omega\alpha$. $l(M) = (\hat{o}(M), k(M))$ is the index of

M. For $(\nu, l) \leq_{\text{lex}} l(M)$, $M|(\nu, l)$ is the initial segment of M with index (ν, l). We write $N \trianglelefteq M$ iff $N = M|(\nu, l)$ for some $(\nu, l) \leq_{\text{lex}} l(M)$. If $\nu \leq \hat{o}(M)$, write $M|\nu$ for $M|(\nu, 0)$. We write $\rho_n(M)$ for the n-th projectum of M and $p_n(M)$ for the n-th standard parameter of M We set $\rho(M) = \rho_{k(M)+1}(M)$ and $p(M) = p_{k(M)+1}(M)$, and call them *the* projectum and parameter of M. We say M is *sound* iff it is $k(M)+1$-sound. An lpm M must be $k(M)$-sound, but it need not be $k(M)+1$-sound. All proper initial segments of an lpm must be sound lpms.

We also use other fine-structural notions from [14] and [4] like $\Sigma_k^{(n)}$ elementarity, and the $\Sigma_1^{(n)}$ Skolem function $\tilde{h}_M^n$ of M, at various places in the paper. We also write $<_M$ for the canonical well-ordering of M.

$\dot{E}^M$ codes the sequence of extenders that go into constructing M. $\dot{F}^M$ if non-empty is the amenable code for a new extender being added; in this case, we say that M is *extender-active* (or just E-active). If $\dot{F}^M = F$ is nonempty, then $M \vDash \text{crt}(F)^+$ exists and $o(M) = i_F^M(\mu)$, where $\mu = \text{crt}(F)^+$. Also F must satisfy the Jensen initial segment condition (ISC), that is, whole initial segments of F must be in $\dot{E}^M$ (see [14] for a detailed discussion of ISC). $\dot{\gamma}$ is the index of the largest whole initial segment of F if exists; otherwise, $\dot{\gamma} = 0$. We also demand M is *coherent*, that is $i_F^M(\dot{E}^M) \upharpoonright o(M)+1 = (\dot{E}^M)^\frown\langle\emptyset\rangle$.

$\dot{\Sigma}^M$ and $\dot{B}^M$ are used to record information about an iteration strategy Ω of M. $\dot{\Sigma}^M$ codes the strategy information added at earlier stages; $\dot{\Sigma}^M$ acts on normal trees. $\dot{\Sigma}^M(s, b)$ implies that $s = \langle \nu, k, \mathcal{T}\rangle$, where $(\nu, k) \leq l(M)$ and $\mathcal{T}$ is a normal tree on $M|(\nu, k)$ in M of limit length and $\mathcal{T}^\frown b$ is according to the strategy. We say that s is an M-tree, and write $s = \langle \nu(s), k(s), \mathcal{T}(s)\rangle$. We write $\dot{\Sigma}_{\nu,k}^M$ for the partial iteration strategy for $M|(\nu, k)$ determined by $\dot{\Sigma}$. We write $\Sigma^M(s) = b$ when $\dot{\Sigma}^M(s, b)$, and we say that s is according to Σ^M if $\mathcal{T}(s)$ is according to $\dot{\Sigma}_{\nu(s),k(s)}^M$.

Now we discuss how to code branch information for a tree $\mathcal{T}(s)$ such that $\Sigma^M(s)$ has not yet been defined into the $\dot{B}^M$ predicate. Here we use the $\mathfrak{B}$-operator in [7]. We are correcting some errors in the original version of [10]. These corrections have been incorporated in its latest version.

M is *branch-active* (or just B-active) iff

(a) there is a largest $\eta < o(M)$ such that $M|\eta \vDash \mathsf{KP}$, and letting $N = M|\eta$,

(b) there is a $<_N$-least N-tree s such that s is by Σ^N, $\mathcal{T}(s)$ has limit length, and $\Sigma^N(s)$ is undefined.

(c) for N and s as above, $o(M) \leq o(N) + lh(\mathcal{T}(s))$.

Note that being branch-active can be expressed by a Σ_2 sentence in $\mathcal{L}_0 - \{\dot{B}\}$. This contrasts with being extender-active, which is not a property of the premouse with its top extender removed. In contrast with extenders, we know when branches must be added before we do so.

Definition 2.1. Suppose that M is branch-active. We set

$$\begin{aligned}
\eta^M &= \text{the largest } \eta \text{ such that } M|\eta \models \mathsf{KP}, \\
\nu^M &= \text{unique } \nu \text{ such that } \eta^M + \nu = o(M), \\
s^M &= \text{least } M|\eta^M\text{-tree such that } \dot{\Sigma}^{M|\eta^M} \text{ is undefined, and} \\
b^M &= \{\alpha \mid \eta + \alpha \in \dot{B}^M\}.
\end{aligned}$$

Moreover,

(1) M is a *potential lpm* iff b^M is a cofinal branch of $\mathcal{T}(s)\restriction\nu^M$.

(2) M is *honest* iff $\nu^M = \mathrm{lh}(\mathcal{T}(s))$, or $\nu^M < lh(\mathcal{T}(s))$ and $b^M = [0, \nu^M)_{T(s)}$.

(3) M is an lpm iff M is an honest potential lpm.

⊣

Note that η^M is a Σ_0^M singleton, because it is the least ordinal in $\dot{B}^M$ (because 0 is in every branch of every iteration tree), and thus s^M is also a Σ_0^M singleton. We have separated honesty from the other conditions because it is not expressible by a Q-sentence, whereas the rest is. Honesty is expressible by a Boolean combination of Σ_2 sentences. See 2.6 below.

The original version of [10] required that when $o(M) < \eta^M + lh(\mathcal{T}(s))$, $\dot{B}^M$ is empty, whereas here we require that it code $[0, o(M))_{T(s)}$, in the same way that $\dot{B}^M$ will have to code a new branch when $o(M) = \eta^M + lh(\mathcal{T}(s))$. Of course, $[0, \nu^M)_{T(s)} \in M$ when $o(M) < \eta^M + lh(\mathcal{T}(s))$ and M is honest, so the current $\dot{B}^M$ seems equivalent to the original $\dot{B}^M = \emptyset$. However, $\dot{B}^M = \emptyset$ leads to Σ_1^M being too weak, with the consequence that a Σ_1 hull of M might collapse to something that is not an lpm.[3] Our current choice for $\dot{B}^M$ solves that problem.

Remark 2.2. Suppose N is an lpm, and $N \models \mathsf{KP}$. It is very easy to see that $\dot{\Sigma}^N$ is defined on all N-trees s that are by $\dot{\Sigma}^N$ iff there are arbitrarily large $\xi < o(N)$ such that $N|\xi \models \mathsf{KP}$. Thus if M is branch-active, then η^M is a successor admissible; moreover, we do add branch information, related to exactly one tree, at each successor admissible. Waiting until the next admissible to add branch information is just a convenient way to make sure we are done coding in the branch information for a given tree before we move on to the next one. One could go faster.

We say that an lpm M is (fully) passive if $\dot{F}^M = \emptyset$ and $\dot{B}^M = \emptyset$. It cannot be the case that M is both E-active and B-active. In the case that M is E-active, using the terminology of [4], the extender $\dot{F}^M$ can be of type A, B, or C.

Suppose that M is an lpm, and $\pi\colon H \to M$. What sort of elementarity for π do we need to conclude that H is an lpm? In the proof of square we have to deal with embeddings that are only

[3] The hull could satisfy $o(H) = \eta^H + lh(\mathcal{T}(s^H))$, even though $o(M) < \eta^M + lh(\mathcal{T}(s^M))$. But then being an lpm requires $\dot{B}^H \neq \emptyset$.

weakly elementary.[4] The possible problem comes when $k(H) = k(M) = 0$. If M is a passive lpm, then so is H, and there is no problem. If M is extender-active, then it could be that H is only a protomouse, in that its last extender predicate is not total. The problem here is solved by the parts of the Schimmerling-Zeman proof related to protomice, which work in our context. Finally, we must consider the case that M is branch-active.

Definition 2.3. A *rQ-formula* of $\mathcal{L}_0$ is a conjunction of formulae of the form

(a) $\forall u \exists v (u \subseteq v \wedge \varphi)$, where φ is a Σ_1 formula of $\mathcal{L}_0$ such that u does not occur free in φ,

or of the form

(b) "$\dot{F} \neq \emptyset$, and for $\mu = \mathrm{crt}(\dot{F})^+$, there are cofinally many $\xi < \mu$ such that ψ", where ψ is Σ_1.

⊣

Formulae of type (a) are usually called Q-formulae. Being a passive lpm can be expressed by a Q-sentence, but in order to express being an extender-active lpm, we need type (b) clauses, in order to say that the last extender is total. rQ formulae are π_2, and hence preserved downward under Σ_1-elementary maps. They are preserved upward under Σ_0 maps that are *strongly cofinal.*

Definition 2.4. Let M and N be $\mathcal{L}_0$-structures and $\pi\colon M \to N$ be Σ_0 and cofinal. We say that π is *strongly cofinal* iff M and N are not extender active, or M and N are extender active, and letting $\pi``(\mathrm{crt}(\dot{F})^+)^M$ is cofinal in $(\mathrm{crt}(\dot{F})^+)^N$. ⊣

It is easy to see that

Lemma 2.5. *rQ formulae are preserved downward under Σ_1-elementary maps, and upward under strongly cofinal Σ_0-elementary maps.*

Lemma 2.6. *(a) There is a Q-sentence φ of $\mathcal{L}_0$ such that for all transitive $\mathcal{L}_0$ structures M, $M \vDash \varphi$ iff M is a passive lpm.*

(b) There is a rQ-sentence φ of $\mathcal{L}_0$ such that for all transitive $\mathcal{L}_0$ structures M, $M \vDash \varphi$ iff M is an extender-active lpm.

(c) There is a Q-sentence φ of $\mathcal{L}_0$ such that for all transitive $\mathcal{L}_0$ structures M, $M \vDash \varphi$ iff M is a potential branch-active lpm.

Proof. (Sketch.) We omit the proofs of (a) and (b). For (c), note that "$\dot{B} \neq \emptyset$" is Σ_1. One can go on then to say with a Σ_1 sentence that if η is least in $\dot{B}$, then $M|\eta$ is admissible, and s^M exists. One can say with a Π_1 sentence that $\{\alpha \mid \dot{B}(\eta + \alpha)\}$ is a branch of $\mathcal{T}(s)$, perhaps of successor order type. One can say that $\dot{B}$ is cofinal in the ordinals with a Q-sentence. Collectively, these sentences express the conditions on potential lpm-hood related to $\dot{B}$. That the rest of M constitutes an extender-passive lpm can be expressed by a Π_1 sentence. □

[4]See section 1.4 of [10] for a discussion of the degrees of elementarity. If $k(H) = k(M) = 0$, then π is weakly elementary iff it is Σ_0 elementary and cardinal-preserving.

Corollary 2.7. *(a) If M is a passive (resp. extender-active, potential branch-active) lpm, and $\text{Ult}_0(M,E)$ is wellfounded, then $\text{Ult}_0(M,E)$ is a passive (resp.extender-active, potential branch-active) lpm.*

(b) Suppose that M is a passive (resp. extender-active, potential branch-active) lpm, and $\pi\colon H \to M$ is Σ_1-elementary; then H is a passive (resp. potential branch-active) lpm.

(c) Let $k(M) = k(H) = 0$, and $\pi\colon H \to M$ be Σ_2 elementary; then H is a branch-active lpm iff M is a branch-active lpm.

Proof. rQ-sentences are preserved upward by strongly cofinal Σ_0 embeddings, so we have (a). They are Π_2, hence preserved downward by Σ_1- elementary embeddings, so we have (b).

It is easy to see that honesty is expressible by a Boolean combination of Σ_2 sentences, so we get (c).

□

Remark 2.8. It could happen that M is a branch-active lpm, $\pi\colon H \to M$ is cofinal and elementary (with $k(M) = k(H) = 0$), and b^M is not cofinal in $\mathcal{T}(s^M)$, but b^H is cofinal in $\mathcal{T}(s^H)$. If we were using the branch coding in the original version of [10], then $\dot{B}^M = \emptyset$, so $\dot{B}^H = \emptyset$, so H is not an lpm.

Part (c) of Lemma 2.6 is not particularly useful. In general, our embeddings will preserve honesty of a potential branch active lpm M because $\dot{\Sigma}^M$ and $\dot{B}^M$ are determined by a complete iteration strategy for M that has strong hull condensation. So the more useful preservation theorem in the branch-active case applies to *hod pairs*, rather than to hod premice.

Least branch hod pairs (lbr). We say that (M, Ω) is a *least branch hod pair (lbr hod pair)* with scope H_δ iff

1. M is an lpm.
2. Ω is a complete iteration strategy for M with scope H_δ (see [10, Section 5.3]).
3. Ω is self-consistent, normalizes well, and has strong hull condensation (again, see [10]), and
4. If s is by Ω with last model N, then $\dot{\Sigma}^N \subseteq \Omega_s$, where $\Omega_s(t) = \Omega(s^\frown t)$.

Included in clause (2) is the requirement that all Ω-iterates of M be least branch premice. Because of our honesty requirement in the branch-active case, this no longer follows automatically from the elementarity of the iteration maps.[5] That the iterates of M are honest comes out of the construction of Ω, as a consequence of self-awareness.

If (M, Ω) is an lbr hod pair and $\pi\colon H \to M$ is weakly elementary, then Ω^π is the *pullback strategy*, given by

$$\Omega^\pi(s) = \Omega(\pi s).$$

[5] Honesty for "branch-anomalous" M does not seem to pass to $\text{Ult}_0(M,E)$ for first-order reasons.

We show now that, except in the protomouse case, (H, Ω^π) is an lbr hod pair.

Lemma 2.9. *Let (M, Ω) be an lbr hod pair with scope H_δ, and let $\pi\colon H \to M$ be weakly elementary. Suppose that one of the following holds:*

(a) M is passive or branch-active, or

(b) H is an lpm.

Then (H, Ω^π) is an lbr hod pair with scope H_δ.

Proof. We show first that H is an lpm. If (b) holds, this is rather easy. If M is passive, we can apply (a) of 2.6, noting that Q sentences go down under weakly elementary embeddings. So let us assume that M is branch-active.

By (b) of 2.6, H is a potential branch active lpm. So we just need to see that H is honest. Let $\nu = \nu^H$, $b = b^H$, and $\mathcal{T} = \mathcal{T}(s^H)$. If $\nu = lh(\mathcal{T})$, there is nothing to show, so assume $\nu < lh(\mathcal{T})$. We must show that $b = [0, \nu)_T$. We have by induction that for $N = H|\eta^H$, (N, Ω^π_N) is an lbr hod pair, and in particular, that it is self-aware. Thus $\mathcal{T}$ is by Ω^π, and so we just need to see that for $\mathcal{U} = \mathcal{T}{\restriction}\nu$ and $\mathcal{W} = \mathcal{U}^\frown b$, $\mathcal{W}$ is by Ω^π, or equivalently, that $\pi\mathcal{W}$ is by Ω. But it is easy to see that $\pi\mathcal{W}$ is a psuedo-hull of $\pi(U)^\frown b^M$, and Ω has strong hull condensation, so we are done.

For the proof that (H, Ω^π) is self-consistent, normalizes well, and has strong hull condensation, the reader should see [10]. We give here the proof that (H, Ω^π) is self-aware, because it extends the honesty proof given above.

Let P be an Ω^π iterate of H via the stack of trees s. Let Q be the corresponding Ω iterate of M via πs, and let $\tau\colon P \to Q$ be the weakly elementary copy map. Then for $\mathcal{U} \in P$,

$$\begin{aligned}\mathcal{U} \text{ is by } \dot{\Sigma}^P &\Rightarrow \tau(\mathcal{U}) \text{ is by } \dot{\Sigma}^Q\\ &\Rightarrow \tau\mathcal{U} \text{ is by } \Omega_{\pi s, Q}\\ &\Rightarrow \mathcal{U} \text{ is by } (\Omega^\pi)_{s,P},\end{aligned}$$

as desired.

□

3. CONDENSATION LEMMA

The main theorem of this section is Theorem 3.7. This theorem will be used in the □-construction, but it is more general than is necessary for those applications.

Our theorem extends Theorem 9.3.2 of [14], which deals with condensation under $\pi\colon H \to M$ for pure extender mice H and M. That theorem breaks naturally into two cases: either (1) $H \notin M$, in which case H is the crt(π)-core of M, or (2) $H \in M$, in which case H is a proper initial segment of either M or an ultrapower of M. The proof in case (1) works for least branch hod mice without much change, so we begin with that case.

Definition 3.1. Let M be an lpm or a pure extender premouse, and $n \leq k(M)$; then

(a) $\tilde{h}_M^{n+1}$ is the $\Sigma_1^{(n)}$-Skolem function of M. We write $\tilde{h}_M$ for $\tilde{h}_M^{k(M)+1}$.

