RESEARCH STATEMENT

NAM TRANG

My research interest in pure mathematics is in set theory and set-theoretic topology. My current research focuses on studying models of various fragments of determinacy for games of infinite length on natural numbers or reals, studying the connections between inner models, structures of sets of reals (like those of determined sets in Polish spaces and quotient spaces of Polish spaces by various equivalence relations), hybrid structures (such as HOD[1] of determinacy models), forcing, and strong combinatorial principles. I'm also interested in applications of strong forcing axioms such as PFA and their connection with other combinatorial principles in set theory. As mentioned, my other interest in pure mathematics is in set-theoretic topology, in particular the study of Ramsey-like spaces on functions on infinite ordinals. Some of the techniques mentioned above are used in combination with combinatorial techniques developed specifically to understand and solve various fundamental topological and combinatorial problems in Ramsey-like spaces of functions on various infinite cardinals. This work has earned me two regular NSF grants and currently an NSF CAREER grant. I describe this work in Section I.

My research interest in applied mathematics includes various aspects of mathematical finance and applied statistics, and in deep learning as applied to computer vision. My work in computer vision, as part of a team of industry expert, is being supported by a DARPA grant. I describe some ongoing work in this area in Section II.

Detailed information about me and my research can be found at https://math.unt.edu/∼ntrang.

I. LOGIC AND SET THEORY

The research described below mostly belongs to an area of set theory called *descriptive inner model theory* (DIMT). DIMT is an emerging field in set theory that explores deep connections between *descriptive set theory* (DST) and *inner model theory* (IMT). DST studies a certain class of well-behaved subsets of the reals and of Polish spaces (e.g. Borel sets, analytic sets) and has its roots in classical analysis, through work of Baire, Borel, Lebesgue, Lusin, Suslin and others. One way a collection Γ of subsets of a Polish space can be well-behaved is that they satisfy various regularity properties, e.g. they have the Baire property, every uncountable set in Γ contains a perfect subset, every set in Γ is Lebesgue measurable. A cornerstone in the history of the subject is the discovery of the Axiom of Determinacy (AD) by Mycielsky and Steinhaus in 1962. AD states that every infinite-length, two-person game of perfect information where players take turns play integers is determined, i.e. one of the players has a winning strategy. If every set in $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is determined, then they have all the regularity properties listed above (and more), and hence very well-behaved. AD contradicts the axiom of choice as the latter implies the existence of very irregular sets like the Vitali set; however, inside a universe of ZFC, there may be many interesting sub-universes (models) that satisfy AD, for instance, $L(\mathbb{R})$ the minimal transitive class model of ZF that contains the reals may satisfy AD. One important and fruitful branch of descriptive set theory studies structure theory of models of AD. IMT forms one of the core subjects in modern set theory; its main objective is study "canonical" models of various extensions of ZFC, called *large cardinal axioms* (or simply large cardinals) and construct such models under various circumstances (e.g. see question (2) below). The large cardinal axioms form a linear hierarchy of axioms (in terms of consistency strength) extending ZFC and every known, natural axiom in mathematics/set theory is decided by one such axiom.[2] The first "canonical model" of large cardinals is *Gödel's constructible universe* L, the minimal model of ZFC. It is well-known that L cannot admit "very large" large cardinals; *the Gödel's inner model program*, a major program in inner model theory, aims to construct and analyze L-like models that can accommodate larger large cardinals under various hypotheses. Benchmark properties that help determine the canonicity of these models include *the Generalized Continuum Hypothesis* (GCH), Jensen's $\square$-principles (see Question (1), more details later).

DIMT uses tools from both DST and IMT to study and deepen the connection between canonical models of large cardinals and canonical models of AD. One of the first significant developments in DIMT comes in the 1980's with works of Martin, Steel, Woodin and others; their work, for instance, shows that one can construct models of AD (e.g. they showed AD holds in $L(\mathbb{R})$) assuming large cardinal axioms (those that involve the crucial notion of

[1] HOD stands for the class of "Hereditarily Ordinal Definable" sets, see [15] for a definition.

[2] By Gödel's incompleteness theorem, given any axiom (A) extending Peano Arithmetic, one can construct a sentence, albeit unnatural, which is not decidable by (A).

Woodin cardinals) and conversely, one can recover models of large cardinal axioms from models of AD. The key to uncover these connections is to analyze structure theory of models of AD. Much of DIMT and my research go along this line (e.g. see Question (1) below). The conjecture that "PFA has the exact consistency strength as that of a supercompact cardinal" is one of the most longstanding and arguably important open problems in set theory. Techniques recently developed in DIMT and to some extent from the research described here also enable us to make significant progress in calibrating the consistency strength lower-bound for PFA (Question (2), to be discussed in detail later).

We now describe these connections in more technical details in the next couple of paragraphs.[3] One way of formalizing this connection is through *the Mouse Set Conjecture* (MSC), which states that, assuming the AD or a more technical version of it (AD^+), then whenever a real x is ordinal definable from a real y, then x belongs to a canonical model of large cardinal (mouse) over y. MSC conjectures that the most complicated form of definability can be captured by canonical structures of large cardinals. An early instance of this is a well-known theorem of Shoenfield that every Δ^1_2 real is in the Gödel's constructible universe L. Another instance is a theorem of W.H. Woodin's that in the minimal class model containing all the reals, $L(\mathbb{R})$, if AD holds, then MSC holds. However, the full MSC is open and is one of the main open problems in DIMT.[4] Instances of MSC have been proved in determinacy models constructed by their sets of reals (much larger than $L(\mathbb{R})$) and these proofs typically obtain canonical models of large cardinals (mice) that capture the relevant ordinal definable real by analyzing HOD of the determinacy models. Hence, the key link between these two kinds of structures (models of large cardinals and models of determinacy) is the HOD of the determinacy models. A central notion in the proof of MSC and the analysis of HOD is the notion of *hod mice* (developed by G. Sargsyan, cf. [27], which built on and generalized earlier unpublished work of H.W. Woodin), which features heavily in my research described here. Hod mice are a type of models constructed from an extender sequence and a sequence of iteration strategies of its own initial segments. The extender sequence allows hod mice to satisfy some large cardinal theory and the strategies allow them to generate models of determinacy. Unlike pure extender models, there are many basic structural questions that are still open for hod mice (to be discussed later).

Another way of exploring the determinacy/large cardinal connection is via the Core Model Induction (CMI), which draws strength from natural theories such as PFA to inductively construct canonical models of determinacy and those of large cardinals in a locked-step process by combining core model techniques (for constructing the core model K) with descriptive set theory, in particular the scales analysis in $L(\mathbb{R})$ and its generalizations. CMI is the only known systematic method for computing lower-bound consistency strength of strong theories extending ZFC and it is hoped that it will allow one to compute the exact strength of important theories such as PFA. CMI is another central theme of my research. Much work has been done the last several years in developing techniques for CMI and it's only recently realized that constructing hod mice in non-AD^+ contexts (e.g. in the context of PFA, in contrast to Sargsyan's constructions) is a key step in CMI.

A large part of my research described here is devoted to studying structural properties of hod mice, developing methods for constructing hod mice in non-AD^+ contexts, and applying these constructions in the core model induction. One of the main goals is to construct determinacy models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular", "$\mathsf{AD}^+ +$ the largest Suslin cardinal is on the Solovay sequence" (LSA) and beyond, from various theories such as PFA, the existence of countably closed guessing models (cf. [47]) (see Section 2), and failures of *the Unique Branch Hypothesis* (UBH) (see Section 2). These lower-bounds are beyond the reach of pure core model methods.

There are many problems in DIMT that I've been interested in and working on but my hope is to make progress toward answering two fundamental questions in the area (to be discussed in details later):

(1) Is HOD of a determinacy model fine-structural (e.g. do the $\mathsf{GCH}, \Box$ hold in HOD)? What large cardinals can HOD accommodate?

(2) What is the consistency strength of PFA?

The three main aspects of my research are as follows: (a) connections between inner models, hybrid structures, and canonical sets of reals; (b) applications of the structure theory of the three hierarchies in (a) and their connections; (c) strong combinatorial principles, determinacy, and large cardinals through the lens of forcing. These three areas will be described in the following three sections. There, I discuss selected problems that will generate interesting results and progress in each area. Many of the important problems listed below are either (indirectly) related to or

[3] Non-logicians may want to simply skim this through.

[4] It is widely believed that a more fundamental property called *generation of pointclasses* or *hod pair capturing*. Current research focuses on proving (instances of) this property.

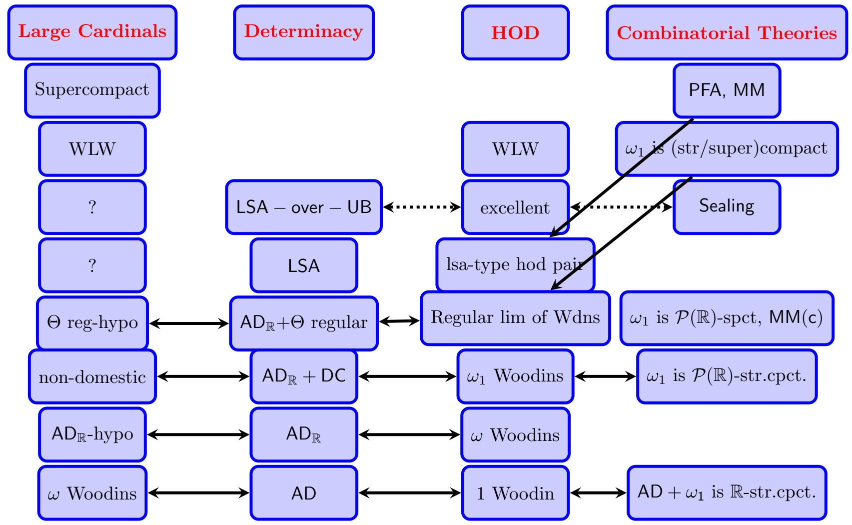


Figure 1: Consistency strength of various theories

(directly) elaborated from problems (1) and (2) above and hence quite ambitious. However, I think these problems are worth pursuing as they are important for advancing the field. I have obtained results related to most of them in the past.

