A BRIEF ACCOUNT OF RECENT DEVELOPMENTS IN INNER MODEL THEORY

NAM TRANG
Department of Mathematics
University of North Texas, Denton, TX, USA
Email: Nam.Trang@unt.edu

The goal of this survey paper is to give an overview of recent developments in inner model theory. We discuss several most important questions in the field and relevant work (partially) answering these questions roughly from the early 2000's to present. Most of the major questions are still open. These questions drive future research directions in inner model theory. We mention some of the most prominent projects pertaining to these.[1]

Inner model theory starts with Gödel's work on *the constructible universe* L in 1938. L is the first example of a "canonical inner model". There is no official definition of "canonical inner models", but in the case of L, it is canonical because it is the minimal transitive proper class model of ZFC and it satisfies various benchmark properties that one would want canonical inner models to satisfy, such as *the Generalized Continuum Hypothesis* (GCH), $\diamond$ and $\square$ (the latter two are due to Jensen in his seminal work [8]). L, however, does not admit very large large cardinals. For instance, Scott, [31], has shown that L cannot contain a measurable cardinal.

A main theme in inner model theory after Gödel's work and Scott's result is the search for canonical inner models that can accommodate large cardinals beyond those accommodated by L. This is sometimes known as the *Gödel program.* An immediate target is a measurable cardinal. Models of the form $L[U]$ that satisfy "U is a κ-complete, normal, nonprincipal measure on a cardinal κ" have been extensively studied. Most notably, it can be shown that $L[U]$ satisfies the GCH (this is due to Silver [32]), $\diamond, \square$. Kunen, in [11], develops the influential theory of M-ultrapowers (based on previous work by H. Gaifman) and uses it to prove fundamental results on the canonicity of $L[U]$. In particular, he shows that $L[U]$ is unique internally (namely U is the unique normal measure on κ in $L[U]$) and for $i \in \{0,1\}$, if U_i is a filter on κ such that $L[U_i] \vDash$ "U_i is a normal measure on κ", then $L[U_0] = L[U_1]$ and $U_0 \cap L[U_0] = U_1 \cap L[U_1]$. At the heart of Kunen's analysis is the comparability of two models of the form $L[U]$, i.e. for $i \in \{0,1\}$, if $M_i = L[U_i] \vDash$ "U_i is a normal measure on κ_i", then either M_0 is an iterated ultrapower of M_1 or M_1 is an iterated ultrapower of M_0 (via the measure U_i and its images).

Mitchell, in [14] and [15], generalizes Kunen's analysis to models of the form $L[\vec{U}]$ where $\vec{U}$ is a sequence of measures. Mitchell's definition of *an extender* (see Section 4 for a definition) then plays a key role in subsequent searches for canonical inner models of larger large cardinals such

[1] This paper is not meant to be a comprehensive encyclopedia of the subject. We apologize for any omissions of recent works that are not mentioned here.

as strong cardinals (cf. [5]) and Woodin cardinals and beyond (cf. [13], [17]).[2] These models are typically called *fine-structural extender models* (*premice*) and are of the form $L[\vec{E}]$ where $\vec{E}$ is a nice sequence of extenders (cf. [41], [51] for the precise meaning of "nice"). The analysis of these models relies on generalizing Jensen's fine structure in [8][3], and proving *iterability* (iterable premice are called *mice*) and *comparison* for countable mice. To produce these models, one typically performs a *backgrounded construction* to produce a sequence $(\mathcal{N}_\xi : \xi \leq \Omega)$ of models (each of which is a fine-structural extender model) and show that some such $\mathcal{N}_\xi$ satisfies the desired large cardinal axiom. Iterability takes the form: suppose for some $\xi \leq \Omega$, $\pi : M \to \mathcal{N}_\xi$ is such that M is countable, transitive, π is sufficiently elementary, then M is sufficiently iterable, that is, there is an *iteration strategy* Λ for M acting on (normal) trees of length $\leq \omega_1$; we say that Λ witnesses M is $\omega_1 + 1$-iterable. We say that $\mathcal{N}_\xi$ is *countably iterable*. This forms of iterability is sufficient to prove *comparison*: if M and N are countable, transitive and embeddable into some $\mathcal{N}_\xi$ and $\mathcal{N}_\zeta$ respectively, letting Σ be an $\omega_1 + 1$-iteration strategy for M and Λ an $\omega_1 + 1$-iteration strategy for N, then there are countable (normal) trees $\mathcal{T}$ according to Σ with last model $\mathcal{M}^{\mathcal{T}}$ and $\mathcal{U}$ according to Λ with last model $\mathcal{M}^{\mathcal{U}}$, such that either $\mathcal{M}^{\mathcal{T}}$ is an *initial segment* of $\mathcal{M}^{\mathcal{U}}$ (denoted $\mathcal{M}^{\mathcal{T}} \trianglelefteq \mathcal{M}^{\mathcal{U}}$) or vice versa.

Inner models in the papers discussed above are typically constructed via a procedure called *the full backgrounded construction*. This means the extenders giving rise to the model $L[\vec{E}]$ come from extenders in V (and these in turn exist because one assumes an appropriate large cardinal hypothesis holds in V). One could also ask whether (canonical) inner models of large cardinals exist without assuming large cardinals exist in V. In which case, *partial backgrounded constructions* or K^c-*constructions* are used. Some remarkable results in computing lower bound consistency strength have been obtained using this kind of constructions, for instance [36] and [9] show Con(ZFC + there is a presaturated ideal on ω_1) implies Con(ZFC + there is a Woodin cardinal)[4], [7] shows *the Proper Forcing Axiom* (PFA) implies the existence of inner models with very large cardinals and many others.

Finally, the discovery that large cardinals and determinacy are intimately connected opens up other venues where some of the most important and exciting research in modern inner model theory is being done. More precisely, the research along this line belongs to an emerging field called *descriptive inner model theory* (DIMT). DIMT uses tools from both *descriptive set theory* (DST) and inner model theory (IMT) to study and deepen the connection between canonical models of large cardinals and canonical models of *the axiom of determinacy* (AD). One of the first significant developments in DIMT comes in the 1980's with works of Martin, Steel, Woodin and others (cf [12] and [41]); their work, for instance, shows that one can construct models of AD (e.g. they showed AD holds in $L(\mathbb{R})$) assuming large cardinal axioms (those that involve the crucial notion of

[2]It has been observed that linear iterations are no longer sufficient for the analysis of models at the level of Woodin cardinals. Iteration trees are what one needs.