(b) Let $\rho(M) \leq \alpha$ and $r = p(M) - \alpha$, and suppose that r is solid. Let $\pi\colon H \to M$ with H transitive be such that $\mathrm{ran}(\pi) = \tilde{h}_M``\alpha \cup r$, and suppose that $\pi^{-1}(r)$ is solid over H. Then we call H the α-*core* of M, and write $H = \mathrm{core}_\alpha(M)$. In addition, if (M, Σ) is a mouse pair, then the α-core of (M, Σ) is (H, Λ), where $H = \mathrm{core}_\alpha(M)$ and $\Lambda = \Sigma^\pi$, where π is the corresponding core map.

(c) M is α-*sound* iff $M = \mathrm{core}_\alpha(M)$.

$\dashv$

According to this definition, if M is α-sound, then $\rho(M) \leq \alpha$. So M could be sound, but not α-sound, which might be confusing at first.

Remark 3.2. Let H be the α-core of M, as witnessed by π. We have $p(M) \subseteq \mathrm{ran}(\pi)$, so the new $\Sigma_{k(M)+1}$ subset of $\rho(M)$ is $\Sigma_{k(M)+1}$ over H. Thus $\rho(H) = \rho(M)$ and $\pi(p(H)) = p(M)$, and $H \notin M$.

One might guess that $P(\alpha)^M \subseteq H$, but this need not be the case, as the following example shows. Let N be sound, and let $M = \mathrm{Ult}(N, E)$, where $\rho(N) \leq \kappa = \mathrm{crt}(E)$, and E has one additional generator α. Let $H = \mathrm{Ult}(N, E{\restriction}\alpha)$, and let $\pi\colon H \to M$ be the factor map. Clearly, π witnesses that H is the α-core of M. But $\alpha = (\kappa^{++})^H < (\kappa^{++})^M$, so H doesn't even have all the bounded subsets of α that are in M.

Theorem 3.3 (AD^+)**.** *Suppose (M, Σ) is a lbr hod pair with scope HC. Suppose $\pi : H \to M$ is nontrivial*[6]*, and letting $n = k(M) = k(H)$ and $\alpha = crt(\pi)$,*[7] *$\alpha < \rho_n(M)$. Suppose also*

(1) H is α-sound,

(2) π is a cardinal-preserving $\Sigma_0^{(n)}$-embedding[8]*, and*

(3) H is an lpm of the same type as M[9]*, and*

(4) $H \notin M$.

Then H is the α-core of M.

Proof. Let $r = p(H) - \alpha$, and $n = k(M)$.

$$T = \mathrm{Th}_{n+1}^H(\alpha \cup r),$$

[6] π is trivial iff $H = M$ and π is the identity.

[7] Here we allow α to be $o(H)$ and π to be the identity.

[8] Letting H^n and M^n be the level n reducts of H and M, this means that $\pi{\restriction}H^n\colon H^n \to M^n$ is Σ_0, and π is the canonical upward extension of $\pi{\restriction}H^n$. See [14, Section 1.7]. If π is weakly elementary, then it is $\Sigma_0^{(n)}$ elementary and cardinal preserving. The converse is probably not true. In our square application, π will in fact be weakly elementary.

[9] This means: H is passive if and only if M is passive; H is B-active if and only if M is B-active; and H is E-active if and only if M is E-active; in the third case, $\dot{F}^H$ is of type A (B, C) if and only if $\dot{F}^M$ is of type A (B, C respectively). All but the last clause are implicit in (2).

so that T codes H.

Suppose first that π is not cofinal $\Sigma_0^{(n)}$. Letting H^n and M^n be the level n reducts, we have that T is Σ_1 over H^n, and hence T is Σ_1 over some proper initial segment of M^n, so that $T \in M^n$. If $n > 0$, then $M|\rho_n(M) \models \mathsf{KP}$ and $T \in M|\rho_n(M)$, so $H \in M$. If $n = 0$ and H is fully passive, then we have $\pi\colon H \to M|\eta$ for some $\eta < o(M)$, and $\mathrm{ran}(\pi)$ in M. Any premouse is closed under transitive collapse, so we again get $H \in M$. If $n = 0$ and H is extender-active, then letting $H^- = H||o(H)$, we get $H^- \in M$ by the argument just given. However, $\dot{F}^H$ can be computed from the fragment $\dot{F}^M \restriction \sup \pi``o(H)$ and π inside M, so $H \in M$. The case that $n = 0$ and H is branch-active can be handled similarly, noting that the proper initial segments of b^M are in M.

So we may assume π is cofinal $\Sigma_0^{(n)}$, and hence $\Sigma_1^{(n)}$. We claim that $\rho(M) \leq \alpha$. For if not, T is a bounded subset of $\rho(M)$ that is Σ_1 over M^n. Thus $T \in M|\rho(M)$, so $H \in M$.

Suppose $r = \emptyset$. If $\gamma \in (p(M) - \alpha)$, then T can be computed easily from the solidity witness W^M_γ, so T in M, and with a bit more work, $H \in M$. So we have $p(M) - \alpha = \emptyset$, which implies that H is the α-core of M, as witnessed by π.

Suppose next that $r = \langle \beta_0, ..., \beta_l \rangle$, and $p(M) - \alpha = \langle \gamma_0, ..., \gamma_m \rangle$, where $\beta_i > \beta_{i+1}$ and $\gamma_i > \gamma_{i+1}$ for all i. We show by induction on $i \leq l$ that $i \leq m$ and $\pi(\beta_i) = \gamma_i$. Suppose we know it for $i \leq k < l$. Let $W = W_{r,\beta^H_{k+1}}$ be the solidity witness for β_{k+1} in H. Since π is $\Sigma_1^{(n+1)}$ elementary, $\pi(W)$ can be used to compute $\mathrm{Th}^M_{n+1}(\pi(\beta_{k+1}) \cup \{\gamma_0, ..., \gamma_k\}$ inside M. But $\rho(M) < \pi(\beta_{k+1})$, so we must have $k < m$. Similarly, $\gamma_{k+1} < \pi(\beta_{k+1})$ is impossible, as otherwise $\pi(W)$ could be used in M to compute the Σ_{n+1} theory of $p(M) \cup \rho(M)$. On the other hand, if $\pi(\beta_{k+1}) < \gamma_{k+1}$, then using the solidity witness $W^M_{p(M),\gamma_{k+1}}$ for γ_{k+1} in M, we get $H \in M$.

It follows that $\pi(r) = p(M) - \alpha$, and thus H is the α_0-core of M.

$\square$

Remark 3.4. In the case H is the core of M, we can also get agreement of Σ and Σ^π up to $(\rho^+)^H = (\rho^+)^M$. See Corollary 3.14. It may be possible to prove strategy condensation in the other cases, but we have not tried to do that.

Next we deal with condensation under $\pi\colon H \to M$ in the case $H \in M$.[10] We shall actually prove a stronger result, one that includes condensation for iteration strategies as well as condensation for the mice themselves. It is convenient here to use the terminology associated to mouse pairs.

Recall from [10, Section 5.3] that (M, Σ) is a *mouse pair* iff either (M, Σ) is a least branch hod pair, or (M, Σ) is a pure extender pair.[11] We say that $(H, \Psi) \trianglelefteq (M, \Sigma)$ iff for some $\langle \nu, l \rangle \leq l(M)$, $H = M|\langle \nu, l \rangle$ and $\Psi = \Sigma_{\langle \nu, l \rangle}$. If (H, Ψ) and (M, Σ) are mouse pairs of the same type, then $\pi\colon (H, \Psi) \to (M, \Sigma)$ is elementary (resp. weakly elementary) iff π is elementary (weakly elementary) as a map from $H \to M$, and $\Psi = \Sigma^\pi$. We say that (M, Σ) is an *iterate of* (H, Ψ) iff there is a stack

[10] If $\pi\colon H \to M$ is elementary, $\alpha = \mathrm{crt}(\pi)$, H is α-sound, and $\alpha < \rho(M)$, then $H \in M$. This is the case with the coarser condensation results of [10, 5.55] and [1, 8.2], where $\alpha = \rho(H)$ and $\pi(\alpha) = \rho(M)$.

[11] That is, M is a pure extender premouse, and Σ is a self-consistent complete iteration strategy for M that normalizes well and has strong hull condensation.

s on H such that s is by Ψ, and $\Sigma = \Psi_s$. It is a *non-dropping iterate* iff the branch H-to-M does not drop. Assuming AD^+ and that our pairs have scope $\mathrm{H}C$, [10] proves the following:

(1) If (M, Σ) is a mouse pair, and $\pi \colon H \to M$ is weakly elementary, then (H, Σ^π) is a mouse pair.

(2) If (H, Ψ) is a mouse pair, and (M, Σ) is a non-dropping iterate of (H, Ψ), then the iteration map $i_s \colon (H, \Psi) \to (M, \Sigma)$ is elementary in the category of pairs.

(3) (Dodd-Jensen) If (H, Ψ) is a mouse pair, (M, Σ) is an iterate of (H, Ψ) via the stack s, and $\pi \colon (H, \Psi) \to (M, \Sigma)$ is weakly elementary, then

   (i) the branch H-to-M of s does not drop, and

   (ii) for all $\eta < o(H)$, $i_s(\eta) \leq \pi(\eta)$, where i_s is the iteration map.

(4) (Mouse order) Let $(H, \Psi) \leq^* (M, \Sigma)$ iff there is a weakly elementary embedding of (H, Ψ) into some iterate of (M, Σ); then $\leq^*$ is a prewellorder of the mouse pairs with scope $\mathrm{H}C$ in each of the two types.

The following is an easy case of condensation for pairs.

Lemma 3.5. *[AD^+] Let (M, Σ) be a mouse pair with scope $\mathrm{H}C$, and let $\pi \colon (H, \Psi) \to (M, \Sigma)$ be elementary, with $\pi =$ identity; then either $(H, \Psi) \trianglelefteq (M, \Sigma)$, or $(H, \Psi) \triangleleft \mathrm{Ult}((M, \Sigma), E^M_\alpha)$, where $\alpha = o(H)$.*

Proof. Suppose first H is extender-active. Let $F = \dot{F}^H$ and $G = \dot{F}^M$, and let $\kappa = \mathrm{crt}(F)$. So $\kappa^{+,H} = \kappa^{+,M} < o(H)$, and $i^H_F``\kappa^{+,H} = i^M_G``\kappa^{+,M}$. Thus $\mathrm{ran}(\pi)$ is cofinal in $o(M)$, which implies $(H, \Psi) = (M, \Sigma)$.

Next, suppose that H is branch active.[12] Since π is the identity, $\eta = \eta^H = \eta^M$ and $s = s^H = s^M$. Let $\mathcal{T} = \mathcal{T}(s)$, and let $\nu = \nu^H$, so that $o(H) = \eta + \nu$. Because π preserves $\dot{B}^H$, $b^H = b^M \cap \nu$. But $b^M \cap \nu = b^{M|o(H)}$ because M is an lpm, so $H = M$. We get $\Sigma^\pi = \Sigma_{l(H)}$ from the self-consistency of (M, Σ), so $(H, \Psi) \trianglelefteq (M, \Sigma)$.

Finally, suppose that H is fully passive. Clearly, $M||\hat{o}(H)$ is branch-passive, and thus $M||\hat{o}(H) = H$. Using self-consistency for (M, Σ), we get $(H, \Psi) \trianglelefteq (M, \Sigma)$, unless $M|\hat{o}(H)$ is extender-active. In that case we get $(H, \Psi) \triangleleft Ult(M, E^M_\alpha)$, where $\alpha = \hat{0}(H)$, using self-consistency and strategy coherence.

$\square$

Definition 3.6. Let M and N either be both pure extender premice or both lpm's with $n = k(M) = k(N)$, and $\pi : M \to N$; then π is **weakly elementary** if π is the n-completion of $\pi \restriction M^n$ (in the sense of [5]), and $\pi \restriction M^n : M^n \to N^n$ is Σ_0 and cardinal preserving.

$\dashv$

[12] Of course, this only applies when M is an lpm. In general, our proofs for pure extender pairs are special cases of the proofs for lbr hod pairs, so it doesnt hurt to assume our mouse pair is an lbr hod pair.

Our main condensation theorem for mouse pairs is:

Theorem 3.7 (AD^+)**.** *Suppose (M, Σ) is a mouse pair with scope HC. Suppose $\pi : (H, \Psi) \to (M, \Sigma)$ is weakly elementary, and not the identity. Let $\alpha = \mathrm{crt}(\pi)$, and suppose*

(1) H is a premouse of the same type as M,

(2) $\rho(H) \leq \alpha < \rho_{k(H)}(H)$, and H is α-sound, and

(3) H is not the α-core of M.

Then exactly one of the following holds.

(a) $(H, \Psi) \lhd (M, \Sigma)$.

(b) $(H, \Psi) \lhd Ult_0((M, \Sigma), \dot{E}_\alpha^M)$.

(c) $(H, \Psi) = Ult((M|(\xi, k), \Sigma_{\langle \xi, k\rangle}), E)$, where $l(M) >_{lex} (\xi, k) >_{lex} (\alpha, n)$, and (ξ, k) is lex least such that $\rho(M|(\xi, k)) < \alpha$, E is on the extender sequence of $M|\xi$, and crt(E) is the cardinal predecessor of α in $M|\xi$ and is the only generator of E.

When one applies Theorem 3.7 in the proof of $\square_\kappa$, it is easy see that $H \in M$, and to rule out conclusions (b) and (c). In that proof, $\rho(H) = \rho(M) = \kappa$, and $\alpha = (\kappa^+)^H$, and both H and M are sound. This implies (c) cannot hold. Moreover, because κ is not subcompact, one can arrange that $\dot{E}_\alpha^M = \emptyset$, so (b) does not hold. So one gets conclusion (a), that $(H, \Psi) \lhd (M, \Sigma)$. In the square proof, what matters then is just that $H \unlhd M$; the full external strategy agreement given by $\Sigma^\pi = \Sigma_{l(H)}$ is not used.

Remark 3.8. It follows from the theorem that the hypothesis $\alpha < \rho_{k(H)}(H)$ can be dropped, if one omits condensation of the external strategy from its conclusion. See 3.13 below.

Remark 3.9. A relatively coarse special case of Theorem 3.7 is sketched in [10, Theorem 5.55]. In that case, π is assumed to be fully elementary and $\mathrm{crt}(\pi) = \rho(H)$.

Proof of Theorem 3.7. Let $\pi \colon (H, \Psi) \to (M, \Sigma)$ be weakly elementary, and let $\alpha_0 = \mathrm{crt}(\pi)$. Suppose H is α_0-sound, and not the α_0-core of M, so that by 3.3, $H \in M$. For definiteness, let us assume that H and M are lpms. The proof in the case that they are pure extender mice is similar.[13][14]

Definition 3.10. A tuple $\langle (N, \Phi), (G, \Lambda), \sigma, \nu\rangle$ is *problematic* iff

(1) (N, Φ) and (G, Λ) are lbr hod pairs of the same type, with scope HC, and $G \in N$,

(2) $\sigma \colon (G, \Lambda) \to (N, \Phi)$ is weakly elementary, with $\mathrm{crt}(\sigma) = \nu$,

[13] Even in the pure extender case, one cannot simply quote 9.3.2 of [14], because we are demanding strategy condensation.

[14] Under AD^+, every countable ω_1-iterable pure extender mouse M has an complete iteration strategy Σ such (M, Σ) is a pure extender pair. Thus our theorems 3.3,3.5, and 3.7 together imply 9.3.2 of [14], modulo some details about where the strategies live, and how elementary π is.

(3) $\nu < \rho_{k(G)}(G)$ and G is ν-sound, and

(4) conclusions (a),(b), and (c) of 3.7 all fail; that is, (G, Λ) is not an initial segment of (N, Φ), nor is it an ultrapower away in the manner described in (b) or (c).

$\dashv$

Claim 1. Let $\langle (N, \Phi), (G, \Lambda), \sigma, \nu \rangle$ be a problematic tuple, and $k = k(G)$; then then $\rho(G) \leq \nu < \rho_k(G) \leq \rho_k(N)$.

Proof. $\rho(G) \leq \nu$ because G is ν-sound. $\rho_k(G) \leq \rho_k(N)$ because σ is weakly elementary. $\square$

We must show that $\langle (M, \Sigma), (H, \Psi), \pi, \alpha_0 \rangle$ is not problematic. Assume toward contradiction that it is, and that (M, Σ) is minimal in the mouse order such that it is the first term in some problematic tuple.

We obtain a contradiction by comparing the phalanx (M, H, α_0) with M, as usual. However, since we are comparing strategies, this must be done indirectly, by iterating both into some sufficiently strong background construction $\mathbb{C}$. It can happen that at some point, the two sides agree with each other (but not with $\mathbb{C}$). This leads to a problem in the argument that the end model on the phalanx side can't be above M. The solution, employed in [10]), is to modify how the phalanx is iterated, moving the whole phalanx (including its exchange ordinal) up at certain stages. Our main new problem here is that because of the restricted elementarity of our maps, if we move up naively, the new phalanx and associated embedding may not be problematic. This forces us to drop to a new problematic phalanx on occasion.

