DODD SOLIDITY FOR MOUSE PAIRS

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Abstract

This is the second of two papers on the fine structure of HOD in models of the Axiom of Determinacy (AD). Let $M \vDash \mathsf{AD}_{\mathbb{R}} + \mathsf{V} = \mathsf{L}(\wp(\mathbb{R}))$. [4] shows that under a natural hypothesis on the existence of iteration strategies, the basic fine structure theory for pure extender models goes over to HOD^M. In this paper, we analyze the Dodd parameters and prove the analogs of [7, Theorems 1.1, 1.2] for lbr hod pairs. The proof of these results relies on the condensation theorems proved in [3]. In a sequel, we shall use these theorems to show that in HOD^M, $\square_\kappa$ holds iff κ is not subcompact.

1. INTRODUCTION

Let (M, Σ) be a mouse pair in the sense of [4]), so that M is either a pure extender premouse or a least branch premouse. Let $F = \dot{F}^M$ be the top extender of M, $\mu = \mathrm{crt}(F)$, $\tau = \mu^{+,M}$, and $\lambda_F = i_F^M(\mu)$ be the image of μ under the F-ultrapower map. One can identify F with $i_F^M \restriction M||\tau$, so we set $\mathrm{dom}(F) = M||\tau$. Let us recall some definitions from [6].

Definition 1.1. (1) If G is a short extender, then η is a *cutpoint of* G if and only if $\eta \leq \lambda_G$ and for all $a \in [\eta]^{<\omega}$ and $f \in \mathrm{dom}(G)$, $i_G(f)(a) < \eta$.

(2) Let M be active, with last extender $F = \dot{F}^M$; then

(i) M (or F) has *type A* iff there is no $\eta < \lambda_F$ such that η is a cutpoint of F,

(ii) M (or F) has *type B* iff there is a largest $\eta < \lambda_F$ such that η is a cutpoint of F. We write λ^*_M for this η.

(iii) M (or F) has *type C* iff λ_F is a limit of cutpoints of F.

(3) M satisfies the Jensen Initial Segment Condition (ISC) if and only if whenever $\eta < \lambda_F$ is a cutpoint of F, then there is a γ such that $E^M_\gamma \restriction \eta = F \restriction \eta$.

⊣

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Dodd solidity is a form of condensation that applies to type B premice, so they are our focus here.

Definition 1.2. Let M be active and $F = \dot{F}^M$; then

(a) $\tau_M = \mathrm{crt}(F)^{+,M}$,

(b) letting $\tau = \tau_M$, for $s \subseteq lh(F) - \tau$ finite and $\alpha \geq \tau$, we say that α *is an s-generator (of F)* iff $\alpha = \mathrm{crt}(\pi)$ where $\pi : \mathrm{Ult}(M||\tau, F \upharpoonright \alpha \cup s) \to \mathrm{Ult}(M||\tau, F)$ is the canonical factor map.

⊣

Definition 1.3. Let M be a pfs premouse of degree 0; then M is *strongly 1-sound* iff M is 1-solid and $M = \mathrm{Hull}_1^M(\rho_1(M) \cup p_1(M))$. ⊣

In the terminology of [4], M is strongly 1-sound iff $M = \bar{\mathfrak{C}}(M)^-$, that is M is its own strong core, but with the degree changed back to 0. M is 1-sound iff M is 1-solid and $M = \mathrm{Hull}_1^M(\rho_1(M) \cup \{\rho_1(M), p_1(M)\})$ iff $M = \mathfrak{C}(M)^-$.

Definition 1.4. Let M be active and 1-sound, and $F = \dot{F}^M$. Let $\tau_M = \mathrm{crt}(F)^{+,M}$. We inductively define the sequence $\langle d_0, \ldots, d_n \rangle$ and ρ^* as follows. Let

$$d_0 = \text{the largest generator of } F\text{, if it exists;}$$

otherwise, let ρ^* be the sup of the generators of F. If $\{d_0, \ldots, d_i\}$ are defined, let

$$d_{i+1} = \text{the largest } \{d_0, \ldots, d_i\} \text{ generator of } F\text{, if it exists;}$$

otherwise, let ρ^* be the sup of the $\{d_0, \ldots, d_i\}$-generators of F.

Let

$$d_M = \langle d_0, \ldots, d_n \rangle$$

for n such that $d_0, \ldots, d_n$ exist and ρ^* is the supremum of $\tau_M \cup$ the $\{d_0, \ldots, d_n\}$-generators of F. d_M is the *Dodd parameter* of M and $\rho^*_M = \rho^*$ is the *Dodd projectum* of M. ⊣

We also write d_F for d_M. d_M may be empty; if not, then it is a strictly decreasing sequence of ordinals. We write τ for τ_M and μ for $\mathrm{crt}(F)$ below. In either case, $\rho^*_M \geq \tau$.

We assumed in 1.4 that M is 1-sound, and this is important. The definition is only appropriate for 1-sound premice.

Remark 1.5. ξ is an s-generator of F iff for all finite $a \subset \xi$ and $f \in M$, $\xi \neq i^M_F(f)(a \cup s)$. By definition, $\min(d_M) > \mu$ and in fact $\min(d_M) > \tau$. $F \upharpoonright \rho^*_M \cup d_M$ generates all of F; more precisely, every $x \in M$ is of the form $i^M_F(f)(a \cup d_M)$ for some finite $a \subset \rho^*_M$ and some $f \in M$.

The following proposition is an immediate corollary of the definitions given above. We shall prove it in the next section.

Proposition 1.6. *Assume M is an active Jensen premouse or an active lpm. Suppose M is 1-sound and* $\deg(M) = 0$*; then*

(a) Suppose $\eta < \rho^*_M$ and $\tau < \rho^*_M$, then $F \restriction (d_M \cup \eta) \in M$, and

(b) $\rho^*_M = max\{\tau, \rho_1(M)\}$.

Definition 1.7. Let M be active and of type B and degree 0; then

(a) M (or d_M) is *Dodd solid at i* iff $i \in \mathrm{dom}(d_M)$, and $\dot{F}^M \restriction (d_M(i) \cup \{d_M \restriction i\}) \in M$;

(b) M (or d_M) is *Dodd solid* iff M is Dodd solid at all $i \in \mathrm{dom}(d_M)$.

$\dashv$

The main result of our paper is the following theorem.

Theorem 1.8 (Dodd solidity)**.** *Assume* AD^+*. Let (M, Σ) be a mouse pair such that M is active of type B, with* $\deg(M) = 0$*; then*

(1) if M is strongly sound, then M is Dodd solid, and

(2) if M is 1-sound, then M is Dodd solid at all i such that $d_M(i) > \rho_1(M)$.

Remark 1.9. (a) The proper initial segments of a pfs premouse must be 1-sound, but they may not be strongly 1-sound. This is why we have isolated conclusion (2) of the theorem.