[3]There are two main such generalizations: the Mitchell-Steel fine structure in [13] and Jensen's $*$-fine structure, cf. [51]. These works define exactly what "nice" means.

[4]The converse had been obtained by S. Shelah.

Woodin cardinals) and conversely, one can recover models of large cardinal axioms from models of AD. The key to uncover these connections is to analyze structure theory of models of AD, and in particular their HOD. The work of Steel and Woodin mentioned above shows that HOD of $L(\mathbb{R})$ is fine-structural (in particular, it satisfies $\mathsf{GCH}, \Box, \diamond$) but it is not an extender model; rather, HOD of $L(\mathbb{R})$ is what's called a *strategic extender model*, a model constructed from a predicate coding a nice extender sequence and a predicate coding a (partial) iteration strategy of its own initial segments. Much of DIMT aims to extend this analysis to larger AD^+ models, studying various kinds of strategic extender models and developing ways of constructing them. For instance, the conjecture that "PFA has the exact consistency strength as that of a supercompact cardinal" is one of the most longstanding and important open problems in set theory. Techniques recently developed in DIMT enable us to make significant progress in calibrating the consistency strength lower bound for PFA (and many other theories). See for instance[37], [22], [44]. Some of the main ingredients that come into these theorems and their applications include the HOD analysis and the core model induction techniques. We will discuss these ongoing developments in more details.

To us, the following are the most fundamental questions and projects facing inner model theorists in the 21-st century. We will address progress and partial answers to these questions in this survey paper.

1. Prove iterability of short-extender models.
2. Does HOD of natural AD^+ models admit fine structure? More generally, analyze the (extender, hybrid) mice in AD^+ models.
3. Compute lower bound consistency strength of PFA and other strong theories.
4. Develop a fine structure theory of long-extender models and understand canonical models at the level of supercompact cardinals and above.

1. ITERABILITY

Proving iterability is arguably the most important problem in inner model theory. Suppose a backgrounded construction has produced models $(\mathcal{N}_\xi : \xi \leq \alpha)$. One typically is interested in continuing the construction, i.e. produce model $\mathcal{N}_{\alpha+1}$. By definition, $\mathcal{N}_{\alpha+1} = \mathcal{J}_1(\mathcal{M}_\alpha)$, where $\mathcal{M}_\alpha$ is the (fine-structural) core of $\mathcal{N}_\alpha$. To prove that $\mathcal{M}_\alpha$ exists, i.e. $\mathcal{M}_\alpha$ is solid and universal, we need to show that $\mathcal{N}_\alpha$ is countably iterable. For *fully backgrounded constructions*, that is extenders used in constructing the models $\mathcal{N}_\alpha$'s come from V-extenders, then the strongest iterability results are proved in [17]. Roughly, [17] shows that assuming there is a Woodin cardinal which is a limit of Woodin cardinals, then there is ξ such that $\mathcal{N}_\xi$ has a Woodin cardinal which is a limit of Woodin cardinals and for every $\xi' \leq \xi$, $\mathcal{N}_{\xi'}$ is countably iterable. For partially backgrounded constructions, or K^c-constructions, similar results are obtained only for a much more modest large cardinal strength. The strongest such result is in [1], where $\mathcal{N}_\xi$ satisfies "there is an inaccessible cardinal λ which is a limit of Woodin cardinals and $<$-λ-strong cardinals". This cardinal strength is typically called *non-domestic*. Improving iterability results proved in [17] for full-backgrounded constructions and in [1] for the K^c-construction is the main challenge going forward for inner model theorists.

Results such as that proved in [1] are used to obtain consistency lower bound for various theories. One examples was by Jensen-Steel and Shelah as mentioned above. In the proof of Theorem 1.1 (1), the crucial notion of *the core model* K is used. The existence of the core model K is given by sufficient iterability of K^c, which in turn follows from an anti-large cardinal assumption; for example, in Theorem 1.1 (1), the anti-large cardinal assumption is "there is no ZFC model with a Woodin cardinal".[5] In many applications, K is used (instead of K^c) because K exhibits many nice properties such as: K is invariant under set generic extensions (for appropriate forcings) while K^c is dependent on which universe it is defined (this is used significantly in the proof of Theorem 1.1 and similar applications), one can prove various useful covering properties involving K such as K computes successors of many cardinals correctly (see more below) etc.

Another example concerns consistency lower bound of PFA: using [1], [7] shows that Con(PFA) implies Con(there is an inaccessible cardinal λ which is a limit of Woodin cardinals and $<$-λ-strong cardinals). We summarize some of the main theorems along this line below.

Theorem 1.1. *1. (Jensen-Steel [9], Shelah) Con(*ZFC+ *there is a presaturated ideal on* ω_1*)* $\Leftrightarrow$ *Con(*ZFC+ *there is a Woodin cardinal).*

*2. (Jensen-Schimmerling-Schindler-Steel [7]) Con(*PFA*) implies Con(*ZFC + *there is a non-domestic premouse).*

In both of the theorems above, the paradigm is roughly as follows: assume the desired large cardinal hypothesis fails to hold. Then K^c is sufficiently iterable. Then there is some appropriate regular cardinal κ such that $\mathrm{cof}((\kappa^+)^{K^c}) \geq \kappa$; in (1), K exists and one can even show $(\kappa^+)^K = \kappa^+$ for every singular cardinal κ. This contradicts the original assumption (PFA or the existence of a presaturated ideal on ω_1).

We will come back to the question concerning the consistency strength of PFA later when we discuss the core model induction and results improving (2) above.

The theories for extender models developed in [13] (Mitchell-Steel fine structure) and [51] (Jensen fine structure) allow one to construct extender models in the region of superstrong cardinals assuming appropriate large cardinals in V and countably substructures of V_δ (for an appropriate δ) are sufficiently iterable. For instance, Neeman and Steel, [19], have shown that assuming there is a cardinal δ which is Woodin, and SBH_δ holds,[6] then PFA implies the existence of extender models of a Π^2_1 subcompact cardinal, which slightly stronger than a superstrong. Another example (to be discussed in more details later) is assuming there are enough iterable (hod) mice, one can analyze HOD of AD^+ models below a superstrong cardinal (cf [42]).

[5] L is the core model if $0^\sharp$ does not exist.

[6] SBH_δ states that countable $H \prec V_\delta$, the transitive collapse of H has an $\omega_1 + 1$-iteration strategy for a certain class of normal, non-overlapping trees using only short extenders.

