NEW LOWERBOUND CONSISTENCY RESULTS FOR FRAGMENTS OF MARTIN'S MAXIMUM

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Abstract

We show that "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$" is a consistency lowerbound for two theories extending $\mathsf{MM}(\mathfrak{c})$:

- $\mathsf{MM}(\mathfrak{c})+\neg\square(\omega_3) + (\dagger)$.
- $\mathsf{MM}(\mathfrak{c})+$ there is a semi-saturated ideal on $\omega_2 + (\dagger)$.

Here $(\dagger)$ is the theory defined in 1.3. As a corollary, we also show that if either of the two theories above holds and M is a class inner model of AD^+ containing all the reals such that $\Theta^M = \omega_3$, then either $\mathsf{AD}_{\mathbb{R}}$ holds in M or else Strong Mouse Capturing (SMC) fails in M. The work in this paper presents some progress towards resolving [28, Problems 8, 12]. We note that the first theory is consistent relative to "$\mathsf{AD}_{\mathbb{R}}$+Θ is Mahlo" and the second is consistent relative to "$\mathsf{AD}_{\mathbb{R}}$+Θ is regular." So our result brings us closer to the exact consistency strength of both theories and may shed light on understanding the strength of $\mathsf{MM}(\mathfrak{c})$.

1. INTRODUCTION

The results of this paper present some progress in determining the consistency strength of Martin's Maximum for posets of size the Continuum ($\mathsf{MM}(\mathfrak{c})$) and its variations. It is a well-known theorem of H.W. Woodin that $\mathsf{MM}(\mathfrak{c})$ holds in generic extensions of models of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular". This, combined with work of G. Sargsyan [9], in turns show that $\mathsf{MM}(\mathfrak{c})$ is weaker, consistency-wise, than $\mathsf{ZFC}+$ " there is a Woodin limit of Woodin cardinals" (WLW). The theories $\mathsf{MM}(\mathfrak{c})$ and $\mathsf{CH}+$"there is an ω_1-dense ideal on ω_1" are two prominent theories conjectured to have consistency strength of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" and have driven major developments in descriptive inner model theory, particularly in the core model induction methods (cf. [28, Problem 12]).

Regarding the problem of determining the consistency of $\mathsf{CH}+$"there is an ω_1-dense ideal on ω_1", [1], built on work of Woodin who has shown that the theories "there is an ω_1-dense ideal on ω_1" and AD are equiconsistent and Ketchersid [5] who shows the existence of models of $\mathsf{AD}_{\mathbb{R}}$ from a strengthening of $\mathsf{CH}+$"there is an ω_1-dense ideal on ω_1", shows that $\mathsf{CH}+$"there is an ω_1-dense ideal on ω_1" and $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" are equiconsistent. This work resolves part of [28, Question 12].

[0] *Key words*: Ideals, Martin's Maximum, hod mice, large cardinals, determinacy, core model induction
[0] *2010 MSC*: 03E15, 03E45, 03E60
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The other part of [28, Question 12] concerning $\mathsf{MM}(\mathfrak{c})$ and its strengthening has seen less progress. Our paper is built on previous work of Steel and Zoble [22], which establishes that $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ follows from $\mathsf{MM}(\mathfrak{c})$. [22], in turns, was built on earlier work of Woodin ([28]) who shows that $\mathsf{MM}(\mathfrak{c})$ implies Projective Determinacy. To construct models of stronger axioms of determinacy (like $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$), we loosely follow the general framework in [5]. At some point in Section 5, we need to construct a hod pair $(\mathcal{P}, \Sigma)$ with Σ having branch condensation from hod pairs with weak condensation; this is precisely the point (†) is used (like in [5]). The other extra hypotheses in the theories $(T1)$ and $(T2)$ below play a role in the proof of Lemma 3.1, which we don't see how to do with just $\mathsf{MM}(\mathfrak{c})$. We note that unlike [5] and this paper, [1] does not use (†) but instead uses a game-theoretic argument developed in [27] to construct such a pair. At this point, we do not see how to adapt arguments in [1] to our situation in this paper.

Definition 1.1. For a cardinal λ, the principle $\neg\square(\lambda)$ asserts that for any sequence $\langle C_\alpha \mid \alpha < \lambda\rangle$ such that

1. for each $\alpha < \lambda$,
   - each C_α is club in α;
   - for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$; and
2. there is a thread through the sequence, i.e., there is a club $E \subseteq \lambda$ such that $C_\alpha = E \cap \alpha$ for each limit point α of E.

$\dashv$

Definition 1.2. Suppose $I \subseteq \wp(\omega_2)$ is a uniform and normal ideal on ω_2. We say that I is *semi-saturated* if whenever U is a V-normal ultrafilter which is set generic over V and such that $U \subseteq \wp(\omega_2)\backslash I$, then $\mathrm{Ult}(V, U)$ is wellfounded. $\dashv$

Definition 1.3. Let (†) be the statement: Whenever A is a set of ordinals that is OD from a countable set of ordinals, for any $X \in \wp_{\omega_1}(A)$, there is a transitive model M of ZFC containing $\{A, X\}$ such that $M \vDash$ "ω_1^V is measurable."

Let $(T1)$ be the theory $\mathsf{MM}(\mathfrak{c}) + \neg\square(\omega_3)$. Let $(T2)$ be the theory $\mathsf{MM}(\mathfrak{c})$ + "there is a semi-saturated ideal on ω_2." $\dashv$

(†) is a variation of a similar hypothesis used in the main theorem of [5] and [20, Theorem 7.1.3]. The main theorem of the paper is.

Theorem 1.4. *Assume the consistency of one of the following theories.*

- $(T1) + (\dagger)$.
- $(T2) + (\dagger)$.

Then $Con(\mathsf{AD}_{\mathbb{R}} + \mathsf{DC})$.

We note that $(T1)$ is a consequence of $\mathsf{MM}(\mathfrak{c}^+)$. It is a theorem of Woodin, see [28], that $\mathrm{Con}(\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular) implies $\mathrm{Con}(T2)$. Moreover, recent advancement in descriptive inner model theory suggests that the strength of $(T1)$ is below that of WLW; for example, [2, Theorems 6.3 and 6.5] shows that $(T1)$ is consistent relative to $\mathsf{AD}_{\mathbb{R}}$ + "Θ is Mahlo". We show in this paper that $(T1) + (\dagger)$ holds in the model of $(T1)$ constructed in [2], and $(T2) + (\dagger)$ holds in a generic extension of any model of $\mathsf{AD}_{\mathbb{R}}$+"Θ is regular." So Theorem 1.4 gets us closer of determining the exact consistency strength of the theories $(T1)$ and $(T2)$, and hopefully that of $\mathsf{MM}(\mathfrak{c})$ and $\mathsf{MM}(\mathfrak{c}^+)$.

The paper introduces several new ideas in the core model induction in the context of $\mathsf{MM}(\mathfrak{c})$. First, Section 3 shows that stationary many elementary substructures of size $\leq \omega_1$ are full with respect to mice in the maximal models (to be defined precisely later). We note that $\mathsf{MM}(\mathfrak{c})$ implies that these elementary substructures are not countably closed; this is an improvement to the results of [24] where the author shows stationary many elementary countably closed substructures are full. Unlike the situations of [5, 20, 1], ideals used in this paper are not in general homogeneous or even quasi-homogeneous;[1] this creates difficulties with constructing strategies for hod mice that have nice properties (like weak condensation as defined in Section 5). We overcome this issue by using the tree projecting to the universal set of the largest Suslin pointclass in the maximal model of AD^+ in $V^{Coll(\omega,\omega_2)}$ as a means to homogenize certain constructions in Sections 4 and 5. Finally, adapting ideas from [13, 1] allows us to extend and restrict hod mice strategies through various generic forcing extensions as in Section 6.

A simple corollary of the proof of the above theorem is as follows.

Corollary 1.5. *Assume $\mathsf{MM}(\mathfrak{c}) + \neg\square(\omega_3)$. Suppose M is a class inner model of AD^+ containing all the reals. Suppose further that $M \vDash \mathsf{SMC} + \neg\mathsf{AD}_{\mathbb{R}}$. Then $\Theta^M < \omega_3$.*

We conjecture that the result of Theorem 1.4 holds without assuming $(\dagger)$ and that we can obtain $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" from both $(T1)$ and $(T2)$. To do this, it appears one needs to refine game-theoretic arguments used in [1, 27, 24].

2. PRELIMINARIES

2.1. Basic facts about AD^+

We start with the definition of Woodin's theory of AD^+. In this paper, we identify $\mathbb{R}$ with ω^ω. We use Θ to denote the sup of ordinals α such that there is a surjection $\pi : \mathbb{R} \to \alpha$. Under AC, Θ is just the successor cardinal of the continuum. In the context of AD, the cardinal Θ is shown to be the supremum of $w(A)$[2] for $A \subseteq \mathbb{R}$ (cf. [17]). The definition of Θ relativizes to any determined pointclass Γ with sufficient closure properties, and we may write Θ^Γ for the supremum of ordinals α such that there is a surjection from $\mathbb{R}$ onto α coded by a set of reals in Γ.

Definition 2.1. AD^+ is the theory $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ plus the following two statements:

[1] It is consistent relative to $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" that $\mathsf{MM}(\mathfrak{c})$ holds and NS_{ω_1} is quasi-homogeneous by [28].

[2] $w(A)$ is the Wadge rank of A.

1. For every set of reals A, there are a set of ordinals S and a formula φ such that $x \in A \iff L[S, x] \models \varphi[S, x]$. The pair (S, φ) is called an ∞-Borel code for A.
2. For every $\lambda < \Theta$, every continuous $\pi : \lambda^\omega \to \omega^\omega$, and every set of reals A, the set $\pi^{-1}[A]$ is determined.

$\dashv$

AD^+ is equivalent to AD + "the set of Suslin cardinals is closed below Θ." Another, perhaps more useful, characterization of AD^+ is AD + "Σ_1 statements reflect into the Suslin co-Suslin sets" (see [21] for the precise statement).

For $A \subseteq \mathbb{R}$, we let θ_A be the supremum of all α such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto α. If Γ is a determined pointclass and $A \in \Gamma$, we write $\Gamma \restriction A$ for the set of all $B \in \Gamma$ that are Wadge reducible to A. If $\alpha < \Theta^\Gamma$, we write $\Gamma \restriction \alpha$ for the set of all $A \in \Gamma$ with Wadge rank strictly less than α.

Definition 2.2 (AD^+)**.** The **Solovay sequence** is the sequence $\langle \theta_\alpha \mid \alpha \leq \lambda \rangle$ where

1. θ_0 is the supremum of ordinals β such that there is an OD surjection from $\mathbb{R}$ onto β;
2. if $\alpha > 0$ is limit, then $\theta_\alpha = \sup\{\theta_\beta \mid \beta < \alpha\}$;
3. if $\alpha = \beta + 1$ and $\theta_\beta < \Theta$ (i.e. $\beta < \lambda$), fixing a set $A \subseteq \mathbb{R}$ of Wadge rank θ_β, θ_α is the sup of ordinals γ such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto γ, i.e. $\theta_\alpha = \theta_A$.

$\dashv$

Note that the definition of θ_α for $\alpha = \beta + 1$ in Definition 2.2 does not depend on the choice of A. One can also make sense of the Solovay sequence of pointclasses that may not be constructibly closed. Such pointclasses show up in core model induction applications. The Solovay sequence $(\theta_\alpha : \alpha < \gamma)$ of a pointclass Ω with the property that if $A \in \Omega$, then $L(A, \mathbb{R}) \models \mathsf{AD}^+$ and $\wp(\mathbb{R}) \cap L(A, \mathbb{R}) \subseteq \Omega$ is defined as follows. First, θ_0 is the supremum of all α such that there is some $A \in \Omega$ and some $OD^{L(A,\mathbb{R})}$ surjection $\pi : \mathbb{R} \to \alpha$. If $\lambda < \gamma$ is limit, then $\theta_\gamma = \sup_{\alpha<\lambda} \theta_\alpha$. If θ_α has been defined and $\alpha + 1 < \gamma$, then letting $A \in \Omega$ be of Wadge rank θ_α, $\theta_{\alpha+1}$ is the supremum of β such that there is some $B \in \Omega$ and some $OD(A)^{L(B,\mathbb{R})}$ surjection $\pi : \mathbb{R} \to \beta$.

Roughly speaking, the longer the Solovay sequence is, the stronger the associated AD^+-theory is. The minimal model of AD^+ is $L(\mathbb{R})$, which satisfies $\Theta = \theta_0$. The theory $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$ implies that the Solovay sequence has limit length. The theory $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ is strictly stronger than $\mathsf{AD}_\mathbb{R}$ since by [17], DC implies $\mathrm{cof}(\Theta) > \omega$ whereas the minimal model[3] of $\mathsf{AD}_\mathbb{R}$ satisfies $\Theta = \theta_\omega$. The theory "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" is much stronger still, as it implies the existence of many models of $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$. We end this section with a theorem of Woodin, which produces models with Woodin cardinals from AD^+. The theorem is important in the HOD analysis of such models.

[3]From here on, whenever we talk about "models of AD^+", we always mean transitive models of AD^+ that contain all reals and ordinals.

Theorem 2.3 (Woodin, see [6]). *Assume* AD^+*. Let* $\langle\theta_\alpha \mid \alpha \leq \Omega\rangle$ *be the Solovay sequence. Suppose* $\alpha = 0$ *or* $\alpha = \beta + 1$ *for some* $\beta < \Omega$*. Then* $\mathrm{HOD} \vDash \theta_\alpha$ *is Woodin.*

2.2. Combinatorial consequences of $\mathsf{MM}(\mathfrak{c})$

Let $g \subseteq Coll(\omega, \omega_1)$ be V-generic. Let $G \subseteq \wp(\omega_1)/NS_{\omega_1}$ be V-generic and let $j_G : V \to M_G \subseteq V[G]$ be the associated embedding. Let $h \subseteq Coll(\omega, \omega_2)$ be V-generic such that $g, G \in V[h]$; we can and do take h to be $V[G \times g]$-generic.