(b) In (2), one cannot demand that d_M be Dodd solid at i when $d_M(i) = \rho_1(M)$. For let M be 1-sound but not strongly 1-sound. Let $N = \bar{\mathfrak{C}}(M)$. Suppose that $\rho_1(M) \geq \tau$. Since $M = \mathrm{Ult}_0(N, D)$ where D is the order zero measure of N on $\rho_1(M)$, we have that $\rho_1(M) > \tau$. The Dodd-solidity of N easily implies that $d_M = i_D(d_N) \cup \{\rho_1(M)\}$, with the solidity witness for $d_N(i)$ being mapped by i_D to the solidity witness for $d_M(i)$. Clearly, $\dot{F}^N \notin M$, so M is not Dodd solid at $\mathrm{dom}(d_N)$.

The argument in 1.9(b) shows that 1.8(1) implies 1.8(2). More generally, one can define d_M for arbitrary possibly unsound active M by

Definition 1.10. For M active type B with $\deg(M) = 0$,

a) $\hat{\rho}_M$ is the least $\alpha \geq \tau$ such that there is a finite d such that $F \restriction (\alpha \cup d) \notin M$.

(b) $\hat{d}_M$ is the $<^*$-least d such that $F \restriction (\hat{\rho}_M \cup d) \notin M$.

$\dashv$

Here $<^*$ is the lexicographic order on descending sequences of ordinals. The definition of $\hat{d}_M$ is parallel to the usual definition of the standard parameter. Our definition of d_M in 1.4 is really only interesting in the case that M is strongly 1-sound, in which case $d_M = \hat{d}_M$. In general, $\hat{d}_M = \pi(d_N)$, where N is the strong core of M and π is the anticore map; moreover π maps the Dodd solidity witnesses for d_N to corresponding witnesses for $\hat{d}_M$. We give the simple proofs of these facts in the last section of the paper.

Dodd solidity is a form of condensation appropriate to type B premice.[1] It is a strengthening of the ms-initial segment condition. The first author proved Dodd solidity for ms-indexed mice below superstrongs in [1][2] and Martin Zeman proved it for λ-indexed pure extender mice in [7].[3] Our proof here applies to λ-indexed mice, and it borrows in significant ways from Zeman's [7]. The main new element is that it incorporates comparison of iteration strategies. Nevertheless, when specialized to the case of pure extender mice, where strategy comparison is not needed, our proof seems to be simpler than Zeman's.[4]

Dodd solidity is important in the proof that Jensen's square principle holds in mice. In a sequel to this paper, we shall use it and the Condensation Lemma in [3] to prove the following.

Theorem 1.11 (AD^+)**.** *Let (M, Σ) be a mouse pair. Let κ be a cardinal of M such that $M \models$ "κ^+ exists"; then in M, the following are equivalent.*

1. $\square_\kappa$.
2. $\square_{\kappa,<\kappa}$.
3. *κ is not subcompact.*
4. *The set of $\nu < \kappa^+$ such that $M|\nu$ is extender-active is non-stationary in κ^+.*

The paper is organized as follows. Section 2 recalls some basic facts about mouse pairs, and proves some elementary facts about Dodd parameters, including Zeman's characterization of them as minimal generating parameters in the language of coherent structures. Section 3 proves Theorem 1.8. In Section 4, we describe the natural generalization of Theorem 1.8 to mice that are not 1-sound.

2. PRELIMINARIES

We recall some basic facts about mouse pairs and Dodd parameters.

2.1. Mouse pairs

Two of the main definitions from [4] are

[1]If M is 1-sound and type A, then $\rho^* = \tau$ and $d_M = p_M$. If M is 1-sound and type C, then $\rho^* = \lambda_F$ and $d_M = \emptyset$. Thus Dodd solidity is trivial (modulo the solidity of p_M!) in these cases.

[2]This would be roughly equivalent to proving parameter solidity for type A Jensen mice, but ms-indexing adds some difficulty, centering on the stronger ms-ISC needed for comparison. Some version of that difficulty re-appears in the proof of Dodd solidity for Jensen mice of type B,

[3]Schlutzenberg proved a strengthening of Dodd solidity for ms-mice in [2, Theorem 10.1].

[4]Formally, we give our proof for pfs premice, but it goes over without change to the λ-indexed pure extender mice of [7].

Definition 2.1. (M, Ω) is a *pure extender pair* with scope H_δ iff

(a) M is a pure extender pfs premouse.

(b) Ω is a complete (ω, δ) iteration strategy for M[5], and

(c) Ω is internally lift-consistent, quasi-normalizes well, and has strong hull condensation.[6]

$\dashv$

Definition 2.2. (M, Ω) is a *least branch hod pair (lbr hod pair)* with scope H_δ iff

(a) M is an least branch premouse (lpm).

(b) Ω is a complete (ω, δ) iteration strategy for M,

(c) Ω is internally lift-consistent, quasi-normalizes well, and has strong hull condensation, and

(d) Ω is pushforward consistent, that is, if s is by Ω with last model N, then $\dot{\Sigma}^N \subseteq \Omega_s$, where $\Omega_s(t) = \Omega(s^\frown t)$.[7]

$\dashv$

Definition 2.3. (M, Ω) is a *mouse pair* iff it is either a pure extender pair or an lbr hod pair. $\dashv$

In our context below, M will be countable, Ω will have scope H_{ω_1}, and we shall assume AD^+. These are the simplest hypotheses under which to develop the theory of mouse pairs, and the main results of [4] are all proved under them.

For the sake of definiteness, we shall prove Theorem 1.8 for lbr hod pairs. The proof also works for pure extender pairs, but by re-arranging a few things[8], one can avoid strategy comparison in that case, and thereby simplify it significantly.

2.2. λ-separated iteration trees

The iteration trees we use below are λ-separated plus trees. The notion is defined in [4, Section 4.4], and we summarize it here. Suppose M is a pfs premouse and E is an extender on the M-sequence, then

- E^+ is the extender with generators $\lambda_E \cup \{\lambda_E\}$ that represents $i_F^{\mathrm{Ult}(M,E)} \circ i_E^M$, where F is the order zero total measure on λ_E in $\mathrm{Ult}(M, E)$.
- $\hat{\lambda}(E^+) = \lambda_E$.
- $\mathrm{lh}(E^+) = \mathrm{lh}(E)$.

[5]See [4, 4.6.3].
[6]See [4, 5.4.4, 7.1.1, 7.1.9].
[7]See [4, 9.2.1].
[8]Mainly, one must replace use of the full Dodd-Jensen Lemma by use of the Weak Dodd-Jensen Lemma. Since we are trying to prove a first order property of our pure extender mouse M, this can be done.

