SUPERCOMPACTNESS CAN BE EQUICONSISTENT WITH MEASURABILITY [*][†]

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Abstract

The main result of this paper, built on work of [19] and [16], is the proof that the theory "$\mathsf{AD}_{\mathbb{R}}$ + DC + there is an $\mathbb{R}$-complete measure on Θ" is equiconsistent with "$\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}$ + there is a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$ + Θ is regular." The result and techniques presented here contribute to the general program of descriptive inner model theory and in particular, to the general study of compactness phenomena in the context of $\mathsf{ZF}+\mathsf{DC}$.

1. INTRODUCTION

Throughout the paper, unless stated otherwise, we assume $\mathsf{ZF}+\mathsf{DC}$. We begin with the following definitions. In the following, a measure on some set Y is an ultrafilter (maximal filter) on Y. If μ is a measure on Y, then for any set $A \subseteq Y$, we say A is μ-measure one if $A \in \mu$ or equivalently $\mu(A) = 1$.

Definition 1.1 (ZF+DC)**.** *Suppose X is an uncountable set and μ is a measure on $\wp_{\omega_1}(X) =_{def} \{\sigma \subseteq X \mid \sigma \text{ is countable}\}$. We say that*

1. *μ is* ***fine*** *if whenever $x \in X$, then the set $A_x =_{def} \{\sigma \mid x \in \sigma\} \in \mu$.*
2. *μ is* ***countably complete*** *if whenever $\langle A_n \mid n < \omega\rangle$ is a sequence of μ-measure one sets then $\bigcap_n A_n \in \mu$.*
3. *μ is* ***normal*** *if whenever $F : \wp_{\omega_1}(X) \to \wp_{\omega_1}(X)$ is such that the set $\{\sigma \mid F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset\} \in \mu$ then there is an $x \in X$ such that the set $\{\sigma \mid x \in F(\sigma)\} \in \mu$.*

If there is a nonprincipal measure μ on $\wp_{\omega_1}(X)$ that satisfies (1)-(3), then we say that ω_1 is X-supercompact. If there is a nonprincipal measure μ on $\wp_{\omega_1}(X)$ that satisfies (1) and (2) then we say ω_1 is X-strongly compact.

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This is a generalization of the notion of supercompactness in the ZFC context. The definition of strong compactness is unchanged. In particular, in clause (3) of Definition 1.1, if we replace "$F(\sigma) \subseteq \sigma$" by "$F(\sigma) \in \sigma$", then we get the standard definition of normality in the ZFC context. Without the full Axiom of Choice, we seem to have to weaken the requirement on F. If X is a set of ordinals then the two notions coincide. Definition 1.1 originates from [11]. The following is not hard to prove (see [17]).

Lemma 1.2 (ZF + DC)**.** *Suppose μ is a fine measure on $\wp_{\omega_1}(X)$. The following are equivalent.*

1. *μ is normal.*
2. *Suppose we have $\langle A_x \mid x \in X \wedge A_x \in \mu\rangle$. Then $\triangle_{x\in X} A_x =_{def} \{\sigma \mid \sigma \in \bigcap_{x\in\sigma} A_x\} \in \mu$.*

From now on, the phrase "μ is a supercompact measure on $\wp_{\omega_1}(X)$" always means "μ is a nonprincipal, normal fine, countably complete measure on $\wp_{\omega_1}(X)$". We will also say "ω_1 is X-supercompact" to mean "there is a supercompact measure on $\wp_{\omega_1}(X)$". When μ is nonprincipal, countably complete, and fine (but not necessarily normal), we say that μ is a *strongly compact* measure. We say that ω_1 *is supercompact* if ω_1 is X-supercompact for all uncountable X and ω_1 *is strongly compact* if ω_1 is X-strongly compact for all uncountable X.

This paper explores aspects of compactness properties of ω_1 under $\mathsf{ZF} + \mathsf{DC}$. In particular, we focus on the consistency strength of the theories:

$$(\mathrm{P}) \equiv \text{“}\mathsf{ZF} + \mathsf{DC} + \omega_1 \text{ is supercompact”},$$

$$(\mathrm{Q}) \equiv \text{“}\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + \omega_1 \text{ is supercompact”}$$

and their variations. From here on, by $\mathsf{AD}_{\mathbb{R}}$, we always mean $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$. See Section 2 for basic terminology and facts about AD^+.

We note that "$\mathsf{ZF} + \omega_1$ is supercompact" implies DC (cf. [4]). We choose to be redundant here since we'll be using DC in many arguments to come. Also, (Q) is equivalent to "$\mathsf{AD}^+ + \omega_1$ is supercompact" by results in [19] and [21].

Woodin (unpublished) has shown that Con(P) and Con(Q) follows from Con(ZFC + there is a proper class of Woodin limits of Woodin cardinals). We conjecture that a (close to optimal) lower-bound consistency strength for the theory (P) is that of (Q) and is "ZFC + there is a Woodin limit of Woodin cardinals."

In the context of $\mathsf{ZF} + \mathsf{DC}$, the papers [16] and [18] study supercompact measures on $\wp_{\omega_1}(\mathbb{R})$ and show that the following theories are equiconsistent:

1. ZFC + there are ω^2 Woodin cardinals.
2. AD^+ + there is a supercompact measure on $\wp_{\omega_1}(\mathbb{R})$.
3. ZF + DC + $\Theta > \omega_2$ + there is a supercompact measure on $\wp_{\omega_1}(\mathbb{R})$.[1]

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[1] The equiconsistency of (1) and (2) is due to H.W. Woodin. The equiconsistency of (2) and (3) is due independently to H.W. Woodin and the author.

It is also well-known that the existence of a supercompact measure on $\wp_{\omega_1}(\mathbb{R})$ is equiconsistent with that of a measurable cardinal (see [18]). Recall that the existence of supercompact measures on $\wp_{\omega_1}(\mathbb{R})$ was first shown by Solovay [11] from $\mathsf{AD}_{\mathbb{R}}$. Consistency-wise, it is known that $\mathsf{AD}_{\mathbb{R}}$ is much stronger than (1) (and hence (2) and (3)).

Surprisingly, [19] shows that having a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$ is much stronger consistency-wise as it implies that there are models of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$. Solovay [11] shows that $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ is strictly stronger than $\mathsf{AD}_{\mathbb{R}}$ consistency-wise.

Theorem 1.3 (Trang-Wilson)**.** *Assume* $\mathsf{ZF} + \mathsf{DC}$. *Suppose there is a supercompact measure on* $\wp_{\omega_1}(\wp(\mathbb{R}))$. *Then there is a transitive* M *containing* $\mathbb{R} \cup \mathrm{OR} \subseteq M$ *such that* $M \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$.

[19] also shows the conclusion of Theorem 1.3 is equiconsistent with "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\wp(\mathbb{R})$-strongly compact". The main conjecture regarding compactness properties of ω_1 under $\mathsf{ZF} + \mathsf{DC}$ is.

Conjecture 1.4. *The following theories are equiconsistent.*

1. (P)
2. *"*$\mathsf{ZF} + \mathsf{DC} + \omega_1$ *is strongly compact"*

Conjecture 1.4's analogue in the ZFC context is perhaps more well-known. However, the above results (e.g. Theorem 1.3) and recent progress in inner model theory suggest that Conjecture 1.4 is more tractable.

Definition 1.5 (ZF+DC)**.** *Let* $\Theta = \sup(\{\alpha \mid \exists \pi : \mathbb{R} \to \alpha \wedge \pi \text{ is onto}\})$ *and* μ *be a measure on* Θ. *We say that* μ *is* ***uniform*** *if sets of the form* $(\alpha, \Theta), [\alpha, \Theta)$ *are in* μ *for all* $\alpha < \Theta$. *We say that* μ *is* $\mathbb{R}$***-complete*** *if* μ *is uniform, and whenever we have* $\langle A_x \mid x \in \mathbb{R} \wedge A_x \in \mu \rangle$ *then* $\bigcap_{x \in \mathbb{R}} A_x \neq \emptyset$.

Let

- $(\mathrm{T}_1) \equiv$ "$\mathsf{ZF} + \mathsf{DC} +$ there is a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R})) + \Theta$ is regular."
- $(\mathrm{T}_2) \equiv$ " $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} +$ there is a nonprincipal $\mathbb{R}$-complete measure on Θ".
- $(\mathrm{T}_3) \equiv$ "$\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} +$ there is a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R})) + \Theta$ is regular."

We will also say "Θ is measurable" in place of "there is a nonprincipal $\mathbb{R}$-complete measure on Θ." The main theorem of this paper is the following.

Theorem 1.6. *Con(*T_2*)* $\Leftrightarrow$ *Con(*T_3*).*

The proof that (T_2) implies (T_3) (and hence (T_1)) is given in [17] (note that by a standard argument, Θ is measurable implies Θ is regular).[2] By [17], we know that (T_2) implies the existence

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[2] Let μ witness Θ is measurable. Suppose Θ is singular. Then it is easy to see that there is a cofinal map $f : \mathbb{R} \to \Theta$. For each $x \in \mathbb{R}$, let $A_x = \langle \alpha < \Theta \mid \alpha \geq f(x) \rangle$. Clearly $A_x \in \mu$ for all $x \in \mathbb{R}$. Let $\alpha \in \bigcap_x A_x \neq \emptyset$. Then $\alpha \geq f(x)$ for all $x \in \mathbb{R}$. This contradicts the fact that f is cofinal.

of a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$, but we do not know the exact consistency strength of this theory. In this paper, we focus on the proof of $\text{Con}(T_3)$ implies $\text{Con}(T_2)$.

Recent developments in the core model induction techniques suggest that the use of AD^+ in the proof of Theorem 1.6 can be omitted. We conjecture the following.

Conjecture 1.7. *Con(T_1) $\Leftrightarrow$ Con(T_2)($\Leftrightarrow$ Con(T_3)). Furthermore, Con(P) implies Con(T_3).*

The outline of the paper is as follows. In Section 2, we summarize some basic facts about descriptive set theory and the theory of AD^+ that we use in this paper. Section 3 introduces the notion of hod mice that we will construct in this paper. Section 4 discusses a variation of the Vopenka algebra that is useful in constructing models of determinacy from hod mice (see Theorem 4.1). Section 5 gives the construction of a proper hod pair, which in turn will generate a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable" and hence completes the proof of Theorem 1.6.

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2. BASIC FACTS ABOUT AD^+

We start with the definition of Woodin's theory of AD^+. In this paper, we identify $\mathbb{R}$ with ω^ω. Recall Θ is the sup of ordinals α such that there is a surjection $\pi : \mathbb{R} \to \alpha$. Under AC, Θ is just the successor cardinal of the continuum. In the context of AD, Θ is shown to be the supremum of $w(A)$ for $A \subseteq \mathbb{R}$.[3] The definition of Θ relativizes to any determined pointclass[4] (with sufficient closure properties). For a pointclass Γ, we denote Θ for the sup of α such that there is a surjection from $\mathbb{R}$ onto α coded by a set of reals in Γ.

Recall that AD_X is determinacy for games in which player I and II take turns to play elements of X for ω many rounds. If $X = \omega$, then $\mathsf{AD}_X = \mathsf{AD}$.

Definition 2.1. AD^+ *is the theory* $\mathsf{ZF} + \mathsf{AD}$+$\mathsf{DC}_{\mathbb{R}}$ *and*

1. *for every set of reals A, there are a set of ordinals S and a formula φ such that $x \in A \Leftrightarrow L[S, x] \vDash \varphi[S, x]$. (S, φ) is called an ∞-Borel code for A;*
2. *for every $\lambda < \Theta$, for every continuous $\pi : \lambda^\omega \to \omega^\omega$, for every $A \subseteq \mathbb{R}$, the set $\pi^{-1}[A]$ is determined.*

AD^+ is equivalent to "AD + the set of Suslin cardinals is closed". Another, perhaps more useful, characterization of AD^+ is "AD+Σ_1 statements reflect into the Suslin co-Suslin sets" (see [15] for

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[3] $w(A)$ is the Wadge rank of A.

[4] See [21] for more backgrounds on descriptive set theory in contexts where determinacy only holds locally.

the precise statement). Recall, our convention is $\mathsf{AD}_{\mathbb{R}}$ is the principle $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$.

Let $A \subseteq \mathbb{R}$, we let θ_A be the supremum of all α such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto α. If Γ is a determined (boldface) pointclass, and $A \in \Gamma$, we write $\Gamma \restriction A$ for the set of $B \in \Gamma$ which is Wadge reducible to A. If $\alpha < \Theta$, we write $\Gamma \restriction \alpha$ for the set of $A \in \Gamma$ with Wadge rank strictly less than α. Occasionally, we will write Γ for a ω-parameterized (lightface) pointclass and write $\underset{\sim}{\Gamma}$ for its corresponding boldface pointclass. We write $\underset{\sim}{\Delta}_{\underset{\sim}{\Omega}}$ for the ambiguous part of the boldface pointclass $\underset{\sim}{\Omega}$, that is $\underset{\sim}{\Delta}_{\underset{\sim}{\Omega}}$ is the collection of A such that both A and $\mathbb{R}\backslash A$ are in $\underset{\sim}{\Omega}$.

Definition 2.2 (AD^+)**.** *The* ***Solovay sequence*** *is the sequence $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ where*

1. *θ_0 is the supremum of ordinals β such that there is an OD surjection from $\mathbb{R}$ onto β;*
2. *$\theta_\Omega = \Theta$;*
3. *if $\alpha > 0$ is limit, then $\theta_\alpha = \sup\{\theta_\beta \mid \beta < \alpha\}$;*
4. *if $\alpha = \beta + 1$ and $\theta_\beta < \Theta$ (i.e. $\beta < \Omega$), fixing a set $A \subseteq \mathbb{R}$ of Wadge rank θ_β, θ_α is the sup of ordinals γ such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto γ, i.e. $\theta_\alpha = \theta_A$.*

Note that the definition of θ_α for $\alpha = \beta + 1$ in Definition 2.2 does not depend on the choice of A. The Solovay sequence is a club set in Θ. Roughly speaking the longer the Solovay sequence is, the stronger the associated AD^+-theory is. For instance the theory $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ is strictly stronger than $\mathsf{AD}_{\mathbb{R}}$ since by [11], DC implies $\mathrm{cof}(\Theta) > \omega$ while the minimal model of $\mathsf{AD}_{\mathbb{R}}$ satisfies $\Theta = \theta_\omega$ ($\mathsf{AD}_{\mathbb{R}}$ implies that the Solovay sequence has limit length). $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular is stronger still as it implies the existence of many models of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

Definition 2.3. *"$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable" is the theory "$\mathsf{AD}_{\mathbb{R}}+$ there is a nonprincipal $\mathbb{R}$-complete measure on Θ".*

It's easy to see that "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable" implies "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular"; in fact, there are unboundedly many $\theta_\alpha < \Theta$ such that $L(\wp(\mathbb{R}) \restriction \theta_\alpha, \mathbb{R}) \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular".

We end this section with a theorem of Woodin, which produces models with Woodin cardinals in AD^+.