2. HYBRID EXTENDER MODELS AND HOD MICE

As mentioned above, one of the most significant developments in inner model theory is through the work of Martin and Steel [12], and Woodin [41] in the 1980's that established the connections between large cardinals and determinacy. One way of formalizing this connection is through *the Mouse Set Conjecture* (MSC), which states that, assuming the AD or a more technical version of it (AD^+),[7] then whenever a real x is ordinal definable from a real y, then x belongs to a canonical model of large cardinal (mouse) over y. MSC conjectures that the most complicated form of definability can be captured by canonical structures of large cardinals. An early instance of this is well-known theorem of Shoenfield that every Δ^1_2 real is in the Gödel's constructible universe L. Another instance is a theorem of Woodin's that in the minimal class model containing all the reals, $L(\mathbb{R})$, if AD holds, then MSC holds. However, the full MSC is open and is one of the main open problems in DIMT. Instances of MSC have been proved in determinacy models constructed by their sets of reals, i.e. those of the form $L(\wp(\mathbb{R}))$, (much larger than $L(\mathbb{R})$) and these proofs typically obtain canonical models of large cardinals (mice) that capture the relevant ordinal definable real by analyzing HOD of the determinacy models. Hence, the key link between these two kinds of structures (models of large cardinals and models of determinacy) is the HOD of the determinacy models. The problem of analyzing HOD of AD^+ models has gradually grown into a central problem of inner model theory and spurs the development of descriptive inner model theory.

In the following, by models of AD^+, we mean models of AD^+ of the form $L(\wp(\mathbb{R}))$. One main reason for the importance of the HOD analysis is that it provides insights into the relationship between canonical inner models of large cardinals (pure extender models) and models of AD^+ (as alluded to earlier). Unlike HOD of ZFC models which are more or less intractable from the point of view of inner model theory, HOD of AD^+ models in some sense are very well-behaved and code up the AD^+ models in a canonical way; hence understanding HOD of such models provide more insights into the models themselves. We will give a more detailed discussion of the HOD analysis shortly. Another reason why studying HOD of AD^+ models is important is because the techniques developed in this study can be used to tackle a variety of problems, even those outside inner model theory. One of the first applications of the HOD analysis is Steel's proof that in $L(\mathbb{R})$, AD implies all regular cardinals below Θ are measurable (see [41]); here Θ is the supremum of ordinals α such that there is a surjection from $\mathbb{R}$ onto α. This result is not known to be proved by any other method (from traditional descriptive set theory or anywhere else). Many other applications of the HOD analysis have been found, including Neeman's proof that ω_1 is $<$-Θ-supercompact in $L(\mathbb{R})$ (cf. [18]), the positive proof of Woodin's conjecture on the uniqueness of natural models of $\mathsf{AD}+\omega_1$ is $\mathbb{R}$-supercompact (cf. [20]), and the construction and analysis of Varsovian models (cf. [26]).

Under *the Axiom of Choice* (AC), Θ is the successor of the Continuum. Under AD, Θ is very

[7] AD^+ is $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}+$ "every set of reals has an ∞-Borel code" + Ordinal Determinacy. AD^+ is arguably a completion of AD in that it completes the structural analysis of Suslin cardinals. In particular, it implies the set of Suslin cardinals is closed below Θ, which is not known to follow from AD. It is not known if AD implies AD^+. However, AD^+ holds in all known models of AD constructed in practice. In $L(\mathbb{R})$, one can show AD implies AD^+.

large, e.g. Θ is a limit of measurable cardinals and more. Steel, [35], is the first person to prove that HOD up to Θ of $L(\mathbb{R})$ (assuming AD) is a fine-structural extender model[8]; Woodin, unpublished but see [40], extends Steel's work and shows that full HOD of $L(\mathbb{R})$ is a fine-structural model, albeit a hybrid one in that it is constructed from an extender sequence and a (partial) strategy of HOD on trees based on HOD$|\Theta$. This is an example of a Σ-mouse for some iteration strategy Σ, which in turns is a special class of *hybrid mice*. For a strategy Σ of a model M that has nice condensation properties such as *hull condensation* or *branch condensation*, one can construct a fine-structural model $\mathcal{P}$ containing M as an element and $\mathcal{P}$ is constructible from a predicate coding an extender sequence and a predicate coding $\Sigma \cap \mathcal{P}$. The general fine structure of such objects has been investigated by various people including Woodin, Steel, Schindler, Sargsyan, Schlutzenberg, Trang and is written in somewhat more generality in [30] and [29].[9] These papers provide partial answers to the following important problem. An operator F is a function typically defined on (a cone over) some H_κ; F is nice if it has sufficient condensation properties, and determines itself on generic extensions. See for example [28], [29] for a more precise definition. Examples of nice operators are: $rud : x \mapsto rud(x)$, $\sharp : x \mapsto \sharp(x)$. Given a nice operator F, one typically is interested in analyzing the scales pattern in models such as $L^F(\mathbb{R})$ (i.e. in this model, given a pointclass Γ, where is the first place in the constructible hierarchy of the model one can show every set in Γ has a scale) much like Steel's analysis of scales in $L(\mathbb{R})$ [34].

Problem 2.1. *Given a nice operator F, develop fine structure theory of hybrid extender models of the form $L[\mathbb{E}, F]$, prove that iterable $L[\mathbb{E}, F]$ models with large cardinals exist (we call these F-mice), and analyze the scales pattern in F-mice over $\mathbb{R}$.*

Problem 2.1 is an integral part of developing the theory of hybrid mice suitable for core model induction applications. As mentioned above, partial answers to Problem 2.1 are given in [30]. The answers in [30] are partial in the sense that the authors assume additionally that F, in some sense, is fine-structural and has some smallness properties. This is enough for applications that aim to construct models up to LSA and a bit beyond (from large cardinals or strong combinatorial principles) but seems hardly complete. Here LSA, which stands for "the Largest Suslin Axiom", is the theory "$\mathsf{AD}^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal" (see the next section for a more detailed discussion of the Solovay sequence $(\theta_\alpha)_\alpha$). LSA is a very strong determinacy principle isolated by W.H. Woodin in the 1990's; LSA implies the existence of many models of $\mathsf{AD}_\mathbb{R} + \Theta$ is regular and much more. It was recently shown to be consistent relative to large cardinals (cf. [24]).