Claim 2. Let $\langle (N, \Phi), (G, \Lambda), \sigma, \nu \rangle$ be a problematic tuple, and $k = k(G)$; then we can decompose $\sigma \upharpoonright G^k$ as

$$\sigma \upharpoonright G^k = \bigcup_{\eta < \rho_k(G)} \sigma^\eta,$$

where each σ^η belongs to N^k.

Proof. Assume first $k = 0$, and that $\hat{o}(G)$ is a limit ordinal. For $\eta < \hat{o}(G)$, let G^η be $G||\eta$, expanded by I^η, where I^η is the appropriate fragment of $\dot{F}^G$ if G is extender active, and the appropriate initial segment of $\dot{B}^G$ if G is branch active. Let N^η be $N||\sigma(\eta)$, expanded by $\sigma(I^\eta)$. Let σ^η be the fragment of σ given by

$$\mathrm{dom}(\sigma^\eta) = h^1_{G^\eta}\text{``}(\nu \cup s),$$

and

$$\sigma^\eta(h^1_{G^\eta}(\delta, s)) = h^1_{N^{\sigma(\eta)}}(\delta, \sigma(s)),$$

for $\delta < \nu$. We have that $\sigma^\eta \in N$, and $\sigma = \bigcup_{\eta < \hat{o}(G)} \sigma^\eta$. If $\hat{o}(G)$ is a successor ordinal, we can ramify using the S-hierarchy.

The case $k > 0$ is similar. We have $G^k = (G||\rho_k(G), A)$ where A is $\mathrm{Th}^G_k(\rho_k(G) \cup p_k(G))$. For $\eta \leq \rho_k(G)$, let

$$G^\eta = (G||\eta, A \cap G||\eta).$$

Let $s = p_1^{G^k} \setminus \nu$, and let $h^1_{G^\eta}$ be the Σ_1 Skolem function, so that $G^k = h^1_{G^k}``(\nu \cup s)$. For $\eta < rho_k(G)$,

$$\mathrm{dom}(\sigma^\eta) = h^1_{G^\eta}``(\nu \cup s),$$

and for $\gamma < \nu$ in $\mathrm{dom}(\sigma^\eta)$,

$$\sigma^\eta(h^1_{G^\eta}(\gamma, s)) = h^1_{N^k||\sigma(\eta)}(\gamma, \sigma(s)).$$

It is easy to see that this works. $\square$

We call $\langle(\sigma^\eta, G^\eta) \mid \nu \leq \eta < \rho_k(G)\rangle$ as above the *natural decomposition* of $\sigma \restriction G^k$.

Using claim 2, we can move a problematic tuple $\langle(N,\Phi),(G,\Lambda),\sigma,\nu\rangle$ of up via an iteration map that is continuous at $\rho_k(G)$. When the iteration map is discontinuous at $\rho_k(G)$, we may have to drop.

Definition 3.11. Let $\Phi = \langle(N,\Phi),(G,\Lambda),\sigma,\nu\rangle$ be a problematic tuple; then Φ is *extender-active* iff $E^N_\nu \neq \emptyset$. $\dashv$

When we move up extender-active tuples, the new exchange ordinal is always the image of the old one, so the new tuple is still extender-active.

Claim 3. Let $\langle(N,\Phi),(G,\Lambda),\sigma,\nu\rangle$ be problematic, and suppose that $(N,\Phi) \leq^* (M,\Sigma)$; then there is no proper initial segment (Q,Ω) of (G,Λ) such that $\nu = \rho(Q)$ and either

(i) $E^N_\nu = \emptyset$, and (Q,Ω) is not an initial segment of (N,Φ), or

(ii) $E^N_\nu \neq \emptyset$, and (Q,Ω) is not a proper initial segment of $\mathrm{Ult}((N,\Phi),E^N_\nu)$.

Proof. This follows from the minimality of (M,Σ) in the mouse order. For if (Q,Ω) is a counterexample, then letting $(R,\Gamma) \lhd (N,\Phi)$ be such that $R = \sigma(Q)$, we have that $(R,\Gamma) <^* (M,\Sigma)$, and $\langle(R,\Gamma),(Q,\Lambda_Q),\sigma \restriction Q,\nu\rangle$ is problematic.

$\square$

So under the hypotheses of claim 3, (N,Φ) agrees with (G,Λ) strictly below $\nu^{+,G}$.

We are ready now to enter the phalanx comparison argument of [10].

Let N^* be a coarse Γ-Woodin mouse, where $\Sigma \in \Delta_\Gamma$ and Γ is an inductive-like, scaled pointclass contained strictly in the Suslin co-Suslin sets. Let $(\vec{F},\Sigma^*,\delta^*)$ satisfy the following.

(i) $N^* \models \vec{F}$ is a coarsely coherent extender sequence.

(ii) $\vec{F}$ witnesses δ^* is Woodin in N^*.

(iii) Σ^* is an (ω_1,ω_1)-$\vec{F}$-strategy for N^*[15] such that the restriction of Σ^* to stacks in $V^{N^*}_{\delta^*}$ is in N^*. In fact, $N^* \models$ "I am uniquely $\vec{F}$-iterable for stacks in V_{δ^*}".

[15]So ultrapowers used in trees according to Σ^* are by extenders in $\vec{F}$ and its images.

(iv) There is a $\mathrm{Coll}(\omega,\delta^*)$-term τ and a universal Γ-set U such that if $i : N^* \to N$ is via Σ^* and $g \subseteq \mathrm{Coll}(\omega, i(\delta^*))$ is N-generic, then $i(\tau)_g = U \cap N[g]$.

Let $\mathbb{C}$ be the maximal $\vec{F}$-construction of N^*, with associated models $M_{\nu,l} = M^{\mathbb{C}}_{\nu,l}$ and induced strategies $\Omega_{\nu,l} = \Omega^{\mathbb{C}}_{\nu,l}$ for $(\nu, l) < (\delta^*, 0)$. By [10, Theorem 5.31], there is a unique pair (η_0, k_0) such that (M,Σ) iterates to $(M_{\eta_0,k_0}, \Omega_{\eta_0,k_0})$, and for all $(\nu, l) <_{\text{lex}} (\eta_0, k_0)$, (M,Σ) iterates strictly past $(M_{\nu,l}, \Omega_{\nu,l})$. Let $\mathcal{U}_{\nu,l}$ be the unique normal tree on M witnessing (M,Σ) iterates past $(M_{\mu,l}, \Omega_{\nu,l})$. [16]

We define trees $\mathcal{S}_{\nu,l}$ on (M, H, α_0) for certain $(\nu, l) \leq (\eta_0, k_0)$. Fix $(\nu, l) \leq (\eta_0, k_0)$ for now, and assume $\mathcal{S}_{\nu',l'}$ is defined whenever $(\nu', l') < (\nu, l)$. Let $\mathcal{U} = \mathcal{U}_{\nu,l}$, and for $\tau < \mathrm{lh}(\mathcal{U})$, let

$$\Sigma^{\mathcal{U}}_\tau = \Sigma_{\mathcal{U}\restriction(\tau+1)}$$

be the tail strategy for $\mathcal{M}^{\mathcal{U}}_\tau$ induced by Σ. We proceed to define $\mathcal{S} = \mathcal{S}_{\nu,l}$, by comparing the phalanx (M, H, α_0) with $M_{\nu,l}$. As we define $\mathcal{S}$, we lift $\mathcal{S}$ to a padded tree $\mathcal{T}$ on M, by copying. Let us write

$$\Sigma^{\mathcal{T}}_\theta = \Sigma_{\mathcal{T}\restriction(\theta+1)}$$

for the tail strategy for $\mathcal{M}^{\mathcal{T}}_\theta$ induced by Σ. For $\theta < lh(\mathcal{S})$, we will have copy map

$$\pi_\theta : \mathcal{M}^{\mathcal{S}}_\theta \to \mathcal{M}^{\mathcal{T}}_\theta.$$

The map π_θ is a weakly elementary. We attach the complete strategy

$$\Lambda_\theta = (\Sigma^{\mathcal{T}}_\theta)^{\pi_\theta}$$

to $\mathcal{M}^{\mathcal{S}}_\theta$. We also define a non-decreasing sequence ordinals $\lambda_\theta = \lambda^{\mathcal{S}}_\theta$ that measure agreement between models of $\mathcal{S}$, and tell us which model we should apply the next extender to.

The following claim will be useful in pushing up problematic tuples.

Claim 4. Suppose $\xi <_S \theta$ and $(\xi, \theta]_S$ does not drop; then $\Lambda_\xi = \Lambda_\theta^{i^{\mathcal{S}}_{\xi,\theta}}$.

Proof. Because Σ is pullback consistent, we have $\Sigma^{\mathcal{T}}_\xi = (\Sigma^{\mathcal{T}}_\theta)^{i^{\mathcal{T}}_{\xi,\theta}}$. But then

$$\begin{aligned}\Lambda_\xi &= (\Sigma^{\mathcal{T}}_\xi)^{\pi_\xi}\\ &= (\Sigma^{\mathcal{T}}_\theta)^{i^{\mathcal{T}}_{\xi,\theta}\circ\pi_\xi}\\ &= (\Sigma^{\mathcal{T}}_\theta)^{\pi_\theta\circ i^{\mathcal{S}}_{\xi,\theta}}\\ &= \Lambda_\theta^{i^{\mathcal{S}}_{\xi,\theta}},\end{aligned}$$

[16] We can work in N^* from now on, and interpret these statements there. But in fact, the strategies $\Omega_{\nu,l}$ are induced by Σ^* in a way that guarantees they extend to Σ^*-induced strategies $\Omega^+_{\nu,l}$ defined on all of HC. $\mathcal{U}_{\nu,l}$ iterates (M,Σ) past $(M_{\nu,l}, \Omega^+_{\nu,l})$ in V.

as desired. $\square$

We start with

$$\mathcal{M}_0^{\mathcal{S}} = M, \mathcal{M}_1^{\mathcal{S}} = H, \lambda_0 = \alpha_0,$$

and

$$\mathcal{M}_0^{\mathcal{T}} = \mathcal{M}_1^{\mathcal{T}} = M, \pi_0 = id, \pi_1 = \pi,$$

and

$$\Lambda_0 = \Sigma,\ \Lambda_1 = \Sigma^{\pi_1}.$$

We say that $0, 1$ are distinct roots of $\mathcal{S}$. We say that 0 is unstable, and 1 is stable. As we proceed, we shall declare additional nodes θ of $\mathcal{S}$ to be unstable. We do so because $(\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta) = (\mathcal{M}_\gamma^{\mathcal{U}}, \Sigma_\gamma^{\mathcal{U}})$ for some γ[17], and when we do so, we shall immediately define $\mathcal{M}_{\theta+1}^{\mathcal{S}}$, as well as σ_θ and α_θ such that

$$\Phi_\theta =_{\text{df}} \langle (\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta), (\mathcal{M}_{\theta+1}^{\mathcal{S}}, \Lambda_{\theta+1}), \sigma_\theta, \alpha_\theta \rangle$$

is a problematic tuple. Here $\Lambda_{\theta+1} = \Lambda_\theta^{\sigma_\theta}$. In this case, $[0, \theta]_S$ does not drop, and all $\xi \leq_S \theta$ are also unstable. We regard $\theta + 1$ as a new root of $\mathcal{S}$. This is the only way new roots are constructed.

Let us also write

$$\Phi_\theta^- =_{\text{df}} \langle \mathcal{M}_\theta^{\mathcal{S}}, \mathcal{M}_{\theta+1}^{\mathcal{S}}, \sigma_\theta, \alpha_\theta \rangle$$

for the part of Φ_θ that is definable over $\mathcal{M}_\theta^{\mathcal{S}}$. We say Φ_θ^- *is problematic* iff it is not the case that either $\mathcal{M}_{\theta+1}^{\mathcal{S}} \lhd \mathcal{M}_\theta^{\mathcal{S}}$ or $\mathcal{M}_{\theta+1}^{\mathcal{S}}$ is an initial segment of an ultrapower of $\mathcal{M}_\theta^{\mathcal{S}}$ in one of the two ways specified in the conclusion of 3.7.

If θ is unstable, then we define

$$\beta_\theta = (\alpha_\theta^+)^{\mathcal{M}_{\theta+1}^{\mathcal{S}}}.$$

If $\xi <_S, \theta$, then we shall have $\beta_\theta \leq i_{\xi,\theta}^{\mathcal{S}}(\beta_\xi)$, and

$$\beta_\theta = i_{\xi,\theta}^{\mathcal{S}}(\beta_\xi) \Rightarrow \Phi_\theta = i_{\xi,\theta}^{\mathcal{S}}(\Phi_\xi),$$

in the appropriate sense. In this connection: it will turn out that $i_{\xi,\theta}(\beta_\xi) = \beta_\theta$ implies $i_{\xi,\theta}^{\mathcal{S}}$ is continuous at $\rho_k(\mathcal{M}_{\xi+1}^{\mathcal{S}})$, where $k = k(\mathcal{M}_{\xi+1}^{\mathcal{S}})$. So we can set

$$i_{\xi,\theta}^{\mathcal{S}}(\sigma_\xi) = \text{ upward extension of } \bigcup_{\eta < \rho_k(\mathcal{M}_{\xi+1}^{\mathcal{S}})} i_{\xi,\theta}^{\mathcal{S}}(\sigma_\xi^\eta),$$

where $\langle \sigma_\xi^\eta \mid \eta < \rho_k(\mathcal{M}_{\xi+1}^{\mathcal{S}}) \rangle$ is the natural decomposition of σ_ξ. This enables us to make sense of $i_{\xi,\theta}^{\mathcal{S}}(\Phi_\xi^-)$.

[17] In the first version of [10] the external strategy agreement was not required for θ to be declared unstable, but it is important to do so here.

The construction of $\mathcal{S}$ takes place in rounds in which we either add one stable θ, or one unstable θ and its stable successor $\theta + 1$. Thus the current last model is always stable, and all extenders used in $\mathcal{S}$ are taken from stable models. If γ is stable, then $\lambda_\gamma = \lambda(E^{\mathcal{S}}_\gamma)$.

In sum, we are maintaining by induction that the last node γ of our current $\mathcal{S}$ is stable, and

Induction hypotheses $(\dagger)_\gamma$**.** If $\theta < \gamma$ and θ is unstable, then

(1) $0 \leq_{\mathcal{S}} \theta$ and $[0,\theta]_{\mathcal{S}}$ does not drop (in model or degree), and every $\xi \leq_S \theta$ is unstable,

(2) there is a γ such that $(\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta) = (\mathcal{M}^{\mathcal{U}}_\gamma, \Sigma^{\mathcal{U}}_\gamma)$,

(3) $\Phi_\theta = \langle (\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta), (M^{\mathcal{S}}_{\theta+1}, \Lambda_{\theta+1}), \sigma_\theta, \alpha_\theta \rangle$ is a problematic tuple,

(4) Φ_θ is extender-active iff Φ_0 is extender-active, and if Φ_θ is extender-active, then $i^{\mathcal{S}}_{0,\theta}(\alpha_0) = \alpha_\theta$,

(5) if $\xi <_S \theta$, then $\alpha_\theta \leq i^{\mathcal{S}}_{\xi,\theta}(\alpha_\xi)$ and $\beta_\theta \leq i^{\mathcal{S}}_{\xi,\theta}(\beta_\xi)$,

6) if $\langle \alpha_\theta, \beta_\theta \rangle = i^{\mathcal{S}}_{\xi,\theta}(\langle \alpha_\xi, \beta_\xi \rangle)$, then

   (a) $\Phi^-_\theta = i^{\mathcal{S}}_{\xi,\theta}(\Phi^-_\xi)$, and

   (b) $i^{\mathcal{S}}_{\xi,\theta}$ is continuous at $\rho_k(\mathcal{M}^{\mathcal{S}}_{\xi+1})$, for $k = k(\mathcal{M}^{\mathcal{S}}_{\xi+1})$,

(7) $\mathcal{M}^{\mathcal{T}}_{\theta+1} = \mathcal{M}^{\mathcal{T}}_\theta$, and $\pi_{\theta+1} = \pi_\theta \circ \sigma_\theta$.

Setting $\sigma_0 = \pi$, we have $(\dagger)_1$.