Theorem 2.4 (Woodin, see [6])**.** *Assume AD^+. Let $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ be the Solovay sequence. Suppose $\alpha = 0$ or $\alpha = \beta + 1$ for some $\beta < \Omega$. Then $\mathrm{HOD} \vDash \theta_\alpha$ is Woodin.*

3. A BRIEF INTRODUCTION TO HOD MICE

In this paper, *a hod premouse $\mathcal{P}$* is one defined as in [8] and [10]. The reader is advised to consult [8] for basic results and notations concerning hod premice and hod mice at the level of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" and [10] for hod mice beyond this.[5] Let us mention some basic first-order properties of a hod premouse $\mathcal{P}$. There are an ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle (\mathcal{P}(\alpha), \Sigma^{\mathcal{P}}_\alpha) \mid \alpha < \lambda^{\mathcal{P}} \rangle$ and $\langle \delta^{\mathcal{P}}_\alpha \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ such that

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[5] We will not deal with short-tree strategy mice in this paper. This is because the hod mice we are constructing is well below the level of lsa hod mice, whose theory is developed in full detail in [10].

1. $\langle \delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \vDash \delta_\alpha^{\mathcal{P}}$ is Woodin;

2. $\mathcal{P}(0) = Lp_\omega(\mathcal{P}|\delta_0)^{\mathcal{P}}$; for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1) = (Lp_\omega^{\Sigma_\alpha^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp_\omega^{\oplus_{\beta<\alpha}\Sigma_\beta^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$;

3. $\mathcal{P} \vDash \Sigma_\alpha^{\mathcal{P}}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$[6]-strategy for $\mathcal{P}(\alpha)$ with hull condensation;

4. if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma_\beta^{\mathcal{P}}$ extends $\Sigma_\alpha^{\mathcal{P}}$.

We will write $\delta^{\mathcal{P}}$ for $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}} = \oplus_{\beta<\lambda^{\mathcal{P}}}\Sigma_\beta^{\mathcal{P}}$. Note that $\mathcal{P}(0)$ is a pure extender model. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ if there is $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We say then that $\mathcal{P}$ is a *hod initial segment* of $\mathcal{Q}$. $(\mathcal{P}, \Sigma)$ is a *hod pair* if $\mathcal{P}$ is a hod premouse and Σ is a strategy for $\mathcal{P}$ (acting on countable stacks of countable normal trees) such that $\Sigma^{\mathcal{P}} \subseteq \Sigma$ and this fact is preserved under Σ-iterations. Typically, we will construct hod pairs $(\mathcal{P}, \Sigma)$ such that Σ has hull condensation, branch condensation, and is Γ-fullness preserving for some pointclass Γ. As a matter of notation, if $(\mathcal{P}, \Sigma)$ is a hod pair and $\mathcal{Q} \triangleleft_{hod} \mathcal{P}$, then $\Sigma_{\mathcal{Q}}$ is Σ restricted to stacks on $\mathcal{Q}$. Also, note that when $\mathcal{Q} = \mathcal{P}(\alpha)$, then $\Sigma_{\mathcal{Q}} = \Sigma_{\mathcal{P}(\alpha)}$ is an extension of the internal strategy $\Sigma_\alpha^{\mathcal{P}}$.

Suppose $(\mathcal{Q}, \Sigma)$ is a hod pair such that Σ has hull condensation. $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$-hod premouse if there are ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle (\mathcal{P}(\alpha), \Sigma_\alpha^{\mathcal{P}}) \mid \alpha < \lambda^{\mathcal{P}} \rangle$ and $\langle \delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ such that

1. $\langle \delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \vDash \delta_\alpha^{\mathcal{P}}$ is Woodin;

2. $\mathcal{P}(0) = Lp_\omega^{\Sigma}(\mathcal{P}|\delta_0)^{\mathcal{P}}$ (so $\mathcal{P}(0)$ is a Σ-premouse built over $\mathcal{Q}$); for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1) = (Lp_\omega^{\Sigma\oplus\Sigma_\alpha^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp_\omega^{\Sigma\oplus_{\beta<\alpha}\Sigma_\beta^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$;

3. $\mathcal{P} \vDash \Sigma \cap \mathcal{P}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$strategy for $\mathcal{Q}$ with hull condensation;

4. $\mathcal{P} \vDash \Sigma_\alpha^{\mathcal{P}}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$strategy for $\mathcal{P}(\alpha)$ with hull condensation;

5. if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma_\beta^{\mathcal{P}}$ extends $\Sigma_\alpha^{\mathcal{P}}$.

Inside $\mathcal{P}$, the strategies $\Sigma_\alpha^{\mathcal{P}}$ act on stacks above $\mathcal{Q}$ and every Σ_α^{P} iterate is a Σ-premouse. Again, we write $\delta^{\mathcal{P}}$ for $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}} = \oplus_{\beta<\lambda^{\mathcal{P}}}\Sigma_\beta^{\mathcal{P}}$. $(\mathcal{P}, \Lambda)$ is a $(\mathcal{Q}, \Sigma)$-hod pair if $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$-hod premouse and Λ is a strategy for $\mathcal{P}$ such that $\Sigma^{P} \subseteq \Lambda$ and this fact is preserved under Λ-iterations. The reader should consult [8] for the definition of $B(\mathcal{Q}, \Sigma)$, and $I(\mathcal{Q}, \Sigma)$. Roughly speaking, $B(\mathcal{Q}, \Sigma)$ is the collection of all hod pairs which are strict hod initial segments of a Σ-iterate of $\mathcal{Q}$ and $I(\mathcal{Q}, \Sigma)$ is the collection of all Σ-iterates of Σ. In the case $\lambda^{\mathcal{Q}}$ is limit, $\Gamma(\mathcal{Q}, \Sigma)$ is the collection of $A \subseteq \mathbb{R}$ such that A is Wadge reducible to some Ψ for which there is some $\mathcal{R}$ such that $(\mathcal{R}, \Psi) \in B(\mathcal{Q}, \Sigma)$. See [8] for the definition of $\Gamma(\mathcal{Q}, \Sigma)$ in the case $\lambda^{\mathcal{Q}}$ is a successor ordinal.

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[6] This just means $\Sigma_\alpha^{\mathcal{P}}$ acts on all stacks of ω-maximal, normal trees in $\mathcal{P}$.

[8] constructs under AD^+ and the hypothesis that there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" hod pairs that are fullness preserving, positional, commuting, and have branch condensation.[7] Such hod pairs are particularly important for our computation as they are points in the direct limit system giving rise to HOD of AD^+ models. For hod pairs $(\mathcal{M}_\Sigma, \Sigma)$, if Σ is a strategy with branch condensation and $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}_\Sigma$ with last model $\mathcal{N}$ (we will denote this model $\mathcal{N}^{\mathcal{T}}$), $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ is independent of $\vec{\mathcal{T}}$ (this property is called *positionality*). Therefore, later on we will omit the subscript $\vec{\mathcal{T}}$ from $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ whenever Σ is a strategy with branch condensation and $\mathcal{M}_\Sigma$ is a hod mouse. We also let $\alpha(\vec{\mathcal{T}})$ denote the supremum of the generators used in $\vec{\mathcal{T}}$.

Suppose AD^+ holds. We fix a simple coding of H_{ω_1} by elements of $\mathbb{R}$. For an (ω_1, ω_1) iteration strategy Λ, we let $\mathrm{Code}(\Lambda)$ be the set of reals coding Λ via the specified coding.[8] Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and is Γ-fullness preserving for some pointclass Γ and suppose $\mathrm{Code}(\Sigma)$ is Suslin co-Suslin, then [8, Corollary 2.44] shows that Σ is positional and commuting. We can then compute the direct limit $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ of all Σ-iterates of $\mathcal{P}$.

In practice (in determinacy models where the HOD analysis can be carried out or in core model induction contexts) we construct hod pairs $(\mathcal{P}, \Sigma)$ such that Σ has branch condensation and is Γ-fullness preserving for some pointclass Γ (if $\Gamma = \wp(\mathbb{R})$ then we simply say "fullness preserving"). In core model induction applications, we construct hod pairs $(\mathcal{P}, \Sigma)$ such that every $(\mathcal{R}, \Lambda) \in B(\mathcal{P}, \Sigma)$ belongs to an AD^+-model. We then can show (using our hypothesis) that the hod pair $(\mathcal{P}, \Sigma)$ we construct belongs to an AD^+-model.

In this paper, $\mathcal{P}$ is a hod premouse if

(i) either $\mathcal{P}$ is a hod premouse below "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable", that is, no hod initial segment $\mathcal{Q}$ of $\mathcal{P}$ satisfies "$\delta^{\mathcal{Q}}$ is a measurable limit of Woodin cardinals" ($\mathcal{P}$ is called *improper* in this case),

(ii) or $\mathcal{P} = (\mathcal{P}^-, E)$ where $\mathcal{P}^-$ is improper hod premouse (or anomalous hod premouse, cf. [8, Section 3.4]), $\mathcal{P} \vDash$ "$\delta^{\mathcal{P}}$ is regular" and E codes (as an amenable predicate) a normal measure over $\mathcal{P}$ with critical point $\delta^{\mathcal{P}}$ ($\mathcal{P}$ is called *proper* in this case).

Suppose $\mathcal{P}$ is a proper hod premouse and suppose Σ is some iteration strategy of $\mathcal{P}$. Suppose $\vec{\mathcal{T}}$ is a stack according to Σ. It's easy to see that $\vec{\mathcal{T}}$ can be decomposed into a sequence of stacks $(\mathcal{T}_\alpha, \mathcal{N}_\alpha : \alpha < \gamma)$ for some γ, where

1. $\mathcal{N}_0 = \mathcal{P} = (\mathcal{N}_0^-, E_0)$, $\mathcal{N}_{\alpha+1}$ is the last model of $\mathcal{T}_\alpha$, and for limit α, $\mathcal{N}_\alpha$ is the direct limit (under the iteration maps) of the $\mathcal{N}_\beta$'s for $\beta < \alpha$;
2. for $\alpha < \gamma - 1$ successor, say $\mathcal{N}_\alpha = (\mathcal{N}_\alpha^-, E_\alpha)$. Then $\mathcal{T}_{\alpha+1}$ is either a stack below $\delta^{\mathcal{N}_\alpha}$ (if $\mathcal{T}_\alpha = \langle \mathcal{N}_{\alpha-1}^-, E_{\alpha-1} \rangle$) or else $\mathcal{T}_{\alpha+1} = \langle \mathcal{N}_\alpha^-, E_\alpha \rangle$.

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[7] Branch condensation does not seem to follow from hull condensation and vice versa. By [8, Theorem 2.42], fullness preserving strategies with branch condensation are positional and hence commuting. In short, we can just write "hod pairs that are fullness preserving and have branch condensation".

[8] Let $\pi : \mathbb{R} \to H_{\omega_1}$ be the coding of elements of H_{ω_1} by elements of $\mathbb{R}$. Then π induces a surjection Code: $\wp(\mathbb{R}) \to \wp(H_{\omega_1})$ as mentioned above. To save space, we will generally not make distinction between Λ and $\mathrm{Code}(\Lambda)$ in this paper.

3. for $\alpha = 0$ or limit, $\mathcal{T}_\alpha$ is either a stack on $\mathcal{N}_\alpha$ below $\mathcal{N}_\alpha$ or else $\mathcal{T}_\alpha = \langle \mathcal{N}_\alpha^-, E_\alpha \rangle$;

Such a sequence is called *the normal form* of $\vec{\mathcal{T}}$. Informally, a stack in normal form on $\mathcal{P}$ consists of stacks below $\delta^{\mathcal{P}}$ and its images and trees of the form $\langle F \rangle$ where F is the predicate coding the normal measure over $\mathcal{R}$ with critical point $\delta^{\mathcal{R}}$. For instance, if $\mathcal{T}_0 = \langle E_0 \rangle$, then $\mathcal{N}_1 = Ult(\mathcal{P}, E_0)$. In constructing a strategy Σ for $\mathcal{P}$, we need to construct strategies for the "new Woodin cardinals" of $\mathcal{N}_1$ (i.e. those Woodin cardinals between $\delta^{\mathcal{P}}$ and $\pi_{E_0}(\delta^{\mathcal{P}})$), cf. the proof of Lemma 5.16.

4. A VOPENKA FORCING

In this section, we prove a theorem concerning a variation of the Vopenka algebra. This theorem will play an important role in the next section. Suppose Γ is such that $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$ and $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$. Let $\mathcal{H}$ be $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$. Woodin has shown that $\mathcal{H} = L[A]$ for some $A \subseteq \Theta$ (see [20]). We write Θ for $\Theta^{L(\Gamma,\mathbb{R})}$. The following theorem comes from many conversations between H.W. Woodin and the author and is due to Woodin. We include a proof here for the reader's convenience. A similar, but less general theorem and its proof can be found in [1]. We note that the version in [1] is enough for our applications in this paper. The more general version as stated in Theorem 4.1 will have applications elsewhere.

Theorem 4.1. *Suppose* $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$ *and* $\mathcal{H} = HOD^{L(\Gamma,\mathbb{R})}$. *Let* $\mathcal{H}^+$ *be a* ZFC *model such that* $A \in \mathcal{H}^+$ *and* $V_\Theta^{\mathcal{H}} = V_\Theta^{\mathcal{H}^+}$, *where* $A \subseteq \Theta$ *is such that* $\mathcal{H} = L[A]$. *There is a forcing* $\mathbb{P} \in \mathcal{H}$ *and a* $h \subseteq \mathbb{P}$ *generic over* $\mathcal{H}^+$ *such that in* $\mathcal{H}^+[h]$*:*

$$\wp(\mathbb{R}) \cap \mathcal{H}^+(\Gamma) = \wp(\mathbb{R}) \cap \mathcal{H}(\Gamma) = \Gamma.[9]$$

In particular, $\mathcal{H}^+(\Gamma) \vDash \mathsf{AD}_{\mathbb{R}}$.

Remark 4.2. $\mathcal{H}^+(\Gamma)$ *can be realized as a certain kind of symmetric model in* $\mathcal{H}^+[h]$*; a similar remark applied to* $\mathcal{H}(\Gamma)$. *The symmetricity is with respect to a certain class of order-preserving maps from* $\mathbb{P}$ *to* $\mathbb{P}$ *specified in Lemma 4.3.*

Proof. First, we define a forcing $\mathbb{Q} \in L(\Gamma, \mathbb{R})$. Let $Z = \wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$, where $\wp_\Theta(\Theta)$ is the collection of bounded subsets of Θ. A condition $q \in \mathbb{Q}$ if $q : n_q \to Z$ for some $n_q < \omega$. The ordering $\leq_{\mathbb{Q}}$ is as follows:

$$q \leq_{\mathbb{Q}} r \Leftrightarrow n_r \leq n_q \wedge \forall i < n_r \; q(i) = r(i).$$

So $\mathbb{Q}$ is simply the Levy collapse forcing $Col(\omega, Z)$. Now we define

$$\mathbb{P}^* = \{A \mid \exists n < \omega \; A \subseteq Z^n \wedge A \in OD^{L(\Gamma,\mathbb{R})} \wedge \text{there is a surjection } \pi : \mathbb{R} \to A\}.$$

For $A \in \mathbb{P}^*$, we let n_A be the unique $n < \omega$ such that $A \subseteq Z^n$. The ordering $\leq_{\mathbb{P}^*}$ is defined as follows:

$$A \leq_{\mathbb{P}^*} B \Leftrightarrow n_B \leq n_A \wedge \forall \, t \in A \; t \restriction n_B \in B.$$

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[9] Here $\mathcal{H}^+(\Gamma)$ is the minimal, transitive ZF model containing $\mathcal{H}^+$ and Γ. $\mathcal{H}(\Gamma)$ is defined similarly.

