Determinacy in $L(\mathbb{R}, \mu)$

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Abstract

Assume $V = L(\mathbb{R}, \mu) \vDash \mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2 + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. We analyze what sets of reals are determined and in fact show that $L(\mathbb{R}, \mu) \vDash \mathsf{AD}$. This arguably gives the most optimal characterization of AD in $L(\mathbb{R}, \mu)$. As a consequence of this analysis, we obtain the equiconsistency of the theories: "ZFC + There are ω^2 Woodin cardinals" and "$\mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2$ + There is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$".

1 Introduction

Suppose X is an uncountable set. A measure μ on $\mathcal{P}_{\omega_1}(X)$ is

- **fine** if for all $x \in X$, $\mu(\{\sigma \in \mathcal{P}_{\omega_1}(X) \mid x \in \sigma\}) = 1$;

- **normal** if for all functions $F : \mathcal{P}_{\omega_1}(X) \to \mathcal{P}_{\omega_1}(X)$ such that $\mu(\{\sigma \mid F(\sigma) \subseteq \sigma\}) = 1$, there is an $x \in X$ such that $\mu(\{\sigma \mid x \in F(\sigma)\}) = 1$.

A normal fine measure μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is often called the Solovay measure. Solovay (in [**?**]) has shown the existence (and uniqueness) of a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ under $\mathsf{AD}_\mathbb{R}$. It is natural to ask whether the existence of a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ has consistency strength that of $\mathsf{AD}_\mathbb{R}$. It is well-known that the existence of an $L(\mathbb{R}, \mu)$[1] that satisfies "$\mathsf{ZF} + \mathsf{DC} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$" is equiconsistent with that of a measurable cardinal; this is much weaker than the consistency strength of $\mathsf{AD}_\mathbb{R}$. The model

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[1] By $L(\mathbb{R}, \mu)$ we mean the model constructed from the reals and using μ as a predicate. We will also use the notation $L(\mathbb{R})[\mu]$ and $L_\alpha(\mathbb{R})[\mu]$ instead of $L(\mathbb{R}, \mu)$ and $L_\alpha(\mathbb{R}, \mu)$ in various places in the paper.

$L(\mathbb{R}, \mu)$ obtained from standard proofs of the equiconsistency satisfies $\Theta = \omega_2$, where Θ is the sup of all α such that there is a surjection from $\mathbb{R}$ onto α. This model fails to satisfy AD because it fails to satisfy the Coding Lemma. So it is natural to consider the situations where $L(\mathbb{R}, \mu) \vDash \Theta > \omega_2$ and try to understand how much determinacy holds in this model; furthemore, one can try to ask what the exact consistency strength of the theory "$\Theta > \omega_2$ + there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$" is. What about the seemingly stronger theory "AD + there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$"?

We attempt to answer the above questions in this paper. First, to analyze the sets of reals that are determined in a model of the form "$V = L(\mathbb{R}, \mu) + \Theta > \omega_2$", which we will call V, we run the core model induction in a certain submodel of V that agrees with V on all bounded subsets of Θ. This model will be defined in the next section. What we'll show is that $Lp(\mathbb{R}) \vDash \mathsf{AD}^+$ where

$$Lp(\mathbb{R}) = \bigcup \{\mathcal{M} \mid \mathcal{M} \text{ is a sound } \mathbb{R}\text{-premouse}, \rho(\mathcal{M}) = \mathbb{R}, \text{ and } \mathcal{M} \text{ is countably iterable}[2]\}.$$

We will then show $\Theta^{Lp(\mathbb{R})} = \Theta$ by an argument like that in Chapter 7 of [?]. Finally, we prove that in V,

$$\mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R}) = \mathcal{P}(\mathbb{R}),$$

which implies $V \vDash \mathsf{AD}^+$. We state the main result of this paper.[3]

Theorem 1.1. *Suppose $V = L(\mathbb{R}, \mu) \vDash$ "$\mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2 + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$". Then $L(\mathbb{R}, \mu) \vDash \mathsf{AD}^+$.*

Woodin (cf. [?]) has shown the following.

Theorem 1.2. *Suppose $L(\mathbb{R}, \mu) \vDash$ "$\mathsf{AD} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$". Then $L(\mathbb{R}, \mu) \vDash$ "$\mathsf{AD}^+ + \mu$ is unique".*

Combining the results in Theorem **??** and Theorem **??**, we get the following.

Corollary 1.3. *Suppose $V = L(\mathbb{R}, \mu) \vDash$ "$\mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2 + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$". Then $L(\mathbb{R}, \mu) \vDash$ "$\mathsf{AD}^+ + \mu$ is unique".*

The equiconsistency result we get from this analysis is stated in Theorem **??**. We should mention that these theories are still much weaker than $\mathsf{AD}_{\mathbb{R}}$ in consistency strength.

Theorem 1.4. *The following theories are equiconsistent.*

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[2]An $\mathbb{R}$-premouse $\mathcal{M}$ is countably iterable if any countable hull of $\mathcal{M}$ is $\omega_1 + 1$ iterable.

[3]This theorem is due independently to Woodin and the author.

1. ZFC + *there are ω^2 Woodin cardinals.*

2. $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+$ + *there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.*

3. $\mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2$ + *there is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.*

Proof. The equiconsistency of (1) and (2) is a theorem of Woodin (see [**?**] for more information). Theorem **??** immediately implies the equiconsistency of (2) and (3). $\qquad\square$

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2 Basic setup

In this section we prove some basic facts about V assuming $V = L(\mathbb{R}, \mu) \vDash$ "$\mathsf{ZF} + \mathsf{DC} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$". First note that we cannot well-order the reals hence full AC fails in this model (this follows from the existence of μ). Secondly, ω_1 is regular; this follows from DC. Now μ induces a countably complete nonprincipal ultrafilter on ω_1 as follows. Let $\pi : \mathbb{R} \to \omega_1$ be defined as follows: $\pi(x) = \alpha$ if and only if x codes a well-ordering on ω of order type α; if x does not code a well-ordering, let $\pi(x) = 0$. Clearly, π is a surjection. We let define $\tau : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \omega_1$ by setting $\tau(\sigma) = \bigcup\{\pi(x) \mid x \in \sigma\}$. Finally, let ν be a filter on ω_1 defined as follows:

$$A \in \nu \Leftrightarrow \tau^{-1}[A] \in \mu.$$

It's easy to verify that ν is a countably complete, non-principal measure on ω_1. Hence, ω_1 is a measurable cardinal. DC also implies that $\mathrm{cof}(\omega_2) > \omega$. We collect these facts into the following proposition.

Proposition 2.1. *Suppose $V = L(\mathbb{R}, \mu) \vDash$ "$\mathsf{ZF} + \mathsf{DC} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$". Then*

1. *ω_1 is regular and in fact measurable;*

2. *$\mathrm{cof}(\omega_2) > \omega$;*

3. *AC fails and in fact, there cannot be an ω_1-sequence of distinct reals.*

Lemma 2.2. Θ *is a regular cardinal.*

Proof. Suppose not. Let $f : \mathbb{R} \to \Theta$ be a cofinal map. Then there is an $x \in \mathbb{R}$ such that f is $\mathrm{OD}(\mu, x)$. For each $\alpha < \Theta$, there is a surjection $g_\alpha : \mathbb{R} \to \alpha$ such that g_α is $\mathrm{OD}(\mu)$ (we may take g_α to be the least such). We can get such a g_α because we can "average over the reals." Now define a surjection $g : \mathbb{R} \to \Theta$ as follows

$$g(y) = g_{f(y_0)}(y_1) \text{ where } y = \langle y_0, y_1 \rangle.$$

It's easy to see that g is a surjection. But this is a contradiction. $\square$

Lemma 2.3. ω_1 *is inaccessible in any (transitive) inner model of choice containing ω_1.*

Proof. This is easy. Let N be such a model. Since $P = L[N][\nu]$ is also a choice model and ω_1 is measurable in P, hence ω_1 is inaccessible in P. This gives ω_1 is inaccessible in N. $\square$

Next, we define two key models that we'll use for our core model induction. Let

$$M = \Pi_{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})} M_\sigma / \mu \text{ where } M_\sigma = \mathrm{HOD}_{\sigma \cup \{\sigma\}}$$

and, for a transitive self-wellordered[4] $a \in H_{\omega_1}$,

$$H_a = \Pi_{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})} H_{a,\sigma} / \mu \text{ where } H_{a,\sigma} = \mathrm{HOD}_{\{a,\sigma\}}^{M_\sigma}.$$

We note that in the definition of M_σ and H_σ above, ordinal definability is with respect to the structure $(L(\mathbb{R}, \mu), \mu)$.

Lemma 2.4. *Lós theorem holds for both of the ultraproducts defined above.*

Proof. We do this for the first ultraproduct. The proof is by induction on the complexity of formulas. It's enough to show the following. Suppose $\phi(x, y)$ is a formula and f is a function such that $\forall_\mu^* \sigma M_\sigma \vDash \exists x \phi[x, f(\sigma)]$. We show that $M \vDash \exists x \phi[x, [f]_\mu)$.

Let $g(\sigma) = \{x \in \sigma \mid (\exists y \in \mathrm{OD}(\mu, x))(M_\sigma \vDash \phi[y, f(\sigma)])\}$. Then $\forall_\mu^* \sigma g(\sigma)$ is a non-empty subset of σ. By normality of μ, there is a fixed real x such that $\forall_\mu^* \sigma x \in g(\sigma)$. Hence we can define $h(\sigma)$ to be the least y in $\mathrm{OD}(\mu, x)$ such that $M_\sigma \vDash \phi[y, f(\sigma)]$. It's easy to see then that $M \vDash \phi[[h]_\mu, [f]_\mu]$. $\square$

By Lemma **??**, M and H_a are well-founded so we identify them with their transitive collapse. First note that $M \vDash \mathsf{ZF} + \mathsf{DC}$ and $H_a \vDash \mathsf{ZFC}$. We then observe that $\Omega = [\lambda \sigma . \omega_1]_\mu$

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[4]This means there is a well-ordering of a in $L_1[a]$.

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is measurable in M and in H_a. This is because ω_1 is measurable in M_σ and $H_{a,\sigma}$ for all σ. Note also that $\Omega > \Theta$ as $\forall^*_\mu \sigma$, Θ^{M_σ} is countable and $\mathcal{P}(\omega_1)^{M_\sigma}$ is countable. The key for this is just an easy fact stated in Lemma **??**: There are no sequences of ω_1 distinct reals. Hence, by a standard Vopenka argument, for any set of ordinals $A \in M$ of size less than Ω, there is an H_a-generic G_A (for a forcing of size smaller than Ω) such that $A \in H_a[G_A] \subseteq M$ and Ω is also measurable in $H_a[G_A]$.

Lemma 2.5. $\mathcal{P}(\mathbb{R}) \subseteq M$.