We say that an extender G is *of plus type* if $G = E^+$ for some extender E on the sequence of a pfs premouse M; we let $G^- = E$. In general, if E is an extender (of plus type or not)

- we let $\varepsilon(E) = \mathrm{lh}(E)$ if E is of plus type; otherwise, $\varepsilon(E) = \lambda(E)$.
- if E is on the sequence of some premouse, then
  (i) $\hat{\lambda}(E) = \lambda(E) = \hat{\lambda}(E^+)$,
  (ii) $E^- = E$.

The *extended M-sequence* consists of all E such that E^- is on the M-sequence.

A *plus tree* $\mathcal{T}$ on a pfs premouse is like an ordinary normal tree, except that

(i) We only require that $E_\alpha^\mathcal{T}$ be on the extended $\mathcal{M}_\alpha^\mathcal{T}$ sequence,

(ii) $E_\alpha^\mathcal{T}$ is applied to the longest possible initial segment of $\mathcal{M}_\beta^\mathcal{T}$, where β is least such that $\mathrm{crt}(E_\alpha^\mathcal{T}) < \hat{\lambda}(E_\beta^\mathcal{T})$, and

(iii) the length-increasing condition is weakened slightly.[9]

See [4, Definition 4.4.3] for the complete definition.

A λ-*separated* tree is a plus tree in which every extender used along the tree is of plus type. The weakening in (iii) above does not affect λ-separated trees; that is, the lengths of the extenders used in a λ-separated tree are strictly increasing. Moreover, quasi-normalization coincides with embedding normalization on stacks of λ-separated plus trees. [4, Section 8.1] shows that λ-separated trees are enough for comparisons. For these and other reasons it is convenient to restrict one's attention to the way an iteration strategy Σ acts on stacks of λ-separated trees. By Lemma 9.3.2 of [4], if (P, Σ) is a mouse pair, then Σ is determined by its action on countable λ-separated trees.

We shall use the notation associated to extender trees from [4, Section 6.3].

Definition 2.4. Let $\mathcal{T}$ be an iteration tree and $\alpha \leq_T \beta$; then $e_{\alpha,\beta}^\mathcal{T}$ is the sequence of extenders $\langle E_{\eta+1}^\mathcal{T} \mid \alpha <_T \eta + 1 \leq_T \beta \rangle$ used in $\mathcal{T}$ on the branch from α to β (listed in increasing order). We let $e_\alpha^\mathcal{T} = e_{0,\alpha}^\mathcal{T}$. ⊣

If $\mathcal{T}$ is λ-separated and its base model is a premouse, then $e_{\alpha,\beta}^\mathcal{T}$ can be recovered from the partial branch embedding $\hat{i}_{\alpha,\beta}^\mathcal{T}$.[10] The recovery process relies on the Jensen ISC holding for the models in $\mathcal{T}$.

[9] The length-increasing condition is enough to guarantee that $T-pred(\alpha+1)$ is the least β such that $\mathrm{crt}(E_\alpha^\mathcal{T}) < \lambda(E_\beta^\mathcal{T})$. Thus none of the generators of a plus extender E, including the generator $\hat{\lambda}(E)$, are moved later on a branch in which E has been used.

[10] By successively extracting E^+, for E the first missing whole initial segment of the current tail factor. See 3.25(2) below.

2.3. The mouse pair order

The basic results of inner model theory, such as the Comparison Lemma and the Dodd-Jensen Lemma, are better stated and proved as results about mouse pairs than as results about mice, with the notions of elementary submodel and iterate adjusted so that this is possible. For example, if (H, Ψ) and (M, Σ) are mouse pairs, then $\pi\colon (H, \Psi) \to (M, \Sigma)$ is elementary (resp. nearly elementary) iff π is elementary (nearly elementary) as a map from H to M, and $\Psi = \Sigma^\pi$. We say that (M, Σ) is an *iterate of* (H, Ψ) iff there is a stack s on H such that s is by Ψ, and $\Sigma = \Psi_s$. It is a *non-dropping iterate* iff the branch H-to-M does not drop. Assuming AD^+ and that our pairs have scope HC, [4] proves the following:

(1) If (M, Σ) is a mouse pair, H is a premouse, and $\pi\colon H \to M$ is nearly elementary, then (H, Σ^π) is a mouse pair.

(2) If (H, Ψ) is a mouse pair, and (M, Σ) is a non-dropping iterate of (H, Ψ), then the iteration map $i_s\colon (H, \Psi) \to (M, \Sigma)$ is elementary in the category of pairs.

(3) (Dodd-Jensen) If (H, Ψ) is a mouse pair, (M, Σ) is an iterate of (H, Ψ) via the stack s, and $\pi\colon (H, \Psi) \to (M, \Sigma)$ is nearly elementary, then

  (i) the branch H-to-M of s does not drop, and

  (ii) for all $\eta < o(H)$, $i_s(\eta) \leq \pi(\eta)$, where i_s is the iteration map.

(4) (Mouse order) Let $(H, \Psi) \leq^* (M, \Sigma)$ iff there is a nearly elementary embedding of (H, Ψ) into some iterate of (M, Σ); then $\leq^*$ is a prewellorder of the mouse pairs with scope HC in each of the two types.

The prelinearity of the mouse pair order is the content of the Comparison Lemma for mouse pairs. For pure extender pairs, it is proved in Theorem 8.4.5 of [4]. The proof for lbr hod pairs is basically the same; it is Theorem 9.5.10 of [4].

2.4. Dodd parameters and coherent structures

The language $\mathcal{L}$ of lpms has symbols $\in, \dot{E}, \dot{F}, \dot{\Sigma}, \dot{B}$, and $\dot{\gamma}$. Here $\dot{F}^M$ is the last (top) extender if M is active, and $\dot{F}^M = \emptyset$ otherwise. $\dot{E}^M$ is the sequence of extenders previously added, and $\dot{\Sigma}^M$ and $\dot{B}^M$ contain information about an iteration strategy for M.

If M is active of type B, then letting

$$\lambda^*_M = \text{largest cutpoint of } \dot{F}^M,$$

$F = \dot{F}^M$, we have

$$\dot{\gamma}^M = (\lambda^*_M)^{+,\mathrm{Ult}(M, F \restriction \lambda^*_M)}.$$

We usually write γ_M for $\dot{\gamma}^M$. The Jensen Initial Segment Condition (Jensen ISC) requires that γ_M indexes the largest cutpoint of F, that is

$$\dot{E}^M_{\dot{\gamma}} \restriction \lambda^*_M = F \restriction \lambda^*_M.$$

If M has type A or type C, then $\dot{\gamma}^M = 0$.