It's easy to see that there is a partial order $(\mathbb{P}, \leq_{\mathbb{P}}) \in \mathcal{H}$ isomorphic to $(\mathbb{P}^*, \leq_{\mathbb{P}^*})$ and in $\mathcal{H}$, $(\mathbb{P}, \leq_{\mathbb{P}})$ has size Θ. Let $\pi : (\mathbb{P}, \leq_{\mathbb{P}}) \to (\mathbb{P}^*, \leq_{\mathbb{P}^*})$ be the isomorphism and π is $OD^{L(\Gamma,\mathbb{R})}$. We will write p^* for $\pi(p)$, where $p \in \mathbb{P}$. $(\mathbb{P}, \leq_{\mathbb{P}})$ is the direct limit of the directed system of complete boolean algebras $\mathbb{P}_n$ in $\mathcal{H}$, where $\mathbb{P}^*_n$ is the "n-dimensional" Vopenka algebra on Z^n and for $n \leq m$, the natural maps $\tau_{n,m}$ from $\mathbb{P}_n$ into $\mathbb{P}_m$ defined as: $\tau_{n,m}(p) = \{t \in Z^m : t \restriction n \in p\}$ are complete embeddings.

$\mathbb{Q}$ is *weakly homogeneous* in the sense that for any $p, q \in \mathbb{Q}$, there is an automorphism $\pi : \mathbb{Q} \to \mathbb{Q}$ such that $\pi(p)$ is compatible with q. In the following, we show that $\mathbb{P}^*$ (and hence $\mathbb{P}$) is fairly closed to being weakly homogeneous.

Lemma 4.3. *Let $p, q \in \mathbb{P}^*$. Let $\mathbb{P}^*_{n_p,n_q} = \{r \in \mathbb{P}^* \mid n_r \geq n_p + n_q\}$. Then there is a map $\pi : \mathbb{P}^* \to \mathbb{P}^*$ such that rng(π) is dense in $\mathbb{P}^*$, $\pi \restriction \mathbb{P}^*_{n_p,n_q}$ is an automorphism of $\mathbb{P}^*_{n_p,n_q}$, and $\pi(p)$ is compatible with q.*

Proof. First, we define a "finite permutation" $\sigma : \omega \to \omega$ as follows.

$$\sigma(n) = \begin{cases} n + n_q & \text{if } n = 0, 1, \ldots, n_p - 1 \\ n - n_p & \text{if } n = n_p, n_p + 1, \ldots, n_p + n_q - 1 \\ n & \text{otherwise} \end{cases} \tag{4.1}$$

Now we proceed to define π. For any $t \in Z^{<\omega}$, for any $n < m < n_r$, by $t \restriction [n, m]$, we mean $\langle t(n), \ldots, t(m)\rangle$; we can define $t \restriction [n, m)$ etc. For any $r \in \mathbb{P}$ such that $n_r < n_p + n_q$, let $r^* = \{t \in Z^{n_p+n_q} : t \restriction n_r \in r\}$; for $r \in \mathbb{P}$ such that $n_r \geq n_p + n_q$, let $r^* = r$. Now let

$$\pi(r) = \{t \circ \sigma : t \in r^*\},$$

where

$$\begin{aligned} t \circ \sigma \restriction [0, n_p + n_q) &= \langle t(\sigma(0)), t(\sigma(1)), \ldots, t(\sigma(n_p + n_q - 1))\rangle \\ &= \langle t(n_q), t(n_q + 1), \ldots, t(n_q + n_p - 1), t(0), \ldots, t(n_p - 1)\rangle, \end{aligned}$$

and if $n_t > n_p + n_q$, then $t \circ \sigma \restriction [n_p + n_q, n_t) = t \restriction [n_p + n_q, n_t)$.

So π permutes the first $n_p + n_q$ coordinates of every $t \in r^*$ for any $r \in \mathbb{P}$ according to σ and does not change coordinates $> n_p + n_q$ (this corresponds to σ being identity above $n_p + n_q$). It is easy to see that π is $\leq_{\mathbb{P}^*}$ order-preserving, is an automorphism of $\mathbb{P}^*_{n_p,n_q}$, and rng(π) is dense in $\mathbb{P}^*$.

Now

$$\pi(p) = \{t \in Z^{n_p+n_q} : t \restriction [n_q, n_p + n_q) \in p\}$$

is compatible with q because $r \leq \pi(p)$ and $r \leq q$, where

$$r = \{t \in Z^{n_p+n_q} : t \restriction [0, n_q - 1] \in q \wedge t \restriction [n_q, n_q + n_p) \in p\}.$$

This completes the proof of the lemma. $\square$

Now let $g^* \subseteq \mathbb{Q}$ be $L(\Gamma, \mathbb{R})$-generic and $g = \bigcup g^*$. By density, $g : \omega \to Z$ is onto. Let $h \subseteq \mathbb{P}$ be defined as follows:

$$p \in h \Leftrightarrow (g \restriction n_{p^*}) \in p^*. \tag{4.2}$$

Also, if $p \in \mathbb{P}$, by n_p, we mean n_{p^*}. The term "symmetric" will be spelled out in during the course of the proof of Lemma 4.4.

Lemma 4.4. *Write h_g for the filter h above. Then the following hold.*

(a) h_g is $\mathbb{P}$-generic over $\mathcal{H}$. In fact, for any condition $p \in \mathbb{P}$, there is a $\mathbb{P}$-generic filter h over $\mathcal{H}$ such that $p \in h$ and $\Gamma \in \mathcal{H}[h]$. Furthermore, $\mathcal{H}(\Gamma)$ is the symmetric extension of $\mathcal{H}$ in $\mathcal{H}[h]$.

(b) Suppose g^* is $L(\mathcal{H}^+, Z)$-generic, then for any $p \in \mathbb{P}$, there is a $\mathbb{P}$-generic h over $\mathcal{H}^+$ such that $p \in h$ and $\Gamma \in \mathcal{H}^+[h]$. Furthermore, $\mathcal{H}^+(\Gamma)$ is the symmetric extension of $\mathcal{H}^+$ in $\mathcal{H}^+[h]$.

Proof. For part (a), to see h_g is generic for $\mathbb{P}$ over $\mathcal{H}$, consider a dense set $D \subseteq \mathbb{P}^*$ which is OD. Let $D' = \bigcup D$. Then D' is dense in $\mathbb{Q}$. Otherwise there would exist a condition $q \in \mathbb{Q}$ which does not extend to a condition in D'. Let

$$p = \{q' \in \mathbb{Q} : n_{q'} = n_q \text{ and } q' \text{ does not extend to a condition in } D'\}$$

then $p \in \mathbb{P}^*$; here p is nonempty as $q \in p$. By density of D we can find some $p' \in D$ extending p. Then any condition $q'' \in p'$ is an extension of a condition in p (namely of $q'' \restriction n_q$) to a condition in D', a contradiction. This proves density of D' in $\mathbb{Q}$. It is now easy to see that if $q^* \in g \cap D'$ then $q^* \in p'$ for some $p' \in D$, witnessing that $p' \in D \cap h$.[10]

In fact, we just proved that given an open dense set $D \subseteq \mathbb{P}$ in $\mathcal{H}$, for any condition $p \in \mathbb{Q}$, there is a $q \leq_{\mathbb{Q}} p$ such that $q \Vdash_{\mathbb{Q}} \dot{h} \cap \check{D} \neq \emptyset$.

Given g and h_g as above, we also can define g from h_g in a simple way. Let $b \in \Theta$ and $n < \omega$. Let $A_{b,n} \in \mathbb{P}$ be such that $A^*_{b,n} = \{s \in Z^{n+1} : b \in s(n)\}$; it is clear that $A^*_{b,n} \in OD$. We take the map $(b, n) \mapsto A_{b,n}$ to be in $\mathcal{H}$. Clearly,

$$b \in g(n) \Leftrightarrow A_{b,n} \in h_g. \tag{4.3}$$

We then can define $\mathbb{P}$-terms for $g(n)$ and $ran(g)$ by

$$\sigma_n = \{\langle p, \check{b}\rangle \mid b < \Theta \wedge p \leq_{\mathbb{P}} A_{b,n}\},$$

and

$$\dot{R} = \{\langle p, \sigma_n\rangle \mid p \in \mathbb{P} \wedge n < \omega\}.$$

Note that $\sigma_n \in \mathcal{H}$ for all n and $\dot{R} \in \mathcal{H}$. The following properties are easy to verify.

Lemma 4.5. *1. For any $g^* \subseteq \mathbb{Q}$ generic over $L(\Gamma, \mathbb{R})$, let $g = \bigcup g^*$ and h_g be defined as in 4.2, then $\sigma_n^{h_g} = g(n)$ for all n and $\dot{R}^{h_g} = ran(g) = Z$.*

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[10] This argument is pointed out by the referee. It is simpler than the author's original argument

2. *For any condition $p \in \mathbb{P}$, there is an $\mathcal{H}$-generic h such that $p \in h$ and $\dot{R}^h = Z$.*

3. *For any finite permutation σ of ω, let π be defined as in Lemma 4.3 from σ. Then $g_\pi =_{def} \pi[g], h_\pi =_{def} \pi[h]$ are $\mathbb{Q}$-generic and $\mathbb{P}$-generic respectively and $\mathcal{H}[h] = \mathcal{H}[h_\pi]$ and $\mathcal{H}[g] = \mathcal{H}[g_\pi]$. Furthermore, letting π^* be the canonical extension of π to $\mathbb{P}$-terms $\dot{R}^h = \pi^*(\dot{R})^h$.*[11]

Remark 4.6. *$\dot{R}$ is "symmetric" with respect to the maps π as in clause 3 of the lemma. We call the models $\mathcal{H}(\Gamma), \mathcal{H}^+(\Gamma)$ symmetric models because they will be shown to be $\mathcal{H}(\dot{R}^h), \mathcal{H}^+(\dot{R}^h)$ respectively for appropriate generics h. It is not true in general that $\pi^*(\sigma_n) = \sigma_n$, but nevertheless, $\{\pi^*(\sigma_n)^h : n < \omega\} \supseteq \{\sigma_n^h : n < \omega\}$; one can see from this that $\pi^*(\dot{R})^h = \dot{R}^h$.*

We can now show that $L(\Gamma, \mathbb{R})$ can be recovered over $\mathcal{H}$ from Z (via the standard Vopenka forcing). This is because for any $A \in \Gamma$:

(i) A has an ∞-Borel code $S \in Z$, and

(ii) S is generic over $\mathcal{H}$ via a forcing of size $< \Theta$.

Both (i) and (ii) follow from $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$ in $L(\Gamma, \mathbb{R})$. For (ii), the forcing is just the standard Vopenka forcing. Suppose $S \subseteq \kappa$ for some $\kappa < \theta_\alpha$, where $\theta_\alpha < \Theta$ is a member of the Solovay sequence of $L(\Gamma, \mathbb{R})$, then by $\mathsf{AD}_{\mathbb{R}}$, the standard Vopenka forcing $\mathbb{P}_0$ adding a subset of κ has size at most θ_α in $\mathcal{H}$. Furthermore, $\mathbb{P}_0$ completely embeds into $\mathbb{P}$ and there is $\mathbb{P}_1$ such that $\mathbb{P} = \mathbb{P}_0 \star \mathbb{P}_1$. [12]

So there is a formula φ such that given any real x, $\mathcal{H}[S][x] \vDash \varphi[S, x]$ if and only if $x \in A$. [13] This equivalence can be computed in $\mathcal{H}[h]$ from $\mathcal{H}$ and $\dot{R}^h$ for any $\mathcal{H}$-generic h such that $\dot{R}^h = Z$. This shows that $\Gamma \in \mathcal{H}[h]$ for any h satisfying (2) of Lemma 4.5. For any such h, we define the symmetric model $\mathcal{S}_{\mathcal{H},h}$ as

$$\mathcal{S}_{\mathcal{H},h} = HOD^{\{\mathcal{H}[h],\mathcal{H}\}}_{\{g \restriction n : n < \omega\}}.$$

Note that $g \restriction n$ is the sequence of $\langle \sigma_0^h, \ldots, \sigma_{n-1}^h \rangle$ in $\mathcal{H}[h]$. We also define

$$h \restriction n = \{p \in h : n_p \leq n\}.$$

In the following, by HOD_x, we mean $HOD_x^{L(\Gamma,\mathbb{R})}$. Let $G(g \restriction n) \subseteq \mathbb{P}(g \restriction n)$ be the generic for the Vopenka algebra adding $g \restriction n$ over $\mathcal{H}$. Note that $\mathcal{H}[h \restriction n]$ and $\mathcal{H}[G(g \restriction n)]$ may differ from $\mathcal{H}[g \restriction n]$[14] but we do have

Lemma 4.7. $\mathcal{H}[h \restriction n] = \mathcal{H}[G(g \restriction n)] = HOD_{\{g \restriction n\}}$.

Proof. Using the equivalence

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[11] g and g_π only differ on finitely many bits, and similarly for h and h_π. Also, in general, $\pi^*(\dot{R}) \neq \dot{R}$ and $\pi^*(\sigma_n) \neq \sigma_n$ for most maps π.

[12] See [7] for a similar observation regarding the ω-dimensional forcing realizing $L(\mathbb{R})$ as a symmetric model over $HOD^{L(\mathbb{R})}$.

[13] We in fact can take S to be in Θ^ω; this is a consequence of $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$.

[14] A proof of the equality seems to require that every OD subset of Z^n has an OD ∞-Borel code. See [7] for the corresponding fact that every OD subset of $\mathbb{R}^n$ has OD ∞-Borel code in $L(\mathbb{R})$.

$$p \in h \restriction n \Leftrightarrow g \restriction n \in p^*,$$

we get that $h \restriction n$ is $OD_{\{g\restriction n\}}$. Hence $\mathcal{H}[h \restriction n] \subseteq HOD_{\{g\restriction n\}}$. A similar argument gives $\mathcal{H}[G(g \restriction n)] \subseteq HOD_{\{g\restriction n\}}$

Conversely, $g \restriction n \in HOD[h \restriction n]$ follows from 4.3, noting that we just need $h \restriction n$ in that equivalence to compute $g \restriction n$. Similarly, $g \restriction n \in HOD[G(g \restriction n)]$. Let X be a set of ordinals in $HOD_{\{g\restriction n\}}$. Say $X \subseteq \gamma$. Let $T \in OD$ be such that for any $\beta < \gamma$,

$$\beta \in X \Leftrightarrow T(\beta, g \restriction n) \text{ holds in } L(\Gamma, \mathbb{R}).$$

Let $\kappa = \max_{i<n}\sup[g(i)]$. Let $\tau : OD \cap \wp([\wp(\kappa)]^n) \to \mathcal{H}$ be the (OD) natural map.