[22] uses the following consequences of $\mathsf{MM}(\mathfrak{c})$ to obtain AD holds in $L(\mathbb{R})^{V[k]}$ for $k \in \{\emptyset, g, h, G, G\times g\}$:

- $2^\omega \leq \omega_2$ and $2^{\omega_1} \leq \omega_2$.
- The nonstationary ideal on ω_1, NS_{ω_1}, is saturated, so the Boolean algebra $\wp(\omega_1)/NS_{\omega_1}$ has the ω_2-cc. In particular, $M_G^\omega \subset M_G$ and $j_G(\omega_1^V) = \omega_2^V$.
- The weak reflection principle $\mathsf{WRP}_2(\omega_2)$ holds, where $\mathsf{WRP}_2(\omega_2)$ asserts that for any stationary subsets S and T of $[\omega_2]^\omega$, there is an ordinal $\delta < \omega_2$ so that $S \cap [\delta]^\omega$ and $T \cap [\delta]^\omega$ are both stationary in $[\delta]^\omega$.

The same consequences (and the exact same proof as done in [22]) show that for each such k, the maximal model of $\mathsf{AD}^+ + \Theta = \theta_0$, $\mathfrak{N}_k$ in $V[k]$ exists and furthermore, Fact 2.16 holds.

However, this is about as far as we see the argument in [22] gives us. To obtain models of stronger forms of determinacy, we need strengthenings of the principles above. More precisely, we use $(T1) + (\dagger)$ or $(T2) + (\dagger)$. None of $(T1), (T2), (\dagger)$ seem to follow from $\mathsf{MM}(\mathfrak{c})$.

By [28, Theorem 9.138], [28, Theorem 9.126] and the discussion after, $(T2)$ is consistent relative to $\mathsf{AD}_\mathbb{R}$+"Θ is regular". More precisely, let $M \vDash \mathsf{AD}_\mathbb{R}$+"$\Theta$ is regular", $G \subseteq \mathbb{P}_{\max}$ be M-generic and $H \subset \mathrm{Add}(\omega_3, 1)^{M[G]}$ be $M[G]$-generic, then $M[G][H] \vDash (T2)$. In the next section, we show $(\dagger)$ also holds in $M[G][H]$.

As mentioned above, [2] shows the consistency of $(T1)$ relative to $\mathsf{AD}_\mathbb{R}$+"Θ is Mahlo." More precisely, the authors of [2] show that if there are models $M_0 \subset M_1$ such that

- $\Theta^{M_0} = \theta_\alpha^{M_1} < \Theta^{M_1}$ for some α with $\mathrm{cof}^{M_1}(\alpha) \geq \omega_2$;
- $M_0, M_1 \vDash \mathsf{AD}_\mathbb{R}$+"$\Theta$ is regular";
- letting $\Gamma_0 = \wp(\mathbb{R}) \cap M_0$, then $M_0 = \mathrm{HOD}_{\Gamma_0}^{M_1}$;

then whenever $G \subset \mathbb{P}_{\max}$ is M_1-generic and $H \subset \mathrm{Add}(\omega_3, 1)^{M_0[G]}$ is $M_1[G]$-generic, then $M_0[G][H] \vDash (T1)$. In the next section, we show that $(T1) + (\dagger)$ also holds in $M_0[G][H]$.

2.3. $(\dagger)$

In this section, we show the consistency of $(\dagger)$. We show $(\dagger)$ holds in $\mathbb{P}_{\max}$-extension of any model N of AD^+. Let N be such a model and $G \subseteq \mathbb{P}_{\max}$ be N-generic. Suppose $A \in N[G]$ is a set of ordinals that is OD from a countable set of ordinals and $X \in \wp_{\omega_1}(A)$. Since $\mathbb{P}_{\max}$ is countably closed and homogeneous, it is clear then that $A, X \in N$. Since $N \vDash \mathsf{AD}^+$, ω_1^N is measurable. Let μ be the unique normal measure on ω_1^N (so μ is just the club filter on ω_1^N). We can let $M = L[A, X][\mu]$. It's clear M has the property in (†).

It is now clear that letting M, G, H be as in the previous section, then $M[G][H] \vDash (T2) + (\dagger)$ and letting M_0, M_1, G, H be as in the previous section, then $M_0[G][H] \vDash (T1) + (\dagger)$. This is because if $A \in M[G][H]$ is ordinal definable from a countable sequence of ordinals s, then first of all $s \in M[G]$ because H is countably closed in $M[G]$; by homogeneity of $Add(\omega_2, 1)^{M[G]}$, $A \in M[G]$ as well. Furthermore, $\wp_{\omega_1}^{M[G][H]}(A) = \wp_{\omega_1}^{M[G]}(A)$. The previous paragraph then shows for any $X \in \wp_{\omega_1}^{M[G][H]}(A)$, there is a model N containing A, X and satisfies "ω_1^V is measurable". So (†) holds in $M[G][H]$. As similar argument shows that (†) also holds in $M_0[G][H]$.

We end this section by proving a variation of (†) at ω_2^V. We will not need this fact in this paper, but it may have potential interest elsewhere. Furthermore, this result does not seem to be published, so it is worth mentioning it here in this paper.[4]

Theorem 2.4. *Suppose $V \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular $+ V = L(\wp(\mathbb{R}))$. Let $G \subseteq \mathbb{P}_{max}$ be V-generic and $H \subseteq Add(\omega_2, 1)^{V[G]}$ be $V[G]$-generic. Suppose A is a set of ordinals that is ordinal definable from a countable sequence of ordinals in $V[G][H]$. Whenever $X \in \wp_{\omega_2}(A)$, there is a model M such that $\{X, A\} \in M$ and $M \vDash$ "$\mathsf{ZFC}+\omega_2^V$ is measurable."*

Proof. Suppose $A \in OD(s)$ in $V[G][H]$, where s is a countable sequence of ordinals. Since both $\mathbb{P}_{\max}$ and $Add(\omega_2, 1)^{V[G]}$ are ω_1-closed and homogeneous (in their respective models), $s \in V$ and furthermore, $A \in V$. Since $V[G] \vDash \omega_2$-DC and $Add(\omega_2, 1)^{V[G]}$ is ω_2-closed in $V[G]$, $(\mathrm{Ord}^{\omega_1})^{V[G]} = (\mathrm{Ord}^{\omega_1})^{V[G][H]}$. By standard theory of $\mathbb{P}_{\max}$ (cf. [28]), for every set of ordinals $X \in V[G]$ with $|X| = \omega_1$, there is some $Y \in V$ with $|Y| = \omega_1$ such that $X \subseteq Y$. Thus, fixing such an X and letting $Y \in V$ that covers X in the manner just described and $Z = (A, Y)$, then $X \in L[Z][B]$ for some $B \subseteq \omega_1$. The following is the main claim.

Claim 2.5. *In $V[G]$, for all set $B \subset \omega_1$, there is a real x and a $HOD_Z^V[x]$-generic for $Coll(\omega, <\omega_1^V)$ such that $B \in HOD_Z^V[x][g]$.*

Proof. Working in V, choose a term τ for B and a $\mathbb{P}_{\max}$ condition of the form (M, I, a) which forces that no such x, g exist. By choosing a strong enough condition, we may assume that there exists a $b \in M$ such that

$$(M, I, a) \Vdash \text{“}\tau \text{ is the image of } b \text{ under the iteration of } (M, I) \text{ given by } \dot{G}.$$

Let x code (M, I, a). In $V[G]$, choose a $g \subseteq Coll(\omega, <\omega_1^V)$ such that

(i) g is $\mathrm{HOD}_Z^V[x]$-generic;

[4] The author would like to thank Hugh Woodin for communications regarding this theorem.

(ii) $\forall \alpha < \omega_1$, $\{\beta < \omega_1 : g_\beta(0) = \alpha\}$ is stationary in ω_1, where g_β is the surjection given by $g \cap Coll(\omega, \beta)$.

(ii) is possible since there is a closed unbounded set of $\gamma < \omega_1$ that is strongly inaccessible in $\mathrm{HOD}^V_Z[x]$.

By a standard argument, cf. [7, Lemma 11], we can use g to construct a generic iteration $\pi : (M, I) \to (M^*, I^*)$ of length ω_1 such that $NS_{\omega_1} \cap M^* = I^*$ and such that

$$\pi \in L[x][g].$$

There is a $\mathbb{P}_{\max}$-generic G^* such that $(M, I, a) \in G^*$ and that π is the iteration of (M, I) given by G^*. By general $\mathbb{P}_{\max}$-theory, $V[G] = V[G^*]$. But then $\tau_{G^*} = \pi(b) \in \mathrm{HOD}_Z[x][g]$. This contradicts the choice of τ.

□

Since ω_2^V is measurable in $\mathrm{HOD}^V_Z[x]$ and $Coll(\omega, <\omega_1^V)$ is a small forcing, $\mathrm{HOD}^V_Z[x][g] \vDash$ "ω_2^V is measurable." By the claim, $B \in \mathrm{HOD}^V_Z[x][g]$, therefore, $X \in \mathrm{HOD}^V_Z[x][g]$; thus letting $M = \mathrm{HOD}^V_Z[x][g]$, then M is the desired model.

□

2.4. Core Model Induction Operators

We summarize some definitions and facts about core model induction operators introduced in the literature (cf. [1, 20]). We refer the reader to [16, 14] for general theory of mouse operators, which core model induction operators are built on, and related concepts omitted in this section.

In the following, a transitive structure N is *closed* under an operator Ω if whenever $x \in \mathrm{dom}(\Omega) \cap N$, then $\Omega(x) \in N$. We are now in a position to introduce the core model induction operators that we will need in this paper. These are particular kinds of mouse operators (in the sense of [16, Example 3.41]) that are constructed during the course of the core model induction. These operators can be shown to satisfy the sort of condensation described in [16, Section 3] (e.g. condense coarsely and condense finely), relativize well, and determine themselves on generic extensions. We will call these operators *nice*. The reader can consult [14] for a detailed treatment of these concepts and [1] for a summary of concepts and terms used in this section.

In core model induction applications, we often have a pair $(\mathcal{P}, \Sigma)$ where $\mathcal{P}$ is a hod premouse and Σ is $\mathcal{P}$'s strategy with branch condensation and is fullness preserving (relative to mice with strategies in some pointclass) or $\mathcal{P}$ is a sound (hybrid) premouse projecting to some countable set a and Σ is the unique (normal) $(\omega_1 + 1)$-strategy for $\mathcal{P}$.

In this section, our main goal is to introduce the main concepts that one uses in the core model induction through the hierarchy $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R}, \Sigma \restriction \mathrm{HC})$[5] [6]. Here $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R}, \Sigma \restriction \mathrm{HC})$ is the union

[5] An equivalent way to define this is to first fix a canonical coding function Code: HC $\to \mathbb{R}$ and consider $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R}, \mathrm{Code}(\Sigma \restriction \mathrm{HC}))$.

[6] Instead of feeding Σ into the hierarchy, we feed in Λ, the canonical strategy of $\mathcal{M}_1^{\Sigma,\sharp}$, into the hierarchy. Roughly

of all sound, Θ-g-organized Σ-premice $\mathcal{M}$ over $(\mathbb{R}, \Sigma \restriction \mathrm{HC})$ such that $\rho_\omega(\mathcal{M}) = \mathbb{R}$ and whenever $\pi : \mathcal{M}^* \to \mathcal{M}$ is sufficiently elementary and $\mathcal{M}^*$ is countable and transitive, then $\mathcal{M}^*$ has a unique $(\omega_1 + 1)$-Σ-iteration strategy Λ.[7] See [15] for a precise definition of g-organized Σ-premice, Θ-g-organized Σ-premice, $\mathrm{Lp}^{\mathsf{g}\Sigma}(x)$, $\mathrm{Lp}_+^{\mathsf{g}\Sigma}(x)$ and other related concepts like operators. When we write $\mathrm{Lp}^{\mathsf{g}\Sigma}$ or $\mathrm{Lp}_+^{\mathsf{g}\Sigma}$, we refer to the hierarchy of g-organized Σ-mice; when we write $\mathrm{Lp}^{\mathsf{G}\Sigma}$ or $\mathrm{Lp}_+^{\mathsf{G}\Sigma}$, we refer to the hierarchy of Θ-g-organized Σ-mice. The g-organized hierarchy of Σ-mice is considered (instead of the traditional "least branch" hierarchy of Σ-mice) because the S-constructions (cf. [12], where they are called P-constructions) work out nicely for this hierarchy.[8] The Θ-g-organized hierarchy, which is a slight modification of the g-organized hierarchy, is considered because the scales analysis under optimal hypotheses can be carried out in $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R}, \Sigma \restriction \mathrm{HC})$ in much the same manner as the scales analysis in $\mathrm{Lp}(\mathbb{R})$.[9] For the purpose of this paper, it will not be important to go into the detailed definitions of these hierarchies. Whenever it makes sense to define $\mathrm{Lp}^{\Sigma}(x)$ and $\mathrm{Lp}^{\mathsf{g}\Sigma}(x)$, [15] shows that $\wp(x) \cap \mathrm{Lp}^{\Sigma}(x) = \wp(x) \cap \mathrm{Lp}^{\mathsf{g}\Sigma}(x)$ (and similarly for $\mathrm{Lp}^{\mathsf{G}\Sigma}(x)$); also in the case it is not clear how to make sense of $\mathrm{Lp}^{\Sigma}(x)$ (say for instance when $x = \mathbb{R}$), it still makes sense to define $\mathrm{Lp}^{\mathsf{g}\Sigma}(x)$ and $\mathrm{Lp}^{\mathsf{G}\Sigma}(x)$ and in that case, [15] shows that $\wp(x) \cap \mathrm{Lp}^{\mathsf{g}\Sigma}(x) = \wp(x) \cap \mathrm{Lp}^{\mathsf{G}\Sigma}(x)$.

Let $\mathcal{F}$ be the operator corresponding to Σ and suppose $\mathcal{M}_1^{\mathsf{g}\mathcal{F},\sharp}$ exists (as a g-$\mathcal{F}$-organized mouse) (see [14]). Then [14] shows that $\mathcal{F}$ condenses finely and $\mathcal{M}_1^{\mathsf{g}\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$. Also, the core model induction will give us that $\mathcal{F} \restriction \mathbb{R}$ is self-scaled (defined below). One final remark is we use the strategy Λ of $\mathcal{M}_1^{\mathsf{g}\mathcal{F},\sharp}$ to define the strategy predicate for the hieararchy of $\mathrm{Lp}^{\Sigma}(\mathbb{R})$ in the manner described in [14]. Since the details of how to define this hierarchy precisely have been fully worked out in [14], the reader is advised to consult it there. In the following, again to simplify the notation, we will write $\mathcal{M}_1^{\Sigma,\sharp}$ for $\mathcal{M}_1^{\mathsf{g}\mathcal{F},\sharp}$.