For a node γ of $\mathcal{S}$, we write $\mathcal{S}$-pred(γ) for the immediate $\leq_{\mathcal{S}}$-predecessor of $\mathcal{S}$. For γ a node in $\mathcal{S}$, we set

$$\mathrm{st}(\gamma) = \text{the least stable } \theta \text{ such that } \theta \leq_{\mathcal{S}} \gamma,$$

and

$$\mathrm{rt}(\gamma) = \begin{cases} \mathcal{S}\text{-pred}(\mathrm{st}(\gamma)) & : \text{ if } \mathcal{S}\text{-pred}(\mathrm{st}(\gamma)) \text{ exists} \\ \mathrm{st}(\gamma) & : \text{ otherwise.} \end{cases}$$

The construction of $\mathcal{S}$ ends when we reach a stable θ such that

(I) $(M_{\nu,l}, \Omega_{\nu,l}) \lhd (\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta)$, or

(II) $(\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta) \trianglelefteq (M_{\nu,l}, \Omega_{\nu,l})$, and $[\mathrm{rt}(\theta), \theta]_{\mathcal{S}}$ does not drop in model or degree.[18]

[18] In [10, Theorem 5.10], there is another way the comparison can end: we reach a stable θ such that for some ξ, $\mathcal{M}^{\mathcal{S}}_\theta = \mathcal{M}^{\mathcal{U}}_\xi$, and neither $[\mathrm{rt}(\theta), \theta]_{\mathcal{S}}$ nor $[0,\xi]_{\mathcal{U}}$ drops in model or degree. Moreover, letting $\mathcal{Q}$ be the result of removing the last extender predicate of $\mathcal{M}^{\mathcal{S}}_\theta$, then $\mathcal{Q} \trianglelefteq M_{\nu,l}$. This is not necessary in our situation, and would cause problems for the strategy-condensation part of our proof.

If case (I) occurs, then we go on to define $\mathcal{S}_{\nu,l+1}$. If case (II) occurs, we stop the construction.

We now describe how to extend $\mathcal{S}$ one more step. First we assume $\mathcal{S}$ has successor length $\gamma+1$ and let $\mathcal{M}^{\mathcal{S}}_\gamma$ be the current last model, so that γ is stable. Suppose$(\dagger)_\gamma$ holds. Suppose (I) and (II) above do not hold for γ, so that we have a least disagreement between $\mathcal{M}^{\mathcal{S}}_\gamma$ and $M_{\nu,l}$. By the proof of [10, Lemma 5.9], the least disagreement involves only an extender E on the sequence of $\mathcal{M}^{\mathcal{S}}_\gamma$. Letting $\tau = \mathrm{lh}(E)$, we have

- $M_{\nu,l}|(\tau,0) = \mathcal{M}^{\mathcal{S}}_\gamma|(\tau,-1)$,[19] and
- $(\Omega_{\nu,l})_{(\tau,0)} = (\Lambda_\gamma)_{(\tau,-1)}$.

Set $\lambda^{\mathcal{S}}_\gamma = \lambda_E$ and ξ be least such that $\mathrm{crt}(E) < \lambda^{\mathcal{S}}_\xi$. We let $\mathcal{S}$-pred$(\gamma+1)=\xi$. Let (β,k) be lex least such that either $\rho(\mathcal{M}^{\mathcal{S}}_\xi|(\beta,k)) \leq \mathrm{crt}(E)$ or $(\beta,k) = (\hat{o}(\mathcal{M}^{\mathcal{S}}_\xi), k(\mathcal{M}^{\mathcal{S}}_\xi))$. Set

$$\mathcal{M}^{\mathcal{S}}_{\gamma+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{S}}_\xi|(\beta,k),E),$$

and let $\hat{i}^{\mathcal{S}}_{\xi,\gamma+1}$ be the canonical embedding. Let

$$\mathcal{M}^{\mathcal{T}}_{\gamma+1} = \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\xi|(\pi_\xi(\beta),k),\pi_\gamma(E)),$$

and let $\pi_{\gamma+1}$ be given by the Shift Lemma. This determines $\Lambda_{\gamma+1}$.

If ξ is stable or $(\beta,k) < (\hat{o}(\mathcal{M}^{\mathcal{S}}_\xi), k(\mathcal{M}^{\mathcal{S}}_\xi))$, then we declare $\gamma+1$ to be stable. $(\dagger)_{\gamma+1}$ follows vacuously from $(\dagger)_\gamma$.[20]

If ξ is unstable, $(\beta,k) = (\hat{o}(\mathcal{M}^{\mathcal{S}}_\xi), k(\mathcal{M}^{\mathcal{S}}_\xi))$, and $(\mathcal{M}^{\mathcal{S}}_{\gamma+1}, \Lambda_\gamma+1)$ is not a model of $\mathcal{U}$, then again we declare $\gamma+1$ stable. Again, $(\dagger)_{\gamma+1}$ follows vacuously from $(\dagger)_\gamma$.

Finally, suppose ξ is unstable, $(\beta,k) = (\hat{o}(\mathcal{M}^{\mathcal{S}}_\xi), k(\mathcal{M}^{\mathcal{S}}_\xi))$, and for some τ,

$$(\mathcal{M}^{\mathcal{S}}_{\gamma+1}, \Lambda_{\gamma+1}) = (\mathcal{M}^{\mathcal{U}}_\tau, \Sigma^{\mathcal{U}}_\tau).$$

We then we declare $\gamma+1$ to be unstable and $\gamma+2$ stable. We must define the problematic tuple needed for $(\dagger)_{\gamma+2}$. Let $i = i^{\mathcal{S}}_{\xi,\gamma+1}$, and

$$\langle (N,\Psi),(G,\Phi),\sigma,\alpha\rangle = \langle (\mathcal{M}^{\mathcal{S}}_\xi,\Lambda_\xi),(\mathcal{M}^{\mathcal{S}}_{\xi+1},\Lambda_{\xi+1}),\sigma_\xi,\alpha_\xi\rangle.$$

We have that $\langle (N,\Psi),(G,\Phi),\sigma,\alpha\rangle$ is problematic. Let $k=k(G)$. (So $k=k(N)=k(M)$.)

Case 1. i is continuous at $\rho_k(G)$.

In this case, we simply let

$$\langle \mathcal{M}^{\mathcal{S}}_{\gamma+2},\alpha_{\gamma+1}\rangle = \langle i(G), i(\alpha)\rangle.$$

[19] Recall $\mathcal{M}^{\mathcal{S}}_\gamma|(\tau,-1)$ is the structure obtained from $\mathcal{M}^{\mathcal{S}}_\gamma|\tau$ by removing E.

[20] It is possible that ξ is unstable, $\lambda_\xi = \alpha_\xi$, $\mathcal{S}$-pred$(\gamma+1)=\xi$, and $\mathrm{crt}(E^{\mathcal{S}}_\gamma) = \lambda_F$ where F is the last extender of $\mathcal{M}^{\mathcal{S}}_\xi|\alpha_\xi$. In this case, $(\beta,k) = (\mathrm{lh}(F),0)$. The problem then is that $\mathcal{M}^{\mathcal{S}}_{\gamma+1}$ is not an lpm, because its last extender $i_{\xi,\gamma+1}(F)$ has a missing whole initial segment, namely F. Schindler and Zeman found a way to deal with this anomaly in [6]. Their method works in the hod mouse context as well. Here we shall not go into the details of this case. The anomaly cannot occur when ξ is stable, because then $\lambda_\xi = \lambda(E^{\mathcal{S}}_\xi)$ is inaccessible in $\mathcal{M}^{\mathcal{S}}_\gamma$.

We must define $\sigma_{\gamma+1}$. Note that by our case hypothesis,

$$i(G^k) = i(G)^k.$$

Let $\langle \sigma^\eta \mid \eta < \rho_k(G)\rangle$ be the natural decomposition of $\sigma \restriction G^k$, and set

$$i(\sigma \restriction G^k) = \bigcup_{\eta<\rho_k(G)} i(\sigma^\eta).$$

Using the continuity of i at $\rho_k(G)$, it is easy to see that $i(\sigma \restriction G^k$ is Σ_0-elementary from $i(G)^k$ to $i(N)^k$. We set

$$\sigma_{\gamma+1} = \text{ completion of } i(\sigma \restriction G^k) \text{ via upward extension of embeddings,}$$

and

$$\Lambda_{\gamma+2} = \Lambda_{\gamma+1}^{\sigma_{\gamma+1}}.$$

This defines $\Phi_{\gamma+1}$. Abusing notation a bit, let us write

$$\Phi_{\gamma+1}^- = \langle i(N), i(G), i(\sigma), i(\alpha)\rangle.$$

We must see that $\Phi_{\gamma+1}$ is problematic. First, it satisfies the hypotheses of the condensation theorem 3.7. For G is α-sound, so $i(G)$ is $i(\alpha)$-sound. By downward extension of embeddings (cf. [5, Lemma 3.3]), $i(\sigma \restriction G^k)$ extends to a unique embedding from some K into $i(N)$, and it is easy to see that $K = i(G)$, because $i(G)$ is k-sound, and that the embedding in question is what we have called $i(\sigma)$. $i(\sigma)$ is weakly elementary: it maps parameters correctly, $i(\sigma) \restriction i(G)^k$ is Σ_0-elementary and cardinal preserving by construction.

Finally, $\mathrm{crt}(i(\sigma)) = i(\alpha)$, because for all sufficiently large $\eta < \rho_k(G)$, $\alpha + 1 \subseteq \mathrm{dom}(\sigma^\eta)$ and $\mathrm{crt}(\sigma^\eta) = \alpha$, so $\mathrm{crt}(i(\sigma^\eta)) = i(\alpha)$.

So we must see that one of the conclusions of 3.7 fails. We show that the conclusion that failed for Φ_ξ fails for $\Phi_{\gamma+1}$.

Suppose first that Φ_ξ^- is problematic. We break into cases. If G is sound and $E_\alpha^N = \emptyset$, then $\neg G \lhd N$. But then $i(G)$ is sound, $E_{i(\alpha)}^{i(N)} = \emptyset$, and $\neg i(G) \lhd i(N)$, so $\Phi_{\gamma+1}^-$ is problematic. If G is sound and $E_\alpha^N \neq \emptyset$, then $\neg G \lhd \mathrm{Ult}(N, E_\alpha^N)$. But then $i(G)$ is sound, $E_{i(\alpha)}^{i(N)} \neq \emptyset$, and $\neg i(G) \lhd \mathrm{Ult}(i(N), E_{i(\alpha)}^{i(N)})$, so $\Phi_{\gamma+1}^-$ is problematic. Finally, there is the case that G is unsound. The Π_1 fact that G is not an initial segment of an appropriate ultrapower of N is preserved by i, so we are done.

So we may assume Φ_ξ^- is not problematic, and hence also $\Phi_{\gamma+1}^-$ is not problematic. Suppose first $G \lhd N$, so $i(G) \lhd i(N)$. We must show $\Lambda_{\gamma+1}^{i(\sigma)} \neq (\Lambda_{\gamma+1})_{i(G)}$, so suppose otherwise. Using claim 4 we get $\Lambda_\xi = \Lambda_{\gamma+1}^i$, so

$$\begin{aligned}\Lambda_\xi^\sigma &= (\Lambda_{\gamma+1})^{i\circ\sigma}\\ &= (\Lambda_{\gamma+1})^{i(\sigma)\circ i}\\ &= (\Lambda_{\gamma+1}^{i(\sigma)})^i\\ &= ((\Lambda_{\gamma+1})_{i(G)})^i\\ &= (\Lambda_\xi)_G,\end{aligned}$$

a contradiction. Equation 2 holds because $i \circ \sigma = i(\sigma) \circ i$, and equation 5 comes from equation 4 using claim 4 again. Thus Φ_ξ is not problematic, contradiction.

A similar argument shows that $\Phi_{\gamma+1}$ is "strategy problematic in the other two cases, when G is an ultrapower away from N.

Thus $\langle \mathcal{M}^{\mathcal{S}}_{\gamma+1}, \mathcal{M}^{\mathcal{S}}_{\gamma+2}, \sigma_{\gamma+1}, \alpha_{\gamma+1}\rangle$ is problematic. Setting

$$\mathcal{M}^{\mathcal{T}}_{\gamma+2} = \mathcal{M}^{\mathcal{T}}_{\gamma+1} \text{ and } \pi_{\gamma+2} = \pi_{\gamma+1} \circ \sigma_{\gamma+1},$$

the rest of $(\dagger)_{\gamma+2}$ is clear.

Case 2. i is discontinuous at $\rho_k(G)$.

Set $\kappa = \mathrm{crt}(E^{\mathcal{S}}_\gamma)$. In case 2, $\rho_k(G)$ has cofinality κ in N. Since $\rho(G) \leq \alpha$ and G is α-sound, we have a Σ_1 over G^k map of α onto $(\alpha^+)^G$. Ramifying this map, we see that $(\alpha^+)^G$ also has cofinality κ in N.

Let $\langle (\sigma^\eta, G^\eta) \mid \alpha \leq \eta < \rho_k(G)\rangle$ be the natural decomposition of $\sigma \restriction G^k$.[21] Let $s = p(G) - \alpha$, so that

$$\mathrm{dom}(\sigma^\eta) = h^1_{G^\eta}``(\alpha \cup s).$$

Let

$$\bar{\tau} = \bigcup_{\eta<\rho_k(G)} i(\sigma^\eta).$$

The domain of $\bar{\tau}$ is no longer all of $i(G)^k$, instead

$$\mathrm{dom}(\bar{\tau}) = \bigcup_{\eta<\rho_k(G)} h^1_{i(G^\eta)}``(i(\alpha) \cup i(s)).$$

But set

$$J = \mathrm{Ult}(G, E_i \restriction \sup i``\alpha),$$

and let $t\colon G \to J$ be the canonical embedding, and $v\colon J \to i(G)$ be the factor map. This is a Σ_k ultrapower, by what may be a long extender. That is, J is the decoding of $J^k = \mathrm{Ult}_0(G^k, E_i \restriction$

[21]We encourage the reader to focus on the case $k = 0$, which has the main ideas.

$$\begin{tikzcd}
 & i(G) \arrow[r, "\bar{\tau}"] & i(N) \\
J \arrow[ur, "v"] & G \arrow[l, "t"] \arrow[u, "i"] \arrow[r, "\sigma"] & N \arrow[u, "i"']
\end{tikzcd}$$

Figure 1: Lift up of (N, G, σ, α) in the case i is discontinuous at $\rho_k(G)$

$\sup i``\alpha)$. Note that t is continuous at α, because α is regular in G (because $\alpha = \mathrm{crt}(\sigma)$), and $\alpha < \rho_k(G)$.

We claim that $\mathrm{ran}(v \restriction J^k) \subseteq \mathrm{dom}(\bar{\tau})$. For let $f \in G^k$ and $b \subseteq \sup i``\alpha$ be finite, so that $t(f)(b)$ is a typical element of $J^k$, and $v(t(f)(b)) = i(f)(b)$. We can find $\eta < \rho_k(G)$ such that $f \in \mathrm{dom}(\sigma^\eta)$ and $\eta > \alpha$, so that $i(f) \in \mathrm{dom}(i(\sigma^\eta))$ and $b \subseteq i(\eta)$. Since $f``(\alpha) \subseteq \mathrm{dom}(\sigma^\eta)$, $i(f)``i(\alpha) \subseteq \mathrm{dom}(i(\sigma^\eta))$, so $i(f)(b) \in \mathrm{dom}(\bar{\tau})$, as desired.

Let τ be the extension of $\bar{\tau}$ given by: for $a \subseteq \sup i``\rho_k(G)$ finite,

$$\tau(h^{k+1}_{i(G)}(a, p_k(i(G))) = h^{k+1}_{i(N)}(\bar{\tau}(a), p_k(i(N))).$$

It is easy to check that $\mathrm{ran}(v) \subseteq \mathrm{dom}(\tau)$.

This gives us the diagram in Figure 1.

The map τ here is only partial on $i(G)$, but $\tau \circ v \colon J \to i(N)$ is total. Also, $i``G \subseteq \mathrm{dom}(\tau)$, so $\tau \circ i$ is total on G. For each $\eta < \rho_k(G)$, and $x \in \mathrm{dom}(\sigma^\eta)$,

$$i \circ \sigma^\eta(x) = i(\sigma^\eta)(i(x)),$$

so $\tau \circ i$ agrees on G^k with $i \circ \sigma$. Since both map $p_k(G)$ to $p_k(i(N))$,

$$\tau \circ i = i \circ \sigma.$$

Clearly $i \restriction G = v \circ t$, so the diagram commutes.

Since $E_i \restriction \sup i``\alpha$ is a long extender, we need some care to show that J is a premouse. The worry is that it could be a protomouse, in the the case that G is extender active and $k = 0$. So suppose $k = 0$ and $\mu = \mathrm{crt}(\dot{F}^G)$; it is enough to see that t is continuous at $\mu^{+,G}$. If not, we have $f \in G$ and $b \subseteq \sup i``\alpha$ finite such that

$$\sup i``\mu^{+,G} < t(f)(b) < i(\mu^{+,G}).$$

We may assume $\mathrm{dom}(f) = \gamma^{|b|}$, where $\gamma < \alpha$, and by Los, $\mathrm{ran}(f)$ is unbounded in $\mu^{+,G}$. It follows that $\mu^{+,G} < \alpha$. But $\operatorname{cof}(\mu^{+,G}) = \operatorname{cof}(\hat{o}(G)) = \kappa$ in N, so $\mu^{+,G}$ is not a cardinal in N, so $\mu^{+,G} < \alpha$ is ruled out by $\sigma \restriction \alpha$ being the identity.