Let $T^*_\beta = \{a \subseteq \kappa^n : T(\beta, a)\}$. Then $Y = \{(\beta, \tau(T^*_\beta)) : \beta < \gamma\} \in \mathcal{H}$ and it's easily checked that

$$\beta \in X \Leftrightarrow g \restriction n \in T^*_\beta \Leftrightarrow (\beta, \tau(T^*_\beta)) \in Y \wedge \tau(T^*_\beta) \in h \restriction n.$$

So $X \in \mathcal{H}[h \restriction n]$. Similarly, $X \in \mathcal{H}[G(g \restriction n)]$. This completes the proof of Lemma 4.7. □

The above calculations show that $\Gamma \in \mathcal{H}(Z)$ and in fact

$$\mathcal{S}_{\mathcal{H},h} = \mathcal{H}(Z) = \mathcal{H}(\Gamma) = L(\Gamma, \mathbb{R}). \tag{4.4}$$

We first verify $\mathcal{S}_{\mathcal{H},h} = \mathcal{H}(Z)$. First note that $Z = \dot{R}^h \in \mathcal{S}_{\mathcal{H},h}$ and $\mathcal{H}$ is an inner model of $\mathcal{S}_{\mathcal{H},h}$, so the $\supseteq$-direction holds. For the converse, let $X \in \mathcal{S}_{\mathcal{H},h}$ be a set of ordinals.

Claim 4.8. *$X \in \mathcal{H}[h \restriction k]$ for some k.*

Proof. Suppose X is defined in $\mathcal{H}[h]$ from $g \restriction n$ for some n by a formula φ. We omit the ordinal parameters for brevity. So for any ordinal α,

$$\alpha \in X \Leftrightarrow \mathcal{H}[h] \vDash \varphi[\alpha, g \restriction n].$$

By Lemma 4.7, $g \restriction n \in \mathcal{H}[h \restriction n]$.

By the discussion above, the canonical Vopenka algebra for $g \restriction n$, $\mathbb{P}(g \restriction n)$ completely embeds into $\mathbb{P}$. Let $G(g \restriction n) \subset \mathbb{P}(g \restriction n)$ be the generic that adds $g \restriction n$ and let $\mathbb{P}/G(g \restriction n)$ be the factor forcing induced by $G(g \restriction n)$, then by Lemma 4.7, we have $G(g \restriction n) \in \mathcal{H}[h \restriction n] = HOD_{\{g\restriction n\}} = \mathcal{H}[G(g \restriction n)]$. Then

$$\alpha \in X \Leftrightarrow \mathcal{H}[G(g \restriction n)] = \mathcal{H}[h \restriction n] \vDash \emptyset \Vdash_{\mathbb{P}/G(g\restriction n)} \varphi[\check{\alpha}, g \restriction n].$$[15]

This gives $X \in \mathcal{H}[h \restriction n]$ as desired.

□

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[15] We use the maps π_{n_p,n_q} as in Lemma 4.3 to get that for any two conditions p, q, it cannot be the case that $p \Vdash_{\mathbb{P}/G(g\restriction n)} \varphi[\check{\alpha}, g \restriction n]$ and $q \Vdash_{\mathbb{P}/G(g\restriction n)} \neg\varphi[\check{\alpha}, g \restriction n]$ and vice versa.

Since for each n, $g \restriction n \in \mathcal{H}(Z)$ and $G(g \restriction n) \in \mathcal{H}(Z)$, and $\mathcal{H}[G(g \restriction n)] = \mathcal{H}[h \restriction n]$, we get $h \restriction n \in \mathcal{H}(Z)$; therefore, $X \in \mathcal{H}(Z)$. This gives $\mathcal{S}_{\mathcal{H},h} \subseteq \mathcal{H}(Z)$.

$L(\Gamma, \mathbb{R}) \subseteq \mathcal{H}(Z)$ follows from the fact that $\mathbb{R} \subset Z$ and Z contains all ∞-Borel codes for sets of reals. To see $\mathcal{H}(Z) \subseteq L(\Gamma, \mathbb{R})$, let X be a set of ordinals in $\mathcal{H}(Z)$. By Claim 4.8, $X \in \mathcal{H}[h \restriction n] = \mathcal{H}[G(g \restriction n)]$ for some n. Since $\mathcal{H} \subseteq L(\Gamma, \mathbb{R})$, $g \restriction n, G(g \restriction n)$ are in $L(\Gamma, \mathbb{R})$, so is X. It's also easy to see that $\mathcal{H}(\Gamma) = L(\Gamma, \mathbb{R})$. This gives 4.4 and completes the proof of Lemma 4.4 (a).

For part (b) of Lemma 4.4, let $g^* \subseteq \mathbb{Q}$ be generic over $L(\mathcal{H}^+, Z)$. Let g, h be defined from g^* as before.

Lemma 4.9. *(i) h is a $\mathbb{P}$-generic over $\mathcal{H}^+$.*

(ii) $\dot{R}^h = Z$ and $\mathcal{H}^+(Z) = \mathcal{S}_{\mathcal{H}^+,h}$.

(iii) $\mathcal{H}^+(Z) \cap \wp_\Theta(\Theta) = Z$ and $\mathcal{H}^+(Z) \cap \wp(\mathbb{R}) = \Gamma$.

Proof. For part (i), suppose not. Then there is a finite sequence $s \in Z^{<\omega}$, $s \in g^*$ and a dense set D in $\mathbb{P}$ such that $D \in \mathcal{H}^+$ and such that $s \Vdash \dot{h} \cap D = \emptyset$. As before (cf. Lemma 4.7), $s \in \mathcal{H}^+[G(s)]$, where $G(s)$ is $\mathcal{H}^+$-generic for the standard Vopenka algebra $\mathbb{P}(s)$. So D must define a dense set D' in the factor forcing $\mathbb{P}/G(s)$. Choose a condition $q \in D'$. q must exist. Now q corresponds to q^*, a nonempty OD_s subset of $Z^{<\omega}$ of finite sequences which extend s; by Lemma 4.7, $q \in \mathcal{H}^+[G(s)]$. Let $t \in q^*$. Then t forces that $\dot{h} \cap D$ is not empty. This is a contradiction.

Clause (ii) follows from the proof that $\mathcal{S}_{\mathcal{H},h} = \mathcal{H}(Z)$, noting that $\mathcal{H}^+[G(g \restriction n)] = \mathcal{H}^+[h \restriction n]$ for all n. Now we want to verify clause (iii) of the lemma. For the first equality, it's clear that the $\supseteq$-direction holds. For the converse, suppose A is a bounded subset of Θ in $\mathcal{H}^+(Z)$. By the proof of Claim 4.8, $X \in \mathcal{H}^+[h \restriction k]$ for some k. But $\mathcal{H}^+[h \restriction k] = \mathcal{H}^+[G(g \restriction k)]$. Since X is a bounded subset of Θ and the forcing $\mathbb{P}(g \restriction k)$ is Θ-c.c. (since $g \restriction k$ is a finite sequence of elements of Z, by $\mathsf{AD}_{\mathbb{R}}$, $\mathbb{P}(g \restriction k)$, the standard Vopenka algebra adding $g \restriction k$, in fact, has size $< \Theta$), so indeed $X \in \mathcal{H}[G(g \restriction k)]$ as $V_\Theta^{\mathcal{H}} = V_\Theta^{\mathcal{H}^+}$.

Now we're onto the second equality of (iii). The $\supseteq$-direction holds since $\mathcal{H}(Z) = L(\Gamma, \mathbb{R}) \subseteq \mathcal{H}^+(Z)$. Let $A \subseteq \mathbb{R}^V$ be in $\mathcal{H}^+(Z)$. First we assume A is definable in $\mathcal{H}^+(Z)$ from an element $a \in \mathcal{H}^+$, via a formula ψ. Let $\dot{x}$ be a $\mathbb{P} \restriction \omega$-name for a real in $\mathcal{H}^+(Z)$ (here $\mathbb{P}^* \restriction \omega$ is the forcing Vop_ω defined in [13, Section 3]; $\mathbb{P}^* \restriction \omega$ consists of nonempty OD subsets of $\mathbb{R}^n$ for some n.). The statement $\psi(\dot{x}, \check{a})$ is decided by $\mathbb{P} \restriction \omega$ by homogeneity of $\mathbb{P} \restriction \omega, \mathbb{P}$ in the sense of Lemma 4.3 (i.e. $\mathcal{H}^+ \vDash$ "$\emptyset \Vdash_{\mathbb{P}\restriction\omega} \emptyset \Vdash_{\mathbb{P}/\mathbb{P}\restriction\omega} \psi[\dot{x}, \check{a}] \vee \emptyset \Vdash_{\mathbb{P}\restriction\omega} \emptyset \Vdash_{\mathbb{P}/\mathbb{P}\restriction\omega} \neg\psi[\dot{x}, \check{a}]$"). Again, by the fact that $\mathbb{P} \restriction \omega$ is Θ-c.c. (in fact $\mathbb{R} \restriction \omega$ has size $< \Theta$ in $\mathcal{H}$ by $\mathsf{AD}_{\mathbb{R}}$), we get that $A \in \mathcal{H}(Z)$, and hence $A \in \Gamma$. [16]

Now suppose A is definable in $\mathcal{H}^+(Z)$ from an $a \in \mathcal{H}^+$ and a $b \in Z$. Using the standard Vopenka algebra and $\mathsf{AD}_{\mathbb{R}}$, we can get a $< \Theta$-generic $G(b)$ over $\mathcal{H}$ and $\mathcal{H}^+$ such that $HOD_b = \mathcal{H}[G(b)] \subseteq \mathcal{H}^+[G(b)]$. Let us use $\mathcal{H}_b$ to denote $\mathcal{H}[G(b)]$ and $\mathcal{H}_b^+$ to denote $\mathcal{H}^+[G(b)]$. Now in $\mathcal{H}_b$, we can define the poset $\mathbb{P}_b$ the same way that $\mathbb{P}$ defined but we replace OD by $OD(b)$ in $L(\Gamma, \mathbb{R})$. Now we get a

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[16] This can be seen by taking a hull $X \prec \mathcal{H}^+$ such that $|X| < \Theta$ in $\mathcal{H}^+$ and $\mathcal{P} \restriction \omega \cup \{\mathbb{P} \restriction \omega, a\} \subset X$. Let M_X be the transitive collapse of X and $\tau : M_X \to X$ be the uncollapse map, then $M_X \in \mathcal{H}$. We get that $x \in A$ if and only if $\mathcal{H}[h] \vDash M_X[h \restriction (\mathbb{P} \restriction \omega)] \vDash \emptyset \Vdash_{\tau^{-1}(\mathbb{P})/\mathbb{P}\restriction\omega} \psi[x, \tau^{-1}(a)]$. This gives $A \in \mathcal{H}(Z)$.

generic h_b over $\mathcal{H}_b^+$ for $\mathbb{P}_b$ as before. A is then definable over $\mathcal{H}_b^+(Z)$ from parameters in $\mathcal{H}_b^+$. Now, we just have to repeat the argument above. This completes the proof of Lemma 4.9. □

Lemma 4.9 completes the proof of Lemma 4.4. □

Lemmata 4.4, 4.5, and 4.9 together prove Theorem 4.1. □

Remark 4.10. *If additionally, $\mathcal{H}^+ \vDash$ "Θ is regular", then $\mathcal{H}^+(Z) \vDash$ "Θ is regular." See [14, Lemma 1].*

5. A PROOF OF THEOREM 1.6

In this section, we assume the hypothesis of Theorem 1.6. We start with some setup and notations. As in [19], we assume $V = L(\wp(\mathbb{R}), \mu)$, where "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC} + \Theta$ is regular" holds and μ is a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$. Suppose N is such that there is a surjection π^* from $\wp(\mathbb{R})$ onto N. Then π^* induced a surjection $\pi : \wp_{\omega_1}(\wp(\mathbb{R})) \to \wp_{\omega_1}(N)$, namely $\pi(\sigma) = \pi^*[\sigma]$. Let μ_N^π be the supercompact measure on $\wp_{\omega_1}(N)$ induced by μ, i.e.

$$A \in \mu_N^\pi \Leftrightarrow \pi^{-1}[A] \in \mu.$$

μ_N^π does not depend on the choice of π. To see this, suppose $\pi_1, \pi_2 : \wp(\mathbb{R}) \to N$ are surjections. Then the set $A = \{\sigma : \exists \tau \in \wp_{\omega_1}(\wp(\mathbb{R}))\ \sigma = \pi_1[\tau] = \pi_2[\tau]\}$ is a strong club subset of $\wp_{\omega_1}(N)$ in the sense of [2, Definition 2.1] and hence by [2, Theorem 2.3], $A \in \mu_N^{\pi_1} \cap \mu_N^{\pi_2}$.[17] Futhermore, $\pi_1^{-1}[A] = \pi_2^{-1}[A] \in \mu$. From this, it follows that $\mu_N^{\pi_1} = \mu_N^{\pi_2}$. We will then denote this measure μ_N and sometimes suppress mentioning the surjection π. We write $\forall^*_{\mu_N}\sigma$ for "for μ_N-a.e. σ".

We assume, for contradiction that

(†) : there is no model M containing all reals and ordinals such that
$M \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable".

Under this smallness assumption, the HOD analysis in V can be carried out as in [8] and [10] to conclude that HOD$|\Theta$ is a union of hod premice and in fact is a direct limit of the directed system $\mathcal{F}$ of hod pairs $(\mathcal{P}, \Sigma)$ such that Σ is fullness preserving and has branch condensation. We then construct a hod premouse $\mathcal{H}^+$ extending HOD$|\Theta$ and a normal measure ν on Θ over $\mathcal{H}^+$ and amenable to $\mathcal{H}^+$. So we have a proper hod premouse $(\mathcal{H}^+, \nu)$. Using the Vopenka forcing in the previous section, we then show that $V = L[\mathcal{H}^+][\nu](\wp(\mathbb{R})) \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable. This contradicts (†). So (†) must be false; equivalently, there must be models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable" after all.

We define a model $\mathcal{H}^+$ extending $\mathcal{H} =_{\text{def}}$ HOD$|\Theta$ as follows: $\mathcal{H}^+$ is the union of sound, countably iterable hod premice $\mathcal{M}$ such that $\mathcal{H} \lhd \mathcal{M}$, $\rho_\omega(\mathcal{M}) \leq \Theta$. Here, $\mathcal{M}$ is said to be countably iterable if whenever $\mathcal{M}^*$ is countable, transitive, embeddable into $\mathcal{M}$ via map π, letting $\mathcal{H}^* = \pi^{-1}(\mathcal{H})$, then $\mathcal{M}^* \lhd \mathrm{Lp}^\Lambda(\mathcal{H}^*)$, where $\Lambda = \oplus_{\alpha < \lambda^{\mathcal{H}^*}} \Sigma_{\mathcal{H}^*(\alpha)}$.

Let N be a transitive structure of a large fragment of $\mathsf{ZF} + \mathsf{DC}$ such that $\wp(\mathbb{R}) \cup \mathcal{H} \subset N$ and such that there is a surjection $\pi : \wp(\mathbb{R}) \to N$. We call such an N *suitable.* We have that

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[17] The reader can also see Lemma 5.9 and the subsequent discussions for a proof.