There are various non-fine-structural operators that are of interest. [29] develops the theory of F-mice without assuming F is fine-structural by carefully formulating the notion of hybrid premouse and F's condensation properties. However, the theory developed there is not general

[8]More precisely, Steel shows that V_Θ^{HOD} is the universe of an extender model.

[9][30] fully resolves fine-structural issues related to feeding in branch information of Σ and is used in the development of the fine structure of least-branch hod mice, cf. [42]. Furthermore, [29] defines a general class of operators F such that F-mice carries fine structure similar to that of extender models. Σ-mice for Σ a strategy with nice condensation properties is an example of F-mice.

enough to cover some important non-fine-structural operators, like the C_Γ operator from the AD^+ world, where $C_\Gamma(x)$ is the largest countable set in Γ containing x.

Woodin proves that *the Mouse Set Conjecture* (MSC)[10] holds in $L(\mathbb{R})$, see [33]. Sargsyan has generalized the aforementioned analyses to much larger AD^+ models, see [21] and [24]. A central notion in the proof of MSC and the analysis of HOD is the notion of *hod mice* (developed by G. Sargsyan, cf. [21], which built on and generalized earlier unpublished work of Woodin). Hod mice are a type of strategic extender models. The extender sequence allows hod mice to satisfy some large cardinal theory and the strategies allow them to generate models of determinacy. Unlike pure extender models and Σ-mice (here we mean F-mice, where the strategy Σ is coded by an operator F), there are many basic structural questions that are still open for hod mice, for example, the extent of condensation and whether combinatorial principles such as $\square$ hold in hod mice.

The approach taken by Sargsyan in [21] and [24] in analyzing HOD is called *rigid layering*. This approach to defining hod mice has many advantages, one of which is that constructing such hod mice in practice (see for example [22], [44], [24]) is relatively easy. The complexity of the rigidly layered hod mice constructed matches that with the complexity of the determinacy pointclasses they generate fairly closely (i.e. the hierarchy does not grow too quickly) However, such hod mice may not satisfty some of the fine structural properties extender models satisfy such as condensation and $\square$[11] due to the non-uniformity in its definition, e.g. rigidly layered hod mice are too extender-biased, in other words, extenders seem to be indexed more often than branches of iteration trees on the model. Further, it is not clear how to define rigidly layered hod mice beyond those defined in [24]; it appears that the complexity in defining rigidly layered hod mice increases along with the complexity of the underlying AD^+ theory.

Steel, in [42], has provided an alternative approach to defining hod mice and analyzing HOD. His approach is called the *least branch* approach. This approach has the advantage that we no longer need to "layer"; instead, strategies and extenders are fed into the models in a uniform way. This ensures that the fine structure of least-branch hod mice is similar to that of extender models. See, for example, [42] and [43] for a proof of solidity and condensation for least branch hod mice. Provided we have enough hod mice in our AD^+ universe, more precisely, *Hod Pair Capturing* (HPC) holds (cf. [42]), where HPC is the alternative version of *Generation of Pointclasses* in the context of least-branch hod mice, which says that for any $A \subseteq \mathbb{R}$ such that A is Suslin co-Suslin, there is a least-branch hod pair $(\mathcal{P}, \Sigma)$ such that $A <_\omega Code(\Sigma)$,[12] then HOD (up to Θ) is the diret limit of least-branch hod mice. The definition of HOD (up to Θ) as a least branch hod mouse is uniform and does not depend on the complexity of the underlying theory. The theory of least-branch hod mice works for AD^+ models roughly up to the level of superstrong cardinals. The fine structure of least branch hod mice mirrors that of extender models. The results in [42] and [43] immediately imply

[10] MSC is the statement: assuming there are no iterable mice with a superstrong cardinal, for any reals x, y such that x is $OD(y)$, there is a countable mouse $\mathcal{M}$ over y such that $\mathcal{M}$ is sound, $\rho_\omega(\mathcal{M}) = \omega$, and $x \in \mathcal{M}$.

[11] We know such hod mice fail to satisfy instances of condensation extender models satisfy. Regarding $\square$, [24] provides a partial answer for hod mice in the minimal model of the theory LSA, which is $\mathsf{AD}^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal.

[12] $Code(\Sigma)$ is a canonical coding of the strategy Σ by a set of reals.

HOD satisfies $\mathsf{GCH} + \diamondsuit$.[13] and move us a step closer to characterizing $\square$ in HOD (under HPC). However, constructing least branch hod mice in core model induction applications is a challenge. It is worthwhile to develop methods for constructing least branch hod mice that can be applied in a variety of applications.

In summary, we believe that it is very important studying both the rigidly layered hierarchy and the least-branch hierarchy as the two approaches complement each other in improving our understanding of the structure theory of hod mice under AD^+. The most important problems for those studying least-branch hod mice to tackle are the following (see Section 4 for the definition of a (long) extender).

Problem 2.2 (Hod Pair Capturing)**.** *Assume* AD^+. *Show that* HPC *is true assuming no mice with a long extender exist.*

The hypothesis "no mice with long extenders" seems necessary here. The theory of "mice" that we are familiar with (i.e. Mitchell-Steel mice and Jensen mice) are called short-extender mice; mice with short-extenders can satisfy large cardinals up to superstrong and subcompact cardinals, but not much more. Long-extender mice can satisfy cardinals in the supercompact region, but our understanding of long-extender mice is limited and their basic theory is still being developed.[14]

In the rigidly layered hierarchy, we can state the corresponding problem to the above. Resolving this is a key step in proving MSC and would allow us to compute HOD (of AD^+ models beyond those considered in [24]). Generation of Pointclasses (GP) was first introduced in [21] for the purpose of showing HOD is the direct limit of hod mice; Steel (cf. [42]) observes that one can simply require the hod pairs are Wadge cofinal in the Suslin coSuslin sets for the HOD analysis. For all intended purposes, the differences between HPC and GP are cosmetic, though as mentioned above, there are difficulties in defining rigidly layered hod mice beyond those defined in [24].