The language of coherent structures is $\mathcal{L}$ but without $\dot{\gamma}$, that is

$$\mathcal{L}^* = \mathcal{L} - \{\dot{\gamma}\}.$$

If M is an lpm, then h_M is its canonical $\Sigma_1^{\mathcal{L}}$ Skolem function, and h^*_M is its canonical $\Sigma_1^{\mathcal{L}^*}$-Skolem function. For $X \subseteq M$, we let

$$h_M[X] = \mathrm{Hull}_1^M(X) = \{h_M(\varphi, a) \mid a \in X^{<\omega} \wedge \varphi \in \Sigma_1^{\mathcal{L}}\},$$
$$h^*_M[X] = \mathrm{Hull}_1^{*,M}(X) = \{h^*_M(\varphi, a) \mid a \in X^{<\omega} \wedge \varphi \in \Sigma_1^{\mathcal{L}^*}\},$$

and let $\mathrm{cHull}_1^M(X)$ and $\mathrm{cHull}_1^{*,M}(X)$ be the transitive collapses of these hulls.[11] A $\Sigma_1^{\mathcal{L}}$ hull of M is just a $\Sigma_1^{\mathcal{L}^*}$ hull that has γ_M in it, of course. Including $\dot{\gamma}$ in $\mathcal{L}$ guarantees that $\Sigma_1^{\mathcal{L}}$ hulls of M continue to satisfy the Jensen ISC.[12] $\Sigma_1^{\mathcal{L}^*}$ hulls may fail the Jensen ISC, and this fact is the main reason that the proof of Dodd solidity involves difficulties beyond those solved by the proof of solidity for the usual standard parameter.

The definitions of h_M and h^*_M involve stratifying Σ_1 relations according to where the witnesses appear. For active M, this involves stratifying $\dot{F}^M$ via its fragments, and the resulting levels M^β are not premice. Here

Definition 2.5. Let M be an active lpm and $\beta < o(M)$, then M^β is the $\mathcal{L}^*$ structure that agrees with $M||\beta$, except that $\dot{F}^{M^\beta} = \dot{F}^M \cap M||\beta$. $\dashv$

The map $\beta \mapsto M^\beta$ is $\Sigma_1^{\mathcal{L}^*}$ over M.

Definition 2.6. For premice M of degree 0, we let $\rho_M = \rho_1(M)$ and $p_M = p_1(M)$. We say that M is *α-sound* iff $\alpha \geq \rho_M$ and $M = h_M[\alpha \cup p_M]$. $\dashv$

The following lemmas characterize the Dodd parameter in terms of $\Sigma_1^{\mathcal{L}^*}$ definability.

If M is an lpm, we write $\mathrm{Th}_1^M(X)$ for the $\Sigma_1^{\mathcal{L}}$-theory in M of parameters in X, and $\mathrm{Th}_1^{*,M}(X)$ for the $\Sigma_1^{\mathcal{L}^*}$-theory in M of parameters in X. When M is clear from the context, we omit it from the notation.

Definition 2.7. Let M be an active lpm; then $\mu_M = \mathrm{crt}(\dot{F}^M)$ and $\tau_M = \mu_M^{+,M}$. $\dashv$

[11]The notation suggests that $h_M[X]$ is the image h_M"X of X under h_M, which is not literally true, but "$h_M[X]$" is less cluttered than "$\mathrm{Hull}_1^M(X)$".

[12]There is some work required to show this in the case that M has type C, but the proof is elementary.

Lemma 2.8. *Let M be an active lpm with $\deg(M) = 0$. Let $F = \dot{F}^M$ and $\tau = \tau_M$. Let $s \subseteq \mathrm{lh}(F) - \tau$ be finite, $\alpha \geq \tau$ and*

$$\pi : \mathrm{Ult}(M||\tau, F \restriction \alpha \cup s) \to \mathrm{Ult}(M||\tau, F)$$

Let $N = \mathrm{Ult}(M||\tau, F \restriction \alpha \cup s)$ and G be the Jensen completion of $F \restriction \alpha \cup s$; then π is $\Sigma_1^{\mathcal{L}^*}$-elementary as a map

$$\pi : (N, G) \to M.$$

Proof. π is $\Sigma_1^{\mathcal{L}}$ elementary from N to $M||o(M)$ by Los's theorem. We need to see that π maps fragments of G to fragments of F. Let $\mathrm{crt}(F) = \mu$, and $\mu < \xi < \tau$ be such $\rho_\omega(M|\xi) = \mu$. For F_μ almost every ν, we have $h_{M|\xi}[\nu \cup p(M|\xi)]$ transitively collapses to $M|\xi_\nu$ for some ξ_ν with

$$\sigma_\nu : M|\xi_\nu \to M|\xi.$$

the inverse of the collapse. Then

$$i_G^N \restriction (M|\xi) = [\{\mu\}, \lambda\nu.\sigma_\nu]_G^N$$

and

$$i_F^M \restriction (M|\xi) = [\{\mu\}, \lambda\nu.\sigma_\nu]_F^M.$$

Hence $\pi(i_G^N \restriction (M|\xi)) = i_F^N \restriction (M|\xi)$ as desired. ☐

Remark 2.9. The proof of Lemma 2.8 shows that $F \restriction (\alpha \cup s)$ is easily intertranslatable with $\mathrm{Th}_1^{*,M}(\alpha \cup s)$. The proof does not show that π is $\Sigma_1^{\mathcal{L}}$-elementary, or even that (N, G) satisfies the Jensen ISC. We shall have to deal with this difficulty in the proof of Theorem 1.8.

Remark 2.10. Suppose $F = \dot{F}^M$ and M is solid at i. It is natural to take $F \restriction (d_i \cup \{d_0, ..., d_{i-1}\})$ to be the Dodd solidity witness for M at i. Equivalently, using the translation in the last remark, we might take $\mathrm{Th}_1^{*,M}(d_i \cup \{d_0, ..., d_{i-1}\})$ to be the witness. Zeman [7] takes $\mathrm{cHull}_1^{*,M}(d_i \cup \{d_0, ..., d_{i-1}\})$ to be the *standard Dodd solidity witness* at i. We don't need to consider "generalized witnesses", because the standard Dodd solidity witnesses are preserved by Σ_0 ultrapowers. That is true because the natural prewellorder on $\mathrm{Th}_1^{*,M}(X)$ has cofinality τ_M if M is active, and τ_M is a successor cardinal in M.

The following lemma gives an alternative characterization of generators for F that we shall use many times.

Lemma 2.11. *For $s \subseteq \mathrm{lh}(F) - \tau$ finite, $\alpha \geq \tau$. The following are equivalent:*

1. *α is an s-generator.*

2. *$\alpha \notin h_M^*[\alpha \cup s]$.*

3. *$\alpha = crt(\pi)$ where $\pi : \mathrm{Ult}(M||\tau, F \restriction \alpha \cup s) \to \mathrm{Ult}(M||\tau, F)$ is the canonical embedding.*

Proof. (1) ⇔ (3) is the definition. (3) ⇔ (2) follows easily from Lemma 2.8. ☐

We characterize ρ^*_M in the case that M is 1-sound:

Proposition 2.12. *Assume M is an active, 1-sound lpm then*

(a) Suppose $\eta < \rho^*_M$ and $\tau < \rho^*_M$, then $F \restriction (d_M \cup \eta) \in M$, and

(b) $\rho^*_M = max\{\tau, \rho_1(M)\}$.