The next 3 sections discuss aspects (a), (b), and (c) respectively of my research. Discussions related to questions (1) and (2) are done at various points in Sections 1, 2, 3. If one is interested in the work done regarding questions (1) and (2), one can simply read Sections 1 and 2. Other results and applications of the basic work described Sections 1 and 2 are discussed in Section 3. To keep the document at a reasonable length, I only mention the main results; a more detailed discussion of my previous work and background of some of the problems described here can be found at https://www.math.unt.edu/∼ntrang. Relative (consistency) strength between various theories I've been studying are summarized in Figure 1.

1. INNER MODELS, HYBRID STRUCTURES, AND CANONICAL SETS OF REALS

The problem of analyzing HOD of AD^+ models has gradually grown into a central problem of inner model theory and spurs the development of descriptive inner model theory. One main reason for its importance is that it provides insights into the relationship between canonical inner models of large cardinals (pure extender models) and models of AD^+ (as alluded to earlier). Unlike HOD of ZFC models which are more or less intractable from the point of view of inner model theory, HOD of AD^+ models in some sense are very well-behaved and code up the AD^+ models in a canonical way; hence understanding HOD of such models provide more insights into the models themselves. It turns out that HOD of an AD^+ model (at least up to model of theories that we have been able to understand and analyze) is a strategic extender model, a kind of hybrid structure that is constructible from a sequence of extenders and iteration strategies of its own initial segments. In other words, HOD contains large cardinal information (coded by its extender sequence) as well as information about the determinacy world (coded by its iteration strategy of its own initial segments). The picture going forward is that one should think of HOD of AD^+ models as a bridge that connects large cardinal universes and determinacy worlds. The three hierarchies: the pure extender models, the AD^+ models, and their HOD (the strategic extender models) are therefore intimately connected. The main goal of descriptive inner model theory is to understand this interconnectedness. This kind of understanding has been giving rise to many types of applications. For example, techniques from the HOD analysis allow one to calibrate the exact consistency strength of determinacy theories by constructing inner models of large cardinals that are in some sense have the same information as the corresponding determinacy model. Another source of applications is through the core model induction, which relies heavily on our understanding of HOD of AD^+ models (see the next section for a more detailed discussion of this topic).

Let NLE be the following statement: "there is no ω_1-iterable pure extender mouse with a long extender".

Question 1.1. *Assume* NLE. *Is HOD of an* AD^+ *model fine-structural? In particular, does HOD satisfy* GCH, $\forall\kappa\ \Box_\kappa$ *and* $\Diamond_\kappa$ *hold?*

This is question (1) above and I regard this as one of the central questions to tackle in his long-term research plan. The conjecture is that the answer to the question is positive. The first breakthrough in answering Question 1.1 is by work of Steel and Woodin in the 1990's [41] for $L(\mathbb{R})$, the minimal model of AD^+ (if there are models of AD^+). In particular, Steel shows that HOD up to Θ in $L(\mathbb{R})$ is a pure extender model of large cardinals and Woodin, building on Steel's work, shows that full HOD of $L(\mathbb{R})$ is a strategic extender model (so HOD knows a fragment of its own iteration strategy). The results and techniques in [41] provide a template for analyzing HOD of bigger models of AD^+. For instance in 2009, Sargsyan [27] gives an analysis of HOD (up to Θ) of all AD^+ models up to $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. After results of Steel and Woodin, [27] is regarded as a landmark in the field as it provides many useful techniques for constructing hod mice (strategic extender models that generate initial segments of HOD). In paper (i) [49], I complete the full HOD analysis and answer positively Question 1.1, for AD^+ models up to $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

There has been progress in answering Question 1.1 for AD^+ models beyond $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. In particular, a very strong determinacy theory that is at the forefront of the subject and has been studied extensively by various authors, including me, in recent years is called LSA or "$\mathsf{AD}^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal". LSA was isolated by Woodin in [55] but has not been known to be consistent until very recently. The general structural theory of LSA as well as the HOD analysis for models of such theories is the subject of the upcoming book by G. Sargsyan, and me [29]. The book proves the consistency of LSA and answers part of Question 1.1 positively by showing that HOD of models of LSA is a strategic extender model of a certain kind and so HOD $\vDash \mathsf{GCH}$. However, the structure theory of HOD developed in [29] is so much more complicated than those in AD^+ models up to $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular that we were unable to fully prove that $\square, \diamond$ hold in HOD. However, I show in [29] that HOD satisfies $\forall\kappa\ \square_{\kappa,2}$[5] by combining the techniques developed in [29] for hod mice and the construction of $\square$ by Schimmerling-Zeman in [33]. The solution, as a result, is not strictly inner model theoretic; it carries some descriptive set theoretic flavor and this seems essential since HOD carries with it information about determinacy world after all. The non-uniformity of the hierarchy developed in [29] makes it very difficult to prove full $\square$ holds in HOD; the hierarchy in [29] is too extender-biased. One way to tackle Question 1.1 is to develop an alternative notion of hod mice in the realm of short-extenders that is simpler, more uniform, and has better condensation properties. Such a work has been done by John Steel in [42]. The advantage of the hierarchy defined in [42] is that we no longer need to "layer"; instead, strategies and extenders are fed into the models in a uniform way. Using the results in [42], Steel and I recently were able to answer Question 1.1 for AD^+ models much stronger than LSA. More precisely, we show that

Theorem 1.2 (AD^+)**.** *Assume* $\mathsf{AD}_{\mathbb{R}} + \mathsf{V} = \mathsf{L}(\mathcal{P}(\mathbb{R}))$ *and* NLE. *Assume Hod Pair Capturing holds (see Question 1.4). Then in HOD, for all κ: κ is not subcompact if and only if $\square_\kappa$ holds.*

[43] (joint with John Steel) proves the full condensation theorem for least branch mouse pairs (i.e. not just condensation for models, but also for strategies)[6]; this is the first important hurdle that has been overcome on the way to the characterization of $\square$ in least branch hod mice. The next major step is to study the *Dodd-parameters* for hod mice. Once this is done, putting together the characterization of $\square$ in HOD is fairly routine (cf. [44]).

However, despite its uniformity and its theory being fully developed up to the level of superstrong, constructing least branch hod mice in core model induction applications is a challenge (roughly because the hierarchy grows too fast). It is worthwhile to develop methods for constructing least branch hod mice that can be applied in a variety of applications (to be discussed more in the next section); Theorem 1.2 will play a crucial role in such applications, much like Schimmerling-Zeman's characterization of $\square$ in extender models has played a crucial role for applications concerning such objects.

These kinds of results are important for various types of applications, one of which is to prove the consistency of LSA from PFA (to be discussed in more details in the next section, cf. Question 2.1).

Since the structures of hod mice in [42] are similar to those of extender models; the proof of Theorem 1.2 naturally relies on the proof of $\square$ in extender models (cf. [33]). Two new ingredients of the proof are: the use of the $\mathfrak{B}$-operator in [35] to feed branch information of various trategies into a hod premouse and a method of phalanx comparisons (similar to that used in [42]) used to prove a key condensation lemma.

In order to carry out the HOD analysis, generally, one has to prove structural theorems about the AD^+ models. The following are two central problems in descriptive inner model theory that one needs to address first before answering Question 1.1 (for any particular AD^+ theory that one is studying). Recent developments in the theory of hod mice (e.g. [42]) suggest that the answer to Question 1.1 can be answered in full generality assuming variations of 1.3 or 1.4.

[5] $\square_{\kappa,2}$ is a slightly weaker combinatorial principle than $\square_\kappa$.

[6] The paper also proves the full strategy condensation for the least branch hod mice and for ordinary $L[E]$ mice with nice enough strategies.

Question 1.3 (Generation of pointclasses, GFP)**.** *Assume* NLE*. Given an* AD^+ *model* M*, suppose* $\Gamma \subsetneq \mathcal{P}(\mathbb{R})^M$ *is a strict Wadge initial segment of the Suslin co-Suslin sets. Is it possible to generate* Γ *by an iterable (hybrid) model?*

Question 1.4 (Hod pair capturing, HPC)**.** *Assume* NLE*. Suppose* A *is Suslin co-Suslin. Is there a hod pair* $(\mathcal{P}, \Sigma)$ *(in the sense of [42]) such that* A *is Wadge reducible to* Σ*?*

Quesion 1.3 asks whether strategic extender iterable structures (like hod mice) can generate complicated pointclasses of AD^+. This is crucial for the HOD analysis since these iterable structures are used in direct limit systems that generate initial segments of HOD; roughly speaking, the more complicated the pointclass generated by the structure, the longer the initial segment of HOD can be captured by the corresponding direct limit system). The positive answer to Quesion 1.3 is given up to LSA by (in chronological order): Steel and Woodin [41] in L($\mathbb{R}$), by Woodin for a broader class of models [40], by Sargsyan [27] for a larger class still (up to minimal models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular), and finally by [29] in the minimal model of LSA.