Definition 2.6. Let Γ be an inductive-like pointclass. For $x \in \mathbb{R}$, $C_\Gamma(x)$ denotes the set of all $y \in \mathbb{R}$ such that for some ordinal $\gamma < \omega_1$, y (as a subset of ω) is $\Delta_\Gamma(\{\gamma, x\})$.

Let $x \in \mathrm{HC}$ be transitive and let $f : \omega \to x$ be a surjection. Then $c_f \in \mathbb{R}$ denotes the code for $(x, \in)$ determined by f. And $C_\Gamma(x)$ denotes the set of all $y \in \mathrm{HC} \cap \wp(x)$ such that for all surjections $f : \omega \to x$ we have $f^{-1}(y) \in C_\Gamma(c_f)$. ⊣

We say that $\vec{A}$ is a *self-justifying-system (sjs)* if for any $A \in \mathrm{rng}(\vec{A})$, $\neg A \in \mathrm{rng}(\vec{A})$ and there is a scale φ on A such that the set of prewellorderings associated with φ is a subset of $\mathrm{rng}(\vec{A})$. A set $Y \subseteq \mathbb{R}$ is *self-scaled* if there are scales on Y and $\mathbb{R} \backslash Y$ which are projective in Y.

The reader should consult [14] for the definition of a Γ-Ω-k-suitable premouse for some pointclass Γ, operator Ω and some integer k. When Γ and Ω are clear form the context, we omit them from the notation; similarly if $k = 1$, we simply say "suitable" instead of "1-suitable". In the following, η is a strong cutpoint of $\mathcal{N}$ if there is no extender E on the sequence of $\mathcal{N}$ such that $\mathrm{crt}(E) \leq \eta \leq \mathrm{lh}(E)$.

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speaking, the trees according to Λ that we feed into $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R}, \mathrm{Code}(\Sigma \restriction \mathrm{HC}))$ are those making the local HOD of $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R}, \mathrm{Code}(\Sigma \restriction \mathrm{HC}))|\alpha$ generically generic, for appropriately chosen ordinals α. See [15].

[7] This means whenever $\mathcal{T}$ is an iteration tree according to Λ with last model $\mathcal{N}$, then $\mathcal{N}$ is a Σ-premouse.

[8] It is not clear how one can perform S-constructions over the least branch hierarchy.

[9] [15] generalizes Steel's scales analysis in [19, 18] to $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R}, \Sigma \restriction \mathrm{HC})$ for various classes of nice strategies Σ. It is not clear that one can carry out the full scales analysis for the hierarchy $\mathrm{Lp}^{\mathsf{g}\Sigma}(\mathbb{R}, \Sigma \restriction \mathrm{HC})$.

Let $\mathcal{N}$ be 1-suitable and let $\xi \in o(\mathcal{N})$ be a limit ordinal such that $\mathcal{N} \vDash$"ξ isn't Woodin". Let $Q \lhd \mathcal{N}$ be the Q-structure for ξ. Let α be such that $\xi = o(\mathcal{N}|\alpha)$. If ξ is a strong cutpoint of $\mathcal{N}$ then $Q \lhd \mathrm{Lp}^{\mathrm{g}\Omega,\Gamma}(\mathcal{N}|\xi)$. Assume now that $\mathcal{N}$ is reasonably iterable. If ξ is a strong cutpoint of Q, our mouse capturing hypothesis gives that $Q \lhd \mathrm{Lp}^{\mathrm{g}\Omega,\Gamma}(\mathcal{N}|\xi)$. If ξ is an $\mathcal{N}$-cardinal then indeed ξ is a strong cutpoint of Q, since $\mathcal{N}$ has only finitely many Woodins. If ξ is not a strong cutpoint of Q, then by definition, we do not have $Q \lhd \mathrm{Lp}^{\mathrm{g}\Omega,\Gamma}(\mathcal{N}|\xi)$. However, using $*$-translation (see [23]), one can find a level of $\mathrm{Lp}^{\mathrm{g}\Omega,\Gamma}(\mathcal{N}|\xi)$ which corresponds to Q (and this level is in $C_\Gamma(\mathcal{N}|\xi)$).

To simplify the notations, from now on, we will simply write $\mathrm{Lp}^{\Sigma}(\mathbb{R})$ for $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R}, \Sigma \restriction \mathrm{HC})$,$\mathrm{Lp}^{\Sigma}(x)$ for $\mathrm{Lp}^{\mathrm{g}\Sigma}(x)$ etc.

If Ω is a nice operator (in the sense of [15]) and Σ is an iteration strategy for a Ω-Γ-1-suitable premouse $\mathcal{P}$ such that Σ has branch condensation and is Γ-fullness preserving (for some pointclass Γ), then we say that $(\mathcal{P}, \Sigma)$ is a Ω-Γ-*suitable pair* or just Γ-*suitable pair* or just *suitable pair* if the pointclass and/or the operator Ω is clear from the context.

Definition 2.7 (Core model induction operators)**.** Suppose $(\mathcal{P}, \Omega)$ is a $\mathcal{G}$-Ω^*-suitable pair for some nice operator $\mathcal{G}$ or a hod pair such that Σ has branch condensation and is Ω^*-fullness preserving for some inductive-like Ω^*. Assume $\mathrm{Code}(\Omega)$ is self-scaled. We say J is a Σ-*core model induction operator* or just a Σ-*cmi operator* if one of the following holds:

1. J is a nice Ω-mouse operator (or g-organized Ω-mouse operator) defined on a cone of HC above some $a \in \mathrm{HC}$. Furthermore, J condenses finely, relativizes well and determines itself on generic extensions.

2. For some $\alpha \in \mathrm{OR}$ such that α ends either a weak or a strong gap in the sense of [18] and [15], letting $M = \mathrm{Lp}^{\Omega}(\mathbb{R})|\alpha$ and $\Gamma = (\Sigma_1)^M$, $M \vDash \mathsf{AD}^+ + \mathsf{MC}(\Omega)$.[10] For some transitive $b \in \mathrm{HC}$ and some 1-suitable (or more fully Ω-Γ-1-suitable) Ω-premouse $\mathcal{Q}$ over b, $J = \Lambda$, where Λ is an (ω_1, ω_1)-iteration strategy for $\mathcal{Q}$ which is Γ-fullness preserving, has branch condensation and is guided by some self-justifying-system (sjs) $\vec{A} = (A_i : i < \omega)$ such that for some real x, for each i, $A_i \in \mathrm{OD}^M_{b,\Sigma,x}$ and $\vec{A}$ seals the gap that ends at α.

When Σ is clear from the context or that we don't want to specify Σ, we simply say J is a cmi operator. $\dashv$

Remark 2.8. Let Γ, M be as in clause 2 above. The (lightface) envelope of Γ is defined as: $A \in \mathrm{Env}(\Gamma)$ iff for every countable $\sigma \subset \mathbb{R}$ there is some A' such that A' is Δ_1-definable over M from ordinal parameters and $A \cap \sigma = A' \cap \sigma$. For a real x, we define $\mathrm{Env}(\Gamma(x))$ similarly: here $\Gamma(x) = \Sigma_1(x)^M$ and $A \in \mathrm{Env}(\Gamma(x))$ iff for every countable $\sigma \subset \mathbb{R}$ there is some A' that is $\Delta_1(x)$-definable over M from ordinal parameters such that $A \cap \sigma = A' \cap \sigma$. We now let $\mathbf{Env}(\Gamma) = \bigcup_{x \in \mathbb{R}} \mathrm{Env}(\Gamma(x))$. Note that $\mathbf{Env}(\Gamma) = \wp(\mathbb{R})^M$ if α ends a weak gap and $\mathbf{Env}(\Gamma) = \wp(\mathbb{R})^{\mathrm{Lp}^{\Omega}(\mathbb{R})|(\alpha+1)}$ if α ends a strong gap.

[10] $\mathsf{MC}(\Omega)$ stands for Mouse Capturing relative to Ω which says that for $x, y \in \mathbb{R}$, x is $\mathrm{OD}(\Omega, y)$ (or equivalently x is $\mathrm{OD}(\Omega, y)$) iff x is in some g-organized Ω-mouse over y. SMC is the statement that for every hod pair $(\mathcal{P}, \Omega)$ such that Σ is fullness preserving and has branch condensation, $\mathsf{MC}(\Omega)$ holds.

In clause 2 above, $\vec{A}$ is Wadge cofinal in $\mathbf{Env}(\Gamma)$ where $\Gamma = \Sigma_1^M$.

The following definitions are obvious generalizations of those defined in [20]. For example, see [20, Definition 3.2.1] for the definition of a coarse (k, U)-Woodin mouse.

Definition 2.9. We say that the coarse mouse witness condition $W_\gamma^{*,\Omega}$ holds if, whenever $U \subseteq \mathbb{R}$ and both U and its complement have scales in $\mathrm{Lp}^\Omega(\mathbb{R})|\gamma$, then for all $k < \omega$ and $x \in \mathbb{R}$ there is a coarse (k, U)-Woodin mouse M containing x and closed under the strategy Λ of $\mathcal{M}_1^{\Omega,\sharp}$ with an $(\omega_1 + 1)$-iteration strategy whose restriction to HC is in $\mathrm{Lp}^\Omega(\mathbb{R})|\gamma$.[11] $\dashv$

Remark 2.10. By the proof of [20, Lemma 3.3.5], $W_\gamma^{*,\Omega}$ implies $\mathrm{Lp}^\Omega(\mathbb{R})|\gamma \vDash \mathsf{AD}^+$.

Definition 2.11. An ordinal γ is a *critical ordinal* in $\mathrm{Lp}^\Omega(\mathbb{R})$ if there is some $U \subseteq \mathbb{R}$ such that U and $\mathbb{R}\backslash U$ have scales in $\mathrm{Lp}^\Omega(\mathbb{R})|(\gamma+1)$ but not in $\mathrm{Lp}^\Omega(\mathbb{R})|\gamma$. In other words, γ is critical in $\mathrm{Lp}^\Omega(\mathbb{R})$ just in case $W_{\gamma+1}^{*,\Omega}$ does not follow trivially from $W_\gamma^{*,\Omega}$. $\dashv$

To any Σ_1 formula $\theta(v)$ in the language of $\mathrm{Lp}^\Omega(\mathbb{R})$ we associate formulae $\theta_k(v)$ for $k \in \omega$, such that θ_k is Σ_k, and for any γ and any real x,

$$\mathrm{Lp}^\Omega(\mathbb{R})|(\gamma + 1) \vDash \theta[x] \iff \exists k < \omega\ \mathrm{Lp}^\Omega(\mathbb{R})|\gamma \vDash \theta_k[x].$$

Definition 2.12. Suppose $\theta(v)$ is a Σ_1 formula (in the language of set theory expanded by a name for $\mathbb{R}$ and a predicate for ${}^{\mathsf{G}}\Omega$), and z is a real; then a $\langle\theta, z\rangle$*-prewitness* is an ω-sound g-organized Ω-premouse N over z in which there are $\delta_0 < \cdots < \delta_9$, S, and T such that N satisfies the formulae expressing

(a) ZFC,

(b) $\delta_0, \ldots, \delta_9$ are Woodin,

(c) S and T are trees on some $\omega \times \eta$ which are absolutely complementing in $V^{\mathrm{Col}(\omega,\delta_9)}$, and

(d) For some $k < \omega$, $p[T]$ is the Σ_{k+3}-theory (in the language with names for each real and predicate for ${}^{\mathsf{G}}\Omega$) of $\mathrm{Lp}^\Omega(\mathbb{R})|\gamma$, where γ is least such that $\mathrm{Lp}^\Omega(\mathbb{R})|\gamma \vDash \theta_k[z]$.

If N is also $(\omega, \omega_1, \omega_1 + 1)$-iterable (as a g-organized Ω-mouse), then we call it a $\langle\theta, z\rangle$*-witness*. $\dashv$

Definition 2.13. We say that the fine mouse witness condition W_γ^Ω holds if whenever $\theta(v)$ is a Σ_1 formula (in the language $\mathcal{L}^+$ of g-organized Ω-premice (cf. [15])), z is a real, and $\mathrm{Lp}^\Omega(\mathbb{R})|\gamma \vDash \theta[z]$, then there is a $\langle\theta, z\rangle$-witness $\mathcal{N}$ whose ${}^g\Omega$-iteration strategy, when restricted to countable trees on $\mathcal{N}$, is in $\mathrm{Lp}^\Omega(\mathbb{R})|\gamma$. $\dashv$

Lemma 2.14. *$W_\gamma^{*,\Omega}$ implies W_γ^Ω for limit γ.*

[11] We demand the strategy has the property that iterates of M according to the strategy are closed under Λ.

The proof of the above lemma is a straightforward adaptation of that of [20, Lemma 3.5.4]. One main point is the use of the g-organization: g-organized Ω-mice behave well with respect to generic extensions in the sense that if $\mathcal{P}$ is a g-organized Ω-mouse and h is set generic over $\mathcal{P}$ then $\mathcal{P}[h]$ can be rearranged to a g-organized Ω-mouse over h.

Remark 2.15. In light of the discussion above, the core model induction (through $\mathrm{Lp}^{\Omega}(\mathbb{R})$) inductively shows $\mathrm{Lp}^{\Omega}(\mathbb{R})|\gamma \vDash \mathsf{AD}^+$ by showing that $W^{*,\Omega}_{\gamma}$ holds for critical ordinals γ. This, in turn, is done by constructing appropriate Ω-cmi operators "capturing" the theory of those levels (as specified in Definitions 2.9 and 2.13).

Later in the paper, we will outline the core model induction showing that $\mathrm{Lp}^{\Omega}(\mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{MC}(\Omega)$ for various nice Ω from our hypotheses. Basically, the arguments in [22] show from $\mathsf{MM}(\mathfrak{c})$ that given a nice Ω (basically Ω has branch condensation, is fullness preserving and determines itself on generic extensions; Ω could be $\emptyset$), then $\mathrm{Lp}^{\Omega}(\mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{MC}(\Omega)$ by showing $W^{*,\Omega}_{\gamma}$ holds for all critical ordinals γ. What we need to do, using the stronger hypotheses $(T1) + (\dagger)$ or $(T2) + (\dagger)$, is to get past the "last gap" of $\mathrm{Lp}^{\Omega}(\mathbb{R})$ by constructing a nice pair $(\mathcal{Q}, \Lambda) \notin \mathrm{Lp}^{\Omega}(\mathbb{R})$ and show $\mathrm{Lp}^{\Lambda}(\mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{MC}(\Lambda)$.