Proof. We first prove (b) implies (a), let $\tau \leq \eta < \rho^*_M$. By part (b), $\rho^*_M = \rho_1(M)$. $F \restriction (\eta \cup d_M)$ can be coded by a $\Sigma_1^{\mathcal{L}}$-subset of η. Since $\eta < \rho(M)$, $F \restriction (\eta \cup d_M) \in M$, as desired.

For part (b), we first claim that $\rho(M) = \rho_1(M) \leq \rho^*_M$. First, recall that $\tau \leq \rho^*_M$. By the definition of d_M, $M = h^*_M[\rho^*_M \cup d_M]$. Thus $\mathrm{Th}_1^{*,M}(\rho^*_M \cup d_M) \notin M$ and hence $\mathrm{Th}_1^M(\rho^* \cup d_M) \notin M$. This implies $\rho_1(M) \leq \rho^*$. Now suppose $\tau < \rho^*_M$. We claim that $\rho^*_M = \rho_1(M)$. Suppose $\rho_1(M) < \rho^*_M$. Since ρ^*_M is a limit of d_M-generators and $M = h^*_M[\rho^*_M \cup d_M]$, there is a d_M-generator $\eta \in (\rho_1(M), \rho^*_M)$ such that $p(M) \cup \{\gamma_M\} \subset h^*_M[\eta \cup d_M]$. This implies $h^*_M[\rho_1(M) \cup \{p(M) \cup \{\gamma_M\}\}] = h_M[\rho_1(M) \cup \{p(M)\}] = M \subseteq h^*_M[\eta \cup d_M]$. The second equality follows from 1-soundness of M. The last inclusion contradicts the fact that η is a d_M-generator.

$\square$

Definition 2.13. The *parameter order* $<^*$ is the lexicographical order on finite descending sequences of ordinals. $\dashv$

If M is 1-sound, then p_M is the $<^*$-least parameter s such that $M = h_M[\rho_M \cup s]$. The lemmas above show that if M is 1-sound and active, then d_M is the $<^*$ least s such that $h^*_M[\rho^*_M \cup s] = M$.[13] This analogy breaks down if M is not 1-sound, because d_M has been defined as a minimal generating parameter, and p_M is defined as a minimal parameter from which one can $\Sigma_1^{\mathcal{L}}$-define a new subset of ρ_M.

One can define d_M in a way that preserves the analogy with $p_1(M)$ in the unsound case. We discuss this in the last section.

2.5. Zeman's exchange lemma

We prove a result of Zeman that shows just how d_M is related to p_M.

Definition 2.14. Let M be a type B lpm; then e_M is the $<^*$-least parameter e such that $\gamma_M \in h^*_M[\rho_M \cup \{e, p_M\}]$. $\dashv$

We shall often abuse notation by identifying e_M with its range, as we do for d_M and p_M.

Remark 2.15. It is easy to see that $e_M \subset \gamma_M + 1$ and is always defined. Clearly $e_M \cap p_M = \emptyset$.

Remark 2.16. Suppose $\eta = e_M(i)$; then η is the least $\beta \geq \rho_M$ such that $\gamma_M \in h^*_M[\beta + 1 \cup \{e_M \restriction i, p_M - (\beta + 1)\}]$.[14]

[13] We often identify p_M and d_M with their ranges.

[14] Proof: The hulls in question all contain $p_M \cup \rho_M$. If $\beta \geq \eta$, then the hull contains e_M, and hence γ_M is in it. If $\rho_M \leq \beta < \eta$ and γ_M is in the hull, then $\gamma_M = h^*_M(p_1(M), c, a)$ for some $a \in \rho_M^{<\omega}$ and $c <^* e_M$, contradiction.

Lemma 2.17. *If $i \in \mathrm{dom}(e_M)$, then $e_M(i)$ is $\Sigma_1^{\mathcal{L}^*}$-definable from parameters in $\{\gamma_M, p_M - e_M(i), \rho_1(M)\}$.*

Proof. By induction on i. Let $\beta = e_M(i)$ and $p = p_1(M) - \beta$. Recalling the stratification $\alpha \mapsto M^\alpha$ in 2.5, for $\alpha < o(M)$, let

$$\xi_\alpha = \text{least } \xi \text{ such that } \gamma_M \in h^*_{M^\alpha}[\xi + 1 \cup \{e_M \restriction i, p\}],$$

and let ξ_α be undefined if there is no such ξ. Then the function $\alpha \mapsto \xi_\alpha$ is $\Sigma_1^{*,M}$ in the parameters γ_M and p, and $\xi_\alpha = \beta$ for all sufficiently large $\alpha < o(M)$. Finally, $\alpha < \theta \Rightarrow \xi_\alpha \geq \xi_\theta$, so ξ_α changes value only finitely often as α increases. Thus each ξ_α is $\Sigma_1^{\mathcal{L}^*}$ definable over M from γ_M and $e_M \restriction i$ and p. Our induction hypothesis finishes the proof. □

Theorem 2.18 (Zeman Exchange Lemma)**.** *Let M be type B and 1-sound; then $d_M = (p_M \cup e_M) - \tau_M$.*

Proof. Let $d = d_M$, $p = p_M$, and $e = e_M$. As we observed above, $p \cap e = \emptyset$. Suppose first that $e = \emptyset$; we claim that then $d = p - \rho^*$. For suppose that $p - (\eta + 1) = d - (\eta + 1)$ and $\eta \geq \rho^*$; then

$$\begin{aligned} \eta \in p &\Leftrightarrow \eta \notin h_M[\eta \cup p - (\eta + 1)] \\ &\Leftrightarrow \eta \notin h^*_M[\eta \cup d - (\eta + 1)] \\ &\Leftrightarrow \eta \in d. \end{aligned}$$

For the second line: let $x = p - (\eta + 1) = d - (\eta + 1)$. Clearly, if $\eta \in h^*_M[\eta \cup x]$ then $\eta \in h_M[\eta \cup x]$. But if $\eta \in h_M[\eta \cup x]$, then $\eta \notin p$, so $p \subseteq \eta \cup x$, so $h^*_M[\eta \cup x] = h_M[\eta \cup x]$ because $e = \emptyset$, so $\eta \in h^*_M[\eta \cup x]$. Now by induction we get that $p = d$, and the theorem holds.

We now show by induction on i

$(*)_i$ If $i \in \mathrm{dom}(e)$, then $d - (e_i + 1) = (p \cup e) - (e_i + 1)$ and $e_i \in d$.