Question 1.4 is, in spirit, similar to Question 1.3 but the notion of hod pairs is precisely given (in [42]). Again, the framework in [42] hopefully will allow us to answer Question 1.4 for very general AD^+ models.

To tackle Questions 1.3, and 1.4 for AD^+ theories beyond LSA, it seems that we need to understand hod mice with certain properties and how to construct them (e.g. from PFA)[7]. We don't seem to have a good understanding of such objects (but see [42]); this topic is therefore at the forefront of descriptive inner model theory and is central to my long-term research plan.

2. CORE MODEL INDUCTIONS, FORCING AXIOMS, AND COMPACTNESS PRINCIPLES

The main sort of applications of the material discussed previously is in calibrating the consistency strength of set-theoretic principles. The main tool for accomplishing this is the core model induction. The core model induction, the only known systematic approach for computing lower-bound consistency strength, is pioneered by Woodin and developed further by Steel, Schindler and others to construct models of AD^+ from strong set-theoretic principles (cf. [34], [37], [3], [38]). The methods of these papers typically show AD^+ holds in $L(\mathbb{R})$ and manage to construct models of slightly stronger AD^+ theories. They are, however, insufficient to construct models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular, LSA and beyond because of the lack of understanding of stronger AD^+ models, F-mice for complicated operators F, and the scales analysis in F-mice over $\mathbb{R}$. There are many ingredients that need to be put together to accomplish this task, some of which are: understanding HOD of complicated AD^+ models, i.e. those with "long" Solovay sequence (e.g. see [27], [49], [29] by Sargsyan, Steel, and me), developing a general enough theory of F-mice and various techniques for constructing new scales and pointclasses of determinacy (cf. T. Wilson [53], F. Schlutzenberg and me [35]).

2.1. LOWER BOUNDS CONSISTENCY COMPUTATIONS AND FUTURE DEVELOPMENT OF THE CORE MODEL INDUCTION

We discuss below several problems involving determining consistency strength of important set-theoretic principles. The arguably most important one is Question (2) above.

Question 2.1. *What is the consistency strength of* PFA*?*

As mentioned above, [37], [30], the paper (i) [47], and the book [29] show that the strength of PFA is at least that of LSA, which is a very strong determinacy principle.

Theorem 2.2 (Sargsyan-Trang)**.** *Assume* PFA*. Then there are inner models* M *such that* $M \vDash \mathsf{LSA}$*.*[8]

However, LSA is shown to be consistent relative to a Woodin limit of Woodin cardinals (cf. [29]) and the consensus amongst set theorists is that PFA should be as strong as a supercompact cardinal (it is well-known that the upper bound for PFA is a supercompact, cf [2]). A complete answer to Question 2.1 is the holy grail of inner model theory. From the point of view of descriptive inner model theory (this is the view I take), to completely solve this problem, one needs to understand HOD of AD^+ models; in particular, one needs to resolve Questions 1.1, 1.3 and generalizations of Conjecture 1.4. Futhermore, at the level of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular and beyond, one needs to do a significant amount of work to construct hod-like objects to eventually generate HOD of the AD^+

[7] i.e. those with extenders overlapping Woodin cardinals.

[8] [29] will appear in the ASL Lecture Notes in Logic series, published by the Cambridge University Press.

models; this is where the bulk of the construction is and seems to be hypothesis-dependent. For example, [29], combining techniques for constructing K^c from core model theory and techniques for analyzing HOD of AD^+ models, introduces a strategic K^c construction, whose outcome is a hod-like object $\mathcal{M}$ that generates HOD of an LSA model (and hence the LSA model itself). The existence of $\mathcal{M}$ is established using covering arguments taking advantage of PFA and my theorem (mentioned above) that $\mathcal{M} \vDash \forall\kappa\ \square_{\kappa,2}$.

One short-coming of the construction in [29] is the fact that we are unable to prove that $\mathcal{M}$ is iterable (we get around this by showing some definable hull of $\mathcal{M}$ is iterable and this suffices to get a model of LSA). However, for going beyond LSA the iterability of $\mathcal{M}$ seems important. I wish to pursue this further as I think the methods introduced by [29] has a lot of mileage in further advancing the solution of Question 2.1. More recently, we have shown the following, which we believe is significant for our understanding of the CMI. In the below theorem, Γ_∞ is the pointclass of universally Baire sets and Γ_∞-*sealing* is the statement that for every set generic extension $V[g]$ of V and every set generic extension $V[g][h]$ of $V[g]$, $L(\Gamma_\infty^{V[g]}) \vDash \mathsf{AD}^+$, $(\Gamma_\infty^{V[g]})^\sharp$ exists, and there is an elementary embedding $j : L(\Gamma_\infty^{V[g]}) \rightarrow L(\Gamma_\infty^{V[g][h]})$.

Theorem 2.3 (Sargsyan-Trang). *(i) (2018) Assume* PFA*. Then there is a hod pair* $(\mathcal{P}, \Sigma)$ *such that* $\mathcal{P}$ *is non-tame.*

(ii) (2019) Γ_∞*-sealing is consistent relative to the existence of a Woodin cardinal which is a limit of Woodin cardinals* (WLW)*.*

We note that the existence of a non-tame hod pair is stronger than LSA. Hence the result in the above theorem improves the conclusion of Theorem 2.2. The construction of a non-tame hod pair in Theorem 2.3 and the above discussion about the difficulty of CMI past LSA appear to suggest that beyond LSA, one needs to construct canonical *third order objects*, i.e. canonical subsets of Γ_∞. The threshold above which the methodology of CMI needs to change from constructing second order objects (i.e. elements of Γ_∞) to third order objects is Γ_∞-sealing. The upcoming paper [31] obtains the exact consistency strength of Γ_∞-sealing, discusses the importance of this principle in set theory, as well as future developments of CMI. In particular, the paper isolates the following principle and shows it is equiconsistent with Γ_∞-sealing (modulo a mild large cardinal assumption).

Definition 2.4 ($\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$, Sargsyan-Trang). *For any set generic extension* $V[g]$ *of* V*, in* $V[g]$*, there is a set of reals* A *such that* $L(A,\mathbb{R}) \vDash \mathsf{LSA}$ *and* Γ_∞ *is the Suslin co-Suslin sets of* $L(A,\mathbb{R})$*,*

$\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ and its variations play a role in clarifying the relationship between the Martin's Maximum (MM) and $(\star)^+$, an extension of Woodin's $(\star)$ axiom. Recent results show that MM^{++}, an strengthening of MM, implies $(\star)$. However, by analyzing models of $\mathsf{MM}^{++}+\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$, one can show that MM^{++} does not imply $(\star)^+$.

The main conjecture from [31], which captures the change in the CMI methodology past Γ_∞-sealing, is the following conjecture. Solving this conjecture is part of my long-term research goal.

Conjecture 2.5. *Assume* NLE *and suppose there are unboundedly many Woodin cardinals and strong cardinals. Let* κ *be a limit of Woodin cardinals and strong cardinals such that either* $cof(\kappa) = \kappa$ *or* $cof(\kappa) = \omega$*. Then there is a transitive model* M *of* $\mathsf{ZFC}-\mathsf{Powerset}$ *such that*

1. $cof(Ord \cap M) \geq \kappa$,
2. M *has a largest cardinal* ν,
3. *for any* $g \subseteq Coll(\omega, <\kappa)$, *letting* $\mathbb{R}^* = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[g\cap Coll(\omega,\alpha)]}$, *in* $V(\mathbb{R}^*)$,

$$L(M, \textstyle\bigcup_{\alpha<\nu}(M|\alpha)^\omega, Hom^\infty, \mathbb{R}) \vDash \mathsf{AD}.$$

In 2017, I observed that

Theorem 2.6. *Assume* $\mathsf{PFA}+$ *there is a Woodin cardinal. Then there are inner models that satisfy* $\mathsf{ZFC}+$ *there is a Woodin cardinal which is a limit of Woodin cardinals.*

So with an extra, mild large cardinal assumption, we can improve the lower-bound of Theorem 2.2 significantly. Unfortunately, the method of Theorem 2.6 does not seem to generalize. So it seems to me the right approach to systematically studying the universe of sets and its canonical structures in the presence of strong forcing axioms like PFA is continue to generalize the approach of [29] and of Theorem 2.3.

In light of the work above, a more concrete conjecture regarding PFA and HOD of models of AD^+ can be made (at least in the region of short-extender models).

Conjecture 2.7. *Assume* PFA *holds. Then there is a model M of* AD^+ *such that* $HOD^M \vDash$ *"there is a superstrong cardinal".*

2.2. OTHER COMBINATORIAL PRINCIPLES

We now discuss other combinatorial principles which are important in their own rights. We first discuss consistency strength of various types of ideals. The following is conjectured by Woodin (part of problem 12 in [55]).

Conjecture 2.8. $\mathsf{AD}_{\mathbb{R}}+\Theta$ *is regular is equiconsistent with* CH *+ there is and ω_1-dense ideal on ω_1.*

In a recent joint-work with G. Sargsyan and T. Wilson, we have shown the following variations of Woodin's conjecture. These are, to the best of my knowledge, the first natural combinatorial principles (extending ZFC) that are equiconsistent with $\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular. We need the following definitions.

Definition 2.9. *An ideal $\mathcal{I}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is strong if*

(a) $\mathcal{I}$ is precipitious, and

(b) whenever G is generic, letting $j_G : V \to Ult(V, G)$ be the ultrapower map, then $j_G(\omega_1) = \mathfrak{c}^+$.