$\forall^*_{\mu_N}\sigma\ \sigma \prec N$. For each such σ, let N_σ be the transitive collapse of σ and π_σ be the uncollapse map. Let $(\Gamma_\sigma, \mathcal{H}_\sigma, \Theta_\sigma) = \pi_\sigma^{-1}(\wp(\mathbb{R}), \mathcal{H}, \Theta)$. We let $\Gamma = \wp(\mathbb{R})$ and $(\theta^\sigma_\alpha : \alpha < \Theta_\sigma)$ be the Solovay sequence defined in Γ_σ. Generally, if $x \in \sigma$, then let $x_\sigma = \pi_\sigma^{-1}(x)$. We also let

$$\mathcal{H}^+_\sigma = \mathrm{Lp}^{\Sigma^-_\sigma}(\mathcal{H}_\sigma).$$[18]

The following gives an alternative characterization of $\mathcal{H}^+$.

Lemma 5.1. *$\mathcal{H}^+ = [\sigma \mapsto \mathcal{H}^+_\sigma]_{\mu_\Omega}$ where Ω is the transitive closure of $\wp(\mathbb{R}) \cap \mathcal{H}$.*[19]

Proof. First, let $\mathcal{M} \lhd \mathcal{H}^+$. Since $\mathcal{M}$ is sound and $\rho_\omega(\mathcal{M}) \leq \Theta$, there is an $A \subset \Theta$ coding $\mathcal{M}$. Then

$$A = [\sigma \mapsto \pi_\sigma^{-1}[A]]_{\mu_\Omega}, \tag{5.1}$$

and

$$\forall^*_{\mu_\Omega}\sigma, \pi_\sigma^{-1}[A] \in \mathcal{H}^+_\sigma. \tag{5.2}$$

To see this, let $\Omega \subset N$ and N is suitable such that $A \in N$. Note any such suitable N, M, $\mu_{N\cap\Omega} = \mu_{M\cap\Omega}$. The main point is for any suitable N: $\forall^*_{\mu_N}\sigma\ \mathcal{H}^+_\sigma$ only depends on $\sigma \cap \Omega$; in fact, $\mathcal{H}^+_\sigma = \mathcal{H}^+_{\sigma\cap\Omega} \in HOD_{\{\sigma\cap\Omega\}}$. Now

$$\forall^*_{\mu_N}\sigma\ A_\sigma = \pi^{-1}_{\sigma\cap\Omega}[A] \wedge A_\sigma \in \mathcal{H}^+_{\sigma\cap\Omega}.$$

This follows from the definition of $\mathcal{H}^+$ and the fact that $\forall^*_{\mu_N}\sigma\ \sigma\cap\mathcal{M} \prec \mathcal{M}$. Finally, A is represented in the μ_N-ultrapower by the collection of "Ω-invariant" functions, i.e.

$$A \cong \{f : \wp_{\omega_1}(N) \to \prod_\sigma A_\sigma/\mu_N : \forall\sigma_1, \sigma_2(\sigma_1 \cap \Omega = \sigma_2 \cap \Omega \Rightarrow f(\sigma_1) = f(\sigma_2)\}.. \tag{5.3}$$

The above discussions give us 5.1 and 5.2. So $\mathcal{M} \lhd [\sigma \mapsto \mathcal{H}_\sigma]_{\mu_\Omega}$.

Let $\mathcal{M} \lhd [\sigma \mapsto \mathcal{H}^+_\sigma]_{\mu_\Omega}$. Let N be suitable such that $\mathcal{M} \in N$. Note that by 5.3, the function $\sigma \mapsto \mathcal{M}_\sigma$ is Ω-invariant and represents $\mathcal{M}$ in the μ_N-ultrapower using only Ω-invariant functions. For any countable transitive $\mathcal{M}^*$ embeddable into $\mathcal{M}$ via τ, there is $\sigma \in \wp_{\omega_1}(N)$ and an embedding $\tau_\sigma : \mathcal{M}^* \to \mathcal{M}_\sigma$ such that $\mathcal{M}_\sigma \lhd \mathcal{H}^+_\sigma$. Therefore, $\mathcal{M}^*$ is iterable. This shows $\mathcal{M} \lhd \mathcal{H}^+$.

□

Lemma 5.2. *No level $\mathcal{M}$ of $\mathcal{H}^+$ is such that $\rho_\omega(\mathcal{M}) < \Theta$.*

Proof. Suppose $\mathcal{M} \lhd \mathcal{H}^+$ is the least such that $\rho_\omega(\mathcal{M}) < \Omega$. Let N be suitable such that $\mathcal{M} \in N$. We start with the following.

Claim 5.3. *For μ_N-a.e. σ, for any $\beta < \lambda_\sigma =_{\mathrm{def}} \lambda^{\mathcal{H}_\sigma}$, $\Sigma_{\mathcal{H}_\sigma(\beta)}$ is fullness preserving and has branch condensation.*

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[18] Note that the Lp-stack is computed in V.

[19] Note that Ω is suitable.

Proof. Fix a σ and a $\beta < \lambda_\sigma$. By the HOD analysis in Γ_σ (which uses $(\dagger)$), there is a hod pair $(\mathcal{P}, \Sigma)$ such that

- Σ is Γ_σ-fullness preserving and has branch condensation;
- $\mathcal{H}_\sigma(\beta)$ is an iterate of Σ.

Using π_σ, we get that $\pi_\sigma(\Sigma)$ is an (ω_1, ω_1) strategy for $\mathcal{P}$ that is fullness preserving and has branch condensation. Since $\Sigma = \pi_\sigma(\Sigma) \restriction \Gamma_\sigma$, $\Sigma^{\mathcal{H}_\sigma(\beta)}$ is the tail of $\pi_\sigma(\Sigma)$ and hence satisfies the conclusion of the claim.[20] $\square$

Fix a σ as in the claim and recall $\mathcal{M}_\sigma = \pi_\sigma^{-1}(\mathcal{M})$. Let Σ_σ be the natural strategy of $\mathcal{M}_\sigma$ defined from π_σ (see [9, Section 11]). The important properties of Σ_σ are:

1. Σ_σ extends $\Sigma_\sigma^- =_{def} \oplus_{\alpha < \lambda^{\mathcal{H}_\sigma}} \Sigma_{\mathcal{H}_\sigma(\alpha)}$;
2. whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{M}_\sigma, \Sigma_\sigma)$, for all $\alpha < \lambda^{\mathcal{Q}}$, $\Sigma_{\mathcal{T},\mathcal{Q}(\alpha)}$ is the pullback of a hod pair $(\mathcal{R}, \Lambda)$ such that Λ has branch condensation and is fullness preserving and hence by [8, Lemma 3.29], $\Sigma_{\mathcal{T},\mathcal{Q}(\alpha)}$ has branch condensation;
3. Σ_σ agrees with Σ_σ^- on stacks below Θ_σ and for each $\alpha < \lambda_\sigma$, the direct limit map $\pi^{\Sigma_\sigma}_{\mathcal{M}_\sigma,\infty} \restriction \theta^\sigma_\alpha$ is the direct limit map $\pi^{\Sigma_\sigma^-}_{\mathcal{H}_\sigma(\alpha),\infty} \restriction \theta^\sigma_\alpha$;
4. suppose $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{M}_\sigma, \Sigma_\sigma)$ and let $i = \pi^{\vec{\mathcal{T}}}$ be the corresponding iteration map, then there is a map $k : \mathcal{Q} \to \mathcal{M}$ such that $k \circ i = \pi_\sigma \restriction \mathcal{M}_\sigma$. k is defined as: $k(i(f)(a)) = \pi_\sigma(f)(\pi^{\Lambda}_{\mathcal{Q},\infty}(a))$ for $f \in \mathcal{M}_\sigma$ and $a \in (\delta^{\mathcal{Q}})^{<\omega}$, where Λ is the $\vec{\mathcal{T}}$-tail of Σ_σ^-. So Σ_σ is $\mathrm{OD}_{\{\pi_\sigma \restriction \mathcal{M}_\sigma\}}$.

(3) above uses the fact that Θ is regular.

Let $\delta = \delta_\alpha^{\mathcal{M}_\sigma} < \Theta_\sigma$ be a Woodin cardinal of $\mathcal{M}_\sigma$ such that $\rho_\omega(\mathcal{M}_\sigma) \leq \delta$. Let $A \subseteq \delta$ witness this. So A is a bounded subset of Θ_σ that is not in $\mathcal{M}_\sigma$. We aim to obtain a contradiction from this.

Now we can construe $(\mathcal{M}_\sigma, \Sigma_\sigma)$ as a $(\mathcal{H}_\sigma(\alpha), \Sigma_{\mathcal{H}_\sigma(\alpha)})$-hod pair. We can define a direct limit system of $(\mathcal{H}_\sigma(\alpha), \Sigma_{\mathcal{H}_\sigma(\alpha)})$ hod pairs as follows:

$$\mathcal{F}^* = \{(\mathcal{Q}', \Lambda') \mid (\mathcal{Q}', \Lambda') \equiv_{DJ} (\mathcal{Q}, \Lambda)\}\ [21].$$

Note that $\mathcal{F}$ does not depend on $(\mathcal{Q}, \Lambda)$ and in fact is $OD_{\Sigma_{\mathcal{H}_\sigma(\alpha)}}$ in $L(\wp(\mathbb{R}))$. This easily implies that A is $OD_{\Sigma_{\mathcal{H}_\sigma(\alpha)}}$ in $L(\wp(\mathbb{R}))$. By $\mathsf{MC}(\Sigma_{\mathcal{H}_\sigma(\alpha)})$ [22] and the fact that $\mathcal{H}_\sigma(\alpha+1)$ is $\Sigma_{\mathcal{H}_\sigma(\alpha)}$-full, $A \in \mathcal{H}_\sigma(\alpha+1)$, so $A \in \mathcal{M}_\sigma$. This contradicts the definition of A. $\square$

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[20] Note that by positionality of $\pi_\sigma(\Sigma)$, which follows from fullness preservation and branch condensation (cf. [8, Theorem 2.42], $\Sigma_{\mathcal{H}_\sigma(\beta)}$ does not depend on any specific iteration from $\mathcal{P}$ to $\mathcal{H}_\sigma(\beta)$.

[21] This means these $(\mathcal{H}^*_\sigma, \Sigma_\sigma)$ hod pairs are Dodd-Jensen equivalent.

[22] This stands for Mouse Capturing with respect to $\Sigma_{\mathcal{H}_\sigma(\alpha)}$, which in turns is the statement that if $x, y \in \mathbb{R}$, and x is $OD_{\Sigma_{\mathcal{H}_\sigma(\alpha)}}(y)$ then x is in a $\Sigma_{\mathcal{H}_\sigma(\alpha)}$-mouse over y.

We define a measure ν on Θ over $\mathcal{H}^+$ as follows. Let $A \in \mathcal{H}^+ \cap \wp(\Theta)$ and N be suitable such that $A \in N$. Then

$$A \in \nu \Leftrightarrow \forall^*_{\mu_N}\sigma \ \sup(\sigma \cap \Theta) \in A. \tag{5.4}$$

First of all, note that for μ_N-a.e. σ, $\sup(\sigma\cap\Theta) < \Theta$ as $\mathrm{cof}(\Theta) > \omega$. Now it appears that whether $A \in \nu$ depends on the choice of suitable N, but it does not. Fix $A \subseteq \Theta$ and suitable N_1, N_2 such that $A \in N_1 \cap N_2$. For μ_{N_1}-a.e. σ, we let A_σ be the transitive collapse of $\sigma \cap A$. Similarly, we define A_σ for μ_{N_2}-a.e. σ. We have that

$$A = [\sigma \mapsto A_\sigma]_{\mu_{N_1}} = [\sigma \mapsto A_\sigma]_{\mu_{N_2}}.$$

Again, as in the proof of Lemma 5.1, here and everywhere else later in the paper, we require that the ultrapowers use only Ω-invariant functions. The point is the transitive collapse of $\sigma \cap A$ only depends on $\sigma \cap \Theta$, not all of σ. Furthermore, letting $N = N_1 \cap N_2$, then N is suitable and $\mathcal{H} \cup \{A\} \in N$. The following equivalences are easy to verify:

$$\begin{aligned} \forall^*_{\mu_{N_1}}\sigma \ \sup(\sigma \cap \Theta) \in A &\Leftrightarrow \forall^*_{\mu_N}\sigma \ \sup(\sigma \cap \Theta) \in A \\ &\Leftrightarrow \forall^*_{\mu_{N_2}}\sigma \ \sup(\sigma \cap \Theta) \in A \end{aligned}$$

The main point is: if $X \in \mu_{N_1}$ (or $X \in \mu_{N_2}$) then the set $\{\sigma \cap N : \sigma \in X\} \in \mu_N$. This shows ν does not depend on the choice of suitable N.[23]

It's clear that ν is a measure. Note also that the above definition makes sense for all $A \in V$ but we only care about those A's in $\mathcal{H}^+$ as we can prove the measure behaves nicely on this collection of sets.

Note that $\mathcal{H}^+$ is a ZFC^- model and $|\mathcal{H}^+| \leq \Theta^+$. Now we show the following.

Lemma 5.4. *ν is amenable to $\mathcal{H}^+$. In other words, for any $\mathcal{M} \lhd \mathcal{H}^+$, $\nu \restriction \mathcal{M} \in \mathcal{H}^+$.*

Proof. Let $\mathcal{M} \lhd \mathcal{H}^+$ be sound and $\rho_\omega(\mathcal{M}) \leq \Theta$ (note that $\mathcal{H}^+$ is the union of such $\mathcal{M}$'s). Let $\nu_\mathcal{M} = \nu \restriction \mathcal{M}$. We show $\nu_\mathcal{M} \in \mathcal{H}^+$.

Again, we fix a suitable N such that $\mathcal{M}, \nu_\mathcal{M} \in N$. Let $\vec{A} = \langle A_\alpha \mid \alpha < \Theta\rangle$ be a definable-over-$\mathcal{M}$ enumeration of $\wp(\Theta) \cap \mathcal{M}$ and let $\mathcal{N} \lhd \mathcal{H}^+$ be least such that $\vec{A} \in \mathcal{N}$.[24] We may choose N so that $\mathcal{N} \in N$.

We use the set-up and notations above. Let $\mathcal{M} = [\sigma \mapsto \mathcal{M}_\sigma]_{\mu_N}$ and note that $\forall^*_{\mu_N}\sigma \ \mathcal{M}_\sigma = \pi_\sigma^{-1}(\mathcal{M})$. Similarly, $\nu_\mathcal{M} = [\sigma \mapsto \nu_\sigma]_{\mu_N}$ where for μ_N-a.e. σ, $\nu_\sigma = \pi_\sigma^{-1}(\nu_\mathcal{M})$. Similar notations are introduced for $\mathcal{N}$. We want to show $\forall^*_{\mu_N}\sigma \ \nu_\sigma \in \mathcal{H}^+_\sigma$. For a μ_N-measure set of σ, we have $(\mathcal{M}_\sigma, \mu_\sigma, \mathcal{N}_\sigma) = \pi_\sigma^{-1}(\mathcal{M}, \nu_\mathcal{M}, \mathcal{N})$ and $\Sigma^-_\sigma(\alpha)$ is fullness preserving for each $\alpha < \lambda_\sigma$. We show the claim holds for all such σ. Let X denote the aforementioned μ_N-measure one set.

Let for each $\sigma \in X$, $\mathcal{R}_\sigma = \mathrm{HOD}_{(\mathcal{H}^+_\sigma, \Sigma^-_\sigma)}$. Note that

$$\wp(\Theta_\sigma) \cap \mathcal{R}_\sigma = \wp(\Theta_\sigma) \cap \mathcal{H}^+_\sigma$$

---

[23] Alternatively, one can define $A \in \nu \Leftrightarrow \forall^*_{\mu_\Omega}\sigma \ \sup(\sigma \cap \Theta) \in A$.