Assume $i = 0$ or $(*)_{i-1}$. Let I be the open interval (e_i, e_{i-1}) if $i > 0$, and $I = (e_0, o(M))$ otherwise. Let $\eta \in I$, and suppose by ("reverse") induction that $(p - (\eta + 1)) \cap I = (d - (\eta + 1)) \cap I$. Then as above we get

$$\begin{aligned} \eta \in p &\Leftrightarrow \eta \notin h_M[\eta \cup p - (\eta + 1)] \\ &\Leftrightarrow \eta \notin h_M[\eta \cup (p \cup e) - (\eta + 1)] \\ &\Leftrightarrow \eta \notin h_M[\eta \cup d - (\eta + 1)] \\ &\Leftrightarrow \eta \notin h^*_M[\eta \cup d - (\eta + 1)] \\ &\Leftrightarrow \eta \in d. \end{aligned}$$

The second line uses Lemma 2.17 to equate the two hulls. The third line uses our induction hypothesis on η, and the fourth uses $e \subseteq \eta \cup (e - (\eta + 1))$ to equate the $\mathcal{L}^*$ hull with the $\mathcal{L}$ hull.

To finish the proof of $(*)_i$ we must show that $e_i \in d$. But if not, let

$$\beta = \max(p \cap e_i, d \cap e_i);$$

then

$$\begin{aligned} h^*_M[\beta + 1, \{e \restriction i, p - e_i\}] &= h^*_M[\beta + 1, d - (\beta + 1)] \\ &= M, \end{aligned}$$

so $\gamma_M \in h^*_M[\beta + 1, \{e \restriction i, p - e_i\}]$, contrary to 2.16.

This proves $(*)_i$ for all $i \in \mathrm{dom}(e)$. Now let i be largest in $\mathrm{dom}(e)$. To finish the proof of the Zeman Exchange Lemma, it will be enough to show that

$$(p \cap e_i) - \rho^* = d \cap e_i.$$

We argue as above. Suppose that $\eta < e_i$ and $p \cap e_i - (\eta + 1) = d \cap e_i - (\eta + 1)$ and $\eta \geq \rho^*$; then

$$\begin{aligned} \eta \in p &\Leftrightarrow \eta \notin h_M[\eta \cup p - (\eta + 1)] \\ &\Leftrightarrow \eta \notin h_M[\eta \cup (p \cup e) - (\eta + 1)] \\ &\Leftrightarrow \eta \notin h_M[\eta \cup d - (\eta + 1)] \\ &\Leftrightarrow \eta \notin h^*_M[\eta \cup d - (\eta + 1)] \\ &\Leftrightarrow \eta \in d. \end{aligned}$$

The second line uses Lemma 2.17 to equate the two hulls. The third line uses our induction hypothesis on η, and the fourth uses $e \subseteq (e - (\eta + 1))$ to equate the $\mathcal{L}^*$ hull with the $\mathcal{L}$ hull. $\square$

3. Proof of the main theorem

This section proves Theorem 1.8. We assume AD^+ throughout the section. By the argument in Remark 1.9(b), it is enough to prove 1.8(a). So let us assume that (M, Σ) is a strongly 1-sound mouse pair of type B such that $\deg(M) = 0$. For definiteness, we assume that M is an lpm.[15]

Before we get to the main comparison arguments, we make some preliminary reductions.

3.1. Preliminary lemmas

Let $F = \dot{F}^M$, $\kappa = \mathrm{crt}(F)$, and $\tau = \tau_M = \kappa^{+,M}$. Let also $\rho^* = \rho^*_M$ and $d_k = d_M(k)$. Assume toward contradiction that M is not Dodd solid, and let i be least such that M is not Dodd solid at i. From parameter solidity for M we get

Lemma 3.1. *$d_i \in e_M$. Hence $d_i \leq \gamma_M$.*

[15]Since $\deg(M) = 0$, various complications in pfs fine structure to do with type 2 pfs premice will not arise.

Proof. Let $\eta = d_M(i)$, and suppose $\eta \notin e_M$. By Theorem 2.18 $\eta \in p_M$, so since M is parameter solid,

$$\mathrm{Th}_1^M(\eta \cup p_M - (\eta+1)) \in M.$$

By Lemma 2.17,

$$e_M - \eta \in h_M[\eta, p_M - (\eta+1)],$$

so $\mathrm{Th}_1^M(\eta \cup \{p_M - (\eta+1), e_M - \eta\}) \in M$, so $\mathrm{Th}_1^{*,M}(\eta \cup \{d_M \restriction i\}) \in M$. Thus M is Dodd solid at i, contradiction. $\square$

We have that $F \restriction (d_i \cup \{d_0, .., d_{i-1}\}) \notin M$, or equivalently, $\mathrm{Th}_1^{*,M}(d_i \cup \{d_0, ..., d_{i-1}\}) \notin M$. Let

$$Q = \mathrm{cHull}_1^{*,M}(d_i \cup \{d_0, ..., d_{i-1}\})$$

and

$$\sigma : Q \to M$$

be the anticollapse. By Lemma 2.8, letting G be the Jensen completion of $F \restriction (d \restriction i \cup d_i)$,

$$Q = (\mathrm{Ult}(M||\tau, G), G),$$

and σ is the factor map

$$\sigma([a, f]_G^{M||\tau}) = [a, f]_F^{M||\tau}. \tag{3.1}$$

Since d_i is a generator of F, $\mathrm{crt}(\sigma) = d_i$. Since $d_i \in e_M$, we have

Lemma 3.2. $\gamma_M \notin \mathrm{ran}(\sigma)$.

Proof. By Lemma 2.17, $d_i \in h_M^*[d_i \cup p_M - (d_i+1) \cup \{\gamma_M\}]$. Since $p_M \subseteq d_M$, if $\gamma_M \in h_M^*[d_i \cup \{d_0, ..., d_{i-1}\}]$, then $d_i \in h_M^*[d_i \cup \{d_0, ..., d_{i-1}\}] = \mathrm{ran}(\sigma)$, contrary to $d_i = \mathrm{crt}(\sigma)$. $\square$

The fact that M satisfies the Jensen ISC gives us some further limitations on d_i and σ.

Lemma 3.3. *(a) $p_M \cap (\gamma_M, \lambda_F) \neq \emptyset$.*

(b) $G = \dot{F}^Q$ has a largest cutpoint λ_Q^*, and $\sigma(\lambda_Q^*) = \lambda_M^*$.

(c) If $d_i = \gamma_M$, then $Q = cHull_1^{*,M}(\lambda_M^* \cup (p_M - \alpha))$.

(d) The Jensen ISC fails for Q.

Proof. For (a): Suppose instead that $p_M \subseteq \gamma_M + 1$; then since $e_0 \leq \gamma_M$, $d_M \subseteq \gamma_M + 1$. If $e_0 = \gamma_M$, then $e_M = \{\gamma_M\}$, so $i = 0$ and $d_i = \gamma_M$. But then M is Dodd solid at i by the Jensen ISC. So $e_0 < \gamma_M$. Clearly $\gamma_M \notin p_M$ because it has an $\mathcal{L}$-name. Thus $d_M \subseteq \gamma_M$. But then M is Dodd solid by the Jensen ISC, contradiction.