Thus J is a premouse. We claim that $\tau \circ v$ is weakly elementary. First, $\bar{\tau}$ is a partial Σ_0 map from $i(G)^k$ to $i(N)^k$, so $\tau \circ v \restriction J^k$ is Σ_0 from J^k to $i(N)^k$.

The diagram shows that

$$\tau \circ v(p_k(J)) = \tau \circ v \circ t(p_k(G)) = i \circ \sigma(p_k(G)) = p_k(i(N)).$$

For $\eta < \rho_k(G)$, we have that σ^η is the identity on $\alpha \cap \eta$, so

$$\sup i``\alpha = \sup t``\alpha \leq \operatorname{crt}(\tau \circ v).$$

But $\alpha < \sigma(\alpha)$, so $i(\alpha) < i \circ \sigma(\alpha) = \tau \circ v \circ t(\alpha)$. Also, $t(\alpha) \leq i(\alpha)$, so $t(\alpha) < \tau \circ v(t(\alpha))$. Thus $\operatorname{crt}(\tau \circ v) \leq t(\alpha)$, and since $t(\alpha) = \sup t``\alpha$, we get

$$\operatorname{crt}(\tau \circ v) = \sup i``\alpha = t(\alpha).$$

We set

$$\begin{aligned} \mathcal{M}^{\mathcal{S}}_{\gamma+2} &= J, \\ \sigma_{\gamma+1} &= \tau \circ v, \text{ and} \\ \alpha_{\gamma+1} &= \operatorname{crt}(\tau \circ v). \end{aligned}$$

We need to see that $\Phi_{\gamma+1}$ is problematic.

Claim.

(a) If Φ^-_ξ is problematic, then $\Phi^-_{\gamma+1}$ is problematic.

(b) Φ_ξ is extender-active iff $\Phi_{\gamma+1}$ is extender-active; moreover, if Φ_ξ is extender-active, then $i^{\mathcal{S}}_{\xi,\gamma+1}(\alpha_\xi) = \alpha_{\gamma+1}$.

Proof. We write

$$\Phi^-_{\gamma+1} = \langle i(N), J, \tau \circ v, \operatorname{crt}(\tau \circ v)\rangle.$$

The reader can easily check that the tuple obeys the hypotheses of 3.7. Clearly J is Σ_{k+1} generated by $\sup i``\alpha \cup t(s)$, and $\operatorname{crt}(\tau \circ v) \geq \sup i``\alpha$. t preserves the solidity of s, so $t(s) = p(J) - \sup i``\alpha$. We have shown that $\tau \circ v$ is weakly elementary. Since $i(G) \in i(N)$, we have $J \in i(N)$, so J is not the $\operatorname{crt}(\tau \circ v)$-core of $i(N)$.

So what we need to see is that none of the conclusions (a)-(c) of 3.7 hold for $\langle i(N), J, \tau \circ v, \operatorname{crt}(\tau \circ v\rangle$. We break into two cases.

Case A. Φ_ξ is not extender-active.

We have $E^N_\alpha = \emptyset$. Since $(N, \Psi) \leq^* (\mathcal{M}^{\mathcal{T}}_\xi, \Sigma^{\mathcal{T}}_\xi) \leq^* (M, \Sigma)$, claim 3 gives

$$G|(\alpha^+)^G = N||(\alpha^+)^G.$$

Since $G \in N$, there is a first level P of N such that $P||(\alpha^+)^G = G|(\alpha^+)^G$ and $\rho(P) \leq \alpha$. Because our tuple is problematic, $P \neq G$. Letting $n = k(P)$, we get by the argument above that in N, $\rho_n(P)$ has the same cofinality as $(\alpha^+)^P = (\alpha^+)^G$, namely κ.

We set

$$Q = \mathrm{Ult}(P, E_i \restriction \sup i``\alpha),$$

and let $t_0 \colon P \to Q$ be the canonical embedding, and $v_0 \colon Q \to i(P)$ be the factor map. Again, we must see that Q is not a protomouse, in the case P is extender active with $\mathrm{crt}(\dot{F}^P) = \mu$, and $n = 0$. If $\alpha \leq \mu^{+,P}$, this follows as above. If $\mu^{+,P} < \alpha$, then because $P|\alpha^{+,P} = G|\alpha^{+,G}$, $\mu^{+,P}$ is a cardinal of G, and hence of N, so i is continuous at $\mu^{+,P}$, as desired.

It is easy to check that the hypotheses of 3.7 hold for $\langle (i(P), \Omega), (Q, \Omega^{v_0}, v_0, \mathrm{crt}(v_0) \rangle$, where $\Omega = (\Lambda_{\gamma+1})_{i(P)}$. Note here that

$$\sup i``\alpha = \mathrm{crt}(v_0) = t_0(\alpha),$$

because t_0 is continuous at α and i is not. Let $s = p(P) \setminus \alpha$; then $P = h_P^{n+1}``\alpha \cup s$ because P is sound and $\rho(P) \leq \alpha$. Thus

$$Q = h_Q^{n+1}``(\sup i``\alpha \cup t_0(s)).$$

Moreover, t_0 maps the solidity witnesses for s to solidity witnesses for $t_0(s)$, so

$$Q \text{ is } \mathrm{crt}(v_0)\text{-sound},$$

with parameter $t_0(s) \setminus \mathrm{crt}(v_0)$.[22] Also,

$$\rho(Q) \leq \sup i``\alpha = \mathrm{crt}(v_0) < t_0(\alpha^{+,P}) \leq \rho_n(Q),$$

where the last inequality comes from $\rho_n(Q) = \sup t_0``\rho_n(P)$ and the fact that t_0 is continuous at $\alpha^{+,P}$. It is easy to verify that v_0 is weakly elementary. Finally, $i(P)$ is sound, so Q cannot be its $\mathrm{crt}(v_0)$-core.

Thus the hypotheses of 3.7 hold for $\langle (i(P), \Omega), (Q, \Omega^{v_0}, v_0, \mathrm{crt}(v_0) \rangle$. But note

$$(i(P), \Omega)) \lhd (i(N), \Lambda_{\gamma+1}) \leq^* (\mathcal{M}^{\mathcal{T}}_{\gamma+1}, \Sigma^{\mathcal{T}}_{\gamma+1}) \leq^* (M, \Sigma).$$

So because (M, Σ) is minimal, one of the conclusions of 3.7 holds, and Q is an initial segment of $i(P)$, or an ultrapower away. (One can argue that $Q \lhd i(P)$, but we don't need this.) However, J and Q agree to $t(\alpha^{+,G}) = t_0(\alpha^{+,P})$, and both project to $\sup i``\alpha$ or below. So if $(i(N), J, \tau \circ v, \mathrm{crt}(\tau \circ v))$ is not problematic, then

$$J = Q.$$

This implies that $k(J) = k(Q)$, and $t(s) = t_0(s_0)$, where $s_0 = p(P) - \alpha$. But $t \restriction \alpha = i \restriction \alpha = t_0 \restriction \alpha$. It follows at once that $G = P$. So $G \lhd N$, and Φ^-_ξ is not problematic, contradiction. This gives (a) of the claim.

[22] Q may not be sound; this happens if $\rho(P) \leq \kappa$.

For (b), we must see that $\mathrm{crt}(\tau \circ v)$ is not an index in $i(N)$. There are two cases. If i is continuous at α, then $\mathrm{crt}(v) > i(\alpha)$ and $\mathrm{crt}(\tau) = i(\alpha)$, so $\mathrm{crt}(\tau \circ v) = i(\alpha)$, which is not an index in $i(N)$. Otherwise, $\mathrm{crt}(\tau \circ v) = \mathrm{crt}(v) = \sup i``\alpha$. But then $\sup i``\alpha$ has cofinality κ in $i(N)$, and since κ is a limit cardinal in $i(N)$, it is not the cofinality of the index of a total extender in $i(N)$.

Case B. Φ_ξ is extender-active.

In this case $\sup i``\alpha = i(\alpha)$, because α has cofinality $\mathrm{crt}(E_\alpha^N)^{+,N}$ in N. So $i(\alpha)$ is an index in $i(N)$, say of F. Moreover, $i(\alpha) = \mathrm{crt}(\tau \circ v)$, so we have (b) of the claim.

Let $R = \mathrm{Ult}(N, E_\alpha^N)$. $G|\alpha^{+,G}$ is an initial segment of R by $(*)(N)$. If $\alpha^{+,R} = \alpha^{+,G}$, then $i(\alpha)^{+,J} = i(\alpha)^{+,\mathrm{Ult}(i(N),F)})$, so $\langle i(N), J, \tau \circ v, i(\alpha)\rangle$ is problematic. (Since $\mathrm{crt}(v) > i(\alpha)$, $i(\alpha) = \mathrm{crt}(\tau \circ v)$).) Otherwise, we have a first initial segment P of R past $\alpha^{+,G}$ that projects to or below α. We can now use P just as we did in Case A, thereby proving (a) of the claim.

$\square$

By the last claim, we may assume that Φ_ξ^- and $\Phi_{\gamma+1}^-$ are not problematic. Suppose for example that $G \lhd N$, so that $J \lhd i(N)$. Since Φ_ξ is problematic, $\Lambda_\xi^\sigma \neq (\Lambda_\xi)_G$.. If $\Phi_{\gamma+1}$ is not problematic, then

$$(\Lambda_{\gamma+1})_J = \Lambda_{\gamma+1}^{\tau\circ v}.$$

Because $(i(G), (\Lambda_{\gamma+1})_{i(G)}) <^* (M, \Sigma)$, we also have

$$(\Lambda_{\gamma+1})_{i(G)}^v = (\Lambda_{\gamma+1})_J.$$

By claim 4, $\Lambda_\xi^\sigma = \Lambda_{\gamma+1}^{i\circ\sigma}$. So we can calculate as above

$$\begin{aligned}
\Lambda_\xi^\sigma &= (\Lambda_{\gamma+1})^{i\circ\sigma} \\
&= (\Lambda_{\gamma+1})^{\tau\circ v\circ t} \\
&= (\Lambda_{\gamma+1}^{\tau\circ v})^t \\
&= (\Lambda_{\gamma+1})_J^t \\
&= ((\Lambda_{\gamma+1})_{i(G)}^v)^t \\
&= (\Lambda_{\gamma+1})_{i(G)}^i \\
&= (\Lambda_\xi)_G.
\end{aligned}$$

This is a contradiction. The cases in which G is an ultrapower away from N are similar, so we conclude that $\Phi_{\gamma+1}$ is problematic in all cases.

This finishes the definition of $\Phi_{\gamma+1}$, and the proof that it is a problematic tuple. We have also proved (4) of $(\dagger)_{\gamma+2}$. We now verify the rest of $(\dagger)_{\gamma+2}$.

For the first part of (5), note that if i is discontinuous at α, then $\sup i``\alpha = \mathrm{crt}(v) = \mathrm{crt}(\tau \circ$

$v) < i(\alpha)$, and if i is continuous at α, then $\mathrm{crt}(\tau) = i(\alpha) = \mathrm{crt}(\tau \circ v)$. Thus in either case, $\alpha_{\gamma+1} \leq i^{\mathcal{S}}_{\xi,\gamma+1}(\alpha_\xi)$.

For the rest of (5) and (6), it is enough to see that $\beta_{\gamma+1} < i(\beta)$, where $\beta = \beta_\xi = (\alpha^+)^G$. If i is discontinuous at α, then α is a limit cardinal of G, and $\beta_{\gamma+1} = (\sup i``\alpha)^{+,J} < i(\alpha) < i(\beta)$, as desired. If i is continuous at α, then since and $(\alpha^+)^G$ has cofinality κ in N, we get

$$(\alpha_{\gamma+1})^{+,J} \leq i(\alpha)^{+,J} = \sup i``\beta < i(\beta),$$

as desired.

(7) of $(\dagger)_{\gamma+2}$ is obvious from our definitions.

Remark 3.12. If Case 2 occurs in the passage from $\Phi_\xi = \langle N, G, \sigma, \alpha\rangle$ to to $\Phi_{\gamma+1} = \langle i(N), J), \tau \circ v, \mathrm{crt}(\tau \circ v)\rangle$, then $\rho_k(J) = \sup t``\rho_k(G)$ has cofinality κ in $i(N)$, where $\kappa = \mathrm{crt}(E^{\mathcal{S}}_\gamma)$. Along branches of $\mathcal{S}$ containing $\gamma+1$, κ can no longer be a critical point. It follows that along any given branch, Case 2 can occur at most once.

If (I) or (II) hold at $\gamma+2$, then the construction of $\mathcal{S}$ is over. Otherwise, we let $E^{\mathcal{S}}_{\gamma+2}$ be the least disagreement between $\mathcal{M}^{\mathcal{S}}_{\gamma+2}$ and $M_{\nu,l}$, and we set

$$\lambda_{\gamma+1} = \inf(\alpha_{\gamma+1}, \lambda(E^{\mathcal{S}}_{\gamma+2})).$$

This completes the successor step in the construction of $\mathcal{S}$.

Now suppose we are given $\mathcal{S} \restriction \theta$, where θ is a limit ordinal. Let $b = \Sigma(\mathcal{T} \restriction \theta)$.

Case 1. There is a largest $\eta \in b$ such that η is unstable.

Fix η. There are two subcases.

(A) for all $\gamma \in b - (\eta+1)$, $\mathrm{rt}(\gamma) = \eta+1$. In this case, $b - (\eta+1)$ is a branch of $\mathcal{S}$. Let $\mathcal{S}$ choose this branch,

$$[\eta+1, \theta)_{\mathcal{S}} = b - (\eta+1),$$

and let $\mathcal{M}^{\mathcal{S}}_\theta$ be the direct limit of the $\mathcal{M}^{\mathcal{S}}_\gamma$ for sufficiently large $\gamma \in b - (\eta+1)$. We define the branch embedding $i^{\mathcal{S}}_{\gamma,\theta}$ a usual and $\pi_\theta : \mathcal{M}^{\mathcal{S}}_\theta \to \mathcal{M}^{\mathcal{T}}_\theta$ is given by the fact that the copy maps commute with the branch embeddings. We declare θ to be stable.

(B) for all $\gamma \in b - (\eta+1)$, $\mathrm{rt}(\gamma) = \eta$. Let $\mathcal{S}$ choose

$$[0, \theta)_{\mathcal{S}} = (b - \eta) \cup [0, \eta]_{\mathcal{S}},$$

and let $\mathcal{M}^{\mathcal{S}}_\theta$ be the direct limit of the $\mathcal{M}^{\mathcal{S}}_\gamma$ for sufficiently large $\gamma \in b$. Branch embeddings $i^{\mathcal{S}}_{\gamma,\theta}$ for $\gamma \geq \eta$ are defined as usual. $\pi_\theta : \mathcal{M}^{\mathcal{S}}_\theta \to \mathcal{M}^{\mathcal{T}}_\theta$ is given by the fact that copy maps commute with branch embeddings. We declare θ to be stable.

Since θ is stable, $(\dagger)_\theta$ follows at once from $\forall\gamma < \theta(\dagger)_\gamma$.

Case 2. There are boundedly many unstable ordinals in b but no largest one.

We let η be the sup of the unstable ordinals in b. Let $\mathcal{S}$ choose

$$[0, \theta)_{\mathcal{S}} = (b - \eta) \cup [0, \eta]_{\mathcal{S}},$$

and define the corresponding objects as in case 1(B). We declare θ stable, and again $(\dagger)_\theta$ is immediate.

Case 3. There are arbitrarily large unstable ordinals in b. In this case, b is a disjoint union of pairs $\{\gamma, \gamma + 1\}$ such that γ is unstable and $\gamma + 1$ is stable. We set

$$[0, \theta)_{\mathcal{S}} = \{\xi \in b \mid \xi \text{ is unstable}\},$$

and let $\mathcal{M}_\theta^{\mathcal{S}}$ be the direct limit of the $\mathcal{M}_\xi^{\mathcal{S}}$'s for $\xi \in b$ unstable. There is no dropping in model or degree along $[0, \theta)_{\mathcal{S}}$. We define maps $i^{\mathcal{S}}_{\xi,\theta}, \pi_\theta$ as usual. If $(\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta)$ is not a pair of the form $(\mathcal{M}_\tau^{\mathcal{U}}, \Sigma_\tau^{\mathcal{U}})$, then we declare θ stable and $(\dagger)_\theta$ is immediate.