2.5. Lifting operators

Let $g \subseteq Coll(\omega, \omega_1)$ be V-generic. Let $G \subseteq \wp(\omega_1)/NS_{\omega_1}$ be V-generic and let $j_G : V \to M_G \subseteq V[G]$ be the associated embedding. Let $h \subseteq Coll(\omega, \omega_2)$ be V-generic such that $g, G \in V[h]$; we can and do take h to be $V[G \times g]$-generic. For $k \in \{\emptyset, g, h, G, G \times g\}$, let κ_k be the largest Suslin cardinal of $\mathfrak{N}_g$, Γ_k be the pointclass of κ_k-Suslin sets in $\mathfrak{N}_k$, and T_k be the tree projecting to the universal κ_k-Suslin set in $\mathfrak{N}_k$. To simplify the notation, we will assume $(\mathcal{P}, \Sigma) = \emptyset$; so in this case $\mathfrak{N}_k \vDash \mathsf{AD}^+ + \Theta = \theta_0$. The arguments in [22] give Σ_1-elementary maps from $\mathfrak{N}_k|\kappa_k$ to $\mathfrak{N}_l|\kappa_l$ for $k, l \in \{\emptyset, g, h, G, G \times g\}$ such that $k \in V[l]$. We isolate this as a fact and will refer to it many times in this paper.

Fact 2.16. *For $k, l \in \{\emptyset, g, h, G, G \times g\}$ such that $k \in V[l]$, there is an Σ_1-elementary embedding $j_{k,l}$ from $\mathfrak{N}_k|\kappa_k$ to $\mathfrak{N}_l|\kappa_l$.*

Of course, if $k = \emptyset$ and $l = G$, then j_G induces a fully elementary from $\mathfrak{N}_k$ to $\mathfrak{N}_l$. Also, if $k = G$ and $l = G \times g$, $j_{k,l}$ is just the Cohen ultrapower and acts on all of $\mathfrak{N}_k$. In the case $l \in \{g, h\}$ and $k = \emptyset$, [22] shows that $j_{k,l}$ is the uncollapse map of $\mathrm{Hull}_1^{\mathfrak{N}_{\kappa_l}}(\mathbb{R}^V)$. Similarly, in the case $l = h$ and $k = g$, $j_{k,l}$ is the uncollapse map of $\mathrm{Hull}_1^{\mathfrak{N}_{\kappa_l}}(\mathbb{R}^{V[g]})$. In the case $k = g$ and $l = G \times g$, we note that $V[l] = V[g \times G]$ is a ccc extension of $V[k]$ and the map $j_{k,l}$ is the identity on the ordinals; this follows from the hypothesis I_{κ_k} proved in [22].

Let us briefly explain how the induction in [22] is carried out and therefore, how the maps $j_{k,l}$'s are constructed. In order to communicate the main ideas, it is necessary to simplify many details of the rather complicated constructions in [22]. We then explain how we extend these ideas and construct our models of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

The arguments in [22] show that W^*_α hold in $\mathfrak{N}$ for all α critical; and therefore $\mathfrak{N} \vDash \mathsf{AD}^+$. Given a critical β and suppose W^*_β holds, we want to show $W^*_{\beta+1}$ holds. The first step is to find a cmi operator J that codes up truth at the level of the first pointclass $\undertilde{\Sigma}_n^{\mathfrak{N}|\beta}$ having the scale property. We then construct the operators $M_n^{J,\sharp}$ for all $n < \omega$ that fully capture truth over $\mathfrak{N}|\beta$; this is where we need to use the core model theory (i.e. constructing K^c and K relative to J). To do this, we need to extend J to J^+ acting on $H(\omega_3)$ and show that J^+ is consistent with $j_G(J)$. J^+ canonically extends to $H(\omega_1)^{V[g]}$. We have to consider W^*_γ in $\mathfrak{N}_g$ where $\gamma < j_G(\beta)$. Part of this extension involves showing that $j_G(\beta)$ is independent of G and W^*_γ holds in $\mathfrak{N}_g$ for all $\gamma \leq j_G(\beta)$. This extension process also gives rise to the maps $j_{k,l}$'s as described above. However, the maps $j_{k,l}$'s are only defined up to $\mathfrak{N}_k|\kappa_k$, where κ_k is the limit of critical ordinals in $\mathfrak{N}_k$. It is true that we get AD^+ holds in $\mathfrak{N}_k$ by Σ_1-reflection, but the argument does not give us a way to extend the map $j_{k,l}$ to all of $\mathfrak{N}_k$. The next paragraph describes how we can do this and construct AD^+ models extending $\mathfrak{N}_k$.

As mentioned, the arguments in [22] does not seem to allow us to construct a model of $\mathsf{AD}^+ + \Theta > \theta_0$.[12] In Section 5, we use our full hypotheses ($(T1) + (\dagger)$ or $(T2) + (\dagger)$) to construct a hod pair $(\mathcal{P}, \Sigma) \in V$ such that $\mathcal{P}$ is countable and Σ is a (ω_1, ω_1)-strategy with branch condensation and is Γ-fullness preserving; furthermore, Σ is guided by a sjs $\mathcal{A}$ consisting sets Wadge cofinal in $\mathfrak{N}$. Section 6 then includes arguments that show we can lift Σ through various generic extensions. First, j_G lifts Σ to an (ω_1, ω_1)-strategy in $\mathrm{Ult}(V, G)$, equivalently in $V[G]$. We then extend Σ_G to a unique strategy $\Sigma_{G\times g}$ in $V[G \times g]$. We then show $\Sigma_{G\times g} \restriction V[g] \in V[g]$ and is an (ω_1, ω_1)-strategy there; call this strategy Σ_g. Σ_g is guided by a sjs $\mathcal{A}_g$, a version of $\mathcal{A}$ in $V[g]$. By a standard boolean valued-comparison, there is an iterate $(\mathcal{R}, \Psi) \in V[g]$ of $(\mathcal{Q}, \Lambda_g)$ such that $\mathcal{R} \in V$, $|\mathcal{R}| \leq \omega_1$, $\Psi \restriction V \in V$ and is an (ω_2, ω_2)-strategy there. By $\mathsf{WRP}_2(\omega_2)$, we can uniquely extend Ψ to a (ω_3, ω_3)-strategy that condenses well. Using the elementarity of j_G, we can show that there is a countable, suitable $\mathcal{S} \in V$ and an (ω_3, ω_3)-strategy Φ guided by a sjs $\mathcal{A}'$. We can use this pair $(\mathcal{S}, \Phi)$ to continue the CMI as in [22] to show $(Lp^\Phi(\mathbb{R}))^{V[k]} \vDash \mathsf{AD}^+ + \Theta = \theta_1$ where $k \in \{\emptyset, g, G, g \times G, h\}$ and maintain the inductive hypotheses as in [22]. We can repeat this process for any $\alpha < \omega_1$. The details are carried out in Section 6.

3. FULL HULLS

Let $g \subseteq Coll(\omega, \omega_1)$ be V-generic and let $(\mathcal{P}, \Sigma) \in V[g]$ be a reasonable hod pair such that $\mathcal{P} \in V$ and $\Sigma \restriction V \in V$. Let $\mathfrak{N}_g$ be the maximal model of $\mathsf{AD}^+ + \Theta = \theta_\Sigma$. By our smallness assumption $(\dagger)$, $\mathfrak{N}_g \vDash \mathsf{SMC}$, in particular,

$$\mathfrak{N}_g \vDash V = L(\mathrm{Lp}^\Sigma(\mathbb{R})).$$

We can also define $\mathfrak{N}_g$ for $g \subseteq \mathbb{P}$ such that $\mathcal{P}$ is countable in $V[g]$. In particular, if $\mathcal{P}$ is countable in V and $g = \emptyset$, we write $\mathfrak{N}$ for $\mathfrak{N}_g$. The next lemma shows that $(T1)$ and $(T2)$ both imply $\Theta^{\mathfrak{N}_g} < \omega_3^V$.

[12] This does not necessarily mean that $\mathrm{Con}(\mathsf{MM}(\mathfrak{c}))$ does not imply $\mathrm{Con}\mathsf{AD}^+ + \Theta > \theta_0$; just that the consequences used in the proofs in [22] seem too weak for this purpose.

Lemma 3.1. *Assume* $(T1)$ *or* $(T2)$. *Assume* g *is either a* V*-generic for* $Coll(\omega,\omega_1)$ *or* $g=\emptyset$. *Then* $\Theta^{\mathfrak{N}_g}<\omega_3^V$. *In fact,* $cof^V(\Theta^{\mathfrak{N}_g})<\omega_2^V$.

Proof. We just prove the lemma for $g\subseteq Coll(\omega,\omega_1)$ being V-generic; the other case is easier. Let $\vec{C}=\langle C_\alpha:\alpha<\Theta^{\mathfrak{N}_g}\rangle$ be the canonical coherent sequence constructed over $\mathfrak{N}_g$ as in [25]. Recall $\vec{C}$ has the property that for all α, C_α is a club subset of α and for all $\beta<\alpha$, if $\beta\in lim(C_\alpha)$, then $C_\alpha\cap\beta=C_\beta$. We note $\vec{C}$ is definable in $V[g]$ from $\Sigma\restriction V$, so by homogeneity of the forcing, $\vec{C}\in V$.

Suppose $\Theta^{\mathfrak{N}_g}=\omega_3^V$. Suppose first $(T1)$ holds. By Todorcevic, [3], there is a thread D through $\vec{C}$. By the construction of $\vec{C}$, D defines an $\mathcal{M}\lhd\mathrm{Lp}^\Sigma(\mathbb{R})$ in $V[g]$ such that $\mathcal{M}\vDash\mathsf{AD}^+$ and $o(\mathcal{M})\geq\omega_3^V$. By soundness of $\mathcal{M}$, we have a surjection from $\mathbb{R}^{V[g]}$ onto ω_3^V. This is a contradiction because $|\mathbb{R}^{V[g]}|=\omega_2^V$ in $V[g]$ and

$$\omega_3^V=\omega_2^{V[g]}>\omega_1^{V[g]}=\omega_2^V.$$

This contradiction shows $\Theta^{\mathfrak{N}_g}<\omega_3^V$. We now use $\mathsf{MM}(\mathfrak{c})$ to show $\mathrm{cof}^V(\Theta^{\mathfrak{N}_g})<\omega_2^V$. This follows from standard arguments in [11]. If $\mathrm{cof}^V(\Theta^{\mathfrak{N}_g})=\omega_2^V$, then any continuous, increasing, and cofinal function $f:\omega_2^V\to\Theta^{\mathfrak{N}_g}$ will "pull back" $\vec{C}$ to a coherent sequence $\vec{D}=\langle D_\alpha:\alpha<\omega_2^V\rangle$; then again by Todorcevic, $\mathsf{MM}(\mathsf{c})$ implies there is a thread E through $\vec{D}$. $f[E]$ is a thread through $\vec{C}$, which induces an $\mathcal{M}\lhd Lp^\Sigma(\mathbb{R})$ in $V[g]$ such that $\mathcal{M}\vDash\mathsf{AD}^+$ and $o(\mathcal{M})\geq\Theta^{\mathfrak{N}_g}$. This is again a contradiction because by maximality of $\mathfrak{N}_g$, $\mathcal{M}\in\mathfrak{N}_g$, but since $o(\mathcal{M})\geq\Theta^{\mathfrak{N}_g}$, $\mathcal{M}\notin\mathfrak{N}_g$.

Now, assume $T(2)$. It suffices to show $\Theta^{\mathfrak{N}_g}<\omega_3^V$. The second clause is exactly as before. Let I_g be the canonical extension of the semi-saturated ideal I, so I_g is a semi-saturated ideal on ω_1 in $V[g]$ by [28, Theorem 9.126]. Let $H\subset\wp(\omega_1)/I_g$ be a $V[g]$-normal ultrafilter and is $V[g]$-generic and $j_H:V[g]\to N\subset V[g,H]$ be the associated ultrapower map. By semi-saturation and [28, Theorem 9.127],

$$j_H(\omega_1^{V[g]})=\omega_2^{V[g]}.$$

Since CH holds in $V[g]$, $\mathbb{R}^{V[g]}\in N$ and is countable there. If $\Theta^{\mathfrak{N}_g}=\omega_3^V=\omega_2^{V[g]}$, then in $j_H(\mathfrak{N}_g)$, there is a ω_1-sequence of distinct reals given by the levels of $\mathfrak{N}_g$. This is a contradiction. □

Remark 3.2. By recent work of M. Zeman and the author, we can replace the hypothesis $\neg\Box(\omega_3)$ in $(T1)$ by a weaker hypothesis $\neg\Box_{\omega_2}$. This is because we can in fact construct a square sequence of length $\Theta^{\mathfrak{N}_g}$ in V in the proof of the above lemma.

Proof of Corollary 1.5. Let M be as in the statement of the lemma. Note that by $\mathsf{MM}(\mathfrak{c})$, $|\mathbb{R}|=\omega_2$, so $\Theta^M\leq\omega_3$. Suppose $\Theta^M=\omega_3$. By standard results, e.g., [10], and the fact that $M\vDash\mathsf{SMC}+\neg\mathsf{AD}_\mathbb{R}$, we have a pair $(\mathcal{P},\Sigma)$[13] such that $M\vDash V=L(\mathrm{Lp}^\Sigma(\mathbb{R}))$. By the argument above, using $\neg\Box(\omega_3)$, we immediately get a contradiction. □

The main result of this section is the following version of the covering lemma for "Lp" stacks. This is what makes this situation different from corresponding versions of the covering lemma for "Lp" stacks, such as those that appear in [24]. The proof of Lemma 3.3 closely resembles that of

[13] $(\mathcal{P},\Sigma)$ may be a hod pair or an sts hod pair.

[4]. However, we note that the elementary substructures that appear in the proof of the lemma are not countably closed, unlike the situations in [24, 4]. Recall that we say an elementary substructure X is *Lp-full* in $\mathfrak{N}_g$ if letting $\mathbb{R}_X = \mathbb{R}^{V[g]} \cap X$, then

- $\mathbb{R}_X = \mathbb{R}^X$,
- $|\mathbb{R}_X| < |\mathbb{R}^{V[g]}|$, and
- letting π_X be the uncollapse map, $\pi_X^{-1}(\mathfrak{N}_g) = (\mathrm{Lp}^\Sigma(\mathbb{R}_X))^{\mathfrak{N}_g}$. [14]

In the following, we say "N is a level of $\mathrm{Lp}(A)$" if $N \lhd \mathrm{Lp}(A)$ is a sound A-mouse such that $\rho_\omega(N) = A$ and similarly for $N \lhd \mathrm{Lp}^\Sigma(A)$.