For (b): By (a), letting ν be least such that $\sigma(\nu) \geq \lambda^*_M$, we have $\sigma(\nu) < \lambda_F$. We claim ν is a cutpoint of G. For suppose that $a \in [\nu]^{<\omega}$ and $f : [\kappa]^{|a|} \to \kappa$ and $i^{Q||\tau}_G(f)(a) > \nu$; then $i^{M||\tau}_F(f)(\sigma(a)) > \lambda^*_M$. Since $\sigma(a) \in [\lambda^*_M]^{<\omega}$, this contradicts λ^*_M being a cutpoint of F.[16]

Since ν is a cutpoint of G, $\sigma(\nu)$ is a cutpoint of F. For otherwise there is a fragment $F \cap ([\sigma(\nu)+1]^{<\omega} \times M||\xi)$ that witnesses $\sigma(\nu)$ is not a cutpoint of F, but this fragment is in $\mathrm{ran}(\sigma)$, so $G \cap ([\nu+1]^{<\omega} \times M||\xi)$ witnesses that ν is not a cutpoint of G.

Since λ^*_M is the largest cutpoint of F below λ_F, we get that $\sigma(\nu) = \lambda^*_M$ and $\nu = \lambda^*_Q$ is the largest cutpoint of $\dot{F}^Q$.

For (c): Let $\psi\colon \mathrm{cHull}_1^{*,M}(\lambda^*_M \cup (p_M - \alpha)) \to M$ be the anticollapse. Then $\psi \restriction \lambda^*_M =$ id, and the proof of (b) shows that $\lambda^*_M \in \mathrm{ran}(\psi)$. So $\mathrm{ran}(\psi)$ collapses to Q, as desired.

For (d): let $\gamma = \nu^{+,Q}$ where $\sigma(\nu) = \lambda^*_M$. σ maps the fragments of $G \restriction \gamma$ to fragments of $F \restriction \gamma_M$, and $\sigma``\gamma$ is cofinal in γ_M because σ is the identity on τ.[17] If the Jensen ISC holds for Q, then $G \restriction \gamma \in Q$, so $\mathrm{cof}^Q(\gamma) = \tau$, so σ is continuous at γ. Thus $\sigma(\gamma) = \gamma_M$, so $\gamma_M \in \mathrm{ran}(\sigma)$, contrary to 3.2.

□

The following notion plays an important role in Zeman's proof of Dodd solidity for Jensen mice in [7]. It will play a similar role in our proof.

Definition 3.4. A potential active Jensen premouse (N, G) has a *strong failure of the ISC at η* if and only if

(a) η is a cutpoint of G, $\eta < \lambda_G$, and the Jensen completion of $G \restriction \eta$ is not on the N-sequence.

(b) Letting $\gamma = (\eta^+)^{\mathrm{Ult}(N, G\restriction\eta)}$, we have $\gamma < (\eta^+)^N$.

⊣

In the situation of 3.4, $G \restriction \gamma$ would collapse γ in N if the Jensen ISC held, but instead, N collapses γ in some way inconsistent with this.

Lemma 3.5. *If $d_i < \gamma_M$, then Q has a strong failure of the ISC at its largest proper cutpoint λ^*_Q.*

Proof. Again, let $G = \dot{F}^Q$ and $\gamma_Q = (\lambda^*_Q)^{+,\mathrm{Ult}(Q, G\restriction\lambda^*_Q)}$. Part (a) of Definition 3.4 was shown in 3.3(c). For (b), we must see that $\gamma_Q < (\lambda^*_Q)^{+,Q}$. We showed in the proof of 3.3(c) that $\sigma \restriction \gamma_Q$ is cofinal in γ_M. It is therefore enough to prove

Claim 3.6. $\exists \xi \in rng(\sigma)(\gamma_M < \xi < (\lambda^*_M)^{+,M})$.

Proof. Let $a \subseteq d_i \cup \{d_0, ..., d_{i-1}\}$ and $h \in M||\tau$ be such that

$$[a, h]^Q_G = \lambda^*_Q,$$

[16] Note here that $M||\tau = Q||\tau$.

[17] For $A \subset \kappa$ in $Q||\tau = M||\tau$, $\sigma(i^{Q||\tau}_G(A)) = i^{M||\tau}_F(A)$, so $\sigma(i^{Q||\tau}_G(A) \cap \nu) = i^{M||\tau}_F(A) \cap \sigma(\nu)$, which implies $\sigma''\gamma$ is cofinal in γ_M as claimed.

so that $[a,h]^M_F = \lambda^*_M$. We may assume $h\colon [\kappa]^{|a|} \to \kappa$. We can assume a is large enough that for some $f\colon [\kappa]^{|a|+1} \to \kappa$,

$$[a \cup \{d_i\}, f]^M_F = \gamma_M.$$

(There is such an f because $\mathrm{Ult}(M||\tau, F \restriction d_i \cup \{d_0, ..., d_i\}) = M||o(M)$.) Now, for $u \in [\kappa]^{|a|}$ let

$$g(u) = \sup\{f(u,v) : f(u,v) < h(u)^{+,M} \wedge (u,v) \in [h(u)]^{|a|+1}\}.$$

It is clear that $g(u) < h(u)^{+,M}$ for a.e. u. This implies $[a,g]^Q_G < (\lambda^*_Q)^{+,Q}$. But also

$$\gamma_M = [a \cup \{d_i\}, f]^M_F < [a,g]^M_F.$$

So $[a,g]^M_F \in (\gamma_M, (\lambda^*_M)^{+,M})$. □

If $\sigma(\nu) \in (\gamma_M, (\lambda^*_M)^{+,M})$, then $\nu \in (\gamma_Q, (\lambda^*_Q)^{+,Q})$, so $\gamma_Q < (\lambda^*_Q)^{+,Q}$, as desired.

Remark 3.7. [6] uses the Interpolation Lemma to prove this claim. Our proof just unpacks the relevant part of that lemma.

3.2. A background construction

Fix a coarse strategy pair $((N^*, \in, w, \mathcal{F}, \Psi), \Psi^*)$ that captures Σ, and let $\mathbb{C}$ be the maximal $(w, \mathcal{F})$ construction, with models $M_{\nu,l}$ and induced strategies $\Omega_{\nu,l}$. Let $\delta^* = \delta(w)$. By [3, Theorem 3.26], $(*)(M,\Sigma)$ holds, so we can fix $\langle \eta_0, 0\rangle$ lex least such that (M,Σ) iterates to $(M_{\eta_0,0}, \Omega_{\eta_0,0})$, and for all $(\nu,l) <_{\text{lex}} (\eta_0, 0)$, (M,Σ) iterates strictly past $(M_{\nu,l}, \Omega_{\nu,l})$. Let $\mathcal{U}_{\nu,l}$ be the unique λ-separated tree on M witnessing (M,Σ) iterates past $(M_{\mu,l}, \Omega_{\nu,l})$.[18]

We shall rule out $d_i < \gamma_M$ by comparing (Q, Σ^σ) with the levels of $\mathbb{C}$. We shall rule out $d_i = \gamma_M$ with a more complicated phalanx comparison.