Proof. Let $A \subseteq \mathbb{R}$. Then there is an $x \in \mathbb{R}$ such that $A \in \mathrm{OD}(x, \mu)$. By fineness of μ, $(\forall^*_\mu \sigma)(x \in \sigma)$ and hence $(\forall^*_\mu \sigma)(A \cap \sigma \in \mathrm{OD}(x, \mu, \sigma))$. So we have $(\forall^*_\mu \sigma)(A \cap \sigma \in M_\sigma)$. This gives us that $A = [\lambda \sigma . A \cap \sigma]_\mu \in M$. $\qquad\square$

Lemma **??** implies that M contains all bounded subsets of Θ.

3 Framework for the core model induction

This section is an adaptation of the framework for the core model induction developed in [**?**], which in turns builds on earlier formulations of the core model induction in [**?**]. For a detailed discussion on basic notions such as model operators, mouse operators, F-mice, Lp^F, Lp^Γ, condenses well, relativizes well, the envelope of an inductive-like pointclass Γ (denoted $\mathrm{Env}(\Gamma)$), iterability, quasi-iterability, see [**?**]. We briefly recall some of these notions here.

Definition 3.1. *Let $\mathcal{L}_0$ be the language of set theory expanded by unary predicate symbols $\dot{E}, \dot{B}, \dot{S}$, and constant symbols $\dot{l}$ and $\dot{a}$. Let a be a given transitive set. A **model with paramemter a** is an $\mathcal{L}_0$-structure of the form*

$$\mathcal{M} = (M; \in, E, B, \mathcal{S}, l, a)$$

such that M is a transtive rud-closed set containing a, the structure $\mathcal{M}$ is amenable, $\dot{a}^\mathcal{M} = a$, $\mathcal{S}$ is a sequence of models $\mathcal{S}_\xi$'s with paramemter a such that letting M_ξ be the universe of $\mathcal{S}_\xi$

- *$\dot{S}^{\mathcal{S}_\xi} = \mathcal{S} \restriction \xi$ for all $\xi \in dom(\mathcal{S})$ and $\dot{S}^{\mathcal{S}_\xi} \in M_\xi$ if ξ is a successor ordinal;*

- *$M_\xi = \cup_{\alpha < \xi} M_\alpha$ for all limit $\xi \in dom(\mathcal{S})$;*

- *if $dom(\mathcal{S})$ is a limit ordinal then $M = \cup_{\alpha \in dom(\mathcal{S})} M_\alpha$ and $l = 0$, and*

- *if $dom(\mathcal{S})$ is a successor ordinal, then $dom(\mathcal{S}) = l$.*

Typically, the predicate $\dot{E}$ codes the top extender of the model; $\dot{S}$ records the sequence of models being built so far; $\dot{B}$ codes the "lower part extenders" or a "branch of some iteration tree". Next, we write down some notations regarding the above definition.

Definition 3.2. *Let $\mathcal{M}$ be the model with parameter a. Then $|\mathcal{M}|$ denotes the universe of $\mathcal{M}$. We let $l(\mathcal{M}) = dom(\dot{S}^{\mathcal{M}})$ denote the **length of** $\mathcal{M}$ and set $\mathcal{M}|\xi = \dot{S}^{\mathcal{M}}_\xi$ for all $\xi < l(\mathcal{M})$. We set $\mathcal{M}|l(\mathcal{M}) = \mathcal{M}$. If $l(\mathcal{M}) = \xi + 1$, then we let $\mathcal{M}^- = \dot{\mathcal{S}}^{\mathcal{M}}_\xi$. We also let $\rho(\mathcal{M}) \leq l(\mathcal{M})$ be the least such that there is some $A \subseteq M$ definable (from parameters in M) over $\mathcal{M}$ such that $A \cap |\mathcal{M}|\rho(\mathcal{M})| \notin M$.*

Definition 3.3. *Let ν be an uncountable cardinal and $a \in H_\nu$ be transitive. A **model operator over** a is a partial function $F : H_\nu \to H_\nu$ such that to each model $\mathcal{M}$ over a, $F(\mathcal{M})$ is a model over a such that*

- $\dot{E}^{F(\mathcal{M})} = \emptyset$;

- $\dot{S}^{F(\mathcal{M})} = (\dot{S}^{\mathcal{M}})^\frown \langle \mathcal{M} \rangle$;

- $F(\mathcal{M}) = Hull^{F(\mathcal{M})}_{\Sigma_1}(|\mathcal{M}|)$ *(here the hull is transitively collapsed)*;

- *if $x \in |F(\mathcal{M})|$ and $y \in |M|\rho(\mathcal{M})|$ then $x \cap y \in |\mathcal{M}|$.*

For a transitive set a, we let $Lp(a)$ be the union of $\mathcal{N}$ such that $\mathcal{N}$ is a sound premouse over a, $\rho_\omega = a$, and for all $\pi : \bar{\mathcal{N}} \to \mathcal{N}$ such that $\bar{\mathcal{N}}$ is countable transitive, π is elementary, then $\bar{\mathcal{N}}$ is $(\omega, \omega_1 + 1)$-iterable, that is $\bar{\mathcal{N}}$ has an iteration strategy (in fact a unique one) that acts on ω-maximal, normal iteration trees of length at most ω_1 on $\bar{\mathcal{N}}$. Let ν be an uncountable cardinal, $a \in H_\nu$ be transtive. We say that J is a **mouse operator on H_ν over** a if there is an rQ-formula $\varphi(v_0, v_1)$ in the language of (Mitchell-Steel) premice such that for all transitive $b \in H_\nu$ such that $a \in b$, $J(b)$ is the least $\mathcal{N} \lhd Lp(b)$ such that $\mathcal{N} \vDash \varphi[b, a]$. We say that J is defined on H_ν over a or on a cone on H_ν above a.

Definition 3.4. *Let J be a mouse operator on H_ν over some transitive $a \in H_\nu$. The model operator F_J induced by J is defined as follows:*

1. *If $J(\mathcal{M})$ is amenable to $\mathcal{M}|\rho(\mathcal{M})$ then*

$$F_J(\mathcal{M}) = (|J(\mathcal{M})|, \in, \emptyset, B, (\dot{\mathcal{S}}^{\mathcal{M}})^\frown \langle \mathcal{M} \rangle, dom(\dot{\mathcal{S}}^{\mathcal{M}}) + 1, a),$$

   where B is the extender sequence for $J(\mathcal{M})$.

2. *Otherwise, let ξ be the least ordinal such that $J(\mathcal{M})|(\xi+1)$ is not amenable to $\mathcal{M}|\rho(\mathcal{M})$ and n be the smallest such that $\rho_{n+1}(J(\mathcal{M})|\xi) = \mathcal{M}$. Then letting $(N, , B)$ be the nth-reduct of $J(\mathcal{M})|\xi$, we set*

$$F_J(\mathcal{M}) = (N, \in, \emptyset, B, (\dot{\mathcal{S}}^{\mathcal{M}})^\frown \langle \mathcal{M} \rangle, dom(\dot{\mathcal{S}}^{\mathcal{M}}) + 1, a).$$

We note in the above that $\rho_1(F_J(\mathcal{M})) = \mathcal{M}$ and $F_J(\mathcal{M}) = Hull_{\Sigma_1}^{F_J(\mathcal{M})}(|\mathcal{M}|)$, and hence F_J is indeed a model operator over a on H_ν. Sometimes when the domain of F_J (or any model operator F) is clear or is not important, we just say that F_J (or F) is a model operator over a.

Definition 3.5. *Suppose F is a model operator over some transitive set a. F **condenses well** if the following hold:*

1. *If $\mathcal{M}, \mathcal{M}', \mathcal{N}$ are models over a such that $\mathcal{M}' = \mathcal{M}^-$, where $dom(\dot{S}^{\mathcal{M}'})$ is a successor ordinal, and $\pi : \mathcal{M} \to \mathcal{F}(\mathcal{N})$ is a 0-embedding or Σ_2-embedding then $\mathcal{M} = F(\mathcal{M}')$.*

2. *If $\mathcal{P}, \mathcal{M}, \mathcal{M}', \mathcal{N}$ are models with $\mathcal{M}' = \mathcal{M}^-$, $\sigma : F(\mathcal{P}) \to \mathcal{M}$ being a 0-embedding, and $\pi : \mathcal{M} \to F(\mathcal{N})$ being a weak 0-embedding, then $\mathcal{M} = F(\mathcal{M}')$.*

Our definition is weaker than Definition 2.1.10 [**?**] in that we don't require (1) and (2) above to hold in $V^{Col(\omega, \mathcal{M})}$; our core model induction will not occur in a generic extension of V hence this is all we need out of the notion of "condenses well". Nevertheless, we can still define the notions of F-premice, projecta, standard parameters, solidity and universality of stadard parameters, iteration trees and stategies for F-premice, and the $K^{c,F}$-construction the same way as in [**?**] (see also [**?**] for a different, and more detailed discussion of F-mice for a fairly general class of operators F).

Definition 3.6. *Suppose F is a model operator on H_ν over some transitive $a \in H_\nu$. We say that F **relativizes well** if there is a formula φ in the language of set theory such that for every pair $\mathcal{P}, \mathcal{Q}$ of models over a such that $\mathcal{P} \in \mathcal{Q}$ and if M is a transitive model of ZF^- such that $F(\mathcal{Q}) \in M$, then $F(\mathcal{P})$ is the unique $x \in M$ such that $M \vDash \varphi[x, \mathcal{P}, \mathcal{Q}, F(\mathcal{Q})]$.*

Definition 3.7. *Suppose F is a model operator on H_ν over some transitive $a \in H_\nu$. We say that F **determines itself on generic extensions** if there is a formula φ (in the language of set theory) such that whenever $M \vDash \mathsf{ZF}^-$ is transitive and is closed under F and $g \in V$ is generic over M, then $M[g]$ is closed under F and φ defines $F \upharpoonright M[g]$ over $M[g]$ from $F \upharpoonright M$.*

Definitions **??**, **??**, and **??** have obvious analogues for mouse operators. The model operators that we encounter during the core model induction in this paper come from mouse operators that condense well, relativize well, and determine themselves on generic extensions. We list examples of such operators. These operators, for the purpose of this paper, are defined on H_{ω_1} above some transitive $a \in H_{\omega_1}$.

1. $F = F_J$ for some mouse operator J defined on H_{ω_1} over some $a \in H_{\omega_1}$. Some examples of J are the $\mathcal{M}_n^\sharp$ operators, the "diagonal operator" defined in 4.2 of [?], and the $\mathcal{A}$-mouse operator $J = J_{\mathcal{A}}$ defined in Definition 4.3.9 of [?], where $\mathcal{A} = (A_i : i < \omega)$ is a self-justifying-system such that $\mathcal{A} \in \mathrm{OD}_{b,\Sigma,x}^{Lp(\mathbb{R})|\alpha}$ for some $x \in b$ and α ends either a weak gap or a strong gap in the sense of [?] and $\mathcal{A}$ seals the gap[5].

2. For some H, H satisfies 1) above, for some $n < \omega$, $F = F_J$, where J is the $x \to \mathcal{M}_n^{\#,H}(x)$ operator.