[24] $\vec{A}$ exists because $\rho_\omega(\mathcal{M}) \leq \Theta$ and $\mathcal{M}$ is sound.

by a similar argument to that used in Lemma 5.2. Let $\vec{A}_\sigma = \langle A^\sigma_\alpha \mid \alpha < \Theta_\sigma\rangle = \pi_\sigma^{-1}(\vec{A})$. We want to show $\langle \alpha \mid A^\sigma_\alpha \in \nu_\sigma\rangle \in \mathcal{R}_\sigma$ which in turns implies $\langle \alpha \mid A^\sigma_\alpha \in \nu_\sigma\rangle \in \mathcal{H}^+_\sigma$.

Let $\sigma \in X$. Let $\gamma_\sigma = \sup(\pi_\sigma[\Theta_\sigma])$ (note that $\pi_\sigma[\Theta_\sigma] = \sigma \cap \Theta$ coincides with the iteration embedding via Σ^-_σ and since $\mathrm{cof}(\Theta) > \omega$, $\gamma_\sigma < \Theta$). Note that

$$\forall \alpha < \Theta_\sigma \ (A^\sigma_\alpha \in \nu_\sigma \Leftrightarrow \gamma_\sigma \in \pi_\sigma(A^\sigma) \cap (\gamma_\sigma + 1)) \tag{5.5}$$

and

$$\langle \pi_\sigma(A^\sigma_\alpha) \cap (\gamma_\sigma + 1) \mid \alpha < \Theta_\sigma\rangle \in \mathcal{R}_\sigma. \tag{5.6}$$

5.5 is true by elementarity and the definition of $\nu_\mathcal{M}$. 5.6 is true because $\langle \pi_\sigma(A^\sigma_\alpha) \cap (\gamma_\sigma + 1) \mid \alpha < \Theta_\sigma\rangle$ is OD from $\pi_\sigma \restriction \Theta_\sigma \cup \{(\Theta_\sigma, \gamma_\sigma)\}$ and $\vec{A}_\sigma$. $\vec{A}_\sigma \in \mathcal{N}_\sigma \in \mathcal{R}_\sigma$. Furthermore, $\pi_\sigma \restriction \Theta_\sigma \cup \{(\Theta_\sigma, \gamma_\sigma)\} = i^{\Sigma^-_\sigma}_{\mathcal{H}_\sigma,\infty} \restriction (\Theta_\sigma + 1)$, hence by the definition of $\mathcal{R}_\sigma$, we have 5.6.

By 5.5 and 5.6, we have $\langle \alpha \mid A_\alpha \in \nu_\sigma\rangle \in \mathcal{R}_\sigma$. The lemma follows from the agreement between $\mathcal{R}_\sigma$ and $\mathcal{H}^+_\sigma$.

□

Remark 5.5. *(i) In the proof of Lemma 5.4, we can't demand that $\mathcal{H}^+ \in N$ because it may be the case that $o(\mathcal{H}^+) = \Theta^+$ and hence there are no surjections from $\wp(\mathbb{R})$ onto $\mathcal{H}^+$.*

(ii) It follows from the fact that Θ is regular and $\mathsf{AD}_\mathbb{R}$ holds that $\mathcal{H}^+ \vDash$ "Θ is regular limit of Woodin cardinals".

Now we want to show that ν is normal and $\wp(\Theta) \cap L[\mathcal{H}^+, \nu] = \wp(\Theta) \cap \mathcal{H}^+$. Let $\mathcal{M} \lhd \mathcal{H}^+$ be sound and $\rho_\omega(\mathcal{M}) \leq \Theta$.

Lemma 5.6. *Let $\mathcal{M} \lhd \mathcal{H}^+$. Then $\nu_\mathcal{M} =_{def} \nu \restriction \mathcal{M}$ is normal.*

Proof. Suppose not. Let N be suitable such that $\mathcal{M}, \nu_\mathcal{M} \in N$. Let $\mathcal{M} = [\sigma \mapsto \mathcal{M}_\sigma]_{\mu_N}$ and note that $\forall^*_{\mu_N}\sigma\ \mathcal{M}_\sigma = \pi_\sigma^{-1}(\mathcal{M})$.

We define a measure ν_σ on Θ_σ over $\mathcal{M}_\sigma$ as follows.

$$A \in \nu_\sigma \Leftrightarrow \gamma_\sigma =_{\text{def}} \sup(\pi_\sigma[\Theta_\sigma]) \in \pi_\sigma(A). \tag{5.7}$$

It's easy to see that

$$\nu_\sigma = \pi_\sigma^{-1}(\nu_\mathcal{M}) \wedge \Pi_\sigma \nu_\sigma/\mu_N = \nu_\mathcal{M}. \tag{5.8}$$

By the assumption on $\nu_\mathcal{M}$, we have that $\forall^*_{\mu_N}\sigma\ \nu_\sigma$ is not normal (in N_σ). This means

$$\forall^*_{\mu_N}\sigma \exists f \in \mathcal{M}_\sigma\ \pi_\sigma(f)(\gamma_\sigma) < \gamma_\sigma \wedge \pi_\sigma(f)(\gamma_\sigma) \notin \sigma \cap \gamma_\sigma. \tag{5.9}$$

By normality of μ_N,

$$\exists f \in \mathcal{M}\ \forall^*_{\mu_N}\sigma\ f(\gamma_\sigma) \notin \sigma \cap \gamma_\sigma \wedge f(\gamma_\sigma) < \gamma_\sigma.$$

Fix such an $f \in \mathcal{M}$ and let

$$A' = \{\sigma \mid f(\gamma_\sigma) \notin \sigma \cap \gamma_\sigma \wedge f(\gamma_\sigma) < \gamma_\sigma\}. \tag{5.10}$$

We have $A' \in \mu_N$. This implies that $B \in \nu_{\mathcal{M}}$ where

$$B = \{\gamma \mid f(\gamma) < \gamma\}. \tag{5.11}$$

Let $\mathcal{M} \lhd \mathcal{M}^* \lhd \mathcal{H}^+$ be such that $\nu_{\mathcal{M}} \in \mathcal{M}^*$. This is possible since $\nu_{\mathcal{M}} \in \mathcal{H}^+$ and $\mathcal{H}^+$ is a limit of such $\mathcal{M}^*$'s. Now we can also assume $\mathcal{M}^* \in N$ by expanding N if necessary. Let then $\forall^*_{\mu_N}\sigma\ \mathcal{M}^*_\sigma = \pi_\sigma^{-1}(\mathcal{M}^*)$.

Claim 5.7. *There is an $\eta < \Theta$ such that $\forall^*_{\mu_N}\sigma\ f(\gamma_\sigma) \le \eta$.*

Proof. $\forall^*_{\mu_N}\sigma$, let Σ_σ be the π_σ-guided strategy for $\mathcal{M}_\sigma$ (as defined in the proof of Lemma 5.2) and $i_\sigma : \mathcal{M}_\sigma \to \mathcal{N}_\sigma$ be the direct limit map, where $\mathcal{N}_\sigma$ is the direct limit of all Σ_σ-iterates of $\mathcal{M}_\sigma$. Note that since $\mathcal{M}_\sigma \vDash$ "Θ_σ is regular", $i_\sigma \restriction \Theta_\sigma = \pi_\sigma \restriction \Theta_\sigma$; also we may and do assume i_σ is cofinal in $\mathrm{o}(\mathcal{N}_\sigma)$. These properties follow from (1)-(4) in the proof of Lemma 5.2. (1)-(4) in the proof of Lemma 5.2 also imply that there is a map $k_\sigma : \mathcal{N}_\sigma \to \mathcal{M}$ such that $k_\sigma \circ i_\sigma = \pi_\sigma \restriction \mathcal{M}_\sigma$ and $\mathrm{crt}(k_\sigma) = i_\sigma(\Theta_\sigma) = \gamma_\sigma$.

Let $\nu^*_\sigma = i_\sigma[\nu_\sigma]$ and $(f_\sigma, B_\sigma) = (\pi_\sigma^{-1}(f), \pi_\sigma^{-1}(B))$. We have then that $\forall^*_{\mu_N}\sigma\ B_\sigma \in \nu_\sigma$, which implies that $i_\sigma(B_\sigma) \in \nu^*_\sigma$. We note that $\mathrm{crt}(k_\sigma) = \gamma_\sigma$ and therefore, ν^*_σ is a subset of the normal measure $\bar{\nu}_\sigma$ induced from k_σ, i.e. for $A \in \mathcal{N}_\sigma$, $A \in \bar{\nu}_\sigma$ iff $\gamma_\sigma \in k_\sigma(A)$.

To prove the lemma, it suffices to show that

$$\forall^*_{\mu_N}\sigma\ \mathcal{M}^*_\sigma \vDash \exists \eta_\sigma < \Theta_\sigma\ i_{\nu_\sigma}(f_\sigma)(\Theta_\sigma) \le \eta_\sigma. \tag{5.12}$$

Fix a σ in the first paragraph. Note that we can extend i_σ to a map $i^+_\sigma : \mathcal{M}^*_\sigma \to \mathcal{N}^*_\sigma$ such that $i^+_\sigma \restriction \Theta_\sigma = i_\sigma \restriction \Theta_\sigma = \pi_\sigma \restriction \Theta_\sigma$ and extend k_σ to a map $k^+_\sigma : \mathcal{N}^*_\sigma \to \mathcal{M}^*$ such that $\mathrm{crt}(k^+_\sigma) = \mathrm{crt}(k_\sigma) = \gamma_\sigma$ and $k^+_\sigma \restriction \mathcal{N}_\sigma = i_\sigma$.

As mentioned above, the measure $\bar{\nu}_\sigma \in \mathcal{N}^*_\sigma$ is normal; so there is some $\eta < \gamma_\sigma$ such that

$$\mathcal{N}^*_\sigma \vDash k_\sigma(i_\sigma(f))(\gamma_\sigma) = \eta. \tag{5.13}$$

By continuity of i_σ at Θ_σ, let η_σ least such that $i_\sigma(\eta_\sigma) \ge \eta$, we get 5.12 from 5.12 and the choice of η_σ.[25] Finally, $\eta = [\sigma \mapsto \eta_\sigma]_{\mu_N}$ satisfies the claim. □

Let now

$$A = \{\sigma \in A' \mid f(\gamma_\sigma) \le \eta\}.$$

By the previous lemma, $A \in \mu_N$.

---

[25] We do not know that $i^+_\sigma(\nu_\sigma) = \bar{\nu}_\sigma$. So from the normality of $\bar{\nu}_\sigma$, we cannot conclude ν_σ is normal using elementarity.

Definition 5.8 (Becker, [2])**.** *Suppose $A \subseteq \wp_{\omega_1}(N)$. We say that A is* ***unbounded*** *if for all $\sigma \in \wp_{\omega_1}(N)$, there is a $\tau \in A$ such that $\sigma \subseteq \tau$. We say that A is a* ***strong club (scub)*** *if A is unbounded and $\forall \sigma \in \wp_{\omega_1}(N) \forall \tau \subseteq \sigma$, if whenever τ is finite, then there is a $\tau' \in A$ such that $\tau \subseteq \tau' \subseteq \sigma$, then $\sigma \in A$. A is a* ***weak club (wcub)*** *if A is unbounded and whenever $\langle \sigma_n \mid n < \omega \rangle$ is a $\subseteq -$increasing sequence of elements of A then $\bigcup_n \sigma_n \in A$.*

Clearly, a strong club is a weak club.

Lemma 5.9. *Suppose $E \in \mu_N$. Then E meets every strong club. In particular, A meets every strong club.*

Proof. Suppose $C \subseteq \wp_{\omega_1}(N)$ is a strong club and $C \cap E = \emptyset$. Let F be defined as follows. $F(\sigma) = \sigma \backslash \bigcup \{\tau \mid \tau \subseteq \sigma \wedge \tau \in C\}$. By our assumption that C is a strong club and $C \cap E = \emptyset$, $\forall^*_{\mu_N} \sigma\ F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset$. By normality, $\exists x \forall^*_{\mu_N} \sigma\ \sigma \in E \backslash C \wedge x \in F(\sigma)$.

We claim that this is a contradiction. Fix such an x. Since C is a strong club, there is a $\sigma^* \in C$ such that $x \in \sigma^*$. By fineness and countable completeness of μ_N, the set $\{\sigma \in E \mid \sigma^* \subsetneq \sigma\} \in \mu_N$. This contradicts the definition of F. ☐

Note also that the above lemma implies that if C is a strong club, then $\mu_N(C) = 1$.

Now let $\mathbb{P}$ be the natural forcing that shoots a weak club through A. Conditions in $\mathbb{P}$ are countable $W \subseteq A$ such that whenever $\langle \sigma_n \mid n < \omega \wedge \sigma_n \in W \rangle$ is $\subseteq -$increasing then $\bigcup_n \sigma_n \in W$. $\forall C_0, C_1 \in \mathbb{P}$, $C_0 \leq_{\mathbb{P}} C_1$ iff $C_1 \subseteq C_0$.

Lemma 5.10. *$\mathbb{P}$ is $(\omega_1, \infty)-$distributive.*

Proof. Fix a condition $C_0 \in \mathbb{P}$ and a sequence $\vec{D} = \langle D_i \mid i < \omega \rangle$ of open dense sets in $\mathbb{P}$. We want to find a condition $C \leq_{\mathbb{P}} C_0$ such that $C \in D_i$ for all i.

Claim 5.11. *The set $D = \{\sigma \mid \sigma \prec N\}$ contains a strong club.*

Proof. D is certainly unbounded (by a standard closure argument using DC). Now let $\sigma \in \wp_{\omega_1}(N)$ and suppose for all finite $\tau \subseteq \sigma$, there is $\tau' \in D$ such that $\tau \subseteq \tau' \subseteq \sigma$. We want to show $\sigma \in D$. We prove by induction that for any n, for any finite $\tau \subseteq \sigma$, whenever $\tau \subseteq \tau' \subseteq \sigma$ and $\tau' \in D$ then $\tau' \prec_{\Sigma_n} \sigma \prec_{\Sigma_n} N$.

This clearly holds for $n = 0$. Now suppose the claim holds for n and let Ψ be a Π_n formula, $\tau \subseteq \sigma$ be finite such that $N \vDash \exists x\ \Psi[x, \tau]$. By our assumption, there is a $\tau' \in D$ such that $\tau \subseteq \tau' \subseteq \sigma$. By definition of D, $\tau' \prec N$, hence $\tau' \vDash \exists x\ \Psi[x, \tau]$. Let $x \in \tau'$ be a witness. We have then $\tau' \vDash \Psi[x, \tau]$. But $x \in \sigma$ and Ψ is Π_n; by the induction hypothesis, $\sigma \vDash \Psi[x, \tau']$. This proves the claim. ☐

Let N' be a transitive model of $\mathsf{ZF}^- + \mathsf{DC}$ such that $\wp(\mathbb{R}) \twoheadrightarrow N'$ and $N, \mathbb{P}, \vec{D} \in N'$. Let N'' be a countable elementary submodel of N' such that $\mathbb{P}, \vec{D} \in N'' \cap N \in D$ (we may assume $\vec{D}$ enumerates all open dense sets in N). Such an N'' exists by the claim. By a standard argument, we can build a $\leq_{\mathbb{P}} -$descending chain of conditions $\langle C_n \mid n < \omega \rangle$ such that

1. $C_{n+1} \in D_n$;

2. $C_n \in N''$ for all n;

3. $\bigcup_n C_n = N'' \cap N$.