Suppose that $(\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta)$ is a pair of $\mathcal{U}$. We declare θ unstable. Note that by clauses (4) and (5) of $(\dagger)$, there is a $\xi <_S \theta$ such that for all γ with $\xi <_S \gamma <_S \theta$, $i^{\mathcal{S}}_{\xi,\gamma}(\langle \alpha_\xi, \beta_\xi \rangle) = \langle \alpha_\gamma, \beta_\gamma \rangle$. So we can set

$$\alpha_\theta = \text{ common value of } i^{\mathcal{S}}_{\gamma,\theta}(\alpha_\gamma), \text{ for } \gamma <_S \theta \text{ sufficiently large.}$$

By clause (5), we can set

$$\mathcal{M}^{\mathcal{S}}_{\theta+1} = \text{ common value of } i^{\mathcal{S}}_{\gamma,\theta}(\mathcal{M}^{\mathcal{S}}_{\gamma+1}), \text{ for } \gamma <_S \theta \text{ sufficiently large.}$$

We also let

$$\sigma_\theta = \text{ common value of } i^{\mathcal{S}}_{\gamma,\theta}(\sigma_\gamma), \text{ for } \gamma <_S \theta \text{ sufficiently large.}$$

Here

$$i^{\mathcal{S}}_{\gamma,\theta}(\sigma_\gamma) = \text{ upward extension of } \bigcup_{\eta<\rho} i^{\mathcal{S}}_{\gamma,\theta}(\sigma^\eta_\gamma),$$

where $\rho = \rho_k(\mathcal{M}^{\mathcal{S}}_{\gamma+1})$ for $k = k(\mathcal{M}^{\mathcal{S}}_{\gamma+1})$, and the σ^η_γ are the terms in the natural decomposition of σ_γ. By (5) of $(\dagger)$, $i^{\mathcal{S}}_{\gamma,\theta}$ is continuous at $\rho_k(\mathcal{M}_{\gamma+1})$ for $\gamma <_S \theta$ sufficiently large, so σ_θ is defined on all of $\mathcal{M}^{\mathcal{S}}_{\theta+1}$. It is easy then to see that $\Phi_\theta = \langle (\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta), (\mathcal{M}^{\mathcal{S}}_{\theta+1}, \Lambda_{\theta+1}), \sigma_\theta, \alpha_\theta \rangle$ is a problematic tuple.

If (I) holds, then we stop the construction of $\mathcal{S} = \mathcal{S}_{\nu,l}$ and move on to $\mathcal{S}_{\nu,l+1}$. If (II) holds, we stop the construction of $\mathcal{S}$ and do not move on. If neither holds, we let $E^{\mathcal{S}}_{\theta+1}$ be the extender on the $\mathcal{M}^{\mathcal{S}}_{\theta+1}$ sequence that represents its first disagreement with $M_{\nu,l}$, and set

$$\lambda_{\theta+1} = \lambda(E^{\mathcal{S}}_{\theta+1}),$$

and

$$\lambda_\theta = \inf(\lambda_{\theta+1}, \alpha_\theta).$$

It then is routine to verify $(\dagger)_{\theta+1}$.

This finishes our construction of $\mathcal{S} = \mathcal{S}_{\nu,l}$ and $\mathcal{T}$. Note that every extender used in $\mathcal{S}$ is taken from a stable node and every stable node, except the last model of $\mathcal{S}$ contributes exactly one extender to $\mathcal{S}$. The last model of $\mathcal{S}$ is stable.

Claim 4. The construction of $\mathcal{S}_{\nu,l}$ stops for one of the reasons (I) and (II).

Proof sketch. This follows at once from the fact that in the comparison above, no strategy disagreements show up, and $M_{\nu,l}$ never moves. That in turn can be shown by the same method by which the other results on comparing phalanxes with background constructions are proved in [10]. See [10, $\mathcal{S}$ 6.2].

$\square$

Claim 5. For some $(\nu, l) \leq (\eta_0, k_0)$, the construction of $\mathcal{S}_{\nu,l}$ stops for reason (II).

Proof. If not, then the construction of $\mathcal{S} = \mathcal{S}_{\eta_0,k_0}$ must reach some $\mathcal{M}_\theta^{\mathcal{S}}$ such that $(M_{\eta_0,k_0}, \Omega_{\eta_0,k_0})$ is a proper initial segment of $(\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta)$. Let $j : M \to M_{\eta_0,k_0}$ be the iteration map given by $\mathcal{U}_{\eta_0,k_0}$. (Note that by the definition of (η_0, k_0), M_{η_0,k_0} is a nondropping iterate of M.). Letting $\mathcal{T} = \mathcal{T}_{\eta_0,k_0}$, we have $\pi_\theta : \mathcal{M}_\theta^{\mathcal{S}} \to \mathcal{M}_\theta^{\mathcal{T}}$ from the copying construction. Let

$$Q = \pi_\theta(M_{\eta_0,k_0});$$

then

$$\pi_\theta \circ j \colon (M, \Sigma) \to (Q, (\Sigma_\theta^{\mathcal{T}})_Q)$$

is an elementary map, because

$$\Sigma = \Omega_{\eta_0,k_0}^j = ((\Lambda_\theta)_{M_{\eta_0,k_0}})^j = ((\Sigma_\theta^{\mathcal{T}})_Q)^{\pi_\theta \circ j}.$$

Thus $\pi_\theta \circ j$ maps (M, Σ) into a proper initial segment of $(\mathcal{M}_\theta^{\mathcal{T}}, \Sigma_\theta^{\mathcal{T}})$, contrary to the Dodd-Jensen Theorem. $\square$

By Claim 5, we may fix $(\nu, l) \leq (\eta_0, k_0)$ such that the construction of $\mathcal{S} = \mathcal{S}_{\nu,l}$ terminates at a stable θ such that for some γ, $\mathcal{M}_\theta^{\mathcal{S}} \trianglelefteq \mathcal{M}_\gamma^{\mathcal{U}_{\nu,l}}$. Let $\mathcal{S} = \mathcal{S}_{\nu,l}$, $\mathcal{U} = \mathcal{U}_{\nu,l}$, and let γ be the least such that $\mathcal{M}_\theta^{\mathcal{S}} \trianglelefteq \mathcal{M}_\gamma^{\mathcal{U}}$. We have $\mathrm{lh}(\mathcal{S}) = \theta + 1$, and $[\mathrm{rt}(\theta), \theta]_{\mathcal{S}}$ does not drop in model or degree. If $0 \leq_{\mathcal{S}} \theta$, then $[0, \theta]_{\mathcal{S}}$ does not drop in model or degree.

Claim 6. For some unstable ξ, $\mathrm{rt}(\theta) = \xi + 1$.

Proof. If not, then $0 \leq_{\mathcal{S}} \theta$ and the branch $[0, \theta]_{\mathcal{S}}$ does not drop.

We claim that $(\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta) = (\mathcal{M}_\gamma^{\mathcal{U}}, \Sigma_\gamma^{\mathcal{U}})$. For otherwise $i_{0,\theta}^{\mathcal{S}}$ maps (M, Σ) to a proper initial segment of a Σ-iterate of (M, Σ), contrary to the Dodd-Jensen Theorem. (Note here that Σ is the pullback of Λ_θ under $i_{0,\theta}^{\mathcal{S}}$, by Claim 4.) For the same reason, $[0, \gamma]_U$ does not drop. We also get by a standard Dodd-Jensen argument that

$$i_{0,\theta}^{\mathcal{S}} = i_{0,\gamma}^{\mathcal{U}}.$$

To see this, note that both are elementary maps from (M, Σ) to $(\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta) = (\mathcal{M}^{\mathcal{U}}_\gamma, \Sigma^{\mathcal{U}}_\gamma)$. Since $i^{\mathcal{U}}_{0,\gamma}$ is an iteration map, for all ξ

$$i^{\mathcal{U}}_{0,\gamma}(\xi) \le i^{\mathcal{S}}_{0,\eta}(\xi).$$

Since $i^{\mathcal{T}}_{0,\theta}$ is also an iteration map, for all ξ

$$i^{\mathcal{T}}_{0,\eta}(\xi) = \pi_\eta \circ i^{\mathcal{S}}_{0,\eta}(\xi) \le \pi_\eta \circ i^{\mathcal{U}}_{0,\tau}(\xi).$$

Multiplying by π_η^{-1}, we get that $i^{\mathcal{S}}_{0,\eta}(\xi) \le i^{\mathcal{U}}_{0,\gamma}(\xi)$ for all ξ. So $i^{\mathcal{S}}_{0,\eta} = i^{\mathcal{U}}_{0,\tau}$.

Since we can recover branch extenders from branch embeddings, we get then that

$$s^{\mathcal{S}}_\theta = s^{\mathcal{U}}_\gamma.^{23}$$

Let $\eta \le_{\mathcal{S}} \theta$ be least such that η is stable. Then $s^{\mathcal{S}}_\eta = s^{\mathcal{S}}_\theta \restriction \delta = s^{\mathcal{U}}_\gamma \restriction \delta$ for some δ. But there is τ such that $s^{\mathcal{U}}_\tau = s^{\mathcal{U}}_\delta \restriction \delta$. Thus $\mathcal{M}^{\mathcal{S}}_\eta = \mathcal{M}^{\mathcal{U}}_\tau$. We have also

$$\Lambda_\eta = \Lambda_\theta^{i^{\mathcal{S}}_{\eta,\theta}} = (\Sigma^{\mathcal{U}}_\gamma)^{i^{\mathcal{U}}_{\tau,\gamma}} = \Sigma^{\mathcal{U}}_\tau,$$

by pullback consistency, since $i^{\mathcal{S}}_{\eta,\theta} = i^{\mathcal{U}}_{\tau,\gamma}$.

If η is a limit ordinal, then by the rules at limit stages of $\mathcal{S}$ above, we declare η unstable. This contradicts our assumption. If $\mathcal{S}$-pred$(\eta) = \mu$, then μ is unstable by our minimality assumption on η; but then we declare η unstable by our rules at successor stages. Again, we reach a contradiction. □

Let ξ be as in Claim 6, and let τ be such that $(\mathcal{M}^{\mathcal{S}}_\xi, \Lambda_\xi) = (\mathcal{M}^{\mathcal{U}}_\tau, \Sigma^{\mathcal{U}}_\tau)$. We have $\mathcal{M}^{\mathcal{S}}_\xi = \mathcal{M}^{\mathcal{U}}_\tau$, so $s^{\mathcal{S}}_\xi = s^{\mathcal{U}}_\tau$ by the proof in claim 6.

Claim 7. $\tau < \gamma$.

Proof. We show first $\tau \le \gamma$. Let

$$\lambda = \sup_i(\mathrm{lh}(s^{\mathcal{S}}_\xi(i))) = \sup_i(\mathrm{lh}(s^{\mathcal{U}}_\tau(i)).$$

We have that $(M^{\mathcal{S}}_\xi, \Lambda_\xi)$, $(\mathcal{M}^{\mathcal{S}}_{\xi+1}, \Lambda_{\xi+1})$, and $(\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta)$ all agree below λ.(Note that $\lambda \le \alpha_\xi$.) However, if $\beta < \tau$, then $\mathcal{M}^{\mathcal{U}}_\beta$ disagrees with $\mathcal{M}^{\mathcal{U}}_\tau$ below λ. Thus $\tau \le \gamma$.

Now suppose $\gamma = \tau$. If $\theta = \xi + 1$, then $(\mathcal{M}^{\mathcal{S}}_{\xi+1}, \Lambda_{\xi+1}) \trianglelefteq (\mathcal{M}^{\mathcal{U}}_\tau, \Sigma^{\mathcal{U}}_\tau) = (\mathcal{M}^{\mathcal{S}}_\xi, \Lambda_\xi)$, so Φ_ξ is not problematic, contradiction. If $\theta > \xi + 1$, then $\mathcal{M}^{\mathcal{S}}_\theta$ is not α_ξ-sound. Since $\mathcal{M}^{\mathcal{S}}_\theta \trianglelefteq \mathcal{M}^{\mathcal{U}}_\gamma$, we must have $\mathcal{M}^{\mathcal{S}}_\theta = \mathcal{M}^{\mathcal{U}}_\tau$. However, $\mathcal{M}^{\mathcal{S}}_{\xi+1}$ belongs to $\mathcal{M}^{\mathcal{S}}_\xi = \mathcal{M}^{\mathcal{U}}_\tau$ because Φ_ξ is problematic, and clearly $\mathcal{M}^{\mathcal{S}}_{\xi+1} \notin \mathcal{M}^{\mathcal{S}}_\theta$, again a contradiction. Thus $\gamma \ne \tau$. □

Note that in fact $(\mathcal{M}^{\mathcal{S}}_\xi, \Lambda_\xi)$, $(\mathcal{M}^{\mathcal{S}}_{\xi+1}, \Lambda_{\xi+1})$, and $(\mathcal{M}^{\mathcal{U}}_\tau, \Sigma^{\mathcal{U}}_\tau)$ all agree with $M_{\nu,l}$ below α_ξ. (Possibly not at α_ξ.) This is because otherwise $\lambda_\xi < \alpha_\xi$, and $\xi + 1$ is a dead node in $\mathcal{S}$.

Claim 8. $(\mathcal{M}^{\mathcal{S}}_\theta, \Lambda_\theta) = (\mathcal{M}^{\mathcal{U}}_\gamma, \Sigma^{\mathcal{U}}_\gamma)$.

[23] $s^{\mathcal{S}}_\theta$ is the sequence of extenders used along the branch $[0, \theta]_{\mathcal{S}}$ and similarly for $s^{\mathcal{U}}_\gamma$.

$$
\begin{tikzcd}
M_\theta^{\mathcal{T}} & M_\theta^{\mathcal{S}} \arrow[l, "\pi_\theta"'] & & M_\gamma^{\mathcal{U}} \\
& & & M_{\tau+1}^{\mathcal{U}} \arrow[u, "i^{\mathcal{U}}_{\tau+1,\gamma}"'] \\
& M_{\xi+1}^{\mathcal{S}} \arrow[uu, "i^{\mathcal{S}}_{\xi+1,\theta}"] \arrow[r, "\sigma_\xi"] & M_\xi^{\mathcal{S}} = M_\tau^{\mathcal{U}} & \\
& & & N \trianglelefteq M_\eta^{\mathcal{U}} \arrow[ul, dashed] \arrow[uu, "E_\tau^{\mathcal{U}}"'] \\
M & H \arrow[l, "\pi_0"] \arrow[r, "\sigma_0"] & M \arrow[uu, "i^{\mathcal{S}}_{0,\xi}"] & M
\end{tikzcd}
$$

Figure 2: Phalanx Comparison

Proof. Otherwise $(\mathcal{M}_\theta^{\mathcal{S}}, \Lambda_\theta) \lhd (\mathcal{M}_\gamma^{\mathcal{U}}, \Sigma_\gamma^{\mathcal{U}})$, so $\mathcal{M}_\theta^{\mathcal{S}}$ is sound, and thus $\theta = \xi + 1$. $\rho(\mathcal{M}_{\xi+1}^{\mathcal{S}}) \leq \alpha_\xi$, so $o(\mathcal{M}_{\xi+1}^{\mathcal{S}}) < (\alpha_\xi^+)^{\mathcal{M}_\gamma^{\mathcal{U}}}$.

Suppose first $\beta = \mathrm{lh}(E_\tau^{\mathcal{U}}) > \alpha_\xi$. Then $(\mathcal{M}_\tau^{\mathcal{U}}, \Sigma_\tau^{\mathcal{U}})$ agrees with $(\mathcal{M}_\gamma^{\mathcal{U}}, \Sigma_\gamma^{\mathcal{U}})$ below β, and $\beta \geq (\alpha_\xi^+)^{\mathcal{M}_\gamma^{\mathcal{U}}}$, so $(\mathcal{M}_{\xi+1}^{\mathcal{S}}, \Lambda_{\xi+1}) \lhd (\mathcal{M}_\tau^{\mathcal{U}}, \Sigma_\tau^{\mathcal{U}}) = (\mathcal{M}_\xi^{\mathcal{S}}, \Lambda_\xi)$. It follows that Φ_ξ is not problematic.