3. $H = F_J$, where for some $a \in HC$ and $\mathcal{M} \lhd Lp(a)$, letting Λ is $\mathcal{M}$'s unique (ω, ω_1)-iteration strategy (in fact $(\omega, \omega_1 + 1)$-strategy by using the measure μ) or $\mathcal{M}$ is a hod premouse (in the sense of [?]) and Λ is $\mathcal{M}$'s $(\omega, \omega_1, \omega_1)$-iteration strategy with branch condensation, for some rQ-formula φ (in the language of Λ-mice), for some $b \in H_{\omega_1}$ coding a, for all $x \in H_{\omega_1}$ coding b, $J(b) \lhd Lp^\Lambda(b)$ is the least that satisfies $\varphi[x, a]$. Typically, the H we need are those that are definable over $Lp^\Lambda(\mathbb{R})|\alpha^6$ for some α.

We remark that the model operators listed above are what we need to satisfy the hypothesis of Theorem ?? to prove AD holds in $Lp(\mathbb{R})$. These are what's called cmi-operators in [?].

The following lemmata are what we need to propagate "nice" properties of a model operator F to those of $\mathcal{M}_1^{\sharp,F}$ and to $F_{\mathcal{M}_1^{\sharp,F}}$. These propagations are needed in the core model induction. The proofs of these lemmata are easy and hence will be ommited.

Lemma 3.8. *Suppose F is a model operator defined on H_ν over a. Suppose F condenses well, relativizes well, and determines itself on generic extensions, then so does the mouse operator $\mathcal{M}_1^{\sharp,F}$ and $dom(\mathcal{M}_1^{\sharp,F}) = dom(F)$.*

Lemma 3.9. *Suppose J is a mouse operator defined on H_ν over a. Suppose J condenses well, relativizes well, and determines itself on generic extensions, then so does the model operator F_J and $dom(J) = dom(F_J)$.*

Remark. Before proceeding, we remark that since $V = L(\mathbb{R}, \mu)$, no $\mathsf{AD}_\mathbb{R}$ model M containing $\mathbb{R}$ can exist in V. If such an M exists, then the Solovay measure ν in M is in fact the club filter in M and by [?], $\nu \subseteq \mu$. This easily implies $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mu)$. But $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \nu) \subsetneq M \subseteq L(\mathbb{R}, \mu)$. Contradiction. Using this fact, [?] implies that for any AD^+ model M containing $\mathbb{R}$ in V, Strong Mouse Capturing (SMC) holds. We will use this fact from now on.

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[5]This means that $\mathcal{A}$ is cofinal in **Env**$(\mathbf{\Gamma})$, where $\mathbf{\Gamma} = \Sigma_1^{Lp(\mathbb{R})|\alpha}$.

[6]See [?] for a definition of $Lp^\Lambda(\mathbb{R})$ and also for a proof that the operators $x \mapsto \mathcal{M}_n^{\sharp,\Lambda}(x)$ condense well, relativize well, and determine themselves on generic extensions.

Finally, we end this section with the following lemma, which will be use at various places in the paper.

Lemma 3.10. *Assume $V = L(\mathbb{R}, \mu)$. Then there is no model M such that*

$$\mathbb{R} \cup \mathrm{OR} \cup Lp(\mathbb{R}) \subseteq M \wedge M = L(Lp^\Sigma(\mathbb{R})) \vDash \mathsf{AD}^+ + \Theta > \theta_0,$$

for some ω_1-strategy Σ with branch condensation.

Proof. Suppose such an M exists. Let Σ be as in the statement of the lemma for M. Let $H = \mathrm{HOD}_{\mathbb{R}}^M$. Note that

$$\mathcal{P}(\mathbb{R})^H = \mathcal{P}(\mathbb{R})^{Lp(\mathbb{R})} = \mathcal{P}_{\theta_0}(\mathbb{R})^M \wedge \Sigma \notin Lp(\mathbb{R}).$$

We aim to show that $L(\mathbb{R}, \mu) \subseteq H$, which is a contradiction. By a similar argument as the proof of Lemma **??** but relativized to Σ, $\nu =_{\mathrm{def}} \mu \restriction \mathcal{P}(\mathbb{R})^H \in M$; in fact, letting ρ be the restriction of μ on the Suslin co-Suslin sets of M, then $\rho \in M$. We show ν is OD in M. Let $\pi : \mathbb{R}^\omega \to \mathcal{P}_{\omega_1}(\mathbb{R})$ be the canonical map, i.e. $\pi(\vec{x}) = \mathrm{rng}(\vec{x})$. Let $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ be in H. There is a natural interpretation of A as a set of Wadge rank less than θ_0^M, that is the preimage $\vec{A}$ of A under π has Wadge rank less than θ_0^M. Fix such an A; note that $\vec{A}$ is invariant in the sense that whenever $\vec{x} \in \vec{A}$ and $\vec{y} \in \mathbb{R}^\omega$ and $\mathrm{rng}(\vec{x}) = \mathrm{rng}(\vec{y})$ then $\vec{y} \in \vec{A}$. Let $G_{\vec{A}}$ and G_A be the Solovay games corresponding to $\vec{A}$ and A respectively. In these games, players take turns and play finite sequences of reals and suppose $\langle x_i \mid i < \omega \rangle \in \mathbb{R}^\omega$ is the natural enumeration of the reals played in a typical play in either game, then the payoff is as follows:

$$\text{Player I wins the play in } G_{\vec{A}} \text{ if } \langle x_i \mid i < \omega \rangle \in \vec{A},$$

and

$$\text{Player I wins the play in } G_A \text{ if } \{x_i \mid i < \omega\} \in A.$$

Claim. G_A is determined.

Proof. For each $\vec{x} \in \mathbb{R}^\omega$, let $\sigma_{\vec{x}} = \mathrm{rng}(\vec{x})$. Consider the games $G_{\vec{A}}^{\vec{x}}$ and $G_A^{\sigma_{\vec{x}}}$ which have the same rules and payoffs as those of $G_{\vec{A}}$ and G_A respectively except that players are required to play reals in $\sigma_{\vec{x}}$. Note that these games are determined and Player I wins the game $G_{\vec{A}}^{\vec{x}}$ iff Player I wins the corresponding game $G_A^{\sigma_{\vec{x}}}$.

Without loss of generality, suppose $\nu(\{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \mid \text{Player I wins } G_A^\sigma\}) = 1$. For each such σ, let τ_σ be the canonical winning strategy for Player I given by the Moschovakis's Third Periodicity Theorem. We can easily integrate these strategies to construct a strategy τ for Player I in G_A. We know

$$\forall_\rho^* \sigma \ \tau_\sigma(\emptyset) \in \sigma.$$

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We have to use ρ since the set displayed above in general does not have Wadge rank less than θ_0 in M. Normality of ρ implies

$$\exists x \in \mathbb{R} \; \forall^*_\rho \sigma \; \tau_\sigma(\emptyset) = x.$$

Let $\tau(\emptyset) = x$ where x is as above. Now let y be II's response in G_A and $G_{\vec{A}}$. Since $\forall^*_\rho \sigma \; y \in \sigma$, by normality,

$$\exists z \in \mathbb{R} \forall^*_\rho \sigma \; \tau_\sigma(x, y) = z.$$

Let then $\tau(x, y) = z$. It's clear that the above procedure defines τ on all finite moves. It's easy to show τ is a winning strategy for Player I in G_A. $\qquad\square$

The claim and standard results of Woodin (see [?]) show that ρ is the unique normal fine measure on the Suslin co-Suslin sets of M and hence $\rho \in \mathrm{OD}^M$. This means $\rho \upharpoonright \mathcal{P}(\mathbb{R})^H = \nu$ is OD in M. This implies $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \nu) \subseteq H \subsetneq M$. But $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mu)$. Contradiction. $\qquad\square$

4 AD in $Lp(\mathbb{R})$

We assume the reader is familiar with at least one core model induction argument proving AD holds in $L(\mathbb{R})$ (or in $Lp(\mathbb{R})$). To save space, in what follows, we give details of some key points of our core model induction, and just outline the parts that follow from the standard theory. Suppose F is a model operator defined on a cone on H_{ω_1} above some $a \in H_{\omega_1}$ that condenses well, relativizes well, and determines itself on generic extensions, then the next theorem shows that $\mathcal{M}_1^{\sharp,F}(x)$ exists (and is (ω_1, ω_1)-iterable) for all $x \in H_{\omega_1}$ such that $a \in x$. By Lemmata ?? and ??, then the mouse operator $\mathcal{M}_1^{\sharp,F}$ and the model operator $F_{\mathcal{M}_1^{\sharp,F}}$ condense well, relativize well, and determine themselves on generic extensions. Hence the same conclusion holds for the corresponding model operator $F_{\mathcal{M}_1^{\sharp,F}}$. Using μ, we get that these operators $F, \mathcal{M}_1^{\sharp,F}, F_{\mathcal{M}_1^{\sharp,F}}$ are defined on the cone above a on H_Ω.

Theorem 4.1. *Suppose F is a nice model operator, where F is nice if it relativizes well, condenses well, and determines itself on generic extensions. Suppose F is defined on the cone on H_{ω_1} above some $a \in H_{\omega_1}$. Then $\mathcal{M}_1^{\sharp,F}(x)$ exists for all $x \in H_{\omega_1}$ coding a. Furthermore, $\mathcal{M}_1^{\sharp,F}(x)$ is $(\omega, \omega_1, \omega_1)$-iterable, hence (ω, Ω, Ω)-iterable in M.*

Proof. To start off, we may assume there is some real $x \in a$ such that F is OD_x, hence

$$\forall^*_\mu \sigma \; F \cap M_\sigma \in M_\sigma.$$

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By Lós theorem, $[\sigma \mapsto F \cap M_\sigma]$ defines a unique model operator on H^M_Ω over a extending F that condenses well, relativizes well, and determines itself on generic extensions. We also call this extension F. Now $F = F_J$ for some mouse operator J, by the same argument, the $F^\sharp$-operator, where $F^\sharp(x)$ is the model operator corresponds to the first active level of $Lp^F(x)$, is nice and is defined on the cone of H^M_Ω above a.

Lemma 4.2. *For each $x \in \mathbb{R}$ coding a, $\mathcal{M}^{\sharp,F}_1(x)$ exists.*

Proof. This is the key lemma. Suppose not, there is some x such that $\mathcal{M}^{\sharp,F}_1(x)$ doesn't exist. Then in $H_x{}^7$, which is closed under $F^\sharp$, the core model $K =_{\mathrm{def}} K^F(x)$ (built up to Ω)^8 exists and is $(\omega, \Omega+1)$ iterable by the unique $F^\sharp$-guided strategy^9. Let $\kappa = \omega^V_1$. By Lemma **??**, κ is inaccessible in H_x and in any $< \Omega$-set generic extension J of H_x such that $J \subseteq M$. By [**?**], $K^{H_x} = K^{H_x[G]}$ for any H_x-generic G for a poset of size smaller than Ω.

Claim. $(\kappa^+)^K = (\kappa^+)^{H_x}$.