Definition 2.10. *An ideal $\mathcal{I}$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is pseudo-homogeneous if whenever $G \subseteq \mathbb{P}_{\mathcal{I}}$ is V-generic with $j_G : V \to Ult(V, G)$ be the corresponding ultrapower map, for all $\alpha \in$, $s \in^{\omega}$, for all $A \subseteq \lambda^{\omega}$ for some $\lambda < \mathfrak{c}^+$, θ is a formula in the language of set theory, the statement*

$$Ult(V, G) \vDash \theta[\alpha, j_G(s), j_G[A]]$$

is independent of the choice of G.

Theorem 2.11. *Con(2) implies Con(1) and Con(3) implies Con(1), where*

1. $\mathsf{AD}_{\mathbb{R}}+\Theta$ *is regular.*
2. CH+ *there is an ω_1-dense ideal $\mathcal{I}$ on ω_1 such that letting G be V-generic filter for the forcing induced by $\mathcal{I}$, and $j : V \to M$ be the generic ultrapower embedding, then $M^{\omega} \subset M$ in $V[G]$, $j \restriction \alpha \in V$ for every ordinal α.*
3. *the nonstationary ideal on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is strong and pseudo-homogeneous.*

Then early this year, the full conjecture was finally resolved (with additional collaborations with D. Adolf and M. Zeman). So we have the following theorem.

Theorem 2.12 (Adolf-Sargsyan-T.-Wilson-Zeman, 2022)**.** *Con(2) implies Con(1) and Con(3) implies Con(1), where*

1. $\mathsf{AD}_{\mathbb{R}}+\Theta$ *is regular.*
2. CH+ *there is an ω_1-dense ideal on ω_1.*
3. *the nonstationary ideal on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is strong and pseudo-homogeneous.*

The converse has been shown by Woodin. So we indeed establish the equiconsistencies of these theories. See [1] for a full account of these theorems.[9]

The following question samples four important theories. Below, guessing models are the strongest known generalizations of the tree property. The existence of κ-guessing models implies the tree property at κ^+. ω_1-guessing models were first introduced by Viale and Weiss [52], and the obvious formulations of κ-guessing models (for $\kappa > \omega_1$) are introduced in [51] and [47]. And $\mathsf{MM}(\kappa)$ is the Martin Maximum for posets of size at most κ for a cardinal κ.

Question 2.13. *(a) What is the consistency strength of $\neg\square_\kappa$ for some singular strong limit κ?*

(b) What is the consistency strength of $\neg\square_\kappa + \neg\square(\kappa)$ for a regular $\kappa \geq \aleph_3$ such that $\kappa^{\omega} = \kappa$ and $2^{<\kappa} = \kappa$?[10]

[9] [1] has been accepted to the Journal of the American Mathematical Society.

[10] $\square(\alpha)$ says that there is a sequence of $(C_\beta : \beta < \alpha)$ such that C_β is a club subset of β, $C_\beta \cap \gamma = C_\gamma$ for every limit point γ of C_β, and there is no "thread" through the sequence, i.e. there is no club C in α such that for any limit point β of C, $C \cap \beta = C_\beta$.

(c) *What is the consistency strength of* $\mathsf{MM}(\mathsf{c}^+)$*?*

(d) *What is the consistency strength of the existence of* ω_1*-guessing models?*

The constructions in [47] and [29] actually can be used to construct models of LSA from the following principles:

(I) GCH + there is a cardinal κ such that κ is countably closed and for all $\alpha \in [\kappa, \kappa^{+4}]$, $\neg\square(\alpha)$;

(II) there exist stationary many ω_2-guessing models that are countably closed;

(III) there exists a strongly compact cardinal.

I hope that these methods can be applied to the situations in Question 2.13. More precisely, since (a), (b), (c) above are local principles, there is less room to work with so the exact methods used above don't seem to work here. However, I expect that appropriate refinements of these methods can be used to make some progress, at least in constructing models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular from (a), (b), and (c). Regarding (d), note that ω_1-guessing models are not countably closed (unlike the situation for ω_2-guessing models), so some of the covering arguments used in [47] don't work for (d). However, I hope that frequent extension techniques and covering arguments without countable closure in [21] can be adapted to tackle (d). To the best of my knowledge, (b), (c), and (d) have lower-bound roughly that of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ (by [16]) and (a) has lower-bound $\mathsf{AD}_{\mathbb{R}}$ (by [37] and [30]).

Conjecture 2.14. *The consistency strength of* $\mathsf{MM}(\mathsf{c})$ *is exactly that of* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular.*

Woodin in [55], using $\mathbb{P}_{\max}$ forcing techniques shows that $\mathsf{MM}(\mathsf{c})$ holds in a $\mathbb{P}_{\max}$-generic extension of any model of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. Neeman and Schimmerling [22], using a different method, also force $\mathsf{MM}(\mathsf{c})$ from a large cardinal property much weaker than supercompact. $\mathsf{MM}(\mathsf{c})$ is of interest by set theorists since it implies various combinatorial principles at H_{ω_2}, for instance, it decides the size of the continuum, the powerset of ω_1 (they both equal $\aleph_2$), it implies the nonstationary ideal on ω_1 is saturated and the weak reflection principle $\mathsf{WRP}_2(\omega_2)$. Steel and Zoble [38] show that $\mathsf{MM}(\mathsf{c})$ implies AD holds in $L(\mathbb{R})$ and this result is one of the first core model induction arguments there was. I expect that the knowledge from recent work can be used to improve the lower-bound of $\mathsf{MM}(\mathsf{c})$. The advantage here is that we now understand models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular very well, much better than when Steel and Zoble proved their result; furthermore, more techniques have been discovered in constructing models of determinacy since.

Question 2.15. *Show that the failure of* UBH *for nice trees imply the existence of models of* LSA.

As mentioned, UBH for nice trees (normal, non-overlapping, and extenders are sufficiently closed in the models they are chosen from) is central in constructing canonical inner models of large cardinals (up to supercompact and beyond). Counterexamples of UBH for non-nice trees have been constructed by Neeman, Steel, and Woodin (discussed above). One way of enforcing the belief that UBH for nice trees is true is to show that the failure of the principle has very high consistency strength. Martin and Steel [18] took the first step along this direction by showing that the failure of UBH implies the existence of a Woodin cardinal (in some inner model). Steel [39] improves this to $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ (and a bit beyond this in some cases). The methods Steel uses are traditional core model theoretic methods. Sargsyan and me in [28] and [32], using the core model induction, improve upon Steel's results significantly and show that the failure of UBH for nice trees implies the existence of models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. As mentioned above, the methods developed in [28] and [32] are different from those developed in [47] and [29] etc. since we cannot use covering-type arguments in this case; in particular, it is not obvious that the strategic K^c construction introduced above converges in this situation (after all, the theorem that $\mathcal{M}$ satisfies $\forall\kappa\ \square_{\kappa,2}$ does not seem useful here). New methods for constructing models of LSA are needed here.

The set of questions concerns compactness principles on ω_1 in the context of $\mathsf{ZF} + \mathsf{DC}$, a topic that has been important to me since his student days at UC Berkeley and plays a major part in his thesis [46]. The first one addresses the uniqueness problem for canonical models of $\mathsf{AD}^+ + \omega_1$ is $\mathbb{R}$-supercompact (i.e. AD^+ models of the form $L(\mathbb{R})[\mu]$).

Conjecture 2.16. *Assume* AD *or* ZFC. *Then there is at most one* AD^+*-model of the form* $L(\mathbb{R})[\mu]$ *where* μ *is the Solovay measure on* $\mathcal{P}_{\omega_1}(\mathbb{R})$ *in the model.*

This conjecture grows out of a question asked by Woodin [54] in the 1980's. The question asks whether assuming determinacy, there must be at most one canonical model of $\mathsf{AD}^+ + \omega_1$ is $\mathbb{R}$-supercompact. Woodin in [54] shows that assuming determinacy, if ω_1 is $\mathbb{R}$-supercompact then there must be a unique measure witnessing this. So the question about the uniqueness of models is natural and resembles the situation regarding uniqueness of minimal models of one measurable cardinal (which Kunen gives a positive answer to).

The question had been open for more than 30 years until very recently. In a joint work with D. Rodríguez [24], we answer the question positively in the AD case and Rodríguez, by refining the techniques in [24], has settled the conjecture fully. The main techniques grow out of the my HOD analysis in [50]; the main point is that the HOD analysis allows us to compare and line up hod mice in different AD^+-models of the form $L(\mathbb{R})[\mu]$ and conclude that on a cone of reals x, HOD_x of the models are the same. This implies the model must be the same via Vopenka-like forcing methods developed by Woodin. In [24], we also make significant progress towards settling Conjecture 2.16 by showing the conjecture is true if one assumes additionally a very mild large cardinal property. Without the additional large cardinal assumption, it seems one needs to understand better the universally Baire sets in V and develops some methods for constructing iterable models with large cardinals in the case that there are distinct such models.

Moving on to higher forms of compactness, in [48], the T. Wilson and I investigate various forms of compactness on ω_1 (beyond $\mathbb{R}$-strong compactness and supercompactness). The main results of [48] show that for a fixed set X, the principles "ω_1 is X-strongly compact" and "ω_1 is X-supercompact" are generally not equiconsistent (this is certainly the case for $X = \mathcal{P}(\mathbb{R})$ as shown in [48]) but they seem to interleave in consistency strength. The paper [48] develops techniques for carrying out the core model induction in contexts where the full axiom of choice fails; these techniques along with the HOD analysis in [27] and [29] can hopefully shed new lights on the full answers to the following question, Question 2.17. The techniques developed in [48] also give unexpected and surprising results. For example, one of the consequences of methods from [48] is a result of T. Wilson's and mine that under a certain smallness assumption, $\mathsf{AD}_{\mathbb{R}}$ is equivalent to Blackwell-$\mathsf{AD}_{\mathbb{R}}$. This is a higher analogue of similar results of Martin and Neeman regarding Blackwell-AD [19].