Lemma 3.3. *Suppose $g \subseteq Coll(\omega, \omega_1)$ is V-generic or $g = \emptyset$, $cof^V(\Theta^{\mathfrak{N}_g}) < \omega_2^V$. Suppose $2^\omega \leq \omega_2$ and $2^{\omega}_1 \leq \omega_2$. Then for any cardinal $\gamma \geq \omega_3^V$, the set*

$$S = \{X : X \prec H_\gamma^{V[g]} \wedge X \textit{ is cofinal in } \mathfrak{N}_\mathfrak{g} \wedge X \textit{ is Lp-full in } \mathfrak{N}_g\}$$

is stationary in $V[g]$.

Proof. We assume $g \subseteq Coll(\omega, \omega_1)$ is V-generic; the other case is similar but simpler. Let $\tau \subseteq H_{\omega_2}$ be a canonical $Coll(\omega, \omega_1)$-name for $\mathbb{R}^{V[g]}$; note that we use our cardinal arithmetic assumption here, i.e. $2^\omega = 2^{\omega_1} = \omega_2$. By S-constructions, $\mathfrak{N}_g = \mathfrak{N}[g]$, where $\mathfrak{N} \trianglelefteq \mathrm{Lp}^\Sigma(\tau)$ in V. Let $\eta = cof^V(\Theta^{\mathfrak{N}_g})$ and $\langle N^*_\alpha : \alpha < \eta\rangle$ be a sequence of $N \lhd \mathrm{Lp}^\Sigma(\mathbb{R}^{V[g]})$ cofinal in $\mathfrak{N}_g$. We let $\langle N_\alpha : \alpha < \eta\rangle \in V$ be the corresponding sequence cofinal in $\mathfrak{N}$, so for each α, $N^*_\alpha = N_\alpha[g]$.

It suffices to show the set X such that

- $X \prec H_\gamma^V$,
- $X \cap \omega_2 \in \omega_2$,
- $\vec{N} = \langle N_\alpha : \alpha < \eta\rangle \in X$,
- $X[g]$ is *Lp*-full in $\mathfrak{N}_g$,

is stationary in V. Suppose not. Let $\langle X_\beta : \beta < \omega_2\rangle$ be an increasing and continuous sequence of elementary substructures of H^V_γ such that $\vec{N} \in X_0$, but for all α, $X_\alpha \cap \omega_2 \in \omega_2$, but $X_\alpha[g]$ is not *Lp*-full in $\mathfrak{N}_g$. For each α, let $\pi_\alpha : M_\alpha \to X_\alpha$ be the uncollapse map. π_α canonically extends to a map from $M_\beta[g]$ to $H_\gamma[g]$, which we also call π_α. We also let

$$N_i^\alpha = \pi_\alpha^{-1}(N_i)$$

for each $i < \eta$. For each α, let

[14] By $(\mathrm{Lp}^\Sigma(\mathbb{R}_X))^{\mathfrak{N}_g}$ we mean the collection of sound Σ-premice $\mathcal{M}$ over $\mathbb{R}_X$ such that $\rho_\omega(\mathcal{M}) = \mathbb{R}_X$ and whenever $\mathcal{M}^*$ is countable, transitive and embeddable into $\mathcal{M}$, then $\mathcal{M}^*$ is ω_1-iterable as a Σ-mouse with its unique iteration strategy in $\mathfrak{N}_g$. Clearly $\pi_\beta^{-1}(\mathfrak{N}_g) \trianglelefteq (\mathrm{Lp}^\Sigma(\mathbb{R}_\alpha))^{\mathfrak{N}_g}$.

(i) $\mathbb{R}_\alpha = \mathbb{R}^{M_\alpha[g]}$,

(ii) P_α be the least level of $(\mathrm{Lp}^\Sigma(\mathbb{R}_\alpha))^{\mathfrak{N}_g}$ such that $P_\alpha \notin \pi_\beta^{-1}(\mathfrak{N}_g)$,

(iii) Q_α be the ultrapower of P_α by the extender of length $\Theta^{\mathfrak{N}_g}$ derived from π_α and for each $\beta > \alpha$, Q_α^β be the ultrapower of P_α by the extender derived from $\pi_{\alpha,\beta}$, where $\pi_{\alpha,\beta} = \pi_\beta^{-1} \circ \pi_\alpha$.

Q_α^β, Q_α may be ill-founded, but see Claim 3.4. To simplify the notation, we assume

$$\rho_1(P_\alpha) = \mathbb{R}_\alpha$$

for all $\alpha < \omega_2$. The general case is handled by going into the reducts just like in the proof of [4, Theorem 3.4]. Fix $\alpha < \omega_2$, let $\langle Y_\beta : \beta < \omega_2 \rangle$ be increasing, continuous such that

(a) $\{P_\alpha, \pi_\alpha, Q_\alpha, \vec{N}\} \in Y_0$,

(b) the set $C_\alpha = \{\beta : Y_\beta[g] \cap \mathfrak{N}_g = \mathrm{rng}(\pi_\beta) \cap \mathfrak{N}_g\}$ is club.

For each β, let $\sigma_\alpha^\beta : M_\beta^*[g] \to H_\gamma[g]$ be the uncollapse map.[15]

Let $\beta \in \triangle_{\alpha<\omega_2} C_\alpha$ be a limit ordinal such that $\mathrm{cof}^V(\beta) \neq \eta$. Such a β exists because $\triangle_{\alpha<\omega_2} C_\alpha$ is a club subset of ω_2 and $\eta < \omega_2$.

Claim 3.4. *There is an $\alpha < \beta$ such that $Q_\alpha^\beta = P_\beta$.*[16]

Proof. Recall we assume $\rho_1(P_\beta) = \mathbb{R}_\beta$; therefore, $\rho_0(P_\beta) = o(P_\beta)$. By [4, Lemma 1.2], we have

$$\eta = cof^V(\Theta^{P_\beta}) = cof^V(\rho_0(P_\beta)) = cof^V(o(P_\beta)).$$

So let $\langle \delta_i : i < \eta \rangle$ be increasing and cofinal in $o(P_\beta)$ and

$$\sigma_i : \mathcal{N}_i^* \to Hull_1^{S_i^{P_\beta}}(\mathbb{R}_\beta \cup \{p_1(P_\beta)\})$$

be the uncollapse maps. By condensation and the minimality of P_β, for each i,

$$\mathcal{N}_i^* \lhd \pi_\beta^{-1}(\mathfrak{N}_g).$$

We also note that $P_\beta | \Theta^{P_\beta} = \pi_\beta^{-1}(\mathfrak{N}_g)$ and $\mathbb{R}_\beta = \bigcup_{\alpha<\beta} \mathbb{R}_\alpha$.

Since $cof^V(\beta) \neq \eta$, there is an $\alpha < \beta$ and unbounded sets $T, T' \subset \eta$ such that

- $i \in T \Rightarrow N_i^\beta, p(N_i^\beta) \in Hull_1^{P_\beta}(\mathbb{R}_\alpha \cup \{p_1(P_\beta)\})$,
- $i \in T' \Rightarrow \mathcal{N}_i^*, \sigma_i^{-1}(p_1(P_\beta)) \in \mathrm{rng}(\pi_{\alpha,\beta})$.

[15] It is not hard to see that $\sigma_\alpha^{\beta,-1}(Q_\alpha) = Q_\alpha^\beta$, but we do not need this fact.

[16] In particular, this shows that Q_α^β is well-founded.

The key equality we need to prove is

$$Hull_1^{P_\beta}(\mathbb{R}_\alpha \cup \{p_1(P_\beta)\}) \cap \Theta^{P_\beta} = \mathrm{rng}(\pi_{\alpha,\beta}) \cap \Theta^{P_\beta}. \tag{3.1}$$

The proof follows closely the corresponding claim in [4, Theorem 3.4]. We give some details here for the reader's convenience. Suppose $\xi \in \mathrm{rng}(\pi_{\alpha,\beta} \cap \Theta^{P_\beta}$. Let $\pi_{\alpha,\beta}(\xi^*) = \xi$. So $\xi^* < \Theta^{P_\alpha}$. So for some $i \in T$,

$$\xi^* \in Hull_1^{N_i^\alpha}(\mathbb{R}_\alpha \cup \{p(N_i^\alpha)\}).$$

Therefore

$$\xi^* \in Hull_1^{N_i^\beta}(\mathbb{R}_\alpha \cup \{p(N_i^\beta)\}) \subseteq Hull_1^{P_\beta}(\mathbb{R}_\alpha \cup \{p_1(P_\beta))\}).$$

The $\subseteq$ above follows from the choice of T. For the other direction, suppose $\xi \in Hull_1^{P_\beta}(\mathbb{R}_\alpha \cup \{p_1(P_\beta)\}) \cap \Theta^{P_\beta}$. So for some $i \in T'$, $\xi \in Hull_1^{S_i^{P_\beta}}(\mathbb{R}_\alpha \cup \{p_1(P_\beta)\})$, say

$$\xi = \epsilon^{S_i^{P_\beta}}[x, p_1(P_\beta)]$$

for some term ϵ and some $x \in \mathbb{R}_\alpha$. It is then easy to see that $\xi = \tau^{\mathcal{N}_i^*}[x, \sigma_i^{-1}(p_1(P_\beta))]$, so by the choice of i, T',

$$\xi \in Hull_1^{\mathcal{N}_i^*}(\mathbb{R}_\alpha \cup \{\sigma_1^{-1}(p_1(P_\beta)\})) \subseteq \mathrm{rng}(\pi_{\alpha,\beta}).$$

This completes the proof of equality (3.1).

Now let $\overline{\sigma} : \overline{P} \to P_\beta$ be the uncollapse of $Hull_1^{P_\beta}(\mathbb{R}_\alpha \cup \{p_1(P_\beta)\})$. By (3.1), we have:

- $\overline{P}|\Theta^{\overline{P}} = P_\alpha|\Theta^{P_\alpha}$,
- $\mathbb{R} \cap \overline{P} = \mathbb{R}_\alpha$,
- $\bar{P} \lhd (\mathrm{Lp}^\Sigma(\mathbb{R}_\alpha))^{\mathfrak{N}_\eth}$ (by condensation),
- no $Q \lhd \overline{P}$ extending $\overline{P}|\Theta^{P_\alpha}$ projects to $\mathbb{R}_\alpha$.

The above easily implies

$$\bar{P} = P_\alpha.$$

This gives us P_β is the ultrapower of $\overline{P}$ by the extender of length Θ^{P_β} derived from $\overline{\sigma}$. Therefore,

$$Q_\alpha^\beta = P_\beta$$

by (3.1). □

By the claim and Fodor's lemma, there is an α such that the set

$$S = \{\beta : Q_\alpha^\beta = P_\beta\}$$

is stationary. Also, Q_α is the direct-limit of the P_β's under the maps π_β's for $\beta \in S$. In particular, this means Q_α is well-founded and countably iterable in $V[g]$. This is because whenever R is countable, transitive in $V[g]$ and there is an elementary embedding $\tau : R \to Q_\alpha$, then there is some $\beta \in S$ and an elementary $\tau' : R \to P_\beta$. This means R is iterable in $\mathfrak{N}_g$. This shows

$$Q_\alpha \lhd \mathfrak{N}_g.$$

We have a contradiction because π_α is cofinal in $\mathfrak{N}_g$ and Q_α extends N_α for all $\alpha < \eta$.

□

Remark 3.5. In the case $\eta = \omega$, the above lemma can be strengthened to give us that there is an ω_1-club of $X \prec H^V_\gamma$ such that $X[g]$ is cofinal and *Lp*-full in $\mathfrak{N}_g$. The proof is an easy modification of the above proof. Here are some main points. Suppose not. Then we can define the sequence $\langle X_\alpha : \alpha < \omega_2\rangle$ as in the proof above and we may assume P_α is defined for a stationary set S of α of cofinality ω_1. Otherwise, the set of ω_1-limit points of the stationary set of α where $\mathrm{cof}(\alpha) = \omega_1$ where P_α is defined is disjoint from an ω_1-club of α such that P_α is not defined. This ω_1-club is what we want. We define the clubs C_α as in the proof. Now we can find a $\beta \in S \cap \triangle_{\alpha<\omega_2} C_\alpha$ since S is stationary and $\triangle_{\alpha<\omega_2} C_\alpha$ is club; furthermore, we can find a β which is a limit point of S. Since $\beta \in S$, $\mathrm{cof}^V(\beta) \neq \eta = \omega$. Since β is a limit point of S, we can find an $\alpha \in S$ that satisfies $Q^\beta_\alpha = P_\beta$ as in the proof of Claim 3.4. The rest of the proof is the same.

We will see in the next section that indeed $\eta = \omega$, and will use the remark at various points in the paper. Using Lemma 3.3 and Remark 3.5, we immediately get the following corollary.

Corollary 3.6. *Assume either $(T1)$ or $(T2)$ and $g \subseteq Coll(\omega, \omega_1)$ is V-generic or $g = \emptyset$. Then for any cardinal $\gamma \geq \omega_3^V$, the set S of $X \prec H^V_\gamma$ such that X is cofinal, and Lp-full in $\mathfrak{N}_\mathfrak{g}$ is stationary in V. Therefore, the set S_g of $X[g]$ for $X \in S$ is stationary in $V[g]$.*

4. THE GENERAL SET UP

Let g, G, h, j_G be the objects introduced above. Let $T = T_h$ and $\Theta = \Theta^{\mathfrak{N}}$. Note that $T \in V$. In the following arguments, we use (often without mention) that SMC holds in $\mathfrak{N}_k$ for $k \in \{G, h, \emptyset, g, G\times g\}$. For notational simplicity, let us assume $\mathfrak{N}_k \vDash \Theta = \theta_0$, i.e. $\mathfrak{N}_k \vDash V = L(\mathrm{Lp}(\mathbb{R}))$. In the general case, which is only more notationally more cumbersome, we have a hod pair $(\mathcal{P}, \Sigma) \in V$ such that $\mathcal{P}$ is countable, Σ is a (ω_3, ω_3)-iteration strategy for $\mathcal{P}$ that is fullness preserving and has branch condensation; furthermore, Σ has canonical interpretation Σ_k for $k \in \{G, h, \emptyset, g, G \times g\}$. These properties of Σ will be shown to hold in Section 6. Then for each such k, $\mathfrak{N}_k \vDash V = L(\mathrm{Lp}^\Sigma(\mathbb{R}))$ and the arguments to follow work for this as well.