Proof. The proof follows that of Theorem 3.1 in [**?**]. Suppose not. Let $\lambda = (\kappa^+)^K$. Hence $\lambda < (\kappa^+)^{H_x}$. Working in H_x, let N be a transitive, $F^\sharp$-closed, power admissible set containing x such that $^\omega N \subseteq N$, $V_\kappa \cup \mathcal{J}^K_{\lambda+1} \subseteq N$, and $\mathrm{card}(N) = \kappa$. We then choose $A \subseteq \kappa$ such that $N \in L^{F^\sharp}[A]^{10}$ and $K^{L^{F^\sharp}[A]}|\lambda = K|\lambda$, $\lambda = (\kappa^+)^{K^{L^{F^\sharp}[A]}}$, and $\mathrm{card}(N)^{L^{F^\sharp}[A]} = \kappa$. Such an A exists by Lemma 3.1.1 in [**?**] and the fact that $\lambda < (\kappa^+)^{H_x}$.

Now, it's easy to see that the sharp of $L^{F^\sharp}[A]$ exists in H_x, and hence $(\kappa^+)^{L^{F^\sharp}[A]} < (\kappa^+)^{H_x}$. By GCH in $L^{F^\sharp}[A]$, $\mathrm{card}^{H_x}(\mathcal{P}(\kappa) \cap L^{F^\sharp}[A]) = \kappa$. So in M, there is an $L^{F^\sharp}[A]$-ultrafilter U over κ that is nonprincipal and countably complete (in M and in V). This is because such a U exists in V as being induced from μ and since U can be coded as a subset of $\omega^V_1 = \kappa$, $U \in M$. Let J be a generic extension of H_x (of size smaller than Ω) such that $U \in J$. From now on, we work in J. Let

$$j : L^{F^\sharp}[A] \to Ult(L^{F^\sharp}[A], U) \cong L^{F^\sharp}[j(A)]$$

be the ultrapower map. We note that since U comes from μ, U moves $F^\sharp$ to itself; that's why it's justified to write $Ult(L^{F^\sharp}[A], U) \cong L^{F^\sharp}[j(A)]$ in the above. Then $\mathrm{crit}(j) = \kappa$, $A = j(A) \cap \kappa \in L^{F^\sharp}[j(A)]$. So $L^{F^\sharp}[A] \subseteq L^{F^\sharp}[j(A)]$. The key point here is that $\mathcal{P}(\kappa) \cap$

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^{7}Technically, we should take a self-wellordered transitive b coding x and consider H_b.

$^8K^F(x)$ is the core model that comes from the relative-to-F $K^{c,F}$-construction over x as defined in [**?**] and [**?**].

^{9}This means that for any normal, ω-maximal tree $\mathcal{T} \in H_x$ of length at most Ω on K, there is a unique cofinal wellfounded branch b such that $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq F^\sharp(\mathcal{T})$.

^{10}Technically, we should write $L^{F^\sharp}_\Omega[A]$ since $F^\sharp$ is only defined on H^M_Ω. All the computations in this claim will be below Ω.

$K^{L^{F^\sharp}[A]} = \mathcal{P}(\kappa) \cap K^{L^{F^\sharp}[j(A)]}$. To see this, first note that the $\subseteq$ direction holds because any κ-strong mouse (in the sense of [?]) in $L^{F^\sharp}[A]$ is a κ-strong mouse in $L^{F^\sharp}[j(A)]$ as $\mathbb{R} \cap L^{F^\sharp}[A] = \mathbb{R} \cap L^{F^\sharp}[j(A)]$ and $L^{F^\sharp}[A]$ and $L^{F^\sharp}[j(A)]$ have the same $< \kappa$-strong mice. To see the converse, suppose not. Then there is a sound mouse $\mathcal{M} \triangleleft K^{L^{F^\sharp}[j(A)]}$ such that $\mathcal{M}$ extends $K^{L^{F^\sharp}[A]}|\lambda$ and $\mathcal{M}$ projects to κ. The iterability of $\mathcal{M}$ is absolute between J and $L^{F^\sharp}[j(A)]$, by the following folklore result

Lemma 4.3. *Let F be a nice model operator defined on all of V. Assume* ZFC + *"there is no F-closed class model with a Woodin." Let M be a transitive class model closed under F that satisfies* ZFC^- + *"there is no class F-closed inner model of a Woodin". Futhermore, assume that $\omega_1 \subseteq M$. Let $\mathcal{P} \in M$ be an F-premouse with no definable Woodin. Then*

$$\mathcal{P} \text{ is a } F\text{-mouse} \iff M \vDash \mathcal{P} \text{ is a } F\text{-mouse.}$$

For a proof of this in the case $F = \mathrm{rud}$, see [?]. The proof of the lemma is just a trivial modification. Again, since we work below Ω in M, we identify Ω with OR when applying the above lemma. By a theorem of R. Schindler, translated into our context, K is just a stack of F-mice above ω_2 (here $\omega_2^J < \kappa$), we have $\mathcal{M} \triangleleft K^J = K$. But $\lambda = (\kappa^+)^K$ and $\mathcal{M} \triangleleft K|\lambda$. Contradiction.

Now the rest of the proof is just as in that of Theorem 3.1 in [?]. Let E_j be the superstrong extender derived from j. Since $\mathrm{card}(N) = \kappa$ and $\lambda < \kappa^+$, a standard argument (due to Kunen) shows that $F, G \in L^{F^\sharp}[j(A)]$ where

$$F = E_j \cap ([j(\kappa)]^{<\omega} \times K^{L^{F^\sharp}[A]})$$

and,

$$G = E_j \cap ([j(\kappa)]^{<\omega} \times N).$$

The key is $\mathrm{card}(N)^{L^{F^\sharp}[A]} = \kappa$ and $\mathrm{card}(K \cap \mathcal{P}(\kappa))^{L^{F^\sharp}[A]} = \kappa$. We show $F \in L^{F^\sharp}[j(A)]$. The proof of $G \in L^{F^\sharp}[j(A)]$ is the same. For $a \in [j(\kappa)]^{<\omega}$, let $\langle B_\alpha \mid \alpha < \kappa \rangle \in L^{F^\sharp}[A]$ be an enumeration of $\mathcal{P}([\kappa]^{|a|}) \cap K^{L^{F^\sharp}[A]} = \mathcal{P}([\kappa]^{|a|}) \cap K^{L^{F^\sharp}[j(A)]}$ and

$$E_a = \{B_\alpha \mid \alpha < \kappa \wedge a \in j(B_\alpha)\}.$$

Then $E_a \in L^{F^\sharp}[j(A)]$ because $\langle j(B_\alpha) \mid \alpha < \kappa \rangle \in L^{F^\sharp}[j(A)]$.

Hence $(K^{L^{F^\sharp}[j(A)]}, F)$ and (N, G) are elements of $L^{F^\sharp}[j(A)]$. In $L^{F^\sharp}[j(A)]$, for cofinally many $\xi < j(\kappa)$, $F|\xi$ coheres with K and (N, G) is a weak $\mathcal{A}$-certificate for $(K, F \restriction \xi)$ (in the sense of [?]), where

$$\mathcal{A} = \bigcup_{n<\omega} \mathcal{P}([\kappa]^n)^K.$$

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By Theorem 2.3 in [**?**], those segments of F are on the extender sequence of $K^{L^{F^\sharp}[j(A)]}$. But then κ is Shelah in $K^{L^{F^\sharp}[j(A)]}$, which is a contradiction. $\qquad\square$

The proof of the claim also shows that $(\kappa^+)^K = (\kappa^+)^J$ for any set (of size smaller than Ω) generic extension J of H_x. In particular, since any $A \subseteq \omega_1^V = \kappa$ belongs to a set generic extension of H_x of size smaller than Ω, we immediately get that $(\kappa^+)^K = \omega_2$. This is impossible in the presence of μ.[11] To see this, let $\vec{C} = \langle C_\alpha \mid \alpha < \omega_2 \rangle$ be the canonical $\square_\kappa$-sequence in K. The existence of $\vec{C}$ follows from the proof of the existence of square sequences in pure $L[E]$-models in [**?**]. Working in V, let ν be the measure on $\mathcal{P}_{\omega_1}(\omega_2)$ induced by μ defined as follows. First, fix a surjection $\pi : \mathbb{R} \to \omega_2$. Then π trivially induces a surjection from $\mathcal{P}_{\omega_1}(\mathbb{R})$ onto $\mathcal{P}_{\omega_1}(\omega_2)$ which we also call π. Then our measure ν is defined as

$$A \in \nu \Leftrightarrow \pi^{-1}[A] \in \mu.$$

Now consider the ultrapower map $j : K \to Ult(K, \nu) = K^*$ (where the ultrapower uses all functions in V). An easy calculation gives us that $j''\omega_2 = [\lambda\sigma.\sigma]_\nu$ and $A \in \nu \Leftrightarrow j''\omega_2 \in j(A)$. So let $\gamma = \sup j''\omega_2$ and $\vec{D} = j(\vec{C}) \in K^*$. Note that $(\kappa^+)^{K^*} = \omega_2$ and since $K^* \vDash \mathsf{ZFC}$, ω_2 is regular in K^*. Also $\gamma < j(\omega_2^V)$. Now consider the set D_γ. By definition, D_γ is an club in γ so it has order type at least ω_2. However, let $C = \langle \alpha < \omega_2 \mid \mathrm{cof}(\alpha) = \omega \rangle$. Then $j(C) \upharpoonright \gamma = j''C$ is an ω-club in γ (C need not be in K but j comes from μ so j acts on any ZFC model containing K, C). Hence $E = \lim D_\gamma \cap j(C)$ is an ω-club in γ. For each $\alpha \in E$, $D_\alpha = D_\gamma \cap \alpha$. Let $F = \pi^{-1}[E]$. By the proof of $\square_\kappa$ in K (cf. [**?**]) and the fact that $(\kappa^+)^K = \omega_2$, the existence of F implies the existence of a κ-sound mouse $\mathcal{M}$ such that $\rho_\omega(\mathcal{M}) = \kappa$ and $\omega_2 \in \mathcal{M}$. So $\mathcal{M}$ defines a surjection from κ onto ω_2. Contradiction. $\qquad\square$

The lemma shows that the mouse operator $\mathcal{M}_1^{\sharp,F}$ and hence the corresponding model operator $F_{\mathcal{M}_1^{\sharp,F}}$ is defined for all $x \in \mathbb{R}$ coding a. Since F, $F^\sharp$ relativize well, $\mathcal{M}_1^{\sharp,F}$ and hence $F_{\mathcal{M}_1^{\sharp,F}}$ are defined on all H_{ω_1} (in M as well as in V) above a. This implies that $\forall_\mu^* \sigma$, $F_{\mathcal{M}_1^{\sharp,F}} \upharpoonright M_\sigma \in M_\sigma$. By Lós, the $F_{\mathcal{M}_1^{\sharp,F}}$-operator is defined on H_Ω^M (above a). Also, for each $y \in H_\Omega^M$, $\mathcal{M}_1^{\sharp,F}(y)$ is $(\omega, \Omega, \Omega + 1)$-iterable. This completes the proof of the theorem. $\qquad\square$

Theorem 4.4. $M_0 =_{def} Lp(\mathbb{R}) \vDash \mathsf{AD}^+ + \Theta = \theta_0$.