Question 2.17. *What is the exact consistency strength of ω_1 is strongly compact (supercompact)? Is "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ strong compact" equiconsistent with "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ supercompact"?*

In a recent work, G. Sargsyan and I, using techniques in [29], have obtained models of LSA from "$\mathsf{AD} + \omega_1$ is supercompact". One should not need AD in the above.

3. COMBINATORICS, FORCING, AND LARGE CARDINALS

Classification of cardinals under determinacy requires deep understanding of combinatorial properties of such cardinals (like the weak and strong partition properties) and sometimes analyses of HOD-like objects. Woodin, under $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$, has classified all cardinals below $[\omega_1]^{<\omega_1}$. For example, Woodin shows that there are exactly 5 uncountable cardinals $\leq [\omega_1]^\omega$, namely $\mathbb{R}, \omega_1, \mathbb{R} \cup \omega_1, \mathbb{R} \times \omega_1, [\omega_1]^\omega$. Under $\mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}}$, the picture is much more complicated. For example, there are $> \Theta$ many distinct cardinals between $\mathbb{R}$ and $\mathbb{R} \times \omega_1$; these cardinals are determined by Turing invariant functions from the Turing degrees into ω_1. Also, not much is know about classifications of cardinals obtained from quotient spaces of the Polish space $\mathbb{R}$ by various equivalence relations on $\mathbb{R}$ (like the Vitali relation E_0, E_1 etc.). We do not know if there are any cardinals strictly between $\mathbb{R} \times \omega_1$ and $[\omega_1]^\omega$.

Problem 3.1. *Classify cardinals under $[\omega_3]^{<\omega_3}$ under AD^+.*

We expect that the zoo of cardinals below $[\omega_3]^{<\omega_3}$ is vast (evidence is given above). Perhaps, under $\mathsf{AD}_{\mathbb{R}}$, it is simplier. Why ω_3? ω_3 is singular (both in V and in HOD) while ω_1 and ω_2 are regular in V and strongly inaccessible in HOD. For instance, one can show $[\omega_1]^{\omega_1}$ is strictly larger than $[\omega_1]^{<\omega_1}$ (and similarly to ω_2) based on this observation and the fact that HOD thinks $[\omega_1]^{\omega_1}$ is strictly larger than $[\omega_1]^{<\omega_1}$ (and similarly to ω_2). One can't do this for ω_3 and hence some new methods are needed here to show $[\omega_3]^{\omega_3}$ is strictly larger than $[\omega_3]^{<\omega_3}$ (or to refute this).

Combinatorial calculations at ω_2 are typically pulled back to combinatorics on ω_1 (e.g. the (weak) partition property on ω_2 is proved using the partition property on ω_1). One important separation in this hierarchy is: show that $[\omega_2]^{<\omega_2}$ is strictly above $[\omega_1]^{\omega_1}$. This is a good test for the intuition above and the fact that we know $[\omega_1]^{<\omega_1}$ is strictly above $[\omega_1]^\omega$. Behind many of these calculations concerning comparing cardinals under AD and studying other combinatorial objects like almost-disjoint families, trees (Suslin, Aronszajn, etc.) on some cardinal $\kappa < \Theta$, we are able to isolate some fundamental properties of functions under AD, like continuity and monotonicity. Here are questions about such properties that we are exploring and have produced partial results.

Ramsey famously showed that the set of natural numbers, ω, satisfies the finite partition relations $\omega \to (\omega)^k_2$ for each $k < \omega$, i.e. for any (coloring) function $\Phi : [\omega]^k \to 2$, there is an infinite homogeneous set for Φ, i.e. an infinite set $C \subset \omega$ such that $\Phi \restriction [C]^k$ is constant. The infinite exponent partition relation $\omega \to (\omega)^\omega_2$ (also called the Ramsey property for all partitions) is a natural generalization that is not compatible with the axiom of choice. However, simple definable partitions such as Borel or analytic partitions always satisfy the Ramsey property (see [12] and [36]).

Mathias, in his celebrated work [20], studied the almost everywhere behavior of functions on the Ramsey space $[\omega]^\omega$ of functions from ω to ω such as when every function $\Phi : [\omega]^\omega \to \mathbb{R}$ is Ramsey almost everywhere continuous or every relation. $R \subseteq [\omega]^\omega \times \mathbb{R}$ has a Ramsey almost everywhere uniformization.

In past and ongoing joint work with W. Chan and S. Jackson, which has resulted in a series of papers (see for example [9, 5, 6]), we attempt to study similar properties for functions on the Ramsey-like spaces $[\kappa]^\kappa$ (or $[\kappa]^\epsilon$) for infinite cardinals $\kappa \geq \epsilon$ and apply these properties to various applications in set theory and set-theoretic topology, in particular the study of cardinalities without the axiom of choice (or with limited choice). This research program not only is fruitful in attacking fundamental problems about these function spaces but also has applications in various problems in studying combinatorial objects in the context of AD. For instance, [7] characterizes the existence of maximal almost disjoint families on every cardinal κ in natural models of AD^+. The main theorem of the paper is the following; here $\mathcal{B}(\kappa)$ is the set of bounded subsets of κ.

Theorem 3.2 (Chan-Jackson-T.). *Assume AD^+. If $\kappa < \Theta$ is a cardinal and $cof(\kappa) > \omega$ then there are no maximal $\mathcal{P}_\kappa(\kappa)$ or $\mathcal{B}(\kappa)$ almost disjoint families $\mathcal{A}$ so that $\neg(|\mathcal{A}| < cof(\kappa))$.*

Assume $\mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathcal{P}(\mathbb{R}))$. If κ is a cardinal so that $\omega < cof(\kappa) < \Theta$, then there are no maximal $\mathcal{P}_\kappa(\kappa)$ or $\mathcal{B}(\kappa)$ almost disjoint families $\mathcal{A}$ so that $\neg(|\mathcal{A}| < cof(\kappa))$.

Unsurprisingly, (almost everywhere) continuity and monotonicity of functions on these spaces are false in ZFC. However, under the Ramsey property, for instance $\kappa \to (\kappa)^\kappa_2$, we can prove monotonicity for functions on the space $[\kappa]^\kappa$. A set $C \subseteq \kappa$ is club in κ if it is closed (under the topology induced by the well-order of the ordinals below κ) and unbounded in κ. ON denotes the class of ordinals. The cofinality of a cardinal κ is the length of the shortest increasing, unbounded, and continuous sequence in κ.

Theorem 3.3 (Chan-Jackson-Trang, [8]). *Suppose $\kappa \to (\kappa)^\kappa_2$. For any function $\Phi : [\kappa]^\kappa \to ON$, there is C club in κ so that for all $f, g \in [C]^\kappa$[11], if for all $\alpha < \kappa, f(\alpha) \leq g(\alpha)$, then $\Phi(f) \leq \Phi(g)$.*

See [8] for a much more detailed analysis and refinements of the above theorem as well as several theorems on (almost everywhere) continuity of functions of Ramsey-like spaces, which is a more fundamental property and is used (along with other combinatorial techniques we developed) to establish monotonicity results like the above. The following is a representative theorem about continuity on these spaces.

Theorem 3.4 (Chan-Jackson-Trang, [8]). *Suppose κ is a cardinal, $\epsilon < \kappa$ is a limit ordinal with cofinality $cof(\epsilon) = \omega$, and $\kappa \to (\kappa)^{\epsilon\times\epsilon}_2$ holds. For any function $\Phi : [\kappa]^\epsilon \to ON$, there is a club $C \subseteq \kappa$ and a $\delta <$ so that for all $f, g \in [C]^\epsilon \to ON$, if $f \restriction \delta = g \restriction \delta$, and $sup(f) = sup(g)$, then $\Phi(f) = \Phi(g)$.*

Lastly, I discuss a recent joint project with D. Ikegami concerning forcings that preserve of the Axiom of Determinacy. The basic questions that we would like to tackle are

Problem 3.5. *Assume $\mathsf{AD}^+ + \mathsf{DC}$. Classify what forcings $\mathbb{P}$ with the property that whenever $g \subseteq \mathbb{P}$ is V-generic:*

1. *there is an elementary embedding $j : V \to V[g]$ definable over $V[g]$;*
2. *$V[g] \vDash \mathsf{AD}^+$.*

We first remark that in $\mathsf{ZF} + \mathsf{DC}$, there are $\mathbb{P}$ and j satisfying (1) above. Such an example has been constructed by Woodin. Also, it is possible for j in (1) to be nontrivial, yet j is the identity on all ordinals.

Clearly, if (1) holds then (2) holds. Some partial answers have been known. For instance, we know that if $V = L(X)$ for some set X and $\mathbb{P}$ adds a real then (1) fails. The method for proving this, unfortunately does not work for general AD^+ models. We hope to bring in tools discussed in Sections 1 and 2 into understanding the model $V[g]$. In particular, in the region where the HOD analysis holds, we hope to have a clearer picture of how the sets of reals in $V[g]$ are related to those in V. Our conjecture regarding (1) above is that no $\mathbb{P}$ has the property that there is an elementary $j : V \to V[g]$. This would be the corresponding result to Kunen's famous theorem for ZFC models.

Results regarding (2) we have obtained are as follows. First, Jackson and Chan, inspired by previous partial results of Ikegami and I, have shown that if AD^+ holds and Θ is regular, then any forcing $\mathbb{P} \subseteq \mathbb{R}$ will destroy AD^+, see [4]. Recently, Ikegami and I have improved their result and get rid of the assumption Theta is regular. So we have the following theorem.