Problem 2.3 (Generation of Pointclasses)**.** *Prove that* AD^+ *implies Generation of Pointclasses* (GP)*, assuming no mice with a long extender exist. Here* GP *is the statement for any sufficiently closed pointclass* Γ *such that every* $A \in \Gamma$ *is Suslin co-Suslin and* Γ *does not contain all Suslin co-Suslin sets, there is a (rigidly layered) hod pair* $(\mathcal{P}, \Sigma)$ *such that* $(\mathcal{P}, \Sigma)$ *generates* Γ.[15]

3. THE CORE MODEL INDUCTION AND ITS APPLICATIONS

Another way of exploring the determinacy/large cardinal connection is via the Core Model Induction (CMI). The core model induction is pioneered by Woodin and developed further by Steel, Schindler and others. CMI draws strength from strong set-theoretic principles such as PFA to inductively construct canonical models of determinacy and those of large cardinals in a locked-step process by

[13] In the $\mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}} + \mathsf{V} = \mathsf{L}(\wp(\mathbb{R}))$ case, it is still open whether HOD up to Θ is a direct limit of least-branch hod mice. However, Steel's analysis shows that HOD up to the largest Suslin cardinal is such a direct limit; and this suffices to show HOD satisfies $\mathsf{GCH} + \diamondsuit$.

[14] At the point this paper is written, the theory of long extenders has been developed at the level of κ being κ^{+n}-supercompact for all $n < \omega$.

[15] Roughly, $(\mathcal{P}, \Sigma)$ generates Γ means that every $A \in \Gamma$ is in the derived model of $\mathcal{P}$ generated by Σ.

combining core model techniques (for constructing the core model K) with descriptive set theory, in particular the scales analysis in $L(\mathbb{R})$ and its generalizations (cf. [34], [38], [30]). CMI is the only known systematic method for computing lower bound consistency strength of strong theories extending ZFC and it is hoped that it will allow one to compute the exact strength of important theories such as PFA.

The Solovay sequence is the sequence $\vec{\theta} = (\theta_\alpha : \alpha \leq \Omega)$ such that $\vec{\theta}$ is continuous (i.e. range of $\vec{\theta}$ is closed), $\sup \vec{\theta} = \Theta$, θ_0 is the supremum of ordinals α such that there is an OD surjection from $\mathbb{R}$ onto α, for $\alpha < \Omega$, $\theta_{\alpha+1}$ is the supremum of ordinals α such that there is a $OD(A)$ surjection from $\mathbb{R}$ onto α, where A is a set of reals of Wadge rank θ_α. $L(\mathbb{R})$ is the minimal (transitive, proper class) model of AD containing all reals. It can be shown that in $L(\mathbb{R})$, assuming AD, $\mathsf{AD}^+ + \Theta = \theta_0$ holds. Here are some AD^+ theories listed in increasing strength: AD^+, $\mathsf{AD}^+ + \Theta > \theta_0$, $\mathsf{AD}_{\mathbb{R}}$, $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$, $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular, LSA. LSA is a very strong determinacy principle, first isolated by Woodin in [48]. [48] conjectures that LSA has strength roughly in the region of a supercompact cardinal. However, [24] shows that LSA is consistent relative to the existence of a Woodin cardinal which is a limit of Woodin cardinals, which is much weaker. However, LSA is still much stronger than $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$, which is roughly the lower bound consistency strength of PFA obtained by [7] via core model (K^c) techniques in [1].

Many of the early work in CMI typically obtains $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ as lower bound of various theories (cf. [28], [3], [39], [37]). Ketchersid, using ideas of Woodin, is the first to have constructed inner models cf $\mathsf{AD}^+ + \Theta > \theta_0$, [10]. [10] contains a wealth of techniques for constructing mice generating models of $\mathsf{AD}^+ + \Theta > \theta_0$ and a bit beyond.[16] They are, however, insufficient to construct models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular, LSA and beyond. To construct models of these stronger theories, there are many ingredients that come into play. One needs to analyze HOD of such theories. Sargsyan (cf. [21], [24]) has solved Problem 2.3 for models of AD^+ up to LSA, computed HOD of such models, and developed various techniques for constructing hod mice in practice. One needs to develop the theory of F-mice for complicated operators F, the scales analysis in F-mice over $\mathbb{R}$, and various techniques for constructing new scales and pointclasses of determinacy. Again, this has been done by various people ([28], [30],[29], [46]).

We discuss below several problems involving determining consistency strength of important set-theoretic principles. The arguably most important one is.

Question 3.1. *What is the consistency strength of* PFA*?*

As mentioned above, several authors have used the core model induction to continually improve the lower bound consistency of PFA. [37], [25], [44], and [24] show that the strength of PFA is at least that of LSA. The strongest theorem along this direction is.

Theorem 3.2 (Sargsyan and Trang [24])**.** *Assume* PFA*. Then there is an inner model M containing $\mathbb{R} \cup ORD$ such that* $M \vDash \mathsf{LSA}$.

[16]These results and techniques can help us construct models of $\mathsf{AD}_{\mathbb{R}}$; however, they do not give lower bound strengths beyond those obtained by K^c-constructions as in [1], which is roughly $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

However, as mentioned above, LSA is shown to be consistent relative to a Woodin limit of Woodin cardinals and the consensus amongst set theorists is that PFA should be as strong as a supercompact cardinal (it is well-known that the upper bound for PFA is a supercompact, cf [2]). A complete answer to Question 3.1 is the holy grail of inner model theory. From the point of view of descriptive inner model theory, to completely solve this problem, one needs to understand HOD of AD^+ models; in particular, one needs to resolve Problems 2.2 and/or 2.3 and their generalizations to the long-extender regions. . Futhermore, at the level of $\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular and beyond, one needs to do a significant amount of work to construct hod-like objects to eventually generate HOD of the AD^+ models; this is where the bulk of the construction is and seems to be hypothesis-dependent, i.e. the construction methods seem to depend on the types of strong hypotheses one assumes as well as the target AD^+ theory one is trying to construct models for. For example, [24], from PFA, combines techniques for constructing K^c from core model theory and techniques for analyzing HOD of AD^+ models, introduces a strategic K^c construction, whose outcome is a hod-like object $\mathcal{M}$ that generates HOD of an LSA model (and hence the LSA model itself).

It has been observed (independently) by I. Neeman and the author that

Theorem 3.3. *Assume* PFA + *there is a Woodin cardinal. Then there are inner models that satisfy* ZFC + *there is a Woodin cardinal which is a limit of Woodin cardinals.*

The proof of the above theorem will appear in an upcoming paper; the basic argument is: suppose not. Let δ be a Woodin cardinal. Then the fully backgrounded construction in V_δ described in [17] will produce an extender model $\mathcal{N}$ of height δ. Let $\mathcal{N}^+$ be the stack of mice over $\mathcal{N}$ (as defined in [7] and [19]), then [19] shows that $\mathrm{cof}(o(\mathcal{N}^+)) \geq \delta$; this contradicts PFA by a standard argument, using the fact that $\square(\delta^+)$ and $\square(\delta)$ both fail.[17]

So with an extra, mild large cardinal assumption, we can improve the lower bound of Theorem 3.2 significantly.[18] Unfortunately, the method of Theorem 3.3 does not seem to generalize; this is because the proof of the theorem is highly dependent on the construction in [17], which has no known improvement. So it seems the right approach to systematically study the universe of sets and its canonical structures in the presence of strong hypotheses like PFA is continue to generalize the approach of [24].