The proof of Theorem **??** is very much like the proof of the core model induction theorems in [**?**], [**?**] (see Chapter 7) and [**?**] using the scales analysis developed in [**?**] and [**?**]. However, there is one point worth going over.

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[11]The argument we're about to give is based on Solovay's proof that square fails above a supercompact cardinal.

Suppose we are doing the core model induction to prove Theorem **??**. During this core model induction, we climb through the levels of $Lp(\mathbb{R})$ some of which project to $\mathbb{R}$ but do not satisfy that "$\Theta = \theta_0$". It is then the case that the scales analysis of [**?**], [**?**] cannot help us in producing the next "new" set. However, such levels can never be problematic for proving that AD^+ holds in $Lp(\mathbb{R})$. This follows from the following lemma.

Lemma 4.5. *Suppose $\mathcal{M} \lhd Lp(\mathbb{R})$ is such that $\rho(\mathcal{M}) = \mathbb{R}$ and $\mathcal{M} \vDash$ "$\Theta \neq \theta_0$". Then there is $\mathcal{N} \lhd Lp(\mathbb{R})$ such that $\mathcal{M} \lhd \mathcal{N}$, $\mathcal{N} \vDash$ "$\mathsf{AD}^+ + \Theta = \theta_0 + \mathsf{MC}$".*

Proof. Since $\mathcal{M} \vDash$ "$\Theta \neq \theta_0$" it follows that $\mathcal{P}(\mathbb{R})^{\mathcal{M}} \cap (Lp(\mathbb{R}))^{\mathcal{M}} \neq \mathcal{P}(\mathbb{R})^{\mathcal{M}}$ by a result of G. Sargsyan and J. Steel, see [**?**] or [**?**]. It then follows that there is some $\alpha < o(\mathcal{M})$ such that $\rho(\mathcal{M}|\alpha) = \mathbb{R}$ but $\mathcal{M}|\alpha \not\trianglelefteq (Lp(\mathbb{R}))^{\mathcal{M}}$. Let $\pi : \mathcal{N} \to \mathcal{M}|\alpha$ be such that $\mathcal{N}$ is countable and its iteration strategy is not in $\mathcal{M}$. Let Λ be the $(\omega, \omega_1, \omega_1)$-iteration strategy of $\mathcal{N}$. Using the measure μ we can lift Λ to an (Ω, Ω)-strategy (in M). Then a core model induction through $L_\Omega^\Lambda(\mathbb{R})$ (using Theorem **??**) shows that $L^\Lambda(\mathbb{R})^{12} \vDash \mathsf{AD}^+$ (this is where we needed clause 3 of the previous section) and so $L(\Lambda, \mathbb{R}) \vDash \mathsf{AD}^+$. Furthermore, it's easy to see that $L(\Lambda, \mathbb{R}) \vDash$ "$V = L(\mathcal{P}(\mathbb{R})) + \Theta = \theta_0$". By the remark at the end of Section 3, $L(\Lambda, \mathbb{R}) \vDash \mathsf{MC}$. It then follows from the remark at the end of Section 3 and the aforementioned result of G. Sargsyan and J. Steel that $L(\Lambda, \mathbb{R}) \vDash \mathcal{P}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})$. Let then $\mathcal{K} \trianglelefteq (Lp(\mathbb{R}))^{L(\Lambda, \mathbb{R})}$ be such that $\rho(\mathcal{K}) = \mathbb{R}$, $\mathcal{K} \vDash \Theta = \theta_0$ and $\Lambda \in \mathcal{K}$ (there is such a $\mathcal{K}$ by an easy application of Σ_1^2 reflection in $L(\Lambda, \mathbb{R})$). Since countable submodels of $\mathcal{K}$ are iterable , we have that $\mathcal{K} \trianglelefteq Lp(\mathbb{R})$. Also we cannot have that $\mathcal{K} \lhd \mathcal{M}$ as otherwise $\mathcal{N}$ would have a strategy in $\mathcal{M}$. Therefore, $\mathcal{M} \trianglelefteq \mathcal{K}$. $\qquad\square$

We can now do the core model induction through the levels of $Lp(\mathbb{R})$ as follows. We prove that if α is a critical ordinal and W_α^* holds (see [**?**] or [**?**] for the definitions of critical ordinals and W_α^*), then $W_{\alpha+1}^*$ holds. In order to show $W_{\alpha+1}^*$, we use the Witness Dichotomy, Theorem 3.6.1 of [**?**]. In order to apply the Witness Dichotomy, we need to define the mouse operator (or hybrid) mouse operator J_α^0 as in the statement of the Witness Dichotomy. If

1. $\alpha = \eta + 1$ for η a critical ordinal, or

2. α is a limit of critical ordinals and $\mathrm{cf}(\alpha) = \omega$, or

3. α is a limit of critical ordinals, $\mathrm{cf}(\alpha) > \omega$, and α is inadmissible,

then arguments in [**?**] define the appropriate J_α^0. Now we use Theorem **??** and the Witness Dichotomy applied to J_α^0 to get $W_{\alpha+1}^*$. Suppose we have reached a gap $[\alpha^*, \alpha]$ such that

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[12]We cut off the model at Ω and pretend that Ω is OR.

$Lp(\mathbb{R})|\alpha \vDash$ "$\Theta = \theta_0$" and $W^*_{\alpha^*}$ (and hence W^*_α) holds, then we can use the scales analysis of [?] and [?] and arguments in [?] to get an appropriate operator J^0_α (in this case, J^0_α is a strategy with branch condensation). We then use Theorem ?? and the Witness Dichotomy applied to J^0_α to get $W^*_{\alpha+1}$ (if $[\alpha^*, \alpha]$ is a strong gap, the scales analysis and arguments in [?] actually define a mouse operator $J^0_{\alpha+1}$; we then use the Witness Dichotomy and Theorem ?? to get $W^*_{\alpha+2}$). If we have reached a gap $[\alpha^*, \alpha]$ such that $Lp(\mathbb{R})|\alpha \vDash$ "$\Theta \neq \theta_0$" then using Lemma ?? we can skip through it and go to the least level beyond it that satisfies "$\Theta = \theta_0$"; this shows there is some $\beta > \alpha$ such that $W^*_{\beta+1}$ holds. At the end, we show that in $Lp(\mathbb{R})$, for all critical α, W^*_α holds; by [?, Lemma 3.3.5], $Lp(\mathbb{R}) \vDash \mathsf{AD}^+$. This completes our proof sketch of Theorem ??.

Remark. $\mathsf{AD}^{Lp(\mathbb{R})}$ is the most amount of determinacy one could hope to prove. This is because if μ comes from the Solovay measure (derived from winning strategies of real games) in an $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R} + \mathsf{SMC}$ universe, call it V (any $\mathsf{AD}_\mathbb{R} + V = L(\mathcal{P}(\mathbb{R}))$-model below "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" would do here), then $L(\mathbb{R}, \mu)^V \cap \mathcal{P}(\mathbb{R}) \subseteq Lp(\mathbb{R})^V$. This is because μ is OD hence $\mathcal{P}(\mathbb{R}) \cap L(\mu, \mathbb{R}) \subseteq \mathcal{P}_{\theta_0}(\mathbb{R})$. Since $\mathsf{AD}^+ + \mathsf{SMC}$ gives us that any set of reals of Wadge rank $< \theta_0$ is contained in an $\mathbb{R}$-mouse (by an unpublished result of Sargsyan and Steel but see [?]), we get that $\mathcal{P}(\mathbb{R}) \cap L(\mu, \mathbb{R}) \subseteq Lp(\mathbb{R})$ (it is conceivable that the inclusion is strict). By Theorem ??, $L(\mathbb{R}, \mu) \vDash \Theta = \theta_0$, which implies $L(\mathbb{R}, \mu) \vDash \mathcal{P}(\mathbb{R}) \subseteq Lp(\mathbb{R})$. Putting all of this together, we get $L(\mathbb{R}, \mu) \vDash Lp(\mathbb{R}) = L(\mathcal{P}(\mathbb{R})) + \mathsf{AD}^+$.

The above remark suggests that we should try to show that every set of reals in $V = L(\mathbb{R}, \mu)$ is captured by an $\mathbb{R}$-mouse, which will prove Theorem ??. This is accomplished in the next three sections.

5   AD in $L(\mathbb{R}, \mu)$

First we show $\Theta = \Theta^{Lp(\mathbb{R})}$. Suppose for contradiction that $\Theta^{Lp(\mathbb{R})} < \Theta$. We first show that there is a model containing $\mathbb{R} \cup \mathrm{OR} \cup Lp(\mathbb{R})$ that satisfies $\mathsf{AD}^+ + \Theta > \theta_0$". The argument will closely follow the argument in Chapter 7 of [?]. All of our key notions and notations come from there unless specified otherwise. Let $\Theta^* = \Theta^{Lp(\mathbb{R})}$. Let $\mathcal{M}_\infty$ be $\mathrm{HOD}^{Lp(\mathbb{R})} \restriction \Theta^*$. Then $\mathcal{M}_\infty = \mathcal{M}^+_\infty \restriction \Theta^*$ where $\mathcal{M}^+_\infty$ is the limit of a directed system (the hod limit system) indexed by pairs $(\mathcal{P}, \vec{A})$ where $\mathcal{P}$ is a suitable premouse, $\vec{A}$ is a finite sequence of OD sets of reals, and $\mathcal{P}$ is strongly $\vec{A}$-quasi-iterable in $Lp(\mathbb{R})$[13]. For more details on how the direct limit system is defined, the reader should consult Chapter 7 of [?]. Let Γ be the collection

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[13]The existence of such $\mathcal{P}$'s and hence the existence of $\mathcal{M}^+_\infty$ follows from the fact that in $Lp(\mathbb{R})$, Mouse Capturing (MC) holds.

of $\mathrm{OD}^{Lp(\mathbb{R})}$ sets of reals. For each $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $Lp(\sigma) \vDash \mathsf{AD}^+$, let $\mathcal{M}_\infty^\sigma$ and Γ^σ be defined the same as $\mathcal{M}_\infty$ and Γ but in $Lp(\sigma)$. Let $\Theta^\sigma = o(\mathcal{M}_\infty^\sigma)$. By $\mathsf{AD}^{Lp(\mathbb{R})}$ and $\Theta^* < \Theta$, we easily get

Lemma 5.1. $\forall_\mu^* \sigma (Lp(\sigma) \vDash \mathsf{AD}^+$ *and there is an elementary map* $\pi_\sigma : (Lp(\sigma), \mathcal{M}_\infty^\sigma, \Gamma^\sigma) \to (Lp(\mathbb{R}), \mathcal{M}_\infty, \Gamma).)$

Proof. First, it's easily seen that $Lp(\mathbb{R}) \vDash \mathsf{AD}^+$ implies $\forall_\mu^* \sigma Lp(\sigma) \vDash \mathsf{AD}^+$. We also have that letting ν be the induced measure on $\mathcal{P}_{\omega_1}(Lp(\mathbb{R}))$

$$\forall_\nu^* X \ X \prec Lp(\mathbb{R}).$$

The second clause of the lemma follows by transitive collapsing the X's above. Note that $\forall_\mu^* \sigma \ Lp(\sigma)$ is the uncollapse of some countable $X \prec Lp(\mathbb{R})$ such that $\mathbb{R}^X = \sigma$. This is because if $\mathcal{M}$ is an $\mathbb{R}$-mouse then $\forall_\nu^* X \ \mathcal{M} \in X$. The π_σ's are just the uncollapse maps. $\square$