[11] We tacitly assume all functions are increasing.

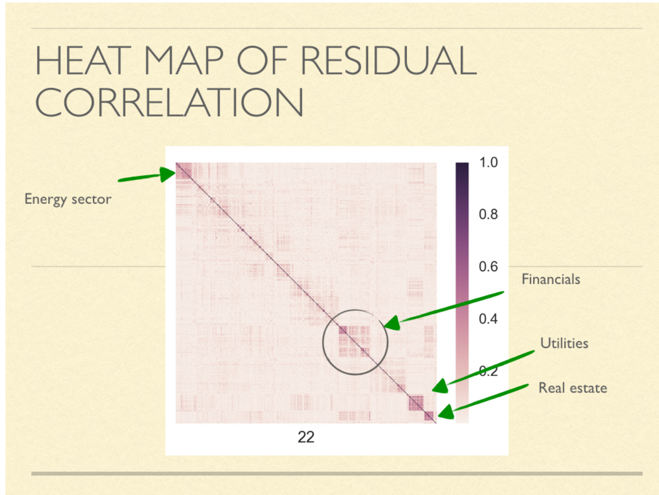


Figure 2: Correlated Sectors

Theorem 3.6 (Chan-Jackson, Ikegami-T.). *Assume* AD^+. *Suppose* $\mathbb{P} \subseteq \mathbb{R}$ *is a forcing poset. Then* $V^{\mathbb{P}} \vDash \neg\mathsf{AD}^+$.

Furthermore, Ikegami and I have shown that if $\mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathcal{P}(\mathbb{R}))$ holds, then letting $\mathbb{P}_\kappa = ADD(\kappa, 2)^{HOD}$ be the forcing for adding a subset of κ using conditions in HOD of length $< \kappa$, then $V^{\mathbb{P}_\kappa} \vDash \mathsf{AD}^+$.

II. APPLIED MATH

4. MATHEMATICAL FINANCE, GRAPHS AND MATRICES

I am interested in various aspects of mathematical finance. In a joint ongoing project with M. Foreman and D. Brownstone, we are interested in: empirically verifying how accurate asset pricing theories such as the Capital Asset Pricing Model (CAPM), the Arbitrage Pricing Theory (APT) and its variations are, and in designing novel methodologies for predicting stock prices using short-time series of stock data (e.g. stock data of the previous 6 months).

The methodology is as follows (this is the first approximation). Suppose we have N independent and identically distributed (iid) random variables $X_1, \ldots, X_N$. X_i represents the data on the i-th stock. In practice, we get a time series of stock returns and these series are (approximately) given by the X_i's. Given a threshold $0 < \delta < 1$, we randomly sample (without replacements) within the set $\mathcal{X} = \{X_1, \ldots, X_N\}$ until we capture $1-\delta$ of the variance of the stocks.[12] In particular, if $\mathcal{X}$ represents the stocks in the $S\&P$ 500 and say $\delta = 0.1$, we use the random sampling method to obtain a set $\mathcal{P} \subset \mathcal{X}$ such that when regress the S&P 500 stocks on $\mathcal{P}$, we get a $R^2 \geq 1 - \delta = 0.9$. We repeat this process to get averages and confidence intervals. The advantage of this method is that we reach the regression threshold with a relatively small number of stocks (e.g. 15) for the S&P 500; this is more feasible than trying to compute the eigenvalues of a 500×500 matrix. However, this methodology is flawed. Figure 2 represents the residuals of other stocks when projected onto the hyperplane spanned by $\mathcal{P}$.

In the above figure, the method did not pick up various sectors that are highly correlated; these correlations are meaningful economically. When we add these stocks into our portfolio, the number of stocks is close to 30 (on average). This suggests better algorithms for constructing random portfolios such as "cluster sampling" ala the census bureau. The high correlations means that our random selections have not touched these sectors – we should artificially pick from those sectors and revise our assumption about the uniform distribution of coupons. We haven't done this study yet. We also hope to apply the same methodology (and its improvements) to other collections of stock index, like the Russell 2000 etc. The hope is to increase the robustness of our methods (i.e. the bigger the stock universe is, the better it approximates the "whole economy") and to empirically verify whether theories like APT hold up when pricing stocks in various sectors of the economy.

The following conjecture is a more general statement that captures our attempts above. The idea is in order to capture enough of the total variance, we need enough "dimensions", i.e. the hyperplane generated by the stocks we sample must be "close enough" to all the stocks in the original collection. Below, for any two random variables

[12] This is a variation of the Coupon Collection Problem.

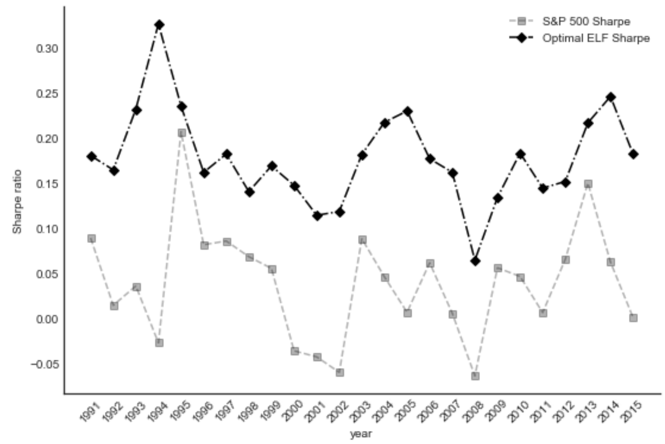


Figure 3: Sharpe ratio comparisons

X, Y, let $Cov(X, Y) = \langle X, Y\rangle$ be the covariance of the X, Y; let $Var(X) = ||X||^2$ be the variance of X.[13] For a hyperplane $\mathcal{P}$ and a random variable X, we let $X^{\mathcal{P}}$ be the projection of X onto $\mathcal{P}$. In the following, we let $\mathcal{C} = (\langle X_i, X_j\rangle)_{1\leq i,j\leq n}$ be the covariance matrix of $\{X_1, \ldots, X_N\}$; by the Singular Valued Decomposition (SVD) theorem, we get a diagonal matrix matrix $\mathcal{D} = diag(\lambda_1^2, \ldots, \lambda_N^2)$ and a unitary matrix U such that $\mathcal{D} = U^T\mathcal{C}U$. We also have an orthogonal set of vectors $\{Y_1, \ldots, Y_N\}$ with the property that $\langle Y_i, Y_i\rangle = \lambda_i^2$.

Conjecture 4.1 (Unindexed version)**.** *For any $0 < \lambda < min_{i\leq N}\lambda_i^2$, for any $I \subseteq \{1, \ldots, N\}$, for any $\delta = \sum_{i\in I}\lambda_i^2 + \lambda$, there is $\epsilon(\delta)$ and $H \subseteq \{X_1, \ldots, X_N\}$ such that letting $\mathcal{P}$ be the hyperplane generated by H, the following are equivalent:*

- $$\frac{\sum_{i=1}^N ||X_i - X_i^{\mathcal{P}}||^2}{\sum_{i=1}^N ||X_i||^2} < \delta$$
- *for some $\vec{Y} \subseteq \{Y_j : j \leq N\}$*

  1.
  $$\sum\nolimits_{Y_j\in\vec{Y}} \frac{||Y_j - Y_j^{\mathcal{P}}||^2}{||Y_j||^2} < \epsilon(\delta)$$
  and

  2.
  $$\frac{\sum_{Y_j\notin\vec{Y}} ||Y_j||^2}{\sum_{i=1}^N ||Y_j||^2} < \delta$$

One can also state the indexed version of the above conjecture. We have reasons to believe the conjectures are true in cases of interest, namely when a small number of eigenvalues of $\mathcal{C}$ are much larger than the rest. We hope that the conjectures (and its generalizations) will shed light on the relationships between the sampling method (better yet, its improvements) and pricing models. In particular, we hope to show that no reasonable number of factors (e.g. Ross [26], Fama-French [10]) can capture (most of) the economy (in terms of returns of stocks and bonds).

The paper [11] is one along this direction. The Capital Asset Pricing Model (CAPM) is a ubiquitous tool for financial applications, from asset management to corporate decision making. It is simply stated, has elegant consequences and has easily applicable corollaries. Unfortunately, due to inherent estimation difficulties it is difficult to check directly. This paper describes a mathematical technique, the Census Taker Method (CTM) which bypasses these estimation difficulties and makes conservative estimates of efficient frontiers. A direct application is to show that the $S\&P$ 500 does not realize returns anywhere close to its efficient frontier (see Figure 3, where it is shown that the Sharpe ratio of the S&P 500 is nowhere closed to that of estimated long efficient frontiers using CTM, hence the Sharpe ratio of the S&P 500 is very far from actual efficient frontiers).[14] Thus a central corollary of the

[13]It is not a coincidence that we denote Cov and Var as inner product and norm. The are indeed the $\mathcal{L}_2$ inner product and norm of real-valued functions on the sample space. We treat a random variable as a vector in the $\mathcal{L}_2$ space.

[14]The figure is the result of running the CTM on S&P 500 stock data.

CAPM, at least as used in practice, is empirically false.

In ongoing joint-work with Prof. K. Song at UNT, we are interested in the following problem.