Let $X \in S$, $\pi_X : M_X \to X$ be the uncollapse map, and $(\mathcal{H}^*_X, \mathfrak{N}_X, \Theta_X) = \pi_X^{-1}(\mathcal{H}, \mathfrak{N}, \Theta^{\mathfrak{N}})$. We let $\mathcal{H}_X = \mathrm{Lp}_\omega(\mathcal{H}^*_X|\Theta_X)$, where the Lp is computed in $\mathfrak{N}$ and $\mathrm{Lp}_\omega(A)$ means we stack Lp ω times over A, more precisely, $\mathrm{Lp}_\omega(A) = \bigcup_n \mathrm{Lp}_n(A)$ where by induction, $\mathrm{Lp}_{n+1}(A) = \mathrm{Lp}(\mathrm{Lp}_n(A))$. So $\mathcal{M} \lhd \mathcal{H}_X$ if whenever $\mathcal{M}^*$ is countable transitive that embeds in to $\mathcal{M}$, then $\mathcal{M}^*$ is iterable (via a unique iteration strategy) in $\mathfrak{N}$. The same proof as the one given in Lemma 3.3, using the fact that $\mathrm{cof}^V((\Theta^{\mathfrak{N}})^{+n,\mathcal{H}}) \leq \omega_1$[17] and the remark after gives.

Lemma 4.1. *$\forall^* X \in S$, $\mathcal{H}_X \in M_X$ and in fact, $\mathcal{H}_X = \pi_X^{-1}(\mathcal{H})$. Furthermore, if $\eta = \omega$, this set contains an ω_1-club.*

Lemma 4.2. *For $X \in S$, $\mathcal{H}_X$ is full in $j_G(\mathfrak{N})$, equivalently in $\mathfrak{N}_G$. Furthermore, $\wp(\Theta_X) \cap L[T, \mathcal{H}_X] = \wp(\Theta_X) \cap \mathcal{H}_X$.*

Proof. Note that $\mathcal{H}_X$ is countable in $M_G = \mathrm{Ult}(V, G)$ and in $V[G]$. Since $j_G(\mathfrak{N})$ and $\mathfrak{N}_G$ have the same largest Suslin pointclass, their notions of fullness and suitability are the same. First note that any sound $\mathcal{H}_X|\Theta_X \lhd \mathcal{M} \lhd \mathcal{H}_X$ is iterable in $j_G(\mathfrak{N})$, equivalently in $\mathfrak{N}_G$. This is because $j_G \restriction \mathcal{M} : \mathcal{M} \to j_G(\mathcal{M})$ is elementary and $j_G(\mathcal{M})$ is countably iterable in $j_G(\mathfrak{N})$.

Suppose $\mathcal{H}_X|\Theta_X \lhd \mathcal{M}$ is the least sound mouse in $\mathfrak{N}_G$ such that $\rho_\omega(\mathcal{M}) \leq \Theta_X$, and $\mathcal{M} \notin \mathcal{H}_X$. Then $\mathcal{M} \in C_{\Gamma_G}(\mathcal{H}_X|\Theta_X)$ and therefore $\mathcal{M} \in C_{\Gamma_h}(\mathcal{H}_X|\Theta_X)$. This is because of the existence of the Σ_1-map $j_{G,h}$. This means $\mathcal{M} \in L[T, \mathcal{H}_X|\Theta_X]$. Since $T \in V$, $\mathcal{M} \in V$ and is (countably) iterable there. This is because if $\mathcal{M}^*$ is countable, transitive and embeds into $\mathcal{M}$, then $\mathcal{M}^*$ is iterable in $\mathfrak{N}_h$ and by Σ_1-reflection, the unique strategy of $\mathcal{M}^*$ is in $\mathfrak{N}_h|\kappa_h$. By the existence of the Σ_1-maps $j_{\emptyset,h}$, $\mathcal{M}^*$ is iterable in $\mathfrak{N}$. So $\mathcal{M} \lhd \mathcal{H}_X$. Contradiction.

The "Furthermore" clause follows easily from the arguments above. First, any sound $\mathcal{H}_X|\Theta_X \lhd \mathcal{M} \lhd \mathcal{H}_X$ is iterable in $j_G(\mathfrak{N})$ and therefore is iterable in $\mathfrak{N}_h$ by the existence of $j_{G,h}$. If $\mathcal{M} \in C_{\Gamma_h}(\mathcal{H}_X|\Theta_X) = L[T, \mathcal{H}_X|\Theta_X] \cap \wp(\Theta_X)$ is a sound mouse extending $\mathcal{H}_X|\Theta_X$ and $\rho_\omega(\mathcal{M}) \leq \Theta_X$, then $\mathcal{M} \in V$ and is countably iterable there. So $\mathcal{M} \lhd \mathcal{H}_X$. □

In the following, we say that $\mathcal{R}$ is full (with respect to mice in $j_G(\mathfrak{N}_X)$) if whenever γ is a strong cutpoint of $\mathcal{R}$, then $\mathcal{R}|(\gamma)^{+,\mathcal{R}} = \mathrm{Lp}(\mathcal{R}|\gamma)$, where Lp is computed in $j_G(\mathfrak{N}_X)$.

Lemma 4.3. *For any $X \in S$, $\mathcal{H}_X$ has the full factor property in $j_G(\mathfrak{N}_X)$ in the sense that whenever $\tau : \mathcal{H}_X \to \mathcal{R}$ and $\sigma : \mathcal{R} \to j_G(\mathcal{H}_X)$ are such that $\mathcal{R}$ is countable in $V[G]$, τ is cofinal in $\tau(\Theta_X)$ and $\sigma \circ \tau = j_G \restriction \mathcal{H}_X$, then $\mathcal{R}$ is full in $j_G(\mathfrak{N}_X)$, equivalently in $j_G(\mathfrak{N})$ and in $\mathfrak{N}_G$.*

Proof. First note that $\mathcal{R}$ is full in $j_G(\mathfrak{N}_X)$ if and only if $\mathcal{R}$ is full in $j_G(\mathfrak{N})$. This is because $j_G(\pi_X) : j_G(\mathfrak{N}_X) \to j_G(\mathfrak{N})$ is elementary, so for any countable $a \in j_G(\mathfrak{N}_X)$, $\mathrm{Lp}(a)$ is computed the same in the two models. Similarly, since Lp is computed the same in $j_G(\mathfrak{N})$ and in $\mathfrak{N}_G$, $\mathcal{R}$ is full in $\mathfrak{N}_G$ if and only if $\mathcal{R}$ is full in $j_G(\mathfrak{N})$.

For $X \in S$, let $\tau : \mathcal{H}_X \to \mathcal{R}$ and $\sigma : \mathcal{R} \to j_G(\mathcal{H}_X)$ be such that $\mathcal{R}$ is countable in M_G and $j_G \restriction \mathcal{H}_X = \sigma \circ \tau$. Let T^* be the ultrapower of T by the extender derived from τ. T^* is well-founded because it embeds into $j_G(T)$. Furthermore, τ lifts to $\tau^+ : L[T, \mathcal{H}_X] \to L[T^*, \mathcal{R}]$; this is because τ is cofinal in $\tau(\Theta_X)$ and in $o(\mathcal{R})$. Since T^* is well-founded, Lemma 4.2 and [27] give us that $\mathcal{R}$ is full in $\mathfrak{N}_h$. By elementarity of τ^+ and Lemma 4.2,

$$\wp(\tau(\Theta_X)) \cap L[T^*, \mathcal{R}] = \wp(\tau(\Theta_X)) \cap \mathcal{R}.$$

[17] Note that $\mathcal{H}$ is a fine-structural model, so for each $n \geq 1$, there is a $\square$-sequence in $\mathcal{H}$ of length $(\Theta^{\mathfrak{N}})^{+n,\mathcal{H}}$. The same proof as the one given in Lemma 3.1 gives the claim.

Now suppose $\mathcal{R}$ is not full in $j_G(\mathfrak{N}_X)$ or equivalently in $j_G(\mathfrak{N})$, then there is a sound mouse $\mathcal{M} \notin \mathcal{R}$ in $j_G(\mathfrak{N})$ such that $\rho_\omega(\mathcal{M}) \leq \tau(\Theta_X)$. Using the Σ_1-map $j_{G,h}$, we get that $\mathcal{M}$ is iterable in $\mathfrak{N}_h$. So $\mathcal{M} \in L[T, \mathcal{R}|\tau(\Theta_X)]$ and therefore, $\mathcal{M} \in L[T^*, \mathcal{R}]$ because T embeds into T^* and by absoluteness (see [27]).

□

Lemma 4.4. *$cof^V(\Theta^{\mathfrak{N}}) = \omega$.*

Proof. Suppose not. Then by the results of the previous section, $\mathrm{cof}^V(\Theta^{\mathfrak{N}}) = \omega_1$. Let $X \prec H_\gamma$ for some large γ be such that X is cofinal and Lp-full in $\mathfrak{N}$. Let $\pi_X : M_X \to X$ be the uncollapse map and $\pi_X(\mathcal{H}_X, \Theta_X) = (\mathcal{H}, \Theta^{\mathfrak{N}})$. Then $\mathrm{cof}^V(\Theta_X) = \omega_1$ and $\mathcal{H}_X$ is full in $j_G(\mathfrak{N})$ by Lemma 4.2. Note then j_G is discontinuous at Θ_X, so let $\gamma = \mathrm{sup} j_G[\Theta_X] < j_G(\Theta_X)$.

Let $\mathcal{R} = \mathrm{Ult}(\mathcal{H}_X, E)$ where E is the (long) extender of length γ derived from j_G. Let $\tau : \mathcal{H}_X \to \mathcal{R}$ be the ultrapower map by E and $\sigma : \mathcal{R} \to j_G(\mathcal{H}_X)$ be the factor map; so $j_G \restriction \mathcal{H}_X = \sigma \circ \tau$. We note that since $\gamma < j_G(\Theta_X)$, there is a $\mathcal{Q}$-structure $\mathcal{Q} \lhd j_G(\mathcal{H}_X)$ for γ. i.e. $\mathcal{R} \lhd \mathcal{Q} \lhd j_G(\mathcal{H}_X)$ and $\mathcal{Q}$ is the least such that $\mathcal{Q}$ is sound, $\rho_\omega(\mathcal{Q}) = \gamma$, $\mathcal{Q}$ defines a witness to non-Woodinness of γ, so $\mathcal{Q} \notin \mathcal{R}$.

Working in $V[G]$, let Y be a countable elementary substructure of H_λ for some large λ such that Y contains the range of $j_G \restriction \mathcal{H}_X$, and the set $\{\mathcal{R}, \mathcal{Q}, \mathcal{H}_X, \sigma, \tau, j_G \restriction \mathcal{H}_X, j_G(\mathcal{H}_X)\}$. Let π_Y be the uncollapse of Y and $\pi_Y(\mathcal{R}_Y, \mathcal{Q}_Y, \sigma_Y, \tau_Y, j_Y, \mathcal{S}) = (\mathcal{R}, \mathcal{Q}, \sigma, \tau, j_G \restriction \mathcal{H}_X, j_G(\mathcal{H}_X))$. By lemma 4.3, $\mathcal{R}_Y$ is full because letting $\sigma^* = \pi_Y \restriction \mathcal{S} \circ \sigma_Y$, then $j_G \restriction \mathcal{H}_X = \sigma^* \circ \tau_Y$. But $\mathcal{Q}_Y$ is a mouse that witnesses $\mathcal{R}_Y$ is not full; $\mathcal{Q}_Y$ is iterable because it embeds into $\mathcal{Q}$ via π_Y. This is a contradiction.

□

Remark 3.5 and Lemma 4.4 then give us the following corollary.

Corollary 4.5. *Assume either $(T1)$ or $(T2)$ and $g = \emptyset$. Then for any cardinal $\gamma \geq \omega_3^V$, the set S of $X \prec H_\gamma^V$ such that X is cofinal, and Lp-full in $\mathfrak{N}_{\mathfrak{g}}$ contains and ω_1-club in V.*

Definition 4.6. $X \in \wp_{\omega_2}(H_\gamma^V)$ is *internally club* if there is a sequence $\langle X_\alpha : \alpha < |X|\rangle$ such that

(a) For any $\alpha < |X|$, $X_\alpha \in X$.

(b) For any limit α, $X_\alpha = \bigcup_{\beta<\alpha} X_\alpha$.

(c) If $|X| = \omega_1$, for any $\alpha < |X|$, $|X_\alpha| = \omega$.

⊣

Lemma 4.7. *The set of internally club $X \in \wp_{\omega_2}(H_\gamma^V)$ such that $|X| = \omega_1$ is stationary.*

Proof. Suppose not. Let $F : [H_\gamma^V]^{<\omega} \to H_\gamma^V$ be such that for any X closed under F, X is not internally club. Let $\lambda > \gamma$ be such that $F \in H_\lambda^V$. Now build a sequence $\langle X_\alpha : \alpha < \omega_1\rangle$ such that for all $\alpha < \omega_1$,

- $F \in X_\alpha$.
- $X_\alpha \prec H_\lambda$.
- $|X_\alpha| = \omega$.
- $X_\alpha \in X_{\alpha+1}$.
- If α is limit, then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.

Let $X = \bigcup_{\alpha<\omega_1} X_\alpha$. Then $X \cap H_\gamma$ is closed under F because $F \in X$ and $X \prec H_\lambda$. Furthermore, $X \cap H_\gamma = \bigcup_{\alpha<\omega_1} X_\alpha \cap H_\gamma$ is internally club by the construction of the sequence $\langle X_\alpha : \alpha < \omega_1 \rangle$. This is a contradiction. □

5. STRATEGIES THAT CONDENSE WELL

Following the argument in [20], we first construct a pair $(\mathcal{P}, \Sigma) \in V$ such that Σ has weak condensation. Again, we assume either $(T1) + (\dagger)$ or $(T2) + (\dagger)$ and use the notations introduced in the previous section. We first show the existence of a pair $(\mathcal{P}, \Sigma) \in V$ such that $\mathcal{P}$ is countable and Σ has weak condensation. As before, let $X \in S$, $\pi_X : M_X \to X$ be the uncollapse map, and $(\mathcal{H}_X, \mathfrak{N}_X) = \pi_X^{-1}(\mathcal{H}, \mathfrak{N})$. As before, we assume $\mathfrak{N} \vDash \Theta = \theta_0$. We use the objects G, g, j_G, h etc. introduced before.