Let $C = \bigcup_n C_n \cup \{N'' \cap N\}$. Then $C \in \mathbb{P}$ and $C \leq_{\mathbb{P}} C_n$ for all n. This means $C \in D_n$ for all n. Hence we're done. $\square$

Let $G \subseteq \mathbb{P}$ be V-generic. In $V[G]$, DC holds and there is a weak club $C \subseteq A$. Let then

$$C^* = \{\gamma_\sigma \mid \sigma \in C\}.$$

Then C^* contains an $\omega-$club in $V[G]$.

Now we proceed to derive a contradiction. First, we use an abstract pointclass argument to generalize Solovay's proof that ω_1 is measurable under AD to show the following.

Lemma 5.12. *In V, there are unboundedly many $\kappa < \Theta$ such that:*

1. *the $\omega-$club filter on κ is an η^+-complete ultrafilter on $\wp(\kappa)$;*

2. *the set $\{\sigma \cap \wp(\mathbb{R}) \mid \sigma \in A \wedge \gamma_\sigma < \kappa\}$ is unbounded in $\wp_{\omega_1}(\wp(\mathbb{R}) \restriction \kappa)$; in particular, $\{\gamma_\sigma \mid \sigma \in A\}$ is unbounded in κ;*

3. *$\forall \xi < \eta$, the set of $\sigma \cap \wp(\mathbb{R})$ such that $\sigma \in A$ and $\xi \in \sigma$ and $\gamma_\sigma < \kappa$ is unbounded in $\wp_{\omega_1}(\wp(\mathbb{R}) \restriction \kappa)$.*

Proof. Since Solovay's proof is well-known, we only highlight the necessary changes needed to run that proof in this situation. Working in V, let $\eta^+ < \rho_1 < \rho_2 < \Theta$ where ρ_1, ρ_2 are regular Suslin cardinals. Furthermore, we assume that there is a prewellordering of length η in $S(\rho_1)$[26]. Fix a prewellordering $\leq$ of length η such that $\leq \in \utilde{\Delta}_{S(\rho_1)}$ and let $f : \mathbb{R} \twoheadrightarrow \eta$ be the natural function induced from $\leq$.

We claim that there is a κ which is a limit of Suslin cardinals of cofinality ρ_2 (in V) and κ satisfies clauses (2) and (3) of the lemma. To see such a κ exists, first note that by Theorem 4.1, $\mathcal{H}^+(\wp(\mathbb{R})) \cap \wp(\mathbb{R}) = \wp(\mathbb{R})$; as discussed in Remark 5.5, $\mathcal{H}^+ \vDash \Theta$ is regular, $\mathcal{H}^+(\wp(\mathbb{R})) \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. Now the set Y of $\sigma \cap \Theta$ such that Σ^-_σ is fullness preserving is in $\mathcal{H}^+(\wp(\mathbb{R}))$ (note that γ_σ is a limit of Suslin cardinals and $\mathrm{cof}(\gamma_\sigma) = \omega$ in $\mathcal{H}^+(\wp(\mathbb{R}))$); also, for each $\xi < \eta$, the set Y_ξ of $\sigma \in Y$ such that $\xi \in \sigma$ is in $\mathcal{H}^+(\wp(\mathbb{R}))$. From these facts and the regularity of Θ in $\mathcal{H}^+(\wp(\mathbb{R}))$, we easily get such a κ.

Fix such a κ. We show that κ satisfies (1) as well. Let $\utilde{\Omega}$ be the (boldface) Steel pointclass at κ (see [12] or [5] for the definition of the Steel pointclass). The properties we need for $\utilde{\Omega}$ are:

1. $\exists^{\mathbb{R}} \utilde{\Delta}_{\utilde{\Omega}} \subseteq \utilde{\Delta}_{\utilde{\Omega}}$ (in fact, $\utilde{\Delta}_{\utilde{\Omega}} = \{Y \mid w(Y) < \kappa\}$);

2. $\utilde{\Omega}$ is closed under $\cap, \cup$ with $S(\rho_1)$-sets.

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[26] For a Suslin cardinal ξ, $S(\xi)$ is the pointclass of $\xi-$Suslin sets.

3. (Boundedness) Let Z be an $\underset{\sim}{\Omega}$-universal set and $\pi : Z \twoheadrightarrow \kappa$ be an $\underset{\sim}{\Omega}$-norm. Then for $A \in \underset{\sim}{\Delta}_{\underset{\sim}{\Omega}}$, $\pi \restriction A$ is bounded in κ.

In the following, we fix Z, π as above and a simple coding of ω-sequences of reals by reals. So a real x codes a sequence of reals $(x_i)_{i<\omega}$. For each $X \in \wp(\kappa)$, we define the Solovay game G_X as follows. Players I and II take turns to play natural numbers. After ω many moves, say player I plays a real x and player II plays a real y. I wins the run of G_X iff either there is an i such that either $x_i \notin Z$ or $y_i \notin Z$ and letting j be the least such then $y_j \notin Z$ or $\sup\{\pi(x_i), \pi(y_j) \mid i, j < \omega\} \in X$.

Now we're ready to prove the $\omega-$club filter at κ, $\mathcal{U}_\kappa$, is an η^+-complete ultrafilter. Note that $\mathcal{U}_\kappa$ is an ultrafilter follows from AD and in fact, $X \in \mathcal{U}_\kappa$ iff player I has a winning strategy in the game G_X. Fix a sequence $\langle A_\alpha \mid \alpha < \eta \wedge A_\alpha \in \mathcal{U}_\kappa \rangle$. We want to show $\bigcap_\alpha A_\alpha \in \mathcal{U}_\kappa$. Since $A_\alpha \in \mathcal{U}_\kappa$, player I has a winning strategy for the game G_{A_α}. Let $g : \eta \to \wp(\mathbb{R})$ be such that for all $\xi < \eta$, $g(\xi) \subseteq \{\tau \mid \tau \text{ is a winning strategy for player I in } G_{A_\xi}\}$ and furthermore $Code(g, \leq) = \{(x, \tau) \mid \tau \in g(f(x))\} \in S(\rho_1)$. Such a g exists by the coding lemma.

For each $\xi < \kappa$, let $Y_\xi = \{(\tau[y])_n \mid n < \omega \wedge \exists x(x, \tau) \in Code(g, \leq) \wedge \forall i(\pi(y_i) < \xi)\}$. It's easy to see from the fact that π is $\underset{\sim}{\Omega}$-norm, Ω is closed under intersection with $S(\rho_1)-$sets that $Y_\xi \in \underset{\sim}{\Delta}_{\underset{\sim}{\Omega}}$. By boundedness, $g(\xi) = \sup\{\pi(z) \mid z \in Y_\xi\} < \kappa$ for all ξ. This easily implies (as in the standard Solovay's proof) that I has a winning strategy in the game $G_{\bigcap_\alpha A_\alpha}$, which in turns implies $\bigcap_\alpha A_\alpha \in \mathcal{U}_\kappa$. □

Let $D = \{\gamma_\sigma \mid \sigma \in A\} \in \nu_{\mathcal{M}}$. Fix a κ as in Lemma 5.12 and let $\mathcal{U}_\kappa$ be the $\omega-$club filter on κ; furthermore, by the choice of κ, $D \cap \kappa$ is unbounded in κ. By the coding lemma, $D \cap \kappa \in L(\wp(\mathbb{R}))$.

We claim that $D \cap \kappa \in \mathcal{U}_\kappa$. Otherwise, $D \cap \kappa$ is disjoint from an $\omega-$club E. Let

$$E' = \{\sigma \mid \gamma_\sigma \in E\}.$$

But in $V[G]$, $D \cap \kappa$ contains an $\omega-$club, namely $C^* \cap \kappa$. In $V[G]$, E remains an $\omega-$club, hence has nonempty intersection with $C^* \cap \kappa$. This is a contradiction.

Finally, since $D \cap \kappa \in \mathcal{U}_\kappa$ and $\mathcal{U}_\kappa$ is η^+-complete, there is a $\xi \leq \eta$ such that $D_\xi = \{\gamma < \kappa \mid f(\gamma) = \xi\} \in \mathcal{U}_\kappa$. But then there is a $\sigma \in C$ such that $\gamma_\sigma < \kappa$, $\xi \in \sigma$, and $f(\gamma_\sigma) = \xi$. This contradicts the fact that $\forall \sigma \in C \; f(\gamma_\sigma) \notin \sigma$. This completes the proof of Lemma 5.6.

□

Let $\mathcal{H}^{+-} = Ult(\mathcal{H}^+, \nu)$, and π_ν be the ultrapower map. Let $\lambda = (\Theta^{++})^{\mathcal{H}^{+-}}$ and E_ν be the (Θ, λ)-extender derived from π_ν, i.e.

$$(a, A) \in E_\nu \Leftrightarrow a \in [\lambda]^{<\omega} \wedge A \in \wp(\Theta)^{|a|} \cap \mathcal{H}^+ \wedge a \in \pi_\nu(A).$$

E_λ is essentially the measure ν.

Lemma 5.13. *$\mathcal{H}^{+-}$ is well-founded. Furthermore, $\wp(\Theta) \cap \mathcal{H}^{+-} = \wp(\Theta) \cap \mathcal{H}^+$.*

Proof. The well-foundedness of $\mathcal{H}^{+-}$ follows from the fact that ν is countably complete in V. The countable completeness of ν follows from the countable completeness of μ. The equality of the powersets follows from Θ-completeness and amenability of ν, cf. Lemmas 5.4 and 5.6.

□

Remark 5.14. *We, as usual, identify $\mathcal{H}^{+-}$ with its transitive collapse. As such, $\mathcal{H}^{+-}$ is a hod premouse. By Lemma 5.13 and Lemma 5.4, E_μ coheres $\mathcal{H}^{+-}$. So $(\mathcal{H}^{+-}|\lambda, E_\nu)$ is a hod premouse.*

Theorem 5.15. *Let $\mathcal{H}^{++} = L[\mathcal{H}^{+-}|\lambda][E_\nu]$.*[27] *Then $\wp(\Theta) \cap \mathcal{H}^{++} = \wp(\Theta) \cap \mathcal{H}^{+}$.*

Proof. Suppose not. Then there is an $\mathcal{M}^* \trianglelefteq \mathcal{H}^{++}$ such that $\rho(\mathcal{M}^*) \leq \Theta$ and $\mathcal{M}^*$ defines a set not in $\mathcal{H}^+$. We may assume $\mathcal{M}^*$ is minimal and $\rho_1(\mathcal{M}^*) \leq \Theta$ (note that $\mathrm{o}(\mathcal{M}^*) > o(\mathcal{H}^+)$). Let $\mathcal{M}$ be the transitive collapse of $Hull_1^{\mathcal{M}^*}(\Theta \cup p_1^{\mathcal{M}^*})$. One can use an argument similar to that in Lemma 5.2 to see that $\rho_1(\mathcal{M}^*) = \Theta$ and therefore, $\mathcal{M}$ is the Σ_1-core of $\mathcal{M}^*$. $\mathcal{M}$ is sound, transitive and $\mathcal{M}$ Σ_1-defines a set not in $\mathcal{H}^+$; so $\mathcal{M}$ has the form $J_\alpha[\mathcal{H}^*][E_\mathcal{M}]$ for some $\mathcal{H}^*, E_\mathcal{M}$. It's easy to see that $E_\mathcal{M} = E_\nu \restriction \mathcal{M}$.

Let N be suitable such that $\mathcal{M}, E_\mathcal{M} \in N$. $\forall^*_{\mu_N}\sigma$, recall that $\pi_\sigma : N_\sigma \to N$ be the uncollapse map. Let

$$\pi_\sigma(\mathcal{M}_\sigma, \mathcal{H}_\sigma, \Theta_\sigma, E_\sigma, \mathcal{H}^*_\sigma, \alpha_\sigma) = (\mathcal{M}, \mathcal{H}, \Theta, E_\mathcal{M}, \mathcal{H}^*, \alpha).$$

Recall the definition of the strategy Σ_σ, which is the π_σ-realizable strategy for $\mathcal{M}_\sigma$ defined after Lemma 5.2 for stacks below Θ_σ (this means Σ_σ does not act on stacks that involve applying E_σ and its images). Our goal is to define a strategy Σ^+_σ extending Σ_σ that acts on all countable stacks of normal form on $\mathcal{M}_\sigma$.

Lemma 5.16. *For μ_N-almost-all σ, there is an iteration strategy Σ^+_σ for $\mathcal{M}_\sigma$ with the following properties:*

1. *Σ^+_σ is a π_σ-realizable strategy that extends Σ_σ. This means $\Sigma_\sigma \subseteq \Sigma^+_\sigma$ and whenever $\vec{\mathcal{T}}$ is a (countable) stack of normal form according to Σ^+_σ, letting $i : \mathcal{M}_\sigma \to \mathcal{P}$ be the iteration embedding, then there is a map $k : \mathcal{P} \to \mathcal{M}$ such that $\pi_\sigma = k \circ i$.*
2. *Whenever $(\mathcal{Q}, \Lambda) \in I(\mathcal{M}_\sigma, \Sigma^+_\sigma)$, $\forall \alpha < \lambda^\mathcal{Q}$, $\Lambda_{\mathcal{Q}(\alpha)}$ is $\Gamma(\mathcal{M}_\sigma, \Sigma^+_\sigma)$-fullness preserving and has branch condensation. Hence Σ^+_σ is $\Gamma(\mathcal{M}_\sigma, \Sigma^+_\sigma)$-fullness preserving.*

Proof. We prove (1) (see Figure 1). The proof of (2) is just the proof of [8, Theorem 3.26] so we omit it; we just mention the key point in proving (2) is that $\Lambda_{\mathcal{Q}(\alpha)}$ for $\alpha < \lambda^\mathcal{Q}$ is a pullback of a strategy that is fullness preserving and has branch condensation.

Fix a σ. Suppose $i : \mathcal{M}_\sigma \to \mathcal{P}$ is the ultrapower map using E_σ. We describe how to obtain a π_σ-realizable strategy $\Sigma_{\mathcal{P}(\alpha)}$ for $\alpha < \lambda^\mathcal{P}$. We then let $\Sigma^-_\mathcal{P} = \oplus_{\alpha<\lambda^\mathcal{P}} \Sigma_{\mathcal{P}(\alpha)}$ and $\vec{\mathcal{T}}$ be a stack on $\mathcal{P}$ according to $\Sigma^-_\mathcal{P}$ with end model $\mathcal{Q}$. Let $j : \mathcal{P} \to \mathcal{Q}$ be the iteration map and $k : \mathcal{Q} \to \mathcal{R}$ be the ultrapower map by $E_\mathcal{Q}$; here we will write $E_\mathcal{P}, E_\mathcal{Q}$ etc for the image of E_σ under the appropriate embeddings. We describe how to obtain π_σ-realizable strategy $\Sigma_{\mathcal{Q}(\alpha)}$ for all $\alpha < \lambda^\mathcal{Q}$ and a π_σ-realizable strategy $\Sigma_{\mathcal{R}(\alpha)}$ for all $\alpha < \lambda^\mathcal{R}$. The construction of the strategy for this special case has all the ideas needed to construct the full strategy as for the general stack (in normal form), we

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[27] Note that E_ν measures all sets in $\mathcal{H}^{+-}|\lambda$ by Lemma 5.13.

simply repeat the arguments given below inductively.