Problem 4.2. *Given an $n \times n$, symmetric matrix $A = (a_{i,j})$. Find a λ such that*

1. *$A + \lambda I_n$ is diagonally dominant.*[15]
2. *$||A + \lambda I_n||_\infty - ||A||_\infty$ is minimal.*[16]
3. *$(A + \lambda I_n)^{-1}$ exists, and $(A + \lambda I_n)^{-1}, \mathcal{D}((A + \lambda I_n)^{-1})$ can be computed or approximated efficiently in a stable fashion.*[17]

This general problem is of interest in many applications, including in machine learning and statistics. The matrix A is typically the Hessian matrix of some appropriate function or the covariance matrix that come from data that arise in applications. In many cases, it is numerically unstable to compute A^{-1}. We may, instead, want to "perturb" A and define a matrix B (in our case $B = A + \lambda I_n$) that is slightly different from A (see requirement (2)) and yet, we can compute or approximate B^{-1} efficiently and the computations are numerically stable.

In many applications, it is often times okay to use B^{-1} as a substitute for A^{-1}. For instance, in optimization or machine learning, one can execute the line search algorithm using the Newton's method applied to a given function f: at step k, the k-th search direction is $x_k = x_{k-1} + \epsilon_k p_k$, where $p_k = H_{k-1}^{-1} \nabla f(x_{k-1})$ and H_{k-1} is the Hessian of f at the point x_{k-1} and ϵ_k is the step size. For convex f, Newton's method guarantees convergence, but it usually is not feasible to compute H^{-1} at each step. By replacing H_k by B_k that satisfies certain conditions (like B_k^{-1} can be computed efficiently and the condition numbers of the B_k's are uniformly bounded, cf [23]), we can still guarantee convergence and efficiency in execution the line search.

The standard algorithm for computing B^{-1} has $O(n^3)$ complexity. Much more efficient algorithms have been designed for various special types of matrices (banded matrices, sparse matrices etc.), see [13]. Our approach to resolving the above problem is via computing the Neumann expansion of $B = A + \lambda I_n$. (cf. [45]). We can show that if λ is chosen such that B is diagonally dominant and positive definite, then we can approximate B^{-1} using the first p many terms of the Neumann expansion of B^{-1} for a relatively small p. p depends on various characteristics of A but our numerical experiments show that in most cases, $p << n$.

In many applications, one often does not care about B^{-1}, but instead computing/approximating $\mathcal{D}(B^{-1})$ becomes the focus. We can achieve this via the *method of probing vectors* (and very efficiently in the case where A is sufficiently sparse, see [45] for a detailed complexity analysis of this method).[18] We just highlight one interesting aspect of this method and how it connects computational linear algebra with graph theory. We define the adjacency graph $G(B) = (V, E)$ associated with the symmetric matrix $B = (b_{i,j})$ as follows: *the vertices $V = \{b_i : i < n\}$, where b_i is the i-th column of B, and $(a_i, a_j) \in E$ if and only if $j \neq i$ and $b_{i,j} \neq 0$.* A key point in the probing vectors method is in coming up with algorithms for coloring the graph $G(B)$. An efficient algorithm using a small number $p << n$ of colors will result in an efficient approximation of $\mathcal{D}(B^{-1})$, see [45]. Our experiments indicate that this method is very numerically stable and has worked very well for certain classes of matrices A (e.g. sparse matrices). Furthermore, I have proven relevant theorems that give bounds on the *chromatic number* of $G(B)$ for a class of matrices B that we are interested in (e.g. for B which are block diagonal with a small number of non-zero entries off of the diagonal blocks, then the chromatic number is bounded by the size of the largest block, or more generally, if B is symmetric with only $2k$ many nonzero entries, then the chromatic number is $O(\sqrt{k})$).

An ongoing work on matrix factorization concerns the following conjecture on the exact cancellations of Cholesky factorizations of symmetric positive definite matrices (SPD). In the following, let $A = LL^T$ be the Cholesky factorization of $A = (a_{ij})$ for $1 \leq i, j \leq n$; here $L = (l_{ij})$ is lower-triangular. We say that an (i, j) element is an *exact cancellation* if $a_{ij} \neq 0$ but $l_{ij} = 0$. We say (i, j) is a *fill-in* if $a_{ij} = 0$ but $l_{ij} \neq 0$. These concepts have received considerable attention from applied mathematicians and computer scientists (see [17, 25]) as they are useful in constructing efficient factorizations of sparse SPDs. In particular, a graph structure (called elimination graph) has been used to prove that assuming the Cholesky factorization of an irreducible matrix A (or its permutation) has no exact cancellations, then some permutation of A produces a Cholesky factorization with no fill-ins. However, the question of whether Cholesky factorizations of some permutation of A will suffer no exact cancellations remains open. In practice, since matrices have entries that are floating point numbers, having no exact cancellations almost always occur. However, from the mathematical perspective, the following conjecture is very interesting.

[15] I_n is the $n \times n$ identity matrix. A matrix $(a_{i,j})$ is diagonally dominant if for every i, $|a_{i,i}| > \sum_{j \neq i} |a_{i,j}|$.

[16] We can replace the ∞-norm by other matrix norms. It is not too important what norm we use.

[17] $\mathcal{D}(A)$ is the matrix whose entries along the diagonal are the diagonal entries of A and whose off-diagonal entries are 0.

[18] In practice, if our matrix A is not sparse, but has many non-zero entries that are very close to 0, we threshold A, by considering considering the matrix A_ϵ for some small ϵ, where the (i, j)-entry of A_ϵ is $a_{i,j}$ if $|a_{i,j}| \geq \epsilon$ and is 0 otherwise.

Conjecture 4.3. *Given an SPD A, there is a permutation $A' = P^T A P$[19] of A such that the Cholesky factorization of A' has no exact cancellations.*

The conjecture has been proven to be true for matrices up to 5×5 (by continuing work of mine and an undergraduate student). With Joseph Prein (a graduate student working under my supervision), we have shown the following general theorem. The first Schur's complement of and $n \times n$ SPD A is defined to be the $(n-1) \times (n-1)$ matrix $S^{(1)} = (a_{ij})_{i,j \geq 2} - \frac{1}{a_{11}}(a_{k1})_{k \geq 2}(a_{k1})^T_{k \geq 2}$.

Theorem 4.4. *Given an irreducible SPD A. There is a permutation $A' = P^T A P$ of A such that there is a column in $S^{(1)}$ that has no exact cancellations.*

Since the Schur's complement is used in the inductive definition of Cholesky factorization that produces the columns of L, we hope this will be important in settling the full conjecture. My undergraduate student is exploring various ideas related to this.

In another joint work with students, we have finished a paper (with a graduate student and two undergrads) on the topic of games on a certain types of graphs, called partition graphs. The paper deals with the games of Cops and Robber on tunnel graphs. We prove several theorems that relate the cop number of the game on a given graph G and the corresponding game on the n-tunnel graph G_n (basically the cop number can increase by at most one). The paper, entitled "On the cop number of subdivisions of graphs", has been submitted to the Journal of Combinatorics, [14].

Many related problems in these areas are suitable as research topics for undergraduate students and early graduate students. For instance, I believe motivated undergraduates can undertake projects similar to the above (on various types of games on graphs). Sparse matrix approximations is another area where graph-theoretic techniques can be applied (approximation of inverse of sparse matrices, efficient approximations of solutions of linear equations of sparse matrices etc.) and undergraduate students can participate in research projects.

5. DEEP LEARNING AND COMPUTER VISION

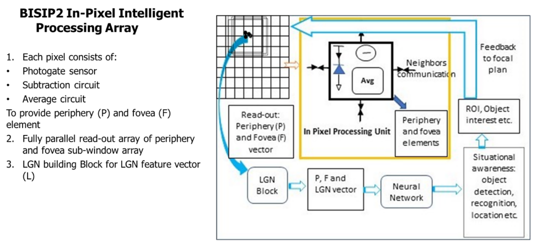
Regarding industry work, I'm part of a team of industry experts working on bio-inspired, adaptive artificial intelligence (AI) systems that are suitable for many types of applications, including image processing, image recognition, feature extractions etc. Our company is working on a DARPA funded project that uses our technology to solve certain problems/tasks DARPA is interested in.

The research project, funded by DARPA, is on designing and building an in-pixel intelligent system as described Figure 5. The system emulates the saccadic eye movement (hence bio-inspired) in processing visual information. It consists of the design of a chip that processes images with resolution $1k \times 1k$, produces a sparsification ($10\times$-reduction) of the image (called the P, F, and the L features), and doing all of this with a constraint in power consumption (at most 250 mWatts per image). The sparse images retain enough visual information so that the back-end (an image detection/recognition system) can be used to perform various tasks on them (like training on these sparse images and feeding the information back into the on-chip memory and the focal plane). The whole system is a close-loop system that achieves high-accuracy (in terms of its primary task, namely recognition, detection and tracking) and at the same time is low-power and fast (at least 100 frames per second). See Figure 5 for a block diagram of the system. The technology used here has potential for commercialization and many other applications in computer vision. This is another key research direction that I plan to pursue in the next several years. The main goal is to refine and improve the above technology and build around that core technology so that it's easy to adapt to various applications.

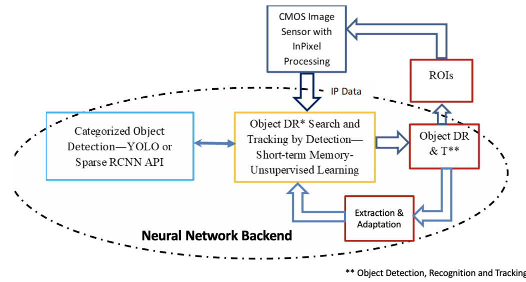
One major potential application in computer vision is to design "compact" neural network models that offer comparable performance to "deep" neural networks but take much less time to train. For instance, one of the sparsification techniques mentioned above have the potential to help reduce the size of the networks for certain computer vision tasks (like image detection). The technology allows us to isolate "good" features as opposed to obtaining them via training convolutional networks with many layers and each layer has many filters (i.e., features) to train. This is the key point of my current research in this direction. If successful, we hope to obtain convolutional neural networks that consist of very few convolutional layers (e.g., 1 to 2 layers) in contrast to current state-of-the-art neural networks (e.g., the YOLO models and its variations) that may contain dozens of such layers.