Lemma 5.2. $(\forall_\mu^* \sigma)(Lp(\sigma) = Lp(\sigma)^{Lp(\mathbb{R})}).$

Proof. Suppose not. Fix a σ such that $Lp(\sigma) \vDash \mathsf{AD}^+$ and $\mathcal{M} \lhd Lp(\sigma)$ a sound mouse over σ, $\rho_\omega(\mathcal{M}) = \sigma$ and $\mathcal{M} \notin Lp(\sigma)^{Lp(\mathbb{R})}$. Let Λ be the strategy of $\mathcal{M}$. Then by a core model induction as in the previous section, we can show that $M =_{\mathrm{def}} L(Lp^\Lambda(\mathbb{R})) \vDash \mathsf{AD}^+ + \Theta > \theta_0$. We only mention a few key points for this induction. First, Λ is a $\omega_1 + 1$ strategy with condensation and $\forall_\mu^* \sigma \ \Lambda \restriction M_\sigma \in M_\sigma$ and $\forall_\mu^* \sigma \ \Lambda \restriction H_\sigma[\mathcal{M}] \in H_\sigma[\mathcal{M}]$. This allows us to lift Λ to a $\Omega + 1$ strategy in M and construct K^Λ up to Ω inside $\prod_\sigma H_\sigma[\mathcal{M}]$. Using this, we can run the proof of Theorem **??** for model operators F relative to Λ. Finally, we run the core model induction as outlined at the end of Section 4 to obtain $M \vDash \mathsf{AD}^+$.

We have shown that there is an $\mathcal{M} \lhd Lp(\sigma)$ such that $\mathcal{M}$'s unique strategy Λ has the property that

$$\Lambda \notin Lp(\mathbb{R}) \wedge M \vDash \mathsf{AD}^+ + \Theta > \theta_0.$$

By the remark before Theorem **??**, there are no AD^+-models M_1, M_2 both containing $\mathbb{R} \cup \mathrm{OR}$ such that $\mathcal{P}(\mathbb{R}) \cap M_1$ and $\mathcal{P}(\mathbb{R}) \cap M_2$ are Wadge incomparable[14]. This means $Lp(\mathbb{R}) \subset M$. Now we can use Lemma **??** applied to M to get a contradiction.

$\square$

Lemma 5.3. $\forall_\mu^* \sigma \ \mathcal{M}_\infty^\sigma$ *is full in* $Lp_2(\mathbb{R}) =_{def} Lp(Lp(\mathbb{R}))^{15}$ *in the sense that* $Lp(\mathcal{M}_\infty^\sigma) \vDash \Theta^\sigma$ *is Woodin.*

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[14]i.e. there is some $A \subseteq \mathbb{R}$ such that $A \in M_1 \backslash M_2$ and there is some $B \subseteq \mathbb{R}$ such that $B \in M_2 \backslash M_1$. By results of Woodin, the existence of Wadge incomparable AD^+-models implies the existence of $\mathsf{AD}_\mathbb{R}$-models, which cannot exist in our universe V.

[15]As shown in the previous section, $Lp(x)$ is the union of $F(x)$ where F comes from a mouse operator that relativizes well.

Proof. First note that $Lp_2(\sigma) =_{def} Lp(Lp(\sigma)) \vDash \mathsf{AD}^+ + \Theta = \theta_0$ because $\mathcal{P}(\mathbb{R})^{Lp_2(\sigma)} = \mathcal{P}(\mathbb{R})^{Lp(\sigma)}$. So suppose $Lp(\mathcal{M}_\infty^\sigma) \triangleright \mathcal{N}^\sigma \triangleright \mathcal{M}_\infty^\sigma$ is the Q-structure. It's easy to see, using footnote 15, that $\mathcal{N}^\sigma \in Lp_2(\sigma)$ and is in fact OD there.

Next we observe that in $Lp_2(\sigma)$, $\Theta = \Theta^\sigma$. By a theorem of Woodin, we know $\mathrm{HOD}^{Lp_2(\sigma)} \vDash \Theta^\sigma$ is Woodin (see Theorem 5.6 of [**?**]). But this is a contradiction to our assumption that $\mathcal{N}^\sigma$ is a Q-structure for Θ^σ. □

The last lemma shows that for a typical σ, $Lp_\omega(\mathcal{M}_\infty^\sigma)$ is suitable in $Lp(\mathbb{R})$. Let $\mathcal{M}_\infty^{\sigma,+}$ be the hod limit computed in $Lp(\sigma)$. Let $(\Gamma^\sigma)^{<\omega} = \{\vec{A}_n \mid n < \omega\}$ and for each $n < \omega$, let $\mathcal{N}_n$ be such that $\mathcal{N}_n$ is strongly $\vec{A}_n$-quasi-iterable in $Lp(\sigma)$ such that $\mathcal{M}_\infty^{\sigma,+}$ is the quasi-limit of the $\mathcal{N}_n$'s in $Lp(\sigma)$. Let $\mathcal{M}_\infty^{\sigma,*}$ be the quasi-limit of the $\mathcal{N}_n$'s in $Lp(\mathbb{R})$. We'll show that $\pi_\sigma''\Gamma^\sigma$ is cofinal in Γ, $\mathcal{M}_\infty^{\sigma,+} = \mathcal{M}_\infty^{\sigma,*} = Lp_\omega(\mathcal{M}_\infty^\sigma)$ and hence $\mathcal{M}_\infty^{\sigma,+}$ is strongly A-quasi-iterable in $Lp(\mathbb{R})$ for each $A \in \pi_\sigma''\Gamma^\sigma$. From this we'll get a strategy Σ_σ for $\mathcal{M}_\infty^{\sigma,+}$ with weak condensation. This proceeds much like the proof in Chapter 7 of [**?**].

Let T be the tree for a universal $(\Sigma_1^2)^{Lp(\mathbb{R})}$-set; let $T^* = \prod_\sigma T/\mu$ and $T^{**} = \prod_\sigma T^*/\mu$. To show $(\forall_\mu^* \sigma)(\pi_\sigma''\Gamma^\sigma$ is cofinal in $\Gamma)$ we first observe that

$$(\forall_\mu^* \sigma)(L[T^*, \mathcal{M}_\infty^\sigma]|\Theta^\sigma = \mathcal{M}_\infty^\sigma),$$

that is, T^* does not create Q-structures for $\mathcal{M}_\infty^\sigma$. This is because $\mathcal{M}_\infty^\sigma$ is countable, ω_1^V is inaccessible in any inner model of choice, $L[T^*, \mathcal{M}_\infty^\sigma]|\omega_1^V = L[T, \mathcal{M}_\infty^\sigma]|\omega_1^V$, and $L[T, \mathcal{M}_\infty^\sigma]|\Theta^\sigma = \mathcal{M}_\infty^\sigma$ by Lemma **??**. Next, let E_σ be the extender derived from π_σ with generators in $[\gamma]^{<\omega}$, where $\gamma = \sup\pi_\sigma''\Theta^\sigma$. By the above, E_σ is a pre-extender over $L[T^*, \mathcal{M}_\infty^\sigma]$.

Lemma 5.4. $(\forall_\sigma^* \sigma)(Ult(L[T^*, \mathcal{M}_\infty^\sigma], E_\sigma)$ is wellfounded$)$.

Proof. The statement of the lemma is equivalent to

$$Ult(L[T^{**}, \mathcal{M}_\infty], \Pi_\sigma E_\sigma/\mu) \text{ is wellfounded. } (*)$$

To see $(*)$, note that

$$\prod_\sigma E_\sigma = E_\mu$$

where E_μ is the extender from the ultrapower map j_μ by μ (with generators in $[\xi]^{<\omega}$, where $\xi = \sup j_\mu''\Theta^*$). This uses normality of μ. We should metion that the equality above should be interpreted as saying: the embedding by $\Pi_\sigma E_\sigma/\mu$ agrees with j_μ on all ordinals (less than Θ).

Since μ is countably complete and DC holds, we have that $Ult(L[T^{**}, \mathcal{M}_\infty], E_\mu)$ is wellfounded. Hence we're done. □

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Theorem 5.5.   *1. $(\forall^*_\mu \sigma)(\pi_\sigma$ is continuous at $\theta^\sigma)$. Hence $cof(\Theta^{Lp(\mathbb{R})}) = \omega$.*

2. If $i : \mathcal{M}^\sigma_\infty \to S$, and $j : S \to \mathcal{M}_\infty$ are elementary and $\pi_\sigma = j \circ i$ and S is countable in $Lp(\mathbb{R})$, then S is full in $Lp(\mathbb{R})$. In fact, if W is the collapse of a hull of S containing $rng(i)$, then W is full in $Lp(\mathbb{R})$.

Proof. The keys are Lemma **??** and the fact that the tree T^*, which enforces fullness for $\mathbb{R}$-mice, does not generate Q-structures for $\mathcal{M}^\sigma_\infty$. To see (1), suppose not. Fix a typical σ for which (1) fails. Let $\gamma = \sup\pi''_\sigma\Theta^\sigma < \Theta^*$. Let E_σ be the extender derived from π_σ with generators in $[\gamma]^{<\omega}$ and consider the ultrapower map

$$\tau : L[T^*, \mathcal{M}^\sigma_\infty] \to N_\sigma =_{def} Ult(L[T^*, \mathcal{M}^\sigma_\infty], E_\sigma).$$

We may as well assume N_σ is transitive by Lemma **??**. We have that τ is continuous at Θ^σ and $N_\sigma \vDash o(\tau(\mathcal{M}^\sigma_\infty))$ is Woodin. Since $o(\tau(\mathcal{M}^\sigma_\infty)) = \gamma < \Theta^*$, there is a Q-structure $\mathcal{Q}$ for $o(\tau(\mathcal{M}^\sigma_\infty))$ in $Lp(\mathbb{R})$. But $\mathcal{Q}$ can be constructed from T^*, hence from $\tau(T^*)$. To see this, suppose $\mathcal{Q} = \Pi_\sigma\mathcal{Q}_\sigma/\mu$ and $\gamma = \Pi_\sigma\gamma_\sigma/\mu$. Then $\forall^*_\mu\sigma$ $\mathcal{Q}_\sigma$ is the Q-structure for $\mathcal{M}^\sigma_\infty|\gamma_\sigma$ and the iterability of $\mathcal{Q}_\sigma$ is certified by T. This implies the iterability of $\mathcal{Q}$ is certified by T^*. But $\tau(T^*) \in N_\sigma$, which does not have Q-structures for $\tau(\mathcal{M}^\sigma_\infty)$. Contradiction.