Let $\mathcal{A}$ be a countable set of OD sets of reals in $\mathfrak{N}$ that is Wadge cofinal in $\mathfrak{N}$; $\mathcal{A}$ exists because $\mathrm{cof}^V(\Theta^{\mathfrak{N}}) = \omega$. Recall the following standard notions from [20]. Given a suitable $\mathcal{P}$[18] with $\delta = \delta^{\mathcal{P}}$ the Woodin cardinal of $\mathcal{P}$ and an OD set of reals A, we let $\tau^{\mathcal{P}}_{A,n}$ be the standard name for a set of reals in $\mathcal{P}^{Coll(\omega,\delta^{+n,\mathcal{P}})}$ witnessing the fact that $\mathcal{P}$ weakly captures A and let

$$\gamma^{\mathcal{P}}_A = sup(\delta^{\mathcal{P}} \cap Hull^{\mathcal{P}}_1(\{\tau^{\mathcal{P}}_{A,n} : n < \omega\})).$$

Here we say $\mathcal{P}$ *weakly term captures* A if letting $\delta = \delta^{\mathcal{P}}$, for each $n < \omega$ there is a term relation $\tau \in \mathcal{P}^{Coll(\omega,(\delta^{+n,\mathcal{P}})}$ such that for comeager many $\mathcal{P}$-generics $g \subseteq Coll(\omega, (\delta^{+n,\mathcal{P}})$, we have $\tau_g = \mathcal{P}[g] \cap A$. We say $\mathcal{P}$ *term captures* A if the equality holds for all generics. We let

$$H^{\mathcal{P}}_A = Hull^{\mathcal{P}}_1(\gamma^{\mathcal{P}}_A \cup \{\tau^{\mathcal{P}}_{A,n} : n < \omega\}).$$

By Lemma 4.1 and the arguments of [5, Lemma 4.55],

$$\mathcal{H} = \bigcup_{A\in\mathcal{A}} H^{\mathcal{H}}_A.$$

[18] This means $\mathcal{P}$ is 1-Γ-suitable for Γ being the largest Suslin pointclass of $\mathfrak{N}$. $\mathcal{P}$ could be the hod limit $\mathcal{H}$ computed in $\mathfrak{N}$.

so by elementarity for $X \in S$,

$$\mathcal{H}_X = \bigcup_{A \in \mathcal{A}} H_A^{\mathcal{H}_X}.$$ [19]

It is clear from [5, Lemma 4.55] that $\text{cof}(\delta^{+n,\mathcal{H}}) = \omega$ for all $n < \omega$ where $\delta = \delta^{\mathcal{H}}$ is the Woodin cardinal of $\mathcal{H}$, and similarly for $\mathcal{H}_X$. For the rest of the paper, we will write $H_{\mathcal{A}}^{\mathcal{H}}$ for $\bigcup_{A \in \mathcal{A}} H_A^{\mathcal{H}}$ and similarly for other objects like $\mathcal{H}_X$.

5.1. Weak Condensation

Let $X \in S$ be internally club as witnessed by $\langle X_\alpha : \alpha < \omega_1 \rangle$. By elementary, $j(X)$ is internally club in M_G as witnessed by $\langle Y_\alpha : \alpha < \omega_2^V \rangle$. Now note that for each $\alpha < \omega_1^V$, $Y_\alpha = j_G(X_\alpha) = j_G[X_\alpha]$, so $Y_{\omega_1^V} = \bigcup_{\alpha < \omega_1^V} Y_\alpha \in j(X)$, is countable in $j_G(X)$ and transitively collapses to M_X. Therefore, by elementarity, there is a countable $X^* \in X$ such that $\mathcal{H}_{X^*}$ has the full factor property in $\mathfrak{N}_X$. By Fodor's lemma, there is a countable X and a stationary set of Y such that

- $X \in Y$;
- $\mathcal{H}_X$ has the full factor property in $\mathfrak{N}_Y$.

We let $\mathcal{P} = \mathcal{H}_X$ and show that there is an iteration strategy for $\mathcal{P}$ that is fullness preserving. First, we record an easy corollary from the arguments above.

Lemma 5.1. *Let $\pi : \mathcal{P} \to \mathcal{H}$ be $\pi_X \restriction \mathcal{P}$. Then π has the full factor property. In particular, $\mathcal{P}$ is suitable.*

Proof. Note that for a stationary Y such that $X \in Y$ as above, $\pi_X = \pi_Y \circ \pi_{X,Y}$ where $\pi_{X,Y}$ is the natural map from M_X to M_Y. For any $\mathcal{R}$ such that there is a map $\tau : \mathcal{P} \to \mathcal{R}$ and $\sigma : \mathcal{R} \to \mathcal{H}$ such that $\pi = \sigma \circ \tau$, there is a Y in the stationary set above such that $\pi_Y^{-1} \circ \sigma : \mathcal{R} \to \mathcal{H}_Y$ is elementary and $\pi_{X,Y} \restriction \mathcal{P} = \pi_Y^{-1} \circ \sigma \circ \tau$. So $\mathcal{R}$ is full in $\mathfrak{N}_Y$ for all such Y. As a result, $\mathcal{R}$ is full (and hence suitable) in $\mathfrak{N}$. In particular, letting τ be the identity function, $\mathcal{P}$ is suitable as well. □

Then exactly as in the proof of [20, Theorem 7.8.9], we obtain a unique (ω_1, ω_1) $\mathcal{A}$-guided iteration strategy Σ for $\mathcal{P}$ that has the Dodd-Jensen property and is fullness preserving. More precisely, Σ has the following properties: whenever $\mathcal{T}$ is of limit length and is according to Σ,

1. if $\mathcal{T}$ is short, $\Sigma(\mathcal{T})$ is the unique branch b such that $\mathcal{Q}(b, \mathcal{T}) \lhd (Lp(\mathcal{M}(\mathcal{T})))^{\mathfrak{N}}$, or else $\Sigma(\mathcal{T})$ is the unique branch b such that $(Lp(\mathcal{M}(\mathcal{T})))^{\mathfrak{N}} = \mathcal{M}_b^{\mathcal{T}}$ and $i_b^{\mathcal{T}}(\tau_A^{\mathcal{P}}) = \tau_A^{\mathcal{M}_b^{\mathcal{T}}}$ for each $A \in \mathcal{A}$.
2. suppose $\Sigma(\mathcal{T}) = b$ does not drop, then there is an embedding $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{H}$ such that $\pi = \sigma \circ i_b^{\mathcal{T}}$ and $\mathcal{M}_b^{\mathcal{T}} = H_{\mathcal{A}}^{\mathcal{M}_b^{\mathcal{T}}}$.

We say that Σ has *weak condensation.*

[19] We abuse the notation a bit here. Technically, we should let $\mathcal{A}_X = \pi_X^{-1}(\mathcal{A})$ and write $\mathcal{H}_X = \bigcup_{A \in \mathcal{A}_{\mathcal{X}}} H_A^{\mathcal{H}_X}$.

5.2. Strong Condensation

In this section, we use the pair $(\mathcal{P}, \Sigma)$ constructed in the previous section and (†) to construct a tail $(\mathcal{Q}, \Lambda)$ of $(\mathcal{P}, \Sigma)$ such that $\mathcal{Q}$ is countable and Λ condenses well. Let $\pi : \mathcal{P} \to \mathcal{H}$ be as in the previous section. The main point is that (†) gives the existence of a model M such that

(i) $M \vDash \mathsf{ZFC}+\omega_1^V$ is measurable.

(ii) $\{\pi, \mathcal{H}\} \in M$.

Using M, the argument in [20, Theorem 7.9.1] gives the existence of $(\mathcal{Q}, \Lambda)$. In particular, $\mathcal{Q} \in V$ is countable, Λ is an (ω_1, ω_1)-iteration strategy of $\mathcal{Q}$ that has the Dodd-Jensen property and satisfies properties (1) and (2) above; furthermore, Λ has *branch condensation*, i.e., whenever $\mathcal{R}$ is a Λ-iterate of $\mathcal{Q}$ and $\mathcal{W}$ is a $\Lambda_{\mathcal{R}}$-iterate of $\mathcal{R}$ with iteration embedding $i : \mathcal{R} \to \mathcal{W}$, whenever $\mathcal{T}$ is according to $\Lambda_{\mathcal{R}}$ and b is a non-dropping cofinal branch of $\mathcal{T}$ such that there is an embedding $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{W}$ and $i = \sigma \circ i_b^{\mathcal{T}}$, then $b = \Lambda_{\mathcal{R}}(\mathcal{T})$.

6. PROOF OF THEOREM 1.4

We start with a pair $(\mathcal{Q}, \Lambda)$ in Section 5.2. In particular, $\mathcal{Q} \in V$ is countable and Λ is guided by a sjs $\mathcal{A}$ and therefore is fullness preserving and has branch condensation. However, Λ at this point is only a (ω_1, ω_1)-iteration strategy. Again, as before, we assume $\mathfrak{N} \vDash \Theta = \theta_0$.

6.1. Lifting and restricting strategies

We first need to lift Λ to a (ω_2, ω_2)-strategy. We let $\Lambda_G = j_G(\Lambda)$.

Lemma 6.1. *ω_1 is inaccessible in $L_{\omega_1}^{\Lambda}[\mathcal{P}, a]$ for any $a \in HC$.*

Proof. This follows from (†). We can take the set of ordinals A to code $\{\mathcal{P}, a\} \cup \Lambda \restriction L_{\omega_1}^{\Lambda}[\mathcal{P}, a]$. ω_1 is measurable in some inner model M containing A, so since $L_{\omega_1}^{\Lambda}[\mathcal{P}, a] \subset M$, ω_1^V must be inaccessible in $L_{\omega_1}^{\Lambda}[\mathcal{P}, a]$. □

Remark 6.2. In general, $\mathfrak{N} \vDash V = L(\mathrm{Lp}^{\Psi}(\mathbb{R}))$ for some Ψ that is OD from a countable sequence of ordinals. Therefore, Λ is also OD from a countable sequence of ordinals. The set A in the proof of Lemma 6.1 is OD from a countable sequence of ordinals as well. (†) can be applied to this set A.

Using the fact that ω_1 is inaccessible in $L_{\omega_1}^{\Lambda}[\mathcal{P}, a]$ for any $a \in HC$ and elementarity of j_G, we can show, using [13, Theorem 7.3], Λ_G extends uniquely to a strategy $\Lambda_{G\times g}$ because g is Cohen generic (hence ccc) over $V[G]$.

Lemma 6.3. *Λ_G extends uniquely to an (ω_1, ω_1)-strategy $\Lambda_{G\times g}$ in $V[G \times g]$.*

Proof. Working in $V[G]$, we first show Λ_G has strong hull condensation. Suppose $\vec{\mathcal{T}}$ is according to Λ_G and $\vec{\mathcal{U}}$ is a pseudo-hull of $\vec{\mathcal{T}}$, with both $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ being countable stacks. We need to see that $\vec{\mathcal{U}}$ is according to Λ_G. This follows from the proof of [1, Lemma 3.44]; we need to observe that running the proof of [1, Lemma 3.44] in $L^{\Lambda_G}_{\omega_2^V}[\mathcal{P},\vec{\mathcal{T}},\vec{\mathcal{U}}]$, which satisfies $\omega_2^V = \omega_1^{V[G]}$ is strongly inaccessible and is closed under Λ_G, gives the desired result. Since $V[G][g]$ is a Cohen generic (hence ccc) forcing extension over $V[G]$ and Λ_G has strong hull condensation, [13, Theorem 7.3] then shows that Λ_G has a unique extension $\Lambda_{G\times g}$ in $V[G\times g]$ as claimed. □

Lemma 6.4. $\Lambda_{G\times g}\restriction V[g]\in V[g]$

Proof. This is by the proof of [13, Lemma 7.6], particularly the observations in the Subclaim 1.1 of [13, Lemma 7.6].[20] We just need to observe that whenever $\mathcal{T}\in V[g]$ is a normal tree according to $\Lambda_{G\times g}$, then whenever $\mathcal{T}$ is short, $b=\Lambda_{G\times g}(\mathcal{T})\in V$ because it is the unique branch given by the $\mathcal{Q}$-structure $\mathcal{Q}(b,\mathcal{T})\lhd (Lp(\mathcal{M}(\mathcal{T})))^{\mathfrak{N}_{G\times g}} = (Lp(\mathcal{M}(\mathcal{T})))^{\mathfrak{N}_g}$; the equality holds because $\mathfrak{N}_g,\mathfrak{N}_{G\times g}$ compute $Lp(a)$ the same way for any $a\in HC^{V[g]}$, due to the existence of the map $j_{g,G\times g}$. If $\mathcal{T}$ is maximal, then first note that letting $b=\Lambda_{G\times g}(\mathcal{T})$, $\mathcal{M}_b^{\mathcal{T}}\in V$ because $\mathcal{M}_b^{\mathcal{T}}=Lp(\mathcal{M}(\mathcal{T}))$ and again "Lp" is computed the same way between the two models. Using the proof of the Subclaim 1.1 of [13, Lemma 7.6] and the fact that $\Lambda_{G\times g}$ has the Dodd-Jensen property, we get that $b\in V$.[21] □

By the above lemma, $\Lambda_{G\times g}\restriction V[g]\in V[g]$ and is an (ω_1,ω_1)-strategy there; call this strategy Λ_g. Λ_g is guided by a sjs $\mathcal{A}_g$ in $V[g]$, where for any $A\in\mathcal{A}_g$, there is some $A^*\in j_g(j_G(\mathcal{A}))$ such that $A=A^*\cap V[g]$, and $j_g:\mathcal{N}_G\to\mathcal{N}_{G\times g}$ is the Cohen ultrapower map. By a standard boolean valued-comparison, there is an iterate $(\mathcal{R},\Psi)\in V[g]$ of $(\mathcal{Q},\Lambda_g)$ such that $\mathcal{R}\in V$, $|\mathcal{R}|\leq\omega_1$, $\Psi\restriction V\in V$ and is an (ω_2,ω_2)-strategy there with branch condensation. Note that $\mathcal{R}$ is countable in $V[G]$ and suitable in $j_G(\mathfrak{N})$.