In closing, we list some sample problems that seem within reach of current core model induction techniques. Below $\mathsf{MM}(\kappa)$ is the Martin's Maximum for posets of size $\leq \kappa$ and c is the size of the continuum.

Question 3.4. *(a) What is the consistency strength of* $\neg\square_\kappa$ *for some singular strong limit* κ*?*

(b) What is the consistency strength of $\neg\square_\kappa + \neg\square(\kappa)$ *for* $\kappa \geq \aleph_3$ *such that* $\kappa^\omega = \kappa$ *and* $2^{<\kappa} = \kappa$*?*

[17] $\square(\alpha)$ says that there is a sequence of $(C_\beta : \beta < \alpha)$ such that C_β is a club subset of β, $C_\beta \cap \gamma = C_\gamma$ for every limit point γ of C_β, and there is no "thread" through the sequence, i.e. there is no club C in α such that for any limit point β of C, $C \cap \beta = C_\beta$.

[18] Note that unlike Neeman-Steel theorem [19, Theorem 1.2], which assumes SBH_δ, which is a very strong iterability assumption, no assumption on iterability is made here.

(c) *What is the consistency strength of* $\mathsf{MM}(\mathsf{c}^+)$*?*

(d) *Is the consistency strength of* $\mathsf{MM}(\mathsf{c})$ *the same as that of* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular?*

(e) *What is the consistency strength of* $\mathsf{ZF} + \mathsf{DC} + \omega_1$ *is supercompact?*

(f) *What is the consistency strength of the theory* $\mathsf{ZF}+$ *every cardinal is singular?*

(g) *What is the consistency strength of the failure of the Unique Branch Hypothesis* (UBH*) for nice trees?*

The first four questions concern consistency strength of certain fragments of MM. Some significant refinements of techniques in [37], [44], [22], and [24] may be needed to make progress on these questions. For example, regarding (a), [37] proves AD holds in $L(\mathbb{R})$ and D. Adolf has recently improved on this result to construct the minimal model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". Regarding (b), we believe the hypothesis is very strong. If we set $\kappa = \omega_2$ in the hypothesis, [4] shows that it is weak. The next two concern developing core model induction methods in the choiceless (or limited choice) context. See [45] and [3]. Concerning the last question, UBH for nice trees is the statement that every nice iteration tree on V has at most one cofinal well-founded branch, where a nice iteration tree is normal, nonoverlapping, and every extender E used in the tree has $length(E) = strength(E) =$ an inaccessible cardinal in the model in which E is chosen from. The motivation for this question is the belief that UBH for nice trees should be true, cf [23] and [27]. Note that Woodin, [49], has constructed nonoverlapping iteration trees on V with distinct cofinal wellfounded branches from a very large cardinal assumption. However, the trees constructed in [49] are not nice.

4. INNER MODEL THEORY AT THE LEVEL OF SUPERCOMPACT CARDINALS

Let M and N be transitive models of (a fragment of) ZFC and $j : M \to N$ be an elementary embedding with critical point κ. Let $\lambda > \kappa$ (λ typically is a limit ordinal). A (κ, λ)-extender $E = E_j$ (over M or M-extender) derived from j is $(E_a : a \in [\lambda]^{<\omega})$,[19] where E_a is a (nonprincipal, countably complete) measure defined as follows: letting κ_a be the least γ such that $j(\gamma) \geq max(a) + 1$, for each $A \subseteq [\kappa_a]^{|a|}$ in M,

$$A \in E_a \Leftrightarrow a \in j(A).$$

A (κ, λ)-extender E is *short* if all measures in E concentrate on κ (i.e. $\kappa_a = \kappa$). Otherwise, E is *long*. For $\gamma < \lambda$, $E|\gamma = (E_a : a \in [\gamma]^{<\omega})$ is an initial segment of E. As mentioned above, fine structure theory has been fully developed for short-extender models (cf. [13] and [51]).

Developing fine structure for long-extender models proves to be very challenging. There are several problems with generalizing fine structure theory for short extenders to long ones. One main problem is the *Failure of the Initial Segment Condition*. The Initial Segment Condition, which

[19] One can define abstractly the notion of an extender by specifying the properties the E_a's must have and how they relate to one another. See e.g. [41] for such a definition.

roughly states that initial segments of an extender F on the extender sequence of a (fine-structural) model are on the sequence, is a staple of inner model theory in the region of short extenders. This condition fails in fine-structural extender models having cardinals just past superstrong cardinals. This and the fact that long generators may be moved in iteration trees further complicate comparison arguments for mice at this level. The basic theory for extender models satisfying the existence of a cardinal κ which is κ^{+n}-supercompact for some $n < \omega$ has been developed by Woodin ([47]) and Neeman, Steel ([16]).[20] Beyond this level of complexity, various attempts have been made with no significant progress. It is worth noting that Woodin (in [49]) has generalized the Martin-Steel theory of coarse extender models to the levels of supercompact cardinals. These are called *suitable extender models*. Woodin also shows that an extender model (not necessarily fine-structural) of a (fully) supercompact cardinal absorbs all large cardinals from V (these are called weak extender models, as defined below, cf Definition 4.4). This remarkable theorem suggests that it is enough to develop fine structure theory for models of a supercompact cardinal. Recent work of Woodin (see [50] for a more detailed discussion) suggests that the right approach for developing such a theory is via going into the strategic-extender hierarchy[21] and again the connection with AD^+ and HOD of models of AD^+ emerges at this level of large cardinal.