(1) shows then that $\pi''_\sigma\Gamma^\sigma$ is cofinal in Γ. The proof of (2) is similar. We just prove the first statement of (2). The point is that i can be lifted to an elementary map

$$i^* : L[T^*, \mathcal{M}^\sigma_\infty] \to L[\overline{T}, S]$$

for some $\overline{T}$ and j can be lifted to

$$j^* : L[\overline{T}, S] \to N_\sigma$$

by the following definition
$$j^*(i^*(f)(a)) = \tau(f)(j(a))$$

for $f \in L[T^*, \mathcal{M}^\sigma_\infty]$ and $a \in [o(\mathcal{S})]^{<\omega}$. By the same argument as above, $\overline{T}$ certifies iterability of mice in $Lp(\mathbb{R})$ and hence enforces fullness for S in $Lp(\mathbb{R})$. This is what we want. $\square$

We can define a map $\tau : \mathcal{M}^{\sigma,+}_\infty \to \mathcal{M}^{\sigma,*}_\infty$ as follows. Let $x \in \mathcal{M}^{\sigma,+}_\infty$. There is an $i < \omega$ and a y such that in $Lp(\sigma)$, $x = \pi^{A_i}_{\mathcal{N}_i,\infty}(y)$, where $\pi^{A_i}_{\mathcal{N}_i,\infty}$ is the direct limit map from $H^{\mathcal{N}_i}_{A_i}$ into $\mathcal{M}^{\sigma,+}_\infty$ in $Lp(\sigma)$. Let

$$\tau(x) = \pi^{A_i}_{\mathcal{N}_i,\mathcal{M}^{\sigma,*}_\infty}(y),$$

where $\pi^{A_i}_{\mathcal{N}_i,\mathcal{M}^{\sigma,*}_\infty}$ witnesses $(\mathcal{N}_i, A_i) \preceq (\mathcal{M}^{\sigma,*}_\infty, A_i)$ in the hod direct limit system in $Lp(\mathbb{R})$.

Lemma 5.6. *1.* $\mathcal{M}_\infty^{\sigma,*} = H_{\pi_\sigma''\Gamma_\sigma}^{\mathcal{M}_\infty^{\sigma,*}}$; *furthermore, for any quasi-iterate $\mathcal{Q}$ of $\mathcal{M}_\infty^{\sigma,*}$, $\mathcal{Q} = H_{\pi_\sigma''\Gamma_\sigma}^{\mathcal{Q}}$ and $\pi_{\mathcal{M}_\infty^{\sigma,*},\mathcal{Q}}^{\pi_\sigma''\Gamma_\sigma}(\tau_A^{\mathcal{M}_\infty^{\sigma,+}}) = \tau_A^{\mathcal{Q}}$ for all $A \in \pi_\sigma''\Gamma_\sigma$.*

2. $\tau = id$ and $\mathcal{M}_\infty^{\sigma,+} = \mathcal{M}_\infty^{\sigma,*}$.

3. $\pi_\sigma = \pi_{\mathcal{M}_\infty^{\sigma,+},\infty}^{\pi_\sigma''\Gamma_\sigma}$.

Proof. The proof is just that of Lemmata 7.8.7 and 7.8.8 in [**?**]. We first show (1). In this proof, "suitable" means suitable in $Lp(\mathbb{R})$. The key is for any quasi-iterate $\mathcal{Q}$ of $\mathcal{M}_\infty^{\sigma,*}$, we have

$$\pi_\sigma|\mathcal{M}_\infty^{\sigma,+} = \pi_{\mathcal{Q},\infty}^{\pi_\sigma''\Gamma_\sigma} \circ \pi_{\mathcal{M}_\infty^{\sigma,*},\mathcal{Q}}^{\pi_\sigma''\Gamma_\sigma} \circ \tau. \quad (*)$$

Using this and Theorem **??**, we get $H_{\pi_\sigma''\Gamma_\sigma}^{\mathcal{Q}} = \mathcal{Q}$ for any quasi-iterate $\mathcal{Q}$ of $\mathcal{M}_\infty^{\sigma,*}$. To see this, first note that $\mathcal{Q}$ is suitable; Theorem **??** implies the collapse $\mathcal{S}$ of $H_{\pi_\sigma''\Gamma_\sigma}^{\mathcal{Q}}$ must be suitable. This means, letting δ be the Woodin of $\mathcal{Q}$, $H_{\pi_\sigma''\Gamma_\sigma}^{\mathcal{Q}}|(\delta + 1) = \mathcal{Q}|(\delta + 1)$. Next, we show $H_{\pi_\sigma''\Gamma_\sigma}^{\mathcal{Q}}|((\delta^+)^{\mathcal{Q}}) = \mathcal{Q}|((\delta^+)^{\mathcal{Q}})$. The proof of this is essentially that of Lemma 4.35 in [**?**]. We sketch the proof here. Suppose not. Let $\pi : \mathcal{S} \to \mathcal{Q}$ be the uncollapse map. Note that $\text{crt}(\pi) = (\delta^+)^{\mathcal{S}}$ and $\pi((\delta^+)^{\mathcal{S}}) = (\delta^+)^{\mathcal{Q}}$. Let $\mathcal{R}$ be the result of first moving the least measurable of $\mathcal{Q}|((\delta^+)^{\mathcal{Q}})$ above δ and then doing the genericity iteration (inside $\mathcal{Q}$) of the resulting model to make $\mathcal{Q}|\delta$ generic at the Woodin of $\mathcal{R}$. Let $\mathcal{T}$ be the resulting tree. Then $\mathcal{T}$ is maximal with $lh(\mathcal{T}) = (\delta^+)^{\mathcal{Q}}$; $\mathcal{R} = Lp(\mathcal{M}(\mathcal{T}))$; and the Woodin of $\mathcal{R}$ is $(\delta^+)^{\mathcal{Q}}$. Since $\{\gamma_A^{\mathcal{R}} \mid A \in \pi_\sigma''\Gamma_\sigma\}$ are definable from $\{\tau_{A,(\delta^+)^{\mathcal{Q}}}^{\mathcal{Q}} \mid A \in \pi_\sigma''\Gamma_\sigma\}$, they are in $\text{rng}(\pi)$. This gives us that $\sup H_{\pi_\sigma''\Gamma_\sigma}^{\mathcal{Q}} \cap (\delta^+)^{\mathcal{Q}} = (\delta^+)^{\mathcal{Q}}$, which easily implies $(\delta^+)^{\mathcal{Q}} \subseteq H_{\pi_\sigma''\Gamma_\sigma}^{\mathcal{Q}}$. The proof that $(\delta^{+n})^{\mathcal{Q}} \subseteq H_{\pi_\sigma''\Gamma_\sigma}^{\mathcal{Q}}$ for $1 < n < \omega$ is similar and is left for the reader.

(2) easily follows from (1). (3) follows using $(*)$ and $\tau = id$. $\qquad\qquad\square$

For each σ such that Theorem **??** and Lemma **??** hold for σ, let Σ_σ be the canonical strategy for $\mathcal{M}_\infty^\sigma$ as guided by $\pi_\sigma''\Gamma^\sigma$. Recall $\pi_\sigma''\Gamma^\sigma$ is a cofinal collection of $\text{OD}^{Lp(\mathbb{R})}$ sets of reals. The existence of Σ_σ follows from Theorem 7.8.9 in [**?**]. Note that Σ_σ has weak condensation, i.e., suppose $\mathcal{Q}$ is a Σ_σ iterate of $\mathcal{M}_\infty^{\sigma,+}$ and $i : \mathcal{M}_\infty^{\sigma,+} \to \mathcal{Q}$ is the iteration map, and suppose $j : \mathcal{M}_\infty^{\sigma,+} \to \mathcal{R}$ and $k : \mathcal{R} \to \mathcal{Q}$ are such that $i = k \circ j$ then $\mathcal{R}$ is suitable (in the sense of $Lp(\mathbb{R})$).

Definition 5.7 (Branch condensation). *Let $\mathcal{M}_\infty^{\sigma,+}$ and Σ_σ be as above. We say that Σ_σ has branch condensation if for any Σ_σ iterate $\mathcal{Q}$ of $\mathcal{M}_\infty^{\sigma,+}$, letting $k : \mathcal{M}_\infty^{\sigma,+} \to \mathcal{Q}$ be the iteration map, for any maximal tree $\mathcal{T}$ on $\mathcal{M}_\infty^{\sigma,+}$, for any cofinal non-dropping branch b of $\mathcal{T}$, letting $i = i_b^{\mathcal{T}}$, $j : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{P}$, where $\mathcal{P}$ is a Σ_σ iterate of $\mathcal{M}_\infty^\sigma$ with iteration embedding k, suppose $k = j \circ i$, then $b = \Sigma_\sigma(\mathcal{T})$.*

Theorem 5.8. $(\forall_\mu^* \sigma)(A \text{ tail of } \Sigma_\sigma \text{ has branch condensation.})$

Proof. The proof is like that of Theorem 7.9.1 in [**?**]. We only mention the key points here. We assume that $\forall^*_\mu \sigma$ no Σ_σ-tails have branch condensation. Fix such a σ. First, let $X_\sigma = \mathrm{rng}(\pi_\sigma \upharpoonright \mathcal{M}^{\sigma,+}_\infty)$ and

$$H = \mathrm{HOD}_{\{\mu, \mathcal{M}^{\sigma,+}_\infty, \mathcal{M}_\infty, \pi_\sigma, \mathrm{T}^*, X_\sigma, x_\sigma\}},$$

where x_σ is a real enumerating $M^{\sigma,+}_\infty$. So $H \vDash \mathsf{ZFC} +$ "$\mathcal{M}^\sigma_\infty$ is countable and ω^V_1 is measurable."

Next, let $\overline{H}$ be a collapse of a countable elementary substructure of a sufficiently large rank-initial segment of H. Let $(\gamma, \rho, \mathcal{N}, \nu)$ be the preimage of $(\omega^V_1, \pi_\sigma, \mathcal{M}_\infty, \mu)$ under the uncollapse map, call it π. We have that $\overline{H} \vDash \mathsf{ZFC}^- +$ "γ is a measurable cardinal as witnessed by ν." This $\overline{H}$ will replace the countable iterable structure obtained from the hypothesis HI(c) in Chapter 7 of [**?**]. Now, in $Lp(\mathbb{R})$, the following hold true:

1. There is a term $\tau \in \overline{H}$ such that whenever g is a generic over $\overline{H}$ for $Col(\omega, < \gamma)$, then τ^g is a $(\rho, \mathcal{M}^{\sigma,+}_\infty, \mathcal{N})$-certified bad sequence. See Definitions 7.9.3 and 7.9.4 in [**?**] for the notions of a bad sequence and a $(\rho, \mathcal{M}^{\sigma,+}_\infty, \mathcal{N})$-certified bad sequence respectively.

2. Whenever $i : \overline{H} \to J$ is a countable linear iteration map by the measure ν and g is J-generic for $Col(\omega, < i(\gamma))$, then $i(\tau)^g$ is truly a bad sequence.

The proof of (1) and (2) is just like that of Lemma 7.9.7 in [**?**]. The key is that in (1), any $(\rho, \mathcal{M}^{\sigma,+}_\infty, \mathcal{N})$-certified bad sequence is truly a bad sequence from the point of view of $Lp(\mathbb{R})$ and in (2), any countable linear iterate J of $\overline{H}$ can be realized back into H by a map ψ in such a way that $\pi = \psi \circ i$.