By $\mathsf{WRP}_2(\omega_2)$, we can uniquely extend Ψ to a (ω_3,ω_3)-strategy that condenses well. Using the elementarity of j_G and the fact that $j_G(\omega_3)=\omega_3$, we can show that there is a countable, suitable $\mathcal{S}\in V$ and an (ω_3,ω_3)-strategy Φ guided by a sjs $\mathcal{A}'$ Wadge cofinal in $\mathfrak{N}$; in particular, $\Phi\notin\mathfrak{N}$.

6.2. The final induction

We can use this pair $(\mathcal{S},\Phi)$ as in the previous section to continue the core model induction as in [22] to show $(\mathrm{Lp}^{\Phi}(\mathbb{R}))^{V[k]}\vDash\mathsf{AD}^++\Theta=\theta_1$ where $k\in\{\emptyset,g,G,g\times G,h\}$ and maintain the inductive hypotheses as in [22]. We can repeat this process for any $\alpha<\omega_1$. At successor α, suppose we have a pair $(\mathcal{S}_\alpha,\Phi_\alpha)$ giving rise to $(Lp^{\Phi_\alpha}(\mathbb{R}))^{V[k]}\vDash\mathsf{AD}^++\Theta=\theta_{\alpha+1}+\mathsf{MC}(\Phi)$, we construct pair $(\mathcal{S}_{\alpha+1},\Phi_{\alpha+1})$ as before. If α is limit, in particular $\mathrm{cof}(\alpha)=\omega$, we can look at a kind of (short) hod pair of the form $(\mathcal{S}_\alpha,\Phi_\alpha)=(\mathcal{S},\Phi)$ where $\mathcal{S}=\bigcup_{\beta<\alpha}\mathcal{S}(\beta)$ and $\Phi=\oplus_{\beta<\alpha}\Phi_{\mathcal{S}(\beta)}$ where $(\mathcal{S}_\alpha(\beta),\Phi_{\mathcal{S}(\beta)})$

[20] We can't literally quote [13, Lemma 7.6] because it assumes ω_1+1-iterability.

[21] The main point is that if $(p,q)\Vdash^{V[g]}_{\mathbb{P}\times\mathbb{P}}\Lambda_{\dot{G_0}\times g}(\mathcal{T})\neq\Lambda_{\dot{G_1}\times g}(\mathcal{T})$ where $\mathbb{P}$ is the forcing $\wp^V(\omega_1)/NS^V_{\omega_1}$, which is ccc over $V[g]$, and here we let $G_0\times G_1$ be $\mathbb{P}\times\mathbb{P}$-generic, then by Dodd-Jensen and absoluteness, in $V[G\times g]$ there is some map $\pi:\mathcal{Q}\to\mathcal{M}_b^{\mathcal{T}}$ such that π is lexicographically less than $i_b^{\mathcal{T}}$. This contradicts Dodd-Jensen.

generates the maximal model of $\Theta = \theta_\beta$. We can then show $(\mathrm{Lp}^\Phi(\mathbb{R}))^{V[k]} \vDash \mathsf{AD}^+ + \Theta = \theta_{\alpha+1} + \mathsf{MC}(\Phi)$ as before.

Now suppose no models of $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ exist, we let Γ be the maximal pointclass of AD^+, i.e.

$$\Gamma = \{A \subseteq \mathbb{R} : L(A, \mathbb{R}) \vDash \mathsf{AD}^+\}.$$

By the argument above, for each $\alpha < \omega_1$, $\Phi_\alpha \in \Gamma$ and furthermore, for every $A \in \Gamma$, there is a scale $\vec{\psi}$ for A such that $\vec{\psi} \in \Gamma$. In other words, the Solovay sequence defined over Γ has limit length. Furthermore, the sequence has uncountable cofinality by the argument from the previous paragraph. We need to see that our maximal pointclass Γ is constructibly closed, i.e. $\Gamma = L(\Gamma) \cap \wp(\mathbb{R})$. Showing this will give that our model has satisfied $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$.

Let $\mathcal{H}$ be the hod limit computed in Γ. We write Θ for Θ^Γ. So $o(\mathcal{H}) = \Theta$. Let $\langle \theta_\alpha : \alpha < \lambda \rangle$ be the Solovay sequence computed in Γ. We know that $\lambda = \omega_1$ in V. The next lemma is the key lemma and its proof is similar to that of [26, Lemma 6.3].

Lemma 6.5. *There is no $\mathcal{N} \lhd L[\mathcal{H}]$ such that $\mathcal{H} \lhd \mathcal{N}$ and $\rho_\omega(\mathcal{N}) < \Theta$.*

Proof. Suppose not. Let $\mathcal{N} \trianglelefteq L[\mathcal{H}]$ be least such that $\rho_\omega(\mathcal{N}) < \Theta$. Let $B \in \Omega^*$ be of Wadge rank θ^*_{n+1} where $n < \lambda$ is such that $\rho_\omega(\mathcal{N}) \leq \theta^*_n$ and $\theta^*_n \geq \upsilon$, where υ is the $\mathcal{N}$-cofinality of λ.[22] Suppose k is the least such that $\rho_{k+1}(\mathcal{N}) < \Theta$; we may assume $\rho_{k+1}(\mathcal{N}) \leq \theta^*_n$. Let $M = L_\gamma(\mathbb{R}, B, \mathcal{N})$, where γ is some sufficiently large cardinal so that $L_\gamma(\mathbb{R}, B, \mathcal{N}) \vDash \mathsf{ZF}^- + \mathsf{DC}$.

For countable $\sigma \prec M$ containing all relevant objects, let $\pi_\sigma : M_\sigma \to M$ be the transitive uncollapse map whose range is σ. Such a σ exists by DC in $L(\mathbb{R}, B, \mathcal{N})$. For each such σ, let $\pi_\sigma(\mathcal{H}_\sigma, \Theta_\sigma, \lambda_\sigma, \mathcal{N}_\sigma, B_\sigma, \upsilon_\sigma) = (\mathcal{H}, \Theta, \lambda, \mathcal{N}, B, \upsilon)$. Let $\Sigma^-_\sigma = \oplus_{\alpha<\lambda_\sigma} \Sigma_{\mathcal{H}_\sigma(\alpha)}$. Note that for each $\alpha < \lambda_\sigma$, $\Sigma_{\mathcal{H}_\sigma(\alpha)}$ acts on all countable stacks as it is the pullback of some hod pair $(\mathcal{R}, \Lambda)$ with the property that $\mathcal{M}_\infty(\mathcal{R}, \Lambda) = \mathcal{H}(\pi_\sigma(\alpha))$.

Let $\sigma \prec M$ be such that $\omega_1^{M_\sigma} > n$; this is possible since $n < \lambda \leq \omega_1$. $\Sigma_{\mathcal{H}_\sigma(n+1)}$ is Γ-fullness preserving and has branch condensation. This follows from the choice of B, which gives that $(\mathcal{H}_\sigma(n+1), \Sigma_{\mathcal{H}_\sigma(n+1)})$ is a tail of some hod pair $(\mathcal{Q}, \Lambda) \in M_\sigma$ such that $\mathcal{Q}$ has $n+1$ Woodin cardinals and Λ has branch condensation and is Ω^*-fullness preserving. We let Σ^n_σ be the fragment of Σ^-_σ for stacks on $\mathcal{N}_\sigma$ above $\delta^{\mathcal{N}_\sigma}_n$. Note that Σ^n_σ is an iteration strategy of $\mathcal{N}_\sigma$ above $\delta^{\mathcal{N}_\sigma}_n$ since Σ^n_σ-iterations are above υ_σ, which may be measurable in $\mathcal{N}_\sigma$, and hence does not create new Woodin cardinals. Σ^n_σ has branch condensation. We then have that $\Sigma^n_\sigma \in \Gamma$; otherwise, by results in the previous sections, we can show $L(\Sigma^n_\sigma, \mathbb{R}) \vDash \mathsf{AD}^+$ and this contradicts the definition of Γ.[23] Also, by [8, Theorem 3.26], Σ^n_σ is Ω-fullness preserving where $\Omega =_{\mathrm{def}} \Gamma(\mathcal{N}_\sigma, \Sigma^n_\sigma)$ is the pointclass generated by $(\mathcal{N}_\sigma, \Sigma^n_\sigma)$.

We then consider the directed system $\mathcal{F}$ of tuples $(\mathcal{Q}, \Lambda)$ where $\mathcal{Q}$ agrees with $\mathcal{N}_\sigma$ up to $\delta^{\mathcal{N}_\sigma}_n$, and $(\mathcal{Q}, \Lambda)$ is Dodd–Jensen equivalent to $(\mathcal{H}_\sigma, \Sigma^n_\sigma)$, that is $(\mathcal{Q}, \Lambda)$ and $(\mathcal{H}_\sigma, \Sigma^n_\sigma)$ coiterate (above $\delta^{\mathcal{N}_\sigma}_n$)

[22] In this case, υ is in fact ω_1^V, which is the least measurable cardinal of $\mathcal{H}$. But this is not relevant for the argument to follow. The only relevant fact we use is that $\upsilon < \Theta$.

[23] We also have that Σ^n_σ is the join of countably many sets of reals, each of which is in Γ and hence is Suslin co-Suslin. This implies that Σ^n_σ is self-scaled.

to a hod pair $(\mathcal{R},\Psi)$. $\mathcal{F}$ can be characterized as the directed system of hod pairs $(\mathcal{Q},\Lambda)$ extending $(\mathcal{N}_\sigma(n),\Sigma_{\mathcal{N}_\sigma(n)})$ such that $\Gamma(\mathcal{Q},\Lambda)=\Omega$, Λ has branch condensation and is Ω-fullness preserving. We note that $\mathcal{F}$ is $\mathrm{OD}_{\Sigma_{\mathcal{H}_\sigma(n)}}$ in $L(C,\mathbb{R})$ for some $C\in\Gamma$. We fix such a C; so $L(C,\mathbb{R})\vDash \mathsf{AD}^+ + \mathsf{SMC}$. Let $A\subseteq\delta_n^{\mathcal{N}_\sigma}$ witness $\rho_{k+1}(\mathcal{N}_\sigma)\leq\delta_n^{\mathcal{N}_\sigma}$. Then A is $\mathrm{OD}_{\Sigma_{\mathcal{H}_\sigma(n)}}$ in $L(C,\mathbb{R})$. By SMC in $L(C,\mathbb{R})$ and the fact that $\mathcal{N}_\sigma(n+1)$ is Γ-full, $A\in\mathrm{Lp}^{\Sigma_{\mathcal{H}_\sigma(n)}}(\mathcal{N}_\sigma|\delta_n^{\mathcal{N}_\sigma})\in\mathcal{N}_\sigma$. This contradicts the definition of A. □

Using Lemma 6.5 and standard arguments, e.g. [26, Section 6], we have then that $L[H](\Gamma)$ is a symmetric extension of $L[\mathcal{H}]$ by the Vopenka algebra $\mathbb{P}\in L[\mathcal{H}]$ for adding all $s\in\Theta^\omega$ in Γ. In particular, $L[\mathcal{H}](\Gamma)\cap\wp(\mathbb{R})=\Gamma$ and therefore,

$$L[\mathcal{H}](\Gamma)\vDash\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}$$

as desired.

7. OPEN PROBLEMS, QUESTIONS, AND CONCLUDING REMARKS

Basically, the hypotheses $(T1)\backslash\mathsf{MM}(\mathfrak{c})$ and $(T2)\backslash\mathsf{MM}(\mathfrak{c})$ are only used in the arguments above to show that whenever M is the maximal model of $\mathsf{AD}^+ + \Theta=\theta_{\alpha+1}$ for some α then $\Theta^M<\omega_3^V$. Recent ongoing joint work with M. Zeman shows that we can replace $\neg\square(\omega_3)$ by $\neg\square_{\omega_2}$ in $(T1)$ for this purpose, and therefore, still get models of $\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}$ as above.

Conjecture 7.1. *Assume either*

- $\mathsf{MM}(\mathfrak{c})+\neg\square_{\omega_2}+(\dagger)$, *or*
- $\mathsf{MM}(\mathfrak{c})+$ *"there is a semi-saturated ideal on* $\omega_2+(\dagger)$*",*

then there is a model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ *is regular" containing all the reals and ordinals.*

It is very plausible that we can do without $(\dagger)$; however, it appears that different methods are required, e.g. along the line of what is done in [1].

Conjecture 7.2. *Assume either*

- $\mathsf{MM}(\mathfrak{c})+\neg\square_{\omega_2}$, *or*
- $\mathsf{MM}(\mathfrak{c})+$ *"there is a semi-saturated ideal on* ω_2*",*

then there is a model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ *is regular" containing all the reals and ordinals.*

As mentioned above, a positive answer to the conjecture shows the equiconsistency of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" and $\mathsf{MM}(\mathfrak{c})$+ "there is a semi-saturated ideal on ω_2". Ultimately, we would like to determine whether $\mathsf{MM}(\mathfrak{c})$ is equiconsistent with $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular". The first step towards resolving this is to answer the following question.

Question 7.3. *Assume* $\mathsf{MM}(\mathfrak{c})$. *Suppose* $\mathfrak{N} \vDash \mathsf{AD}^+ + \Theta = \theta_{\alpha+1}$ *for some* α *and is maximal with respect to this property. Then* $o(\mathfrak{N}) < \omega_3$.

Finally, ongoing joint work with M. Zeman suggests the following theories are equiconsistent.

Conjecture 7.4. *The following theories are equiconsistent.*

1. $\mathsf{MM}(\mathfrak{c}) + \neg\square_{\omega_2} + (\dagger)$,
2. $\mathsf{MM}(\mathfrak{c}) + \neg\square_{\omega_2}$,
3. $\mathsf{AD}_{\mathbb{R}}$+"Θ *is regular" and the set* $\{\theta_\alpha : cof(\theta_\alpha) > \omega \wedge HOD_{\wp(\mathbb{R})\restriction\theta_\alpha} \vDash \theta_\alpha \text{ is regular}\}$ *is stationary.*

Acknowledgements. The author would like to thank Hugh Woodin for communications regarding the content of Section 2.3. The author also would like to thank Farmer Schlutzenberg for his help clarifying various parts of his paper [13] that are used substantially in Section 6.

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