Suppose $\mathrm{lh}(E_\tau^{\mathcal{U}}) = \alpha_\xi$. Let us write

$$F = E_\tau^{\mathcal{U}} = E_{\alpha_\xi}^{\mathcal{M}_\tau^{\mathcal{U}}} = E_{\alpha_\xi}^{\mathcal{M}_\xi^{\mathcal{S}}}.$$

We have $(M_{\xi+1}^{\mathcal{S}}, \Lambda_{\xi+1}) \lhd (\mathcal{M}_{\tau+1}^{\mathcal{U}}, \Sigma_{\tau+1}^{\mathcal{U}})$, because $(\mathcal{M}_{\tau+1}^{\mathcal{U}}, \Sigma_{\tau+1}^{\mathcal{U}})$ agrees sufficiently with $(\mathcal{M}_\gamma^{\mathcal{U}}, \Sigma_\gamma^{\mathcal{U}})$. Thus $\gamma = \tau + 1$ and $\theta = \xi + 1$. Let $\kappa = \mathrm{crt}(F)$ and $\mu = \lambda(F)$. Since $\sigma_\xi(\mu) = \mu$, μ is a cardinal of $\mathcal{M}_\xi^{\mathcal{S}}$, and F is total on $\mathcal{M}_\xi^{\mathcal{S}}$. We shall show that $(\mathcal{M}_{\xi+1}^{\mathcal{S}}, \Lambda_{\xi+1}) \lhd \mathrm{Ult}_0((\mathcal{M}_\xi^{\mathcal{S}}, \Lambda_\xi), F)$, so that Φ_ξ is not problematic. Note that $\mathrm{Ult}_0((\mathcal{M}_\xi^{\mathcal{S}}, \Lambda_\xi), F) = \mathrm{Ult}_0((\mathcal{M}_\tau^{\mathcal{U}}, \Sigma_\tau^{\mathcal{U}}), F)$.

Let $\eta = \mathcal{U}\text{-pred}(\tau + 1)$. By (4) of $(\dagger)_\xi$, $\alpha_\xi = i^{\mathcal{S}}_{0,\xi}(\alpha_0) = i^{\mathcal{U}}_{0,\tau}(\alpha_0)$. Thus $\kappa \in \mathrm{ran}(i^{\mathcal{U}}_{0,\tau})$. From this we get that $\eta \leq_U \tau$.

Let $n = k(M) = k(M_\tau^{\mathcal{U}}) = k(M_\eta^{\mathcal{U}})$. We have $\alpha_0 < \rho_n(M)$ by hypothesis, so $\alpha_\xi < \rho_n(M_\tau^{\mathcal{U}})$. If $\eta = \tau$, then $\mathcal{M}_{\tau+1}^{\mathcal{U}} = \mathrm{Ult}_n(\mathcal{M}_\tau^{\mathcal{U}}, F)$ agrees with $\mathrm{Ult}_0(\mathcal{M}_\tau^{\mathcal{U}}, F)$ to $\sup i``\rho_n(\mathcal{M}_\tau^{\mathcal{U}})$, which is well past $\mathrm{lh}(F)^+$ as computed in the ultrapower, so we are done. So assume $\eta <_U \tau$, and let G be the extender applied to $\mathcal{M}_\eta^{\mathcal{U}}$ in $(\eta, \tau]_U$. We must have $\mathrm{crt}(G) < \rho_n(\mathcal{M}_\eta^{\mathcal{U}})$, as otherwise $[0, \tau]_U$ drops. But also $\kappa < \mathrm{crt}(G)$, because $\kappa < \lambda(G)$ by the definition of η, and $\kappa \in \mathrm{ran}(i^{\mathcal{U}}_{0,\tau})$. Thus $(\kappa)^{+++})^{\mathcal{M}_\eta^{\mathcal{U}}} < \mathrm{crt}(G) < \rho_n(\mathcal{M}_\eta^{\mathcal{U}})$. It follows that $\mathrm{Ult}_n(\mathcal{M}_\eta^{\mathcal{U}}, F)$, $\mathrm{Ult}_0(\mathcal{M}_\eta^{\mathcal{U}}, F)$, and $\mathrm{Ult}_0(\mathcal{M}_\tau^{\mathcal{U}}, F)$ all agree to their common value for $\mathrm{lh}(F)^+$. This is what we need.

$\square$

Claim 9. α_ξ is a successor cardinal of $\mathcal{M}_{\xi+1}^{\mathcal{S}}$.

Proof. Suppose not. It follows that α_ξ is a limit cardinal of $\mathcal{M}_\xi^{\mathcal{S}}$, and that $\rho(\mathcal{M}_{\xi+1}^{\mathcal{S}}) = \alpha_\xi$. Thus

$\mathcal{M}^{\mathcal{S}}_{\xi+1}$ is sound, and it is the core of $\mathcal{M}^{\mathcal{S}}_{\theta}$. Moreover, $i^{\mathcal{S}}_{\xi+1,\theta}$ is the uncoring embedding, and

$$\Lambda_{\xi+1} = \Lambda_{\theta}^{i^{\mathcal{S}}_{\xi+1,\theta}}$$

by Claim 4.

So $(\mathcal{M}^{\mathcal{S}}_{\xi+1}, \Lambda_{\xi+1})$ is the core of $(\mathcal{M}^{\mathcal{U}}_{\gamma}, \Sigma^{\mathcal{U}}_{\gamma})$. It follows that there is a $\beta \in [0, \gamma]_U$ such that either $\mathcal{M}^{\mathcal{S}}_{\xi+1} = \mathcal{M}^{\mathcal{U}}_{\beta}$ or $\mathcal{M}^{\mathcal{S}}_{\xi+1} \lhd \mathcal{M}^{\mathcal{U}}_{\beta}$. In either case,

$$\hat{i}_{\beta,\gamma^{\mathcal{U}}} = i^{\mathcal{S}}_{\xi+1,\theta}$$

is again the uncoring map. By pullback consistency in $\mathcal{U}$, setting $Q = \mathcal{M}^{\mathcal{S}}_{\xi+1}$,

$$\begin{aligned}
(\Sigma^{\mathcal{U}}_{\beta})_Q &= (\Sigma^{\mathcal{U}}_{\gamma})^{\hat{i}^{\mathcal{U}}_{\beta,\gamma}} \\
&= \Lambda_{\theta}^{i^{\mathcal{S}}_{\xi+1,\theta}} \\
&= \Lambda_{\xi+1}.
\end{aligned}$$

Thus $(\mathcal{M}^{\mathcal{S}}_{\xi+1}, \Lambda_{\xi+1}) \trianglelefteq (\mathcal{M}^{\mathcal{U}}_{\beta}, \Sigma^{\mathcal{U}}_{\beta})$.

Clearly $\beta \geq \tau$. $\beta = \tau$ is impossible because Φ_ξ is problematic, and $(\mathcal{M}^{\mathcal{U}}_{\tau}, \Sigma^{\mathcal{U}}_{\tau}) = (\mathcal{M}^{\mathcal{S}}_{\xi}, \Lambda_\xi)$. So suppose $\beta > \tau$.

Since α_ξ is a limit cardinal of $\mathcal{M}^{\mathcal{S}}_{\xi}$, $\mathrm{lh}(E^{\mathcal{U}}_{\tau}) > \alpha_\xi$. $\mathrm{lh}(E^{\mathcal{U}}_{\tau})$ is a cardinal of $\mathcal{M}^{\mathcal{U}}_{\beta}$, so if $(\mathcal{M}^{\mathcal{S}}_{\xi+1}, \Lambda_{\xi+1}) \lhd (\mathcal{M}^{\mathcal{U}}_{\beta}, \Sigma^{\mathcal{U}}_{\beta})$, then $(\mathcal{M}^{\mathcal{S}}_{\xi+1}, \Lambda_{\xi+1}) \lhd (\mathcal{M}^{\mathcal{U}}_{\tau}, \Sigma^{\mathcal{U}}_{\tau})$, contrary to Φ_ξ being problematic. So $\mathcal{M}^{\mathcal{S}}_{\xi+1} = \mathcal{M}^{\mathcal{U}}_{\beta}$.

Now let F be the first extender used on the branch $[0, \beta]_U$ such that $\mathrm{lh}(F) > \alpha_\xi$. Since $\rho(\mathcal{M}^{\mathcal{U}}_{\beta}) = \alpha_\xi$, $\mathrm{crt}(F) \geq \alpha_\xi$. But then $\mathcal{M}^{\mathcal{U}}_{\beta} = \mathcal{M}^{\mathcal{S}}_{\xi+1}$ is not α_ξ-sound, contradiction. □

Let μ be the cardinal predecessor of α_ξ in $\mathcal{M}^{\mathcal{S}}_{\xi+1}$, or equivalently, in $M^{\mathcal{S}}_{\xi}$. Let also $\rho = \rho(\mathcal{M}^{\mathcal{S}}_{\xi+1})$. We have $\rho \in \{\mu, \alpha_\xi\}$, and

$$\begin{aligned}
\rho &= \rho(\mathcal{M}^{\mathcal{S}}_{\xi+1}) \\
&= \rho(\mathcal{M}^{\mathcal{S}}_{\theta}) \\
&= \rho(\mathcal{M}^{\mathcal{U}}_{\gamma}).
\end{aligned}$$

Claim 10. $E^{\mathcal{M}^{\mathcal{S}}_{\xi}}_{\alpha_\xi} = \emptyset$.

Proof. Otherwise $E^{\mathcal{M}^{\mathcal{S}}_{\xi}}_{\alpha_\xi} = E^{\mathcal{U}}_{\tau}$, $\mathrm{lh}(E^{\mathcal{U}}_{\tau}) = \alpha_\xi$, and $\mu = \lambda(E^{\mathcal{U}}_{\tau})$.

Let F be the first extender used in $[0, \gamma]_U$ such that $\mathrm{lh}(F) \geq \alpha_\xi$. We claim that $F = E^{\mathcal{U}}_{\tau}$. For if not, then by the rules of normal trees, $\mathrm{crt}(F) < \lambda(E^{\mathcal{U}}_{\tau}) < \lambda(F)$. Since $\rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) \leq \alpha_\xi < \lambda(F)$, we must have $\rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) \leq \mathrm{crt}(F) < \mu$. However,

$$\rho = \rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) = \rho(\mathcal{M}^{\mathcal{S}}_{\theta}) = \rho(\mathcal{M}^{\mathcal{S}}_{\xi+1}) \geq \mu,$$

because μ is a cardinal of $\mathcal{M}^{\mathcal{S}}_{\xi}$ and $\mathcal{M}^{\mathcal{S}}_{\xi+1} \in \mathcal{M}^{\mathcal{S}}_{\xi}$. This is a contradiction, so we have $F = E^{\mathcal{U}}_{\tau}$, and

$\tau + 1 \leq_U \gamma$.

We claim that

$$\rho = \rho(\mathcal{M}^{\mathcal{U}}_{\tau+1}).$$

We remarked above that this holds if $\tau + 1 = \gamma$, so suppose $\tau + 1 <_U \gamma$. Let $\delta = \mathrm{crt}(\hat{i}^{\mathcal{U}}_{\tau+1,\gamma})$, so that $\mu \leq \delta$. Let $Q \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\tau+1}$ be such that $Q = \mathrm{dom}(\hat{i}^{\mathcal{U}}_{\tau+1,\gamma})$. If $\rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) > \delta$, then $\rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) > (\delta^+)^{\mathcal{M}^{\mathcal{U}}_{\gamma}} \geq \alpha_\xi$; thus $\rho \leq \delta$. It follows that $\rho = \rho(Q)$.

If $Q \triangleleft \mathcal{M}^{\mathcal{U}}_{\tau+1}$, then $\rho(Q) \geq \alpha_\xi$ since $\alpha_\xi = \mathrm{lh}(F)$ is a cardinal of $\mathcal{M}^{\mathcal{U}}_{\tau+1}$, so $\rho(Q) = \rho = \alpha_\xi$. It follows that

$$(Q, (\Sigma^{\mathcal{U}}_{\tau+1})_Q) = \alpha_\xi\text{-core of } \mathcal{M}^{\mathcal{U}}_{\gamma} = (\mathcal{M}^{\mathcal{S}}_{\xi+1}, \Lambda_{\xi+1}),$$

so $(\mathcal{M}^{\mathcal{S}}_{\xi+1}, \Lambda_{\xi+1}) \triangleleft (\mathcal{M}^{\mathcal{U}}_{\tau+1}, \Sigma^{\mathcal{U}}_{\tau+1})$, contrary to Φ_ξ being problematic. (As we showed above, $\mathrm{Ult}_0((\mathcal{M}^{\mathcal{U}}_{\tau}, \Sigma^{\mathcal{U}}_{\tau}), F)$ is in sufficient agreement with $(\mathcal{M}^{\mathcal{U}}_{\tau+1}, \Sigma^{\mathcal{U}}_{\tau+1})$ that we can conclude this.) Thus $Q = \mathcal{M}^{\mathcal{U}}_{\tau+1}$, and $\rho = \rho(Q)$.

We cannot have $\rho = \mu$ because $\lambda(E^{\mathcal{U}}_{\tau})$ is not a possible value of $\rho(\mathcal{M}^{\mathcal{U}}_{\tau+1})$, and thus $\rho = \alpha_\xi$. Let η be the U-predecessor of $\tau + 1$ and $\kappa = \mathrm{crt}(F)$. If $[0, \tau + 1]_U$ drops, then $\rho \leq \kappa$, so $[0, \tau + 1]_U$ does not drop. Since $\rho(\mathcal{M}^{\mathcal{U}}_{\tau+1}) = \mathrm{lh}(F)$, $\rho(\mathcal{M}^{\mathcal{U}}_{\eta}) = (\kappa^+)^{\mathcal{M}^{\mathcal{U}}_{\eta}}$. Let

$$Z = \mathrm{Th}^{\mathcal{M}^{\mathcal{U}}_{\eta}}_n((\kappa^+)^{\mathcal{M}^{\mathcal{U}}_{\eta}} \cup r),$$

where $n = k(\mathcal{M}^{\mathcal{U}}_{\eta}) + 1 = k(M) + 1$, and $r = p_n(\mathcal{M}^{\mathcal{U}}_{\eta})$. Z is not in $\mathcal{M}^{\mathcal{U}}_{\eta}$, and hence Z is not in $\mathcal{M}^{\mathcal{U}}_{\tau}$. But Z can be computed inside $\mathcal{M}^{\mathcal{U}}_{\tau}$ from F and $\mathcal{M}^{\mathcal{S}}_{\xi+1}$, both of which belong to $\mathcal{M}^{\mathcal{U}}_{\tau}$. This is because

$$i^{\mathcal{U}}_{\eta,\gamma}(r) = p(\mathcal{M}^{U}_{\gamma}) = i^{S}_{\xi+1,\theta}(t),$$

where $t = p(\mathcal{M}^{\mathcal{S}}_{\xi+1})$[24], and

$$\mathrm{crt}(i_{\tau+1,\gamma}) > \alpha_\xi,$$

because otherwise $\rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) > \alpha_\xi$, so for $\nu < (\kappa^+)^{\mathcal{M}^{\mathcal{U}}_{\eta}}$,

$$\begin{aligned} \langle \varphi, \nu, r \rangle \in Z &\Leftrightarrow \mathcal{M}^{\mathcal{U}}_{\gamma} \models \varphi[i^{\mathcal{U}}_{\eta,\tau+1}(\nu), p(\mathcal{M}^{\mathcal{U}}_{\gamma})] \\ &\Leftrightarrow \mathcal{M}^{\mathcal{S}}_{\xi+1} \models \varphi[i^{\mathcal{U}}_{\eta,\tau+1}(\nu), t]. \end{aligned}$$

Since $i^{\mathcal{U}}_{\eta,\tau+1} \upharpoonright \kappa^+$ can be computed from F, we get $Z \in \mathcal{M}^{\mathcal{U}}_{\tau}$, a contradiction.

This completes the proof of claim 10. □

Claim 11. Let $F = E^{\mathcal{U}}_{\nu}$ be the first extender used in $[0, \gamma]_U$ such that $\mathrm{lh}(F) \geq \alpha_\xi$; then $\mathrm{crt}(F) = \mu$.

Proof. $\mathrm{lh}(F) > \alpha_\xi$ by claim 10, so $\lambda(F) > \alpha_\xi$. ρ is not in the interval $(\mathrm{crt}(F), \lambda(F)]$, so $\mathrm{crt}(F) \geq \mu$. Let $\eta = \mathrm{pred}_U(\nu + 1)$, and let $Q \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\eta}$ be such that

$$\mathcal{M}^{\mathcal{U}}_{\nu+1} = \mathrm{Ult}(Q, F).$$

[24] $\mathcal{M}^{\mathcal{S}}_{\xi+1}$ is sound, because $\rho = \alpha_\xi$.

$$\begin{tikzcd}
Q \arrow[r, "i^{\mathcal{U}}_{\tau,\nu+1}"] \arrow[dr, "i_D"] & \mathcal{M}^{\mathcal{U}}_{\nu+1} \arrow[r, "i^{\mathcal{U}}_{\nu+1,\gamma}"] & \mathcal{M}^{\mathcal{U}}_{\gamma} \\
& Ult(Q,D) \arrow[u, "k"] &
\end{tikzcd}$$

Figure 3: $\mathcal{M}^{\mathcal{S}}_{\xi+1} = \mathrm{Ult}(Q, D)$, where the trivial completion of D is on the Q-sequence

Note that $\eta \leq \tau$, as otherwise some extender G such that $lh(G) \geq \mathrm{lh}(E^{\mathcal{U}}_{\tau}) > \alpha_{\xi}$ is used on $[0, \eta)_U$. If $\eta < \tau$, then $\mathrm{crt}(F) < \lambda(E^{\mathcal{U}}_{\eta}) < \mu$, contradiction. Thus $\eta = \tau$. Note Q, $\mathcal{M}^{\mathcal{U}}_{\gamma}$, and $\mathcal{M}^{\mathcal{S}}_{\xi+1}$ have the same subsets of α_{ξ}. Since $\mathcal{M}^{\mathcal{S}}_{\xi+1} \in \mathcal{M}^{\mathcal{U}}_{\tau}$, this implies that Q is a proper initial segment of $\mathcal{M}^{\mathcal{U}}_{\tau}$.