Finally, using (1), (2), the iterability of $\overline{H}$, and an AD^+-reflection in $Lp(\mathbb{R})$ like that in Theorem 7.9.1 in [**?**], we get a contradiction. $\qquad\square$

Fix a σ as in Theorem **??** and let Σ'_σ be a tail of Σ_σ with branch condensation. By the construction of Σ_σ, $\Sigma'_\sigma \notin Lp(\mathbb{R})$. Using μ and the fact that $\forall^*_\mu \tau \ \Sigma'_\sigma \upharpoonright M_\tau \in M_\tau$, we can lift Σ'_σ to a strategy on H^M_Ω with branch condensation, which we also call Σ'_σ. An argument as in Lemma **??** shows that $Lp^{\Sigma'_\sigma}(\mathbb{R})^{16} \vDash \mathsf{AD}^+$. This along with the fact that $\Sigma'_\sigma \notin Lp(\mathbb{R})$ imply

$$L(Lp^{\Sigma'_\sigma}(\mathbb{R})) \vDash \Theta > \theta_0.$$

Furthermore, since Σ'_σ is $Lp(\mathbb{R})$-fullness preserving (in fact is guided by a self-justifying system cofinal in $\mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})$), $\mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})$ is a Wadge initial segment of $\mathcal{P}(\mathbb{R}) \cap L(\Sigma'_\sigma, \mathbb{R})$.

Recall that T is the tree for a universal Σ^2_1-in-$Lp(\mathbb{R})$-set and $T^* = \prod_\sigma T/\mu$.

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[16] $K^{\Sigma'_\sigma}(\mathbb{R})$ is the union of all sound Σ'_σ-$\mathbb{R}$-premice $\mathcal{M}$ such that $\rho_\omega(\mathcal{M}) = \mathbb{R}$ and every countable $\mathcal{M}^*$ embeddable into $\mathcal{M}$ has a unique $(\omega, \omega_1 + 1)$-iteration Σ'_σ strategy. See [**?**] or [**?**] for a constructive definition of $K^{\Sigma'_\sigma}(\mathbb{R})$.

Lemma 5.9. *Let* $\mathcal{P}(\mathbb{R}) \cap L(T^*, \mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})$.

Proof. By MC in $Lp(\mathbb{R})$, we have

$$(\forall_\mu^* \sigma)(\mathcal{P}(\sigma) \cap L(T, \sigma) = Lp(\sigma) \cap \mathcal{P}(\sigma)).$$

This proves the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now show that μ is amenable to $Lp(\mathbb{R})$ in the sense that μ restricting to any Wadge initial segment of $\mathcal{P}(\mathbb{R})^{Lp(\mathbb{R})}$ is in $Lp(\mathbb{R})$.

Lemma 5.10. *Suppose* $S = \{(x, A_x) \mid x \in \mathbb{R} \ \wedge \ A_x \in \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))\} \in Lp(\mathbb{R})$. *Then* $\mu \upharpoonright S = \{(x, A_x) \mid \mu(A_x) = 1\} \in Lp(\mathbb{R})$.

Proof. Let A_S be an ∞-Borel code[17] for S in $Lp(\mathbb{R})$. The existence of A_S follows from the fact that $Lp(\mathbb{R}) \models \mathsf{AD}^+$. We may pick A_S such that it is a bounded subset of Θ^*. We may as well assume that A_S is $\mathrm{OD}^{Lp(\mathbb{R})}$ and A_S codes T. This gives us

$$(\forall_\mu^* \sigma)(\mathcal{P}(\sigma) \cap L(A_S, \sigma) = \mathcal{P}(\sigma) \cap L(T, \sigma)),$$

or equivalently letting $A_S^* = \prod_\sigma A_S$,

$$\mathcal{P}(\mathbb{R}) \cap L(A_S^*, \mathbb{R}) = L(T^*, \mathbb{R}).$$

We have the following equivalences:

$$
\begin{aligned}
(x, A_x) \in \mu \upharpoonright S \quad &\Leftrightarrow \quad (\forall_\mu^* \sigma)(\sigma \in A_x \cap \mathcal{P}_{\omega_1}(\sigma)) \\
&\Leftrightarrow \quad (\forall_\mu^* \sigma)(L(A_S, \sigma) \models \emptyset \Vdash_{Col(\omega,\sigma)} \check{\sigma} \in A_x \cap \mathcal{P}_{\omega_1}(\sigma)) \\
&\Leftrightarrow \quad L(A_S^*, \mathbb{R}) \models \emptyset \Vdash_{Col(\omega,\mathbb{R})} \check{\mathbb{R}} \in A_x.
\end{aligned}
$$

The above equivalences show that $\mu \upharpoonright S \in L(S^*, \mathbb{R})$. But by Lemma **??** and the fact that $\mu \upharpoonright S$ can be coded as a set of reals in $L(S^*, \mathbb{R})$, hence $\mu \upharpoonright S \in L(T^*, \mathbb{R})$, we have that $\mu \upharpoonright S \in Lp(\mathbb{R})$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Recall from our assumption that $\Theta > \Theta^{Lp(\mathbb{R})} = \theta_0^{L(\Sigma'_\sigma, \mathbb{R})}$, we construct the model $M = Lp^{\Sigma'_\sigma}(\mathbb{R})$ such that

$$\mathbb{R} \cup \mathrm{OR} \cup Lp(\mathbb{R}) \subseteq L(Lp^{\Sigma'_\sigma}(\mathbb{R})) \wedge L(Lp^{\Sigma'_\sigma}(\mathbb{R})) \models \mathsf{AD}^+ + \Theta > \theta_0.$$

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[17]If $S \subseteq \mathbb{R}$, A_S is an ∞-Borel code for S if $A_S = (T, \psi)$ where T is a set of ordinals and ψ is a formula such that for all $x \in \mathbb{R}$, $x \in S \Leftrightarrow L[T, x] \models \psi[T, x]$.

Lemma **??** shows that $M = L(Lp^{\Sigma'_\sigma}(\mathbb{R}))$ cannot exist. This in turns implies $\Theta^{Lp(\mathbb{R})} = \Theta$. We finish by showing $\mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R}) = \mathcal{P}(\mathbb{R})$.

Lemma 5.11. $\mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R}) = \mathcal{P}(\mathbb{R})$. *Hence* $L(\mathbb{R}, \mu) \vDash$ AD.

Proof. First we observe that if α is such that there is a new set of reals in $L_{\alpha+1}(\mathbb{R})[\mu] \backslash L_\alpha(\mathbb{R})[\mu]$ then there is a surjection from $\mathbb{R}$ onto $L_\alpha(\mathbb{R})[\mu]$. This is because the predicate μ is a predicate for a subset of $\mathcal{P}(\mathbb{R})$, which collapses to itself under collapsing of hulls of $L_\alpha(\mathbb{R})[\mu]$ that contain all reals. With this observation, the usual proof of condensation (for L) goes through with one modification: one must put all reals into hulls one takes.

Now suppose for a contradiction that there is an $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \mu)$ such that $A \notin Lp(\mathbb{R})$. Let α be least such that $A \in L_{\alpha+1}(\mathbb{R})[\mu] \backslash L_\alpha(\mathbb{R})[\mu]$. We may assume that $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subseteq Lp(\mathbb{R})$. By the above observation, $\alpha < \Theta = \Theta^{Lp(\mathbb{R})}$ because otherwise, there is a surjection from $\mathbb{R}$ on Θ, which contradicts the definition of Θ. Now if $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \subsetneq \mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})$, then by Lemma **??**, $\mu \upharpoonright \mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] \in Lp(\mathbb{R})$. But this means $A \in Lp(\mathbb{R})$. So we may assume $\mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu] = \mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})$. But this means that we can in $L_\Theta(\mathbb{R})[\mu]$ use $\mu \upharpoonright \mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})[\mu]$ compute $\Theta^{Lp(\mathbb{R})}$ and this contradicts the fact that $\Theta^{Lp(\mathbb{R})} = \Theta$. $\qquad\square$

6 Open problems and questions

We first mention the following

Conjecture: Suppose $L(\mathbb{R}) \vDash$ DC $+ \Theta$ is inaccessible. Then $L(\mathbb{R}) \vDash$ AD.

This is arguably the analogous statement in $L(\mathbb{R})$ of our main theorem. It is tempting to conjecture that if $L(\mathbb{R}) \vDash$ DC $+ \Theta > \omega_2$ then $L(\mathbb{R}) \vDash$ AD but this is known to be false by theorems of Harrington [**?**]. Next, we mention the following uniqueness problem which concerns the relationship between AD models of the form $L(\mathbb{R}, \mu)$.

Open problem: Suppose $L(\mathbb{R}, \mu_i) \vDash$ "ZF $+$ DC $+$ AD $+ \mu_i$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$" for $i = 0, 1$. Must $L(\mathbb{R}, \mu_0) = L(\mathbb{R}, \mu_1)$?

We suspect that the answer is no but haven't been able to construct two distinct models of the form $L(\mathbb{R}, \mu)$ that satisfy AD. By Theorem **??**, if $L(\mathbb{R}, \mu_0)$ and $L(\mathbb{R}, \mu_1)$ are the same model then $\mu_0 \cap L(\mathbb{R}, \mu_0) = \mu_1 \cap L(\mathbb{R}, \mu_1)$. A generalization of the problem proved in this paper is to consider determinacy in models of the form $L(S, \mathbb{R}, \mu)$ where S is a set of ordinals and $L(S, \mathbb{R}, \mu) \vDash$ "ZF $+$ DC $+ \Theta > \omega_2 + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$". Here is a

(vague) conjecture.

Conjecture: Let $L(S, \mathbb{R}, \mu)$ be as above. Let $\Theta = \Theta^{L(S,\mathbb{R},\mu)}$ and M_∞ be the maximal model of determinacy in $L(S, \mathbb{R}, \mu)$. Then either $\Theta^{M_\infty} = \Theta$ or there is a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" containing $\mathbb{R} \cup \mathrm{OR}$.

In another direction, we could ask about how to identify the first stage in the core model induction (under appropriate hypotheses) that reaches $\mathsf{AD}^{L(\mathbb{R},\mu)}$ where μ comes from some filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and $L(\mathbb{R}, \mu) \vDash$ "μ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$". A problem of this kind is the following

Open problem: Suppose NS_{ω_1} is saturated and $\mathsf{WRP}_2^*(\omega_2)$[18]. Must there be a filter μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that $L(\mathbb{R}, \mu) \vDash$ "$\mathsf{AD} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$"?

This problem is discussed in [**?**] though in a slightly different formulation. The point is that the hypothesis of the problem is obtained in a $\mathbb{P}_{\max}$-extension of a model of the form $L(\mathbb{R}, \mu) \vDash$ "$\mathsf{AD} + \mu$ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$".

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[18]NS_{ω_1} is the nonstationary ideal on ω_1 and $\mathsf{WRP}_2^*(\omega_2)$ is defined in section 9.5 of [**?**].