Let $\tau \prec N$ be such that $\sigma, \vec{\mathcal{T}} \in \tau$.[28] μ_N-allmost-all τ have this property. Let $\pi_{\sigma,\tau} = \pi_\tau^{-1} \circ \pi_\sigma$. Working in N_τ, let $\mathcal{F}_{\sigma,\tau}$ be the direct limit system consisting of all non-dropping iterates of $(\mathcal{H}_\sigma, \Sigma_\sigma^- \cap N_\tau)$, let

$$\gamma_0 = i^{\Sigma_\sigma^-}_{\mathcal{H}_\sigma,\infty}(\lambda^{\mathcal{M}_\sigma}),$$

where $i^{\Sigma_\sigma^-}_{\mathcal{H}_\sigma,\infty}$ is the corresponding direct limit map.[29] Let $i^* : \mathcal{P} \to \mathcal{M}_\tau$ be such that

$$i^*(i(f)(\lambda^{\mathcal{M}_\sigma})) = \pi_{\sigma,\tau}(f)(\gamma_0).$$

By the definition of ν_σ, it's not hard to show i^* is elementary and $\pi_{\sigma,\tau} = i^* \circ i$ (so $\pi_\sigma = \pi_\tau \circ i^* \circ i$).

Note also that $i^*(E_\mathcal{P}) = E_\tau$. Now, let $(\mathcal{N}, \Lambda)$ be a point in the direct limit system giving rise to $\mathcal{H}_\tau$ such that $ran(i^* \restriction \lambda^\mathcal{P}) \subseteq ran(i^\Lambda_{\mathcal{N},\infty})$. There is some $s : \mathcal{P}|\lambda^\mathcal{P} \to \mathcal{N}$ such that $i^\Lambda_{\mathcal{N},\infty} \circ s = i^* \restriction \lambda^\mathcal{P}$. Then $\Sigma^-_\mathcal{P}$, the strategy of $\mathcal{P}$ for stacks that do not use $E_\mathcal{P}$ or its images, is simply the s-pullback of Λ. Note that by the choice of $(\mathcal{N}, \Lambda)$, Λ is a fullness preserving strategy with branch condensation. It's not hard to show that the definition of $\Sigma^-_\mathcal{P}$ doesn't depend on the choice of $(\mathcal{N}, \Lambda)$ and the choice of τ. We show why $\Sigma^-_\mathcal{P}$ doesn't depend on the choice of $(\mathcal{N}, \Lambda)$. Suppose $(\mathcal{N}, \Lambda)$, $(\mathcal{N}', \Lambda')$, $s : \mathcal{P}|\lambda^\mathcal{P} \to \mathcal{N}$, and $s' : \mathcal{P}|\lambda^\mathcal{P} \to \mathcal{N}'$ are as in the definition of $\Sigma^-_\mathcal{P}$, then we can compare $(\mathcal{N}, \Lambda), (\mathcal{N}', \Lambda')$ and get a common iterate $(\mathcal{S}, \Psi)$, where Ψ is the common tail of Λ and Λ'; this follows from positionality of Λ, Λ'. Let $i_{\mathcal{N},\mathcal{S}} : \mathcal{N} \to \mathcal{S}$ and $i_{\mathcal{N}',\mathcal{S}} : \mathcal{N}' \to \mathcal{S}$ be iteration maps. Note that $i_{\mathcal{N},\mathcal{S}} \circ s = i_{\mathcal{N}',\mathcal{S}} \circ s' =_{\text{def}} t$ and

$$\Lambda^s = (\Lambda')^{s'} = \Psi^t.$$

A similar argument shows that $\Sigma^-_\mathcal{P}$ does not depend on the choice of τ. Let $\mathcal{P}_\infty$ be the direct limit of $\Sigma^-_\mathcal{P}$ iterates of $\mathcal{P}|\delta^\mathcal{P}$ and $\pi_\mathcal{P} : \mathcal{P}_\infty \to \mathcal{H}_\tau$ be the natural map such that $\pi_\mathcal{P} \circ i^{\Sigma^-_\mathcal{P}}_{\mathcal{P},\infty} \restriction (\mathcal{P}|\delta^\mathcal{P}) = i^* \restriction (\mathcal{P}|\delta^\mathcal{P})$.

Now every element of $\mathcal{Q}$ has the form $j(f)(a)$ for some $f \in \mathcal{P}$ and $a \in \alpha(\vec{\mathcal{T}})^{<\omega}$, where $\alpha(\vec{\mathcal{T}})$ is the supremum of the generators used in $\vec{\mathcal{T}}$. We let $j^* : \mathcal{Q} \to \mathcal{M}_\tau$ be such that $j^*(j(f)(a)) = i^*(f)(\pi_\mathcal{P}(i^{\Sigma^-_\mathcal{Q}}_{\mathcal{Q},\infty}(a)))$. Hence $i^* = j^* \circ j$ and $\pi_\sigma = j^* \circ j \circ i$.

Finally, every element of $\mathcal{R}$ has the form $k(f)(\lambda^\mathcal{Q})$ for some $f \in \mathcal{Q}$. Let $h : \mathcal{M}_\tau \to Ult(\mathcal{M}_\tau, \nu_\tau)$ be the ultrapower map and $h^* : Ult(\mathcal{M}_\tau, \nu_\tau) \to \mathcal{M}$ be such that $\pi_\tau = h^* \circ h$. Then let $k^* : \mathcal{Q} \to Ult(\mathcal{M}_\tau, \nu_\tau)$ be such that $k^*(k(f)(\lambda^\mathcal{Q})) = h(j^*(f))(\lambda^{\mathcal{M}_\tau})$. It's easy to see that $h \circ j^* = k^* \circ k$. We can now derive the strategy $\Sigma^-_\mathcal{R}$ using $h^* \circ k^* \restriction \lambda^\mathcal{R}$ the same way we used $i^* \restriction \lambda^\mathcal{P}$ to derive the strategy $\Sigma^-_\mathcal{P}$. Again, it's easy to show that $\Sigma^-_\mathcal{R}$ is a π_σ-realizable strategy. The definition of $\Sigma^-_\mathcal{R}$ does not depend on the choice of τ.

In general, suppose $\vec{\mathcal{T}} = (\mathcal{T}_\alpha, \mathcal{N}_\beta : \alpha < \gamma, \beta \leq \gamma)$ is a countable stack on $\mathcal{M}_\sigma$ in normal form according to Σ^+_σ and $\mathcal{T}_\gamma$ is on $\mathcal{N}_\gamma$. We want to define Σ^+_σ on $\mathcal{T}_\gamma$. As part of the definition of Σ^+_σ, we have iteration map $i_{\mathcal{M}_\sigma,\mathcal{N}_\alpha} : \mathcal{M}_\sigma = \mathcal{N}_0 \to \mathcal{N}_\alpha$, a map $i : \mathcal{N}_\alpha \to \mathcal{M}_\tau$ for a sufficiently large τ that contains all relevant objects, i-pullback strategy Σ_α for $\mathcal{N}_\alpha|\lambda^{\mathcal{N}_\alpha}$, here $\lambda^{\mathcal{N}_\alpha} = i_{\mathcal{M}_\sigma,\mathcal{N}_\alpha}(\Theta_\sigma)$. If

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[28] Note that $\sigma, \vec{\mathcal{T}}$ are countable in τ.

[29] Here $\lambda^{\mathcal{M}_\sigma} = \lambda^{\mathcal{H}_\sigma} = \Theta_\sigma = \delta^{\mathcal{H}_\sigma}$ by the regularity of Θ_σ in $\mathcal{M}_\sigma, \mathcal{H}_\sigma$.

$$\begin{tikzcd}
 & & & \mathcal{M}, \nu_{\mathcal{M}} \\
 & & \mathcal{M}_\tau, \nu_\tau \arrow[ur, "\pi_\tau"] \arrow[r, "h"] & Ult(\mathcal{M}_\tau, \nu_\tau) \arrow[u, "h^*"'] \\
\mathcal{M}_\sigma, \nu_\sigma \arrow[urr, "\pi_{\sigma,\tau}"] \arrow[r, "i"'] & \mathcal{P}, \nu_{\mathcal{P}} \arrow[ur, "i^*"'] \arrow[r, "{\vec{\mathcal{T}}, j}"'] & \mathcal{Q}, \nu_{\mathcal{Q}} \arrow[u, "j^*"] \arrow[r, "k"'] & \mathcal{R}, \nu_{\mathcal{R}} \arrow[u, "k^*"']
\end{tikzcd}$$

Figure 1: The construction of Σ_σ^+

$\mathcal{T}_\gamma = \langle \mathcal{N}_\alpha, E_\alpha \rangle$, where $E_\alpha = i_{\mathcal{M}_\sigma, \mathcal{N}_\alpha}(E_\sigma)$, then we can define maps $k^* : \mathrm{Ult}(\mathcal{N}_\alpha, E_\alpha) \to \mathrm{Ult}(\mathcal{M}_\tau, E_\tau)$, $h : \mathcal{M}_\tau \to \mathrm{Ult}(\mathcal{M}_\tau, E_\tau)$, and $h^* : \mathrm{Ult}(\mathcal{M}_\tau, E_\tau) \to \mathcal{M}$ as above and derive a strategy $\Sigma_{\alpha+1}$ for $\mathcal{N}_{\alpha+1} | \lambda^{\mathcal{N}_{\alpha+1}}$, where $\mathcal{N}_{\alpha+1} = \mathrm{Ult}(\mathcal{N}_\alpha, E_\alpha)$. We then let $\Sigma_{\alpha+1} \subset \Sigma_\sigma^+$. Suppose $\mathcal{T}_\gamma$ is below $\lambda^{\mathcal{N}_\alpha}$. Then we use $\Sigma_\alpha \subset \Sigma_\sigma^+$ to choose a branch b for $\mathcal{T}_\gamma$ and a map $j^* : \mathcal{N}^{\mathcal{T}^\frown b} \to \mathcal{M}_\tau$ such that $j^* \circ i_b^{\mathcal{T}} = i_\alpha$.

This completes the construction of Σ_σ^+ and hence the proof of Lemma 5.16. Note it also follows that Σ_σ^+ extends Σ_σ.

□

By a ZFC-comparison argument ([8, Section 2.7]) and the fact that Σ_σ^+ is $\Gamma(\mathcal{M}_\sigma, \Sigma^+)$-fullness preserving, an iterate of Σ_σ^+ has branch condensation. Without loss of generality, we may assume Σ_σ^+ has branch condensation.

Since $\rho_1(\mathcal{M}_\sigma) \leq \Theta_\sigma$, we let $A \subseteq \Theta_\sigma$ be a set Σ_1 definable over $\mathcal{M}_\sigma$ but not in $\mathcal{H}_\sigma^+$.[30] Say

$$\alpha \in A \Leftrightarrow \mathcal{M}_\sigma \vDash \psi[\alpha, s, p_1^{\mathcal{M}_\sigma}], \tag{5.14}$$

for some $s \in \Theta_\sigma^{<\omega}$. Recall that $\mathcal{M}_\sigma \vDash \Theta_\sigma$ is measurable as witnessed by E_σ. We can define a direct limit system $\mathcal{F} = \{(\mathcal{Q}, \Lambda) \mid (\mathcal{Q}, \Lambda) \equiv_{DJ} (\mathcal{M}_\sigma, \Sigma_\sigma^+)\}$ [31]. Let $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$ and let $i_{\mathcal{M}_\sigma, \infty} : \mathcal{M}_\sigma \to \mathcal{M}_\infty$ be the iteration embedding. We have that $\mathrm{HOD}|\gamma_\sigma \lhd \mathcal{M}_\infty \in \mathrm{HOD}$ and $\rho_1(\mathcal{M}_\infty) \leq \gamma_\sigma$. Let A_∞ be defined over $\mathcal{M}_\infty$ the same way A is defined over $\mathcal{M}_\sigma$, i.e.

$$\alpha \in A_\infty \Leftrightarrow \mathcal{M}_\infty \vDash \psi[\alpha, i_{\mathcal{M}_\sigma, \infty}(s), p_1^{\mathcal{M}_\infty}]. \tag{5.15}$$

Since A_∞ is OD, A is ordinal definable from $(\mathcal{H}_\sigma, \Sigma_\sigma^-)$. This is because from 5.14 and 5.15, $\alpha \in A$ if and only if $i_{\mathcal{H}_\sigma, \infty}^{\Sigma_\sigma^-}(\alpha) \in A_\infty$. By $\mathsf{MC}(\Sigma_\sigma^-)$ (which follows from our smallness assumption (†) and the HOD analysis done in [10]), $A \in \mathcal{H}_\sigma^+$. Contradiction. □

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[30] From the fact that $\mathcal{H}^+ = [\sigma \mapsto \mathcal{H}_\sigma^+]_{\mu_\Omega}$ and Los theorem, we can conclude that $\forall^*_{\mu_N} \sigma$ there is A Σ_1-definable over $\mathcal{M}_\sigma = \mathcal{M}_{\sigma \cap \Omega}$ such that $A \notin \mathcal{H}_\sigma^+$.

[31] We take Σ_0-ultrapowers for extenders with critical points $\geq$ the image of Θ_σ under iteration embeddings by Σ_σ and Σ_1-ultrapowers otherwise

Lemma 5.17. $\mathcal{H}^{++}(\Gamma) \cap \wp(\mathbb{R}) = \Gamma$ *and* $\mathcal{H}^{++}(\Gamma) \vDash \mathsf{AD}_{\mathbb{R}}+$ *there is an* $\mathbb{R}$*-complete normal measure on* Θ.

Proof. First note that no $\mathcal{H}^{++}|\lambda \lhd \mathcal{M} \lhd \mathcal{H}^{++}$ is such that $\rho_\omega(\mathcal{M}) \leq \Theta$. The equality in the conclusion of the lemma follows from Theorem 4.1 with $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ playing the role of $\mathcal{H}$ and $\mathcal{H}^{++}$ playing the role of $\mathcal{H}^{+}$. Note that $\mathcal{H}^{++} \vDash$ "Θ is regular" and in fact $\mathcal{H}^{++}(\Gamma) \vDash$ "Θ is regular" since Θ is regular in V. The $\mathbb{R}$-complete normal measure on Θ in $\mathcal{H}^{++}(\Gamma)$ comes from ν from the proof of Theorem 2.4 in [3]. The proof uses the fact that every $A \in \Gamma$ can be added to $\mathcal{H}^{++}$ via a forcing of size $< \Theta$. This means every $A \subseteq \Theta$ in $\mathcal{H}^{++}(\Gamma)$ is in some generic extension of $\mathcal{H}^{++}$ via a forcing of size $< \Theta$ and hence is measured by the canonical extension of ν. The normality comes from normality of ν. The $\mathbb{R}$-completeness of the induced measure then follows from [3, Theorem 2.4]. □

This completes the proof of Theorem 1.6.

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