This is another area that undergraduates can take part in research. For instance, I am mentoring an intern, who is a senior undergraduate at UC Irvine, at the company I'm working with on some work related to computer vision and deep learning. I would envision many projects that undergraduates can participate in (e.g., build convolutional

[19] P is called a permutation matrix.



figureA high level diagram of the In-pixel intelligent processing system



figureBlock diagram of the system

neural networks that perform certain tasks in computer vision). Students will learn very interesting mathematical ideas (like convolutions, back-propagation, logistic regression, and many more that go into the building blocks of a neural network) and practical programming skills. These are absolutely important skills, I believe, that a STEM student (in most disciplines) should acquire.

References

[1] Sargsyan Grigor Trang Nam Wilson Trevor Adolf, Dominik and Martin Zeman. Ideals and strong axioms of determinacy. *arXiv preprint arXiv:2111.06220*, 2022.

[2] James E Baumgartner. Iterated forcing. *Surveys in set theory*, 87:1–59, 1983.

[3] Daniel Busche and Ralf Schindler. The strength of choiceless patterns of singular and weakly compact cardinals. *Annals of Pure and Applied Logic*, 159(1):198–248, 2009.

[4] William Chan and Stephen Jackson. The destruction of the axiom of determinacy by forcings on when θ is regular. *Israel Journal of Mathematics*, 241(1):119–138, 2021.

[5] William Chan, Stephen Jackson, and Nam Trang. More definable combinatorics around the first and second uncountable cardinal. *submitted, available at https://math.unt.edu/∼ntrang*, 2020.

[6] William Chan, Stephen Jackson, and Nam Trang. Countable length everywhere club uniformization. *to appear on the Journal of Symbolic Logic, available at https://math.unt.edu/∼ntrang*, 2021.

[7] William Chan, Stephen Jackson, and Nam Trang. Almost disjoint families under determinacy. 2022. submitted, available at https://math.unt.edu/∼ntrang.

[8] William Chan, Stephen Jackson, and Nam Trang. Almost everywhere behavior of functions according to partition measures. *submitted, available at https://math.unt.edu/∼ntrang*, 2022.

[9] William Chan, Stephen Jackson, and Nam Trang. The size of the class of countable sequences of ordinals. *Transactions of the American Mathematical Society*, 375(03):1725–1743, 2022.

[10] Eugene F Fama and Kenneth R French. Common risk factors in the returns on stocks and bonds. *Journal of financial economics*, 33(1):3–56, 1993.

[11] Matt Foreman, Nam Trang, and David Brownstone. An empirical refutation of CAPM. *In preparation*, 2020-2021.

[12] Fred Galvin and Karel Prikry. Borel sets and ramsey's theorem1. *The Journal of Symbolic Logic*, 38(2):193–198, 1973.

[13] Gene H Golub and Charles F Van Loan. Matrix computations johns hopkins university press. *Baltimore and London*, 1996.

[14] STEPHEN JACKSON, CODY OLSEN, JOSIAH SWEATT, NAM TRANG, and ANGELA YUAN. On the cop number of subdivisions of graphs. 2022. submitted to the Journal of Combinatorics.

[15] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[16] R. Jensen, E. Schimmerling, R. Schindler, and J. R. Steel. Stacking mice. *J. Symbolic Logic*, 74(1):315–335, 2009.

[17] Joseph WH Liu. The role of elimination trees in sparse factorization. *SIAM journal on matrix analysis and applications*, 11(1):134–172, 1990.

[18] D. A. Martin and J. R. Steel. Iteration trees. *J. Amer. Math. Soc.*, 7(1):1–73, 1994.

[19] Donald A Martin, Itay Neeman, and Marco Vervoort. The strength of blackwell determinacy. *The Journal of Symbolic Logic*, 68(02):615–636, 2003.

[20] Adrian RD Mathias. Happy families. *Annals of Mathematical logic*, 12(1):59–111, 1977.

[21] William J Mitchell and Ernest Schimmerling. Weak covering without countable closure. *Mathematical Research Letters*, 2:595–610, 1995.

[22] Itay Neeman and Ernest Schimmerling. Hierarchies of forcing axioms i. *The Journal of Symbolic Logic*, 73(01):343–362, 2008.

[23] Jorge Nocedal and Stephen Wright. *Numerical optimization*. Springer Science & Business Media, 2006.

[24] D. Rodríguez and N. Trang. $L(\mathbb{R},\mu)$ is unique. *Advances in Mathematics*, 2018.

[25] Donald J Rose. A graph-theoretic study of the numerical solution of sparse positive definite systems of linear equations. In *Graph theory and computing*, pages 183–217. Elsevier, 1972.

[26] Stephen A Ross. *Neoclassical finance*. Princeton University Press, 2009.

[27] G. Sargsyan. *Hod mice and the mouse set conjecture*, volume 236 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2014.

[28] G. Sargsyan and N. Trang. Non-tame mice from tame failures of the unique branch hypothesis. *Canadian Journal of Mathematics*, 66(4):903–923, 2014.

[29] G. Sargsyan and N. Trang. *The largest Suslin axiom*. 2016.

[30] Grigor Sargsyan. Nontame mouse from the failure of square at a singular strong limit cardinal. *Journal of Mathematical Logic*, 14(01):1450003, 2014.

[31] Grigor Sargsyan and Nam Trang. The exact consistency strength of generic absoluteness for universally Baire sets. 2019. Available at math.rutgers.edu/∼gs481/.

[32] S. Sargsyan and N. Trang. Tame failures of the unique branch hypothesis and models of $\mathsf{AD}_{\mathbb{R}} + \theta$ is regular. *submitted to the Journal of Mathematical Logic*, 2014.

[33] Ernest Schimmerling and Martin Zeman. Characterization of $\square_\kappa$ in core models. *Journal of Mathematical Logic*, 4(01):1–72, 2004.

[34] R. Schindler and J. R. Steel. *The core model induction*. available at http://www.math.uni-muenster.de/logik/Personen/rds/. 2013.

[35] F. Schlutzenberg and N. Trang. Scales in hybrid mice over $\mathbb{R}$. *arXiv preprint arXiv:1210.7258v4*, 2016.

[36] Jack Silver. Every analytic set is ramsey. *The Journal of Symbolic Logic*, 35(1):60–64, 1970.

[37] J. R. Steel. PFA implies $\mathsf{AD}^{L(\mathbb{R})}$. *J. Symbolic Logic*, 70(4):1255–1296, 2005.

[38] John Steel and Stuart Zoble. Determinacy from strong reflection. *Transactions of the American Mathematical Society*, 366(8):4443–4490, 2014.

[39] John R Steel. Core models with more woodin cardinals. *The Journal of Symbolic Logic*, 67(03):1197–1226, 2002.

[40] John R Steel. Derived models associated to mice. *Computational prospects of infinity. Part I. Tutorials*, 14:105–193, 2008.

[41] John R Steel and W Hugh Woodin. HOD as a core model. In *Ordinal definability and recursion theory: Cabal Seminar*, volume 3, 2012.

[42] J.R. Steel. Normalizing iteration trees and comparing iteration strategies. *available at https://math.berkeley.edu/∼steel/papers/strategycomparejuly2016.pdf*, 2016.

[43] J.R. Steel and Nam Trang. Condensation for mouse pairs. *submitted to the Journal of Symbolic Logic*, 2018.

[44] J.R. Steel and Nam Trang. Characterization of $\square$ in hod of AD^+ models. *In preparation*, 2022.

[45] Jok M Tang and Yousef Saad. A probing method for computing the diagonal of a matrix inverse. *Numerical Linear Algebra with Applications*, 19(3):485–501, 2012.

[46] N. Trang. *Generalized Solovay Measures, the HOD Analysis, and the Core Model Induction*. PhD thesis, UC Berkeley, 2013.

[47] N. Trang. PFA and guessing models. *Israel Journal of Mathematics*, pages 607–667, 2016.

[48] N. Trang and T. Wilson. Determinacy from strong compactness of ω_1. *submitted to the Annals of Pure and Applied Logic*, 2016.

[49] Nam Trang. HOD in natural models of AD^+. *Annals of Pure and Applied Logic*, 165(10):1533–1556, 2014.

[50] Nam Trang. Structure theory of $L(\mathbb{R}, \mu)$ and its applications. *The Journal of Symbolic Logic*, 80(01):29–55, 2015.

[51] Matteo Viale. Guessing models and generalized laver diamond. *Annals of Pure and Applied Logic*, 163(11):1660–1678, 2012.

[52] Matteo Viale and Christoph Weiß. On the consistency strength of the proper forcing axiom. *Advances in Mathematics*, 228(5):2672–2687, 2011.

[53] Trevor M Wilson. The envelope of a pointclass under a local determinacy hypothesis. *Annals of Pure and Applied Logic*, 2015.

[54] W. H. Woodin. AD and the uniqueness of the supercompact measures on $\mathcal{P}_{\omega_1}(\lambda)$. In *Cabal Seminar 79–81*, pages 67–71. Springer, 1983.

[55] W. H. Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, volume 1 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 1999.