[6] defines *the Ultrapower Axiom* (UA), which states that suppose M_0, M_1 are two ultrapowers of the universe V, with $j_0 : V \to M_0$, $j_1 : V \to M_1$ be the respective ultrapower maps, then there are internal ultrapower maps $i_0 : M_0 \to N$, $i_1 : M_1 \to N$ such that $i_0 \circ j_0 = i_1 \circ j_1$. This axiom was isolated by Goldberg from Woodin's previous work on weak comparisons. UA is known to hold in all fine-structural extender models,[22] has many consequences that suggests the universe in which UA holds resembles that of a fine-structural extender model (e.g. UA implies the normal fine κ-complete ultrafilters on $\wp_\kappa(\lambda)$ are well-ordered by the (generalized) Mitchell order, GCH holds above a supercompact cardinal κ), and permits the development of a semi-fine structural theory of large cardinals in the region of supercompact cardinals. This work suggests UA must be consistent with supercompacts. This in turn suggests that inner model theory must reach the levels of many supercompact cardinals since (at this stage) it is hard to conceive how one can build models satisfying UA which do not come from fine-structural models. Most importantly, UA suggests non-strategic extender hierarchy cannot reach the infinite levels of supercompact (cf. [6, Conjecture 8.3]).

Finally, one way of formalizing the development discussed above is the axiom $V = \mathit{Ultimate}\text{-}L$. In the following, a set $A \subseteq \mathbb{R}$ is universally Baire if for each γ, there are trees T_γ, U_γ on $\omega \times \mathit{Ord}$ such that $A = p[T_\gamma] = \mathbb{R} \backslash p[U_\gamma]$, and furthermore, for any forcing extension $V[g]$ for V such that the forcing has size $< \gamma$, in $V[g]$, $p[T_\gamma] = \mathbb{R}\backslash p[U_\gamma]$. In the presence of a proper class of Woodin

[20] There are many difficulties in developing inner model theory at the level of supercompactness. See, for example, [50] for more details.

[21] Models in the strategic-extender hierarchy are constructed from a sequence of partial extenders and partial strategies of their own initial segments, much like the hod mice discussed above. Woodin also suggests that at this level of complexity, the strategic-extender hierarchy has to be different from the layered hierarchy in [21] and [24].

[22] This is the only known proof of consistency of UA. Forcing UA holds (along with a large cardinal) seems very difficult. See [6, Proposition 3.4].

cardinals, if A is universally Baire, then $L(A,\mathbb{R}) \models \mathsf{AD}^+$.

Definition 4.1 (V = Ultimate-L, Woodin [50]).

1. There is a proper class of Woodin cardinals.
2. For each Σ_2-sentence Φ, if Φ holds in V then there exists a universally Baire set $A \subseteq \mathbb{R}$ such that

$$(HOD \cap V_\Theta)^{L(A,\mathbb{R})} \models \Phi.$$

$\dashv$

There are many variations of the version of the axiom stated above (cf. [50]). One main motivation for this axiom is that $V = Ultimate - L$, when supplemented by large cardinals, is expected to axiomatize the structure of canonical inner models of a supercompact cardinal (if they exist). Another important feature of this axiom is that it suggests the universe looks like a strategic extender model, which is the HOD of $L(A,\mathbb{R})$ where A is a universally Baire set. The following is a version of Woodin's *Ultimate-L Conjecture*. We need a few definitions.

Definition 4.2. A sequence $N = (N_\alpha : \alpha \in Ord)$ is *weakly* Σ_2*-definable* if there is a formula $\varphi(x)$ such that:

1. For all $\beta < \eta_1 < \eta_2 < \eta_3$, if $(N_\varphi)^{V_{\eta_1}}|\beta = (N_\varphi)^{V_{\eta_3}}|\beta$, then

$$(N_\varphi)^{V_{\eta_1}}|\beta = (N_\varphi)^{V_{\eta_2}}|\beta = (N_\varphi)^{V_{\eta_3}}|\beta.$$

2. For all $\beta \in Ord$, $N|\beta = (N_\varphi)^{V_\eta}|\beta$ for all sufficiently large η.

Here for $\gamma \in Ord$, $(N_\varphi)^{V_\gamma} = \{a : V_\gamma \models \varphi[a]\}$. $\dashv$

Definition 4.3. Suppose that $N \subseteq V$ is an inner model and $N \models \mathsf{ZFC}$. Then N is *weakly* Σ_2*-definable* if the sequence

$$(N \cap V_\alpha : \alpha \in Ord)$$

is weakly Σ_2-definable. $\dashv$

Inner models of V which are Σ_2-definable are weakly Σ_2-definable. For instance, HOD is Σ_2-definable, hence it is weakly Σ_2-definable. The increasing enumeration of supercompact cardinals is weakly Σ_2-definable.

Definition 4.4. A transitive class $N \models \mathsf{ZFC}$ is a *weak extender model for δ is supercompact* if for every $\gamma > \delta$, there exists a δ-complete normal fine measure U on $\wp_\delta(\gamma)$ such that

1. $N \cap \wp_\delta(\gamma) \in N$,
2. $U \cap N \in N$.

$\dashv$

By the remarkable [50, The Universality Theorem], weak extender models for δ is supercompact absorb all large cardinal properties given by extenders or elementary embeddings in the universe.

Conjecture 4.5 (Woodin, [50]). *Suppose that δ is an extendible cardinal. Then there exists a weak extender model N for the supercompactness of δ such that:*

1. *N is weakly Σ_2-definable and $N \subset HOD$,*
2. *$N \vDash$ "$V =$ Ultimate-L".*

Proving this conjecture or its variants would show in a decisive fashion that the strategic-extender hierarchy is the correct hierarchy for the canonical universe of sets.

Acknowledgement. The author would like to thank the NSF for its generous support through grant DMS-1849295. The author would also like to thank the referee for several helpful comments on an older version of the paper.

References

[1] Alessandro Andretta, Itay Neeman, and John Steel. The domestic levels of K^c are iterable. *Israel Journal of Mathematics*, 125(1):157–201, 2001.

[2] James E Baumgartner. Iterated forcing. *Surveys in set theory*, 87:1–59, 1983.

[3] Daniel Busche and Ralf Schindler. The strength of choiceless patterns of singular and weakly compact cardinals. *Annals of Pure and Applied Logic*, 159(1):198–248, 2009.

[4] Andrés Eduardo Caicedo, Paul Larson, Grigor Sargsyan, Ralf Schindler, John Steel, and Martin Zeman. Square principles in $\mathbb{P}_{max}$ extensions. *Israel Journal of Mathematics*, 217(1):231–261, 2017.

[5] A. Dodd and R. Jensen. The core model. *Ann. Math. Logic*, 20(1):43–75, 1981.

[6] G. Goldberg. The ultrapower axiom. 2018. Same Volume.