The branch $(\nu + 1, \gamma]_U$ can have no drops, since otherwise $\rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) \geq \lambda(F) > \alpha_{\xi}$, whereas $\rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) \in \{\mu, \alpha_{\xi}\}$. It follows that Q is the core of $\mathcal{M}^{\mathcal{U}}_{\gamma}$. (The full core, not the α_{ξ}-core.)

We claim $\mathrm{crt}(F) = \mu$. For otherwise, $\mathrm{crt}(F) > \alpha_{\xi}$, which implies that Q is the α_{ξ}-core of $\mathcal{M}^{\mathcal{U}}_{\gamma}$, so that $Q = \mathcal{M}^{\mathcal{S}}_{\xi+1}$. One also has that $\hat{i}^{\mathcal{U}}_{\tau,\gamma} = i^{\mathcal{S}}_{\xi+1,\theta}$ is the uncoring map, so

$$\begin{aligned}
(\Sigma^{\mathcal{U}}_{\tau})_Q &= (\Sigma^{\mathcal{U}}_{\gamma})^{\hat{i}^{\mathcal{U}}_{\tau,\gamma}} \\
&= (\Sigma^{\mathcal{S}}_{\theta})^{i^{\mathcal{S}}_{\xi+1,\theta}} \\
&= \Lambda_{\xi+1}.
\end{aligned}$$

The last equation holds because $\Lambda_{\xi+1} = (\Sigma^{\mathcal{T}}_{\xi})^{\pi_{\xi+1}} = (\Sigma^{\mathcal{T}}_{\theta})^{i^{\mathcal{T}}_{\xi,\theta} \circ \pi_{\xi+1}} = (\Sigma^{\mathcal{T}}_{\theta})^{\pi_{\theta} \circ i^{\mathcal{S}}_{\xi,\theta}} = (\Sigma^{\mathcal{S}}_{\theta})^{i^{\mathcal{S}}_{\xi+1,\theta}}$.
So if $\mathrm{crt}(F) > \alpha_{\xi}$, then Φ_{ξ} is not problematic, contradiction. Thus $\mathrm{crt}(F) = \mu$.

□

Claims 10 and 11 give that $\lambda(E^{\mathcal{U}}_{\nu}) > \alpha_{\xi}$. Claim 11 implies also that $\rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) = \mu$, as otherwise $\rho(\mathcal{M}^{\mathcal{U}}_{\gamma}) = \alpha_{\xi}$ is in the forbidden interval $(\mathrm{crt}(E^{\mathcal{U}}_{\nu}), \lambda(E^{\mathcal{U}}_{\nu}))$. From this we get that $\rho(Q) = \mu$ as well.

This implies that Q is the core of $\mathcal{M}^{\mathcal{S}}_{\xi+1}$, and in fact leting D be the normal measure defined by F, we have the diagram in Figure 3, where $i_D : Q \to Ult(Q, D)$ is the ultrapower map and $k : Ult(Q, D) \to \mathcal{M}^{\mathcal{U}}_{\nu+1}$ is the factor map. We have that $k \circ i_D = i^{\mathcal{U}}_{\tau,\nu+1}$ and $\mathrm{crt}(k) > \alpha_{\xi}$. We have then that

$$\begin{aligned}
Ult(Q, D) &= Core_{\alpha_{\xi}}(\mathcal{M}^{\mathcal{U}}_{\nu+1}) \\
&= Core_{\alpha_{\xi}}(\mathcal{M}^{\mathcal{U}}_{\gamma}) \\
&= \mathcal{M}^{\mathcal{S}}_{\xi+1}
\end{aligned}$$

Claim 12. $\tau \in [0, \nu]_U$.

Proof. We assume $\nu > \tau$; otherwise, there is nothing to prove. Let $N = \mathcal{M}^{\mathcal{U}}_\tau | \alpha_\xi$. Suppose that $D \in Ult(N, F)$. Then D witnesses that the first generator above μ of the extender given by the branch embedding $\hat{i}^{\mathcal{U}}_{\tau,\gamma}$ is $\beta =_{def} \text{crt}(k)$ and $\beta < (\mu^{++})^{\mathcal{M}^{\mathcal{U}}_\gamma}$. On the other hand, the first generator $> \mu$ of the extender given by the branch embedding $\hat{i}^{\mathcal{S}}_{\xi+1,\theta}$ is an inaccessible in $\mathcal{M}^{\mathcal{U}}_\gamma$. This is a contradiction.

The above paragraph implies that $\rho(\mathcal{M}^{\mathcal{U}}_\nu) \leq \alpha_\xi$ since D codes a subset of α_ξ missing from $\mathcal{M}^{\mathcal{U}}_\nu$. Now we proceed to prove $\tau \in [0, \nu]_U$. First we claim that F is the top extender of $\mathcal{M}^{\mathcal{U}}_\nu$. Otherwise, $(\mathcal{M}^{\mathcal{U}}_\nu || lh(F), F) \lhd \mathcal{M}^{\mathcal{U}}_\nu$ is sound. By the above paragraph, $D \notin Ult(N, F)$. This implies that F is a type A extender, by the initial segment condition. Note also that $Ult(N, F) = \mathcal{M}^{\mathcal{U}}_\nu || lh(F)$. Now consider the factor map

$$\tilde{k} : W =_{def} (Ult(N, D^*), D^*) \to Y =_{def} (Ult(N, F), F).$$

We note that $\text{crt}(\tilde{k}) = \beta \geq \alpha_\xi$, W, Y are premice of the same type, and $\tilde{k}$ is weakly elementary. Since $D \notin Ult(N, F)$, $W \notin Y$ and $\rho(Y) \leq \alpha_\xi$. Also, W, Y are α_ξ-sound, and $\rho_1(W) = \alpha_\xi \leq \beta$. We can apply Theorem 3.3 and conclude that W is the α_ξ-core of Y. Since $Y = \mathcal{M}^{\mathcal{U}}_\nu | lh(F) \lhd \mathcal{M}^{\mathcal{U}}_\nu$, Y is sound. We then conclude that $W = Y$ and $D^* = F$. This means F is on the extender sequence of $\mathcal{M}^{\mathcal{U}}_\tau$ (by the agreement of $\mathcal{M}^{\mathcal{U}}_\tau$ and $\mathcal{M}^{\mathcal{U}}_\nu$). So $\tau = \nu$, which contradicts our assumption $\nu > \tau$.

Let G be the first extender used on the branch $[0, \nu]_U$ that has length $> \alpha_\xi$. Then $\text{crt}(G) \geq \alpha_\xi$. Otherwise, $\mu \notin \text{rng}(\hat{i}^{\mathcal{U}}_{0,\nu})$, but we know $\mu \in \text{rng}(\hat{i}^{\mathcal{U}}_{0,\nu})$ as μ is the critical point of the top extender of $\mathcal{M}^{\mathcal{U}}_\nu$. Then G has to be applied to (an initial segment of) $\mathcal{M}^{\mathcal{U}}_\tau$ since τ is the least τ' such that $\text{crt}(G) < \lambda(E^{\mathcal{U}}_{\tau'})$.[25] We have shown $\tau \in [0, \nu]_U$.

□

Claim 13. D^* is on the sequence of Q.

Proof. The proof of Claim 12 implies that if $\nu > \tau$, then $\text{crt}(\hat{i}^{\mathcal{U}}_{\tau,\nu}) > \alpha_\xi$ and if $D^* = F$ then F is on the extender sequence of $\mathcal{M}^{\mathcal{U}}_\tau$. In this case, using the fact that $\rho(Q) = \mu$ and $\alpha_\xi = (\mu^+)^Q$, we get F must be on the Q-sequence.

We assume $D^* \neq F$. In this case F is the top extender of $\mathcal{M}^{\mathcal{U}}_\nu$. The proof of Claim 12 gives that if $\mathcal{M}^{\mathcal{U}}_\nu$ is α_ξ-sound, then $D^* = F$. So we may assume that $\mathcal{M}^{\mathcal{U}}_\nu$ is not α_ξ-sound. So the branch $[0, \nu]_U$ must have truncation points. We let $\epsilon \in [0, \nu]_U$ be last truncation point of $[0, \nu]_U$. So there is $Y \lhd \mathcal{M}^{\mathcal{U}}_\epsilon$ such that $\hat{i}^{\mathcal{U}}_{\epsilon,\nu} : Y \to \mathcal{M}^{\mathcal{U}}_\nu$ has critical point $> \alpha_\xi$. Y is sound with top extender E such that $\hat{i}^{\mathcal{U}}_{\epsilon,\nu}(E) = F$. Using the same argument as in Claim 12, we get that $D^* = E$ is on the Q-sequence.

□

The above claims show Φ^-_ξ is not problematic. To show that Φ_ξ is not problematic (contradiction), we must show that $\Lambda_{\xi+1} = (\Lambda_\xi)_s$, where s is the stack that consists of dropping to Q, and

[25] Recall we already know that for any $\tau' < \tau$, $lh(E^{\mathcal{U}}_{\tau'}) \leq \alpha_\xi$.

then forming $\mathrm{Ult}(Q, D)$. But

$$\begin{aligned}\Lambda_{\xi+1} &= \Lambda_\theta^{i^{\mathcal{S}}_{\xi+1,\theta}}\\ &= (\Sigma^{\mathcal{U}}_\gamma)^{i^{\mathcal{U}}_{\nu+1,\gamma}\circ k}\\ &= (\Sigma^{\mathcal{U}}_{\nu+1})^k\\ &= (\Sigma^{\mathcal{U}}_\tau)_s\\ &= (\Lambda_\xi)_s.\end{aligned}$$

The fourth equation requires an explanation, as we do not know k is an iteration map and hence we cannot directly apply pullback consistency to k. Let $\mathcal{W}_0 = \mathcal{U} \restriction (\tau+1)^\frown\langle D\rangle$ and $\mathcal{W}_1 = \mathcal{U} \restriction (\nu+1)^\frown\langle F\rangle$. Then using the notation in [10, Definition 2.26] and using the fact that $i^{\mathcal{U}}_{\tau,\nu}(D^*) = F$, $\mathcal{W}_0$ pseudo-hull embeds into $\mathcal{W}_1$ as witnessed by $(u, \vec{t}, p)$ (see Figure 4), where

(a) $u \restriction \tau = id = v \restriction \tau$, $u(\tau) = \nu$ and $v(\tau) = \tau$.

(b) p is identity on $(\mathcal{U} \restriction \tau)^{ext}$ and $p(D) = F$. Here $D = E^{\mathcal{W}_0}_\tau$ and $F = E^{\mathcal{W}_1}_{u(\tau)}$.

(c) $\vec{t}$ is determined by u, v, p as prescribed in [10, Definition 2.26], with $t^0_{\tau+1} = k$, $t^0_\tau = id$, $t^1_\tau = i^{\mathcal{U}}_{\tau,\nu}$, and for $\xi < \tau$, $t^0_\xi = t^1_\xi = id$.

Benjamin Siskind has recently shown, using the methods developed in [10], that iteration strategies pull back under pseudo-hull embeddings. That is, if $\mathcal{T}$ is a normal tree on a mouse pair (P, Σ), and Φ is a psuedo-hull embedding from $\mathcal{T}$ into $\mathcal{U}$, and $t = t^0_\alpha \colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{U}}_{v(\alpha)}$ is one of the maps of Φ, then for $R = \mathcal{M}^{\mathcal{T}}_\alpha$ and $S = \mathcal{M}^{\mathcal{U}}_{v(\alpha)}$, $\Sigma^t_S = \Sigma_R$. The same is then true for t^1_α and $\mathcal{M}^{\mathcal{U}}_{u(\alpha)}$, by pullback consistency in $\mathcal{U}$. In other words, the t-maps of a psuedo-hull embedding are elementary in the category of mouse pairs.

In our situation, we get that for every $\xi \leq \tau+1$,

$$\Sigma^{\mathcal{W}_0}_\xi = (\Sigma^{\mathcal{W}_1}_{v(\xi)})^{t^0_\xi}.$$

In particular, this gives the fourth equation above. This in turns implies Φ_ξ is not problematic. This contradiction completes of proof of Theorem 3.7.

□

We can drop the hypothesis that $\mathrm{crt}(\pi) < \rho_{k(H)}(H)$ from Theorem 3.7, at the cost of omitting its conclusions concerning condensation of the external strategies. This will be useful in the proof of square.

Corollary 3.13 (AD^+). *Suppose (M, Σ) is a mouse pair with scope HC. Suppose $\pi : H \to M$ is weakly elementary, and not the identity. Let $\alpha = \mathrm{crt}(\pi)$, and suppose*

(1) H is a premouse of the same type as M,

(2) H is α-sound, and

$$
\begin{tikzcd}
 & \mathcal{M}^{\mathcal{W}_1}_{\nu+1} \\
\mathcal{M}^{\mathcal{W}_0}_{\tau+1} \arrow[ur, "t^0_{\tau+1} = k"] & \mathcal{M}^{\mathcal{W}_1}_{\nu}, F \\
\mathcal{M}^{\mathcal{W}_0}_{\tau} \arrow[u, "i^{\mathcal{W}_0}_{\tau,\tau+1} = \pi_D"] \arrow[ur, "t^1_\tau = i^{\mathcal{U}}_{\tau,\nu}"] \arrow[r, "t^0_\tau = id"] & \mathcal{M}^{\mathcal{W}_1}_{\tau}, D \arrow[u, "i^{\mathcal{W}_1}_{v(\tau),u(\tau)} = i^{\mathcal{U}}_{\tau,\nu}"'] \arrow[uu, bend right=60, "i^{\mathcal{W}_1}_{\tau,\nu+1}"']
\end{tikzcd}
$$

Figure 4: $\mathcal{W}_0$ is a pseudo-hull of $\mathcal{W}_1$

(3) H is not the α-core of M.

Then exactly one of the following holds.

(a) $H \lhd M$.

(b) $H \lhd Ult_0(M, \dot{E}^M_\alpha)$.

(c) $H = Ult(M|(\xi,k), E)$, where $l(M) >_{lex} (\xi,k) >_{lex} (\alpha,n)$, and (ξ,k) is lex least such that $\rho(M|(\xi,k)) < \alpha$, E is on the extender sequence of $M|\xi$, and $crt(E)$ is the cardinal predecessor of α in $M|\xi$ and is the only generator of E.

Proof. Let n be largest such that $\alpha < \rho_n(H)$, and $n \leq k(H)$. Let G and N be the same as H and M, except that $k(G) = n = k(N)$. Let $\Psi = \Sigma^\pi_N$. The hypotheses of 3.7 hold of (G,Ψ), (N,Σ_N), and π. (We have $H \in M$ by 3.3, hence $G \in N$, hence G is not the α-core of N.) Hence one of the conclusions of 3.7 holds of them.

If it is conclusion (a), then $G \lhd N$, which easily implies $H \lhd M$. If it is (b), then $G \lhd Ult_0(N, \dot{E}^M_\alpha)$ yields $H \lhd Ult_0(M, \dot{E}^M_\alpha)$. Finally, (c) for G and N clearly implies (c) for H and M; in fact G is not sound in this case, so $G = H$. $\square$

Corollary 3.14 (AD$^+$)**.** *Let $(M,\Sigma), \alpha, \pi$ etc. be as in the hypothesis of Theorem 3.3. Assume additionally that H is sound, $p(M) \in rng(\pi)$ and $\alpha = \rho(H) = \rho(M)$. Then π is the core map. Furthermore, letting $\rho = \rho(H)$, then*

1. $H|(\rho^+)^H = M|(\rho^+)^M$.

2. *Letting* $\Psi = \Sigma^{\pi}$ *and* $K = H|(\rho^+)^H$, *then* $\Sigma_K = \Psi_K$.

Proof sketch. It is clear that π is the core map. The first conclusion follows from [10, Theorem 5.57].

For the second conclusion, we apply Theorem 3.7 to the map $\pi \restriction N : N \to \pi(N)$ for each $N \lhd K$ such that $\rho(N) = \rho$, noting that $N \in \pi(N)$. We get that $\Sigma_N = \Psi_N$. Therefore,

$$\Sigma_K = \oplus_{N \lhd K, \rho(N)=\rho} \Sigma_N = \oplus_{N \lhd K, \rho(N)=\rho} \Psi_N = \Psi_K.$$

$\square$

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