[7] R. Jensen, E. Schimmerling, R. Schindler, and J. R. Steel. Stacking mice. *J. Symbolic Logic*, 74(1):315–335, 2009.

[8] R. Björn Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308; erratum, ibid. 4 (1972), 443, 1972. With a section by Jack Silver.

[9] Ronald Jensen and John Steel. K without the measurable. *The Journal of Symbolic Logic*, 78(3):708–734, 2013.

[10] R. Ketchersid. *Toward $AD_{\mathbb{R}}$ from the Continuum Hypothesis and an ω_1-dense ideal.* PhD thesis, Ph. D. thesis, Berkeley, 2000.

[11] Kenneth Kunen. Some applications of iterated ultrapowers in set theory. *Ann. Math. Logic*, 1:179–227, 1970.

[12] Donald A Martin and John R Steel. A proof of projective determinacy. *Journal of the American Mathematical Society*, 2(1):71–125, 1989.

[13] W. J. Mitchell and J. R. Steel. *Fine structure and iteration trees*, volume 3 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1994.

[14] William J. Mitchell. Sets constructible from sequences of ultrafilters. *J. Symbolic Logic*, 39:57–66, 1974.

[15] William J. Mitchell. Sets constructed from sequences of measures: revisited. *J. Symbolic Logic*, 48(3):600–609, 1983.

[16] I. Neeman and J.R. Steel. Fine structure of plus-one premice. 2014. handwritten notes, available at math.berkeley.edu/∼steel/papers/Publications.html.

[17] Itay Neeman. Inner models in the region of a Woodin limit of Woodin cardinals. *Ann. Pure Appl. Logic*, 116(1-3):67–155, 2002.

[18] Itay Neeman. Inner models and ultrafilters in $L(\mathbb{R})$. *Bulletin of Symbolic Logic*, 13(1):31–53, 2007.

[19] Itay Neeman and John Steel. Equiconsistencies at subcompact cardinals. *Archive for Mathematical Logic*, 55(1-2):207–238, 2016.

[20] D. Rodríguez and N. Trang. $L(\mathbb{R}, \mu)$ is unique. *Advances in Mathematics*, 324:355–393, 2018.

[21] G. Sargsyan. *Hod mice and the mouse set conjecture*, volume 236 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2014.

[22] G. Sargsyan. Covering with universally Baire operators. *Advances in Mathematics*, 268:603–665, 2015.

[23] G. Sargsyan and N. Trang. Non-tame mice from tame failures of the unique branch hypothesis. *Canadian Journal of Mathematics*, 66(4):903–923, 2014.

[24] G. Sargsyan and N. Trang. *The largest Suslin axiom, submitted.* 2017. available at math.unt.edu/∼ntrang.

[25] Grigor Sargsyan. Nontame mouse from the failure of square at a singular strong limit cardinal. *Journal of Mathematical Logic*, 14(01):1450003, 2014.

[26] Grigor Sargsyan and Ralf Schindler. Varsovian models I. *The Journal of Symbolic Logic*, 83(2):496–528, 2018.

[27] S. Sargsyan and N. Trang. Tame failures of the unique branch hypothesis and models of $\mathsf{AD}_{\mathbb{R}} + \theta$ is regular. *Journal of Mathematical Logic*, 12(2), 2016.

[28] R. Schindler and J. R. Steel. *The core model induction.* available at http://www.math.uni-muenster.de/logik/Personen/rds/. 2013.

[29] Farmer Schlutzenberg and Nam Trang. The fine structure of operator mice. *arXiv preprint arXiv:1604.00083*, 2016.

[30] Farmer Schlutzenberg and Nam Trang. Scales in hybrid mice over $\mathbb{R}$. *arXiv preprint arXiv:1210.7258*, 2016.

[31] Dana Scott. Measurable cardinals and constructible sets. In *Mathematical Logic In The 20th Century*, pages 407–410. World Scientific, 2003.

[32] Jack Silver. The consistency of the GCH with the existence of a measurable cardinal. Axiomatic Set Theory, Proc. Sympos. Pure Math. 13, Part I, 391-395 (1971)., 1971.

[33] J. R. Steel. A theorem of Woodin on mouse sets, available at http://math.berkeley.edu/∼steel/.

[34] J. R. Steel. Scales in $\mathrm{L}(\mathbb{R})$. In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 107–156. Springer, Berlin, 1983.

[35] J. R. Steel. $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ is a core model below Θ. *Bull. Symbolic Logic*, 1(1):75–84, 1995.

[36] J. R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1996.

[37] J. R. Steel. PFA implies $\mathsf{AD}^{L(\mathbb{R})}$. *J. Symbolic Logic*, 70(4):1255–1296, 2005.

[38] J. R. Steel. Scales in $K(\mathbb{R})$ at the end of a weak gap. *J. Symbolic Logic*, 73(2):369–390, 2008.

[39] John Steel and Stuart Zoble. Determinacy from strong reflection. *Transactions of the American Mathematical Society*, 366(8):4443–4490, 2014.

[40] John R Steel and W Hugh Woodin. HOD as a core model. In *Ordinal definability and recursion theory: Cabal Seminar*, volume 3, 2012.

[41] J.R. Steel. An outline of inner model theory. *Handbook of set theory*, pages 1595–1684, 2010.

[42] J.R. Steel. Normalizing iteration trees and comparing iteration strategies. 2018. available at math.berkeley.edu/∼steel/papers/Publications.html.

[43] J.R. Steel and N. Trang. Condensation for mouse pairs. 2018. submitted, available at math.uci.edu/∼ntrang.

[44] N. Trang. PFA and guessing models. *Israel Journal of Mathematics*, 215(2):607–667, 2016.

[45] N. Trang and T. Wilson. Supercompact measures on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$. 2016. submitted, available at math.unt.edu/∼ntrang.

[46] Trevor M Wilson. The envelope of a pointclass under a local determinacy hypothesis. *Annals of Pure and Applied Logic*, 2015.

[47] H.W. Woodin. The fine structure of suitable extender models I. 2014.

[48] W. H. Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, volume 1 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 1999.

[49] W Hugh Woodin. Suitable extender models I. *Journal of Mathematical Logic*, 10(01n02):101–339, 2010.

[50] W Hugh Woodin. In search of ultimate-L the 19th Midrasha Mathematicae lectures. *Bulletin of Symbolic Logic*, 23(1):1–109, 2017.

[51] Martin Zeman. *Inner models and large cardinals*, volume 5 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2002.