3.3. Ruling out $d_i < \gamma_M$

Suppose that $d_i < \gamma_M$, so that Q has a strong failure of the Jensen ISC at its largest cutpoint. We compare (Q, Σ^σ) with the levels of $\mathbb{C}$. Non-dropping iterates of (Q, Σ^σ) also have strong failures of the ISC, so (Q, Σ^σ) cannot iterate into any such level. This will lead to a contradiction.

Remark 3.8. This part of the argument does not require a phalanx comparison, so it is simpler than the $d_i = \gamma_M$ case.

Q is not an lpm because the ISC fails. Nevertheless the strategy-regularity properties that define lbr hod pairs make sense for (Q, Σ^σ),

Lemma 3.9. (Q, Σ^σ) *has strong hull condensation, normalizes well, and is internally lift consistent and pushforward consistent.*

[18]Since $\deg(M) = 0$, M is strongly stable, so all iterates of (M,Σ) have type 1.

Proof. (Sketch.) (M, Σ) is an lbr hod pair, so it has these properties. Lemma 9.2.3 of [4] shows that if $\pi\colon (N, \Psi) \to (M, \Sigma)$ is $\Sigma_1^{\mathcal{L}}$-elementary and cofinal, then (N, Ψ) has them, that is, (N, Ψ) is an lbr hod pair. In our case, σ is cofinal, but only $\Sigma_1^{\mathcal{L}^*}$-elementary. However, this is enough for the proof that the strategy regularity properties hold for (Q, Σ^σ). (But not enough to conclude that Q is an lpm, of course.) $\square$

Continuing to adapt [4], we get

Lemma 3.10. *Let $\langle \nu, l\rangle \leq_{\text{lex}} \langle \eta_0, 0\rangle$, and suppose (Q, Σ^σ) iterates strictly past $(M_{\beta,k}, \Omega_{\beta,k})$ for all $\langle \beta, k\rangle <_{\text{lex}} \langle \nu, l\rangle$; then (Q, Σ^σ) iterates past $(M_{\nu,l}, \Omega_{\nu,l})$.*

Proof. We adapt the proof of [4, 9.5.2]. The proof that no strategy disagreements show up when (Q, Σ^σ) is compared with $(M_{\nu,l}, \Omega_{\nu,l})$ goes through without change. The proof that only the Q-side moves involves the ISC for Q, so we must look at it.

Let $\mathcal{S}$ be the tree on the Q-side, and

$$(P_\gamma, \Sigma_\gamma) = (\mathcal{M}_\gamma^{\mathcal{S}}, \Sigma_{\mathcal{S}\restriction\gamma+1})$$

be the pair at γ in $\mathcal{S}$. (So $(P_0, \Sigma_0) = (Q, \Sigma^\sigma)$.) Suppose toward contradiction that $(P_\gamma, \Sigma_\gamma)||\eta = (M_{\nu,l}, \Omega_{\nu,l})||\eta$ and $P_\gamma|\eta \neq M_{\nu,l}|\eta$, and $M_{\nu,l}|\eta$ is active. Let ν, l be the lex least counterexample. As in [4], we get that $l = 0$ and $\eta = \nu$. We have assumed that $\langle \nu, l\rangle \leq_{\text{lex}} \langle \eta_0, 0\rangle$, so we may set

$$\mathcal{U} = \mathcal{U}_{\nu,0}.$$

Let $F = \dot{F}^{M_{\nu,0}}$ and let F^* be the background extender for F. Let $j = i^V_{F^*}$ and $\kappa = \text{crt}(F) = \text{crt}(j)$. We have

(a) F^* backgrounds F^+.

(b) $j(M_{\nu,0})|\langle \nu, 0\rangle = M_{\nu,0}||\nu$ and $\text{lh}(F) = o(M_{\nu,0})$ is a cardinal in $j(M_{\nu,0})$.[19]

(c) $\mathcal{S} \restriction \gamma + 1 = j(\mathcal{S}) \restriction \gamma + 1$.

(d) $j \restriction \mathcal{M}_\kappa^{\mathcal{S}} = i^{j(\mathcal{S})}_{\kappa, j(\kappa)}$.

(e) $j \restriction \mathcal{M}_\kappa^{\mathcal{U}} = i^{j(\mathcal{U})}_{\kappa, j(\kappa)}$.

(f) $\mathcal{M}_\kappa^{\mathcal{S}}|(\kappa^+)^{\mathcal{M}_\kappa^{\mathcal{S}}} = \mathcal{M}_\kappa^{\mathcal{U}}|(\kappa^+)^{\mathcal{M}_\kappa^{\mathcal{U}}}$.

Let

$$\begin{aligned} E_j &= \{(a, X) \mid a \in [j(\kappa)]^{<\omega} \wedge X \in \mathcal{M}_\kappa^{\mathcal{S}} \wedge a \in j(X)\}, \\ N &= \mathcal{M}^{j(\mathcal{S})}_{j(\kappa)}||j(\kappa) \\ &= j(M_{\nu,0}))||j(\kappa). \end{aligned}$$

[19]See [4, Theorem 10.4.1].

By (a) and (b), $F^+ \lhd E_j$, $F \notin N$, and every whole proper initial segment of F is in $M_{\nu,0}$, and hence in N. That is

$$F \upharpoonright \lambda_F = \text{ first whole initial segment } K \text{ of } E_j \text{ such that } K \notin N.$$

Let

$$G^+ = E_\theta^{j(\mathcal{S})},$$

where $\theta + 1 <_{j(S)} j(\kappa)$ and $\mathrm{pd}_{j(S)}(\theta+1) = \kappa$. We have that $G^+ \lhd E_j$, and $G \notin N$, so $F \unlhd G$. If F is on the sequence of $\mathcal{M}_\theta^{j(\mathcal{S})}$ then since $P_\gamma = \mathcal{M}_\gamma^{j(\mathcal{S})}$ and $\gamma \leq \theta$ and $\mathrm{lh}(E_\gamma^{j(\mathcal{S})}) > \mathrm{lh}(F)$, F is on the sequence of P_γ, contradiction. Thus

$$F \upharpoonright \lambda_F \lhd G,$$

and F witnesses that $\mathcal{M}_\theta^{j(\mathcal{S})}|\mathrm{lh}(G)$ does not satisfy the Jensen ISC.

It follows that $[0,\theta]_{j(S)}$ does not drop, $G = \dot{F}^{\mathcal{M}_\theta^{j(\mathcal{S})}}$, and $\mathcal{M}_\theta^{j(\mathcal{S})}$ has a strong failure of the ISC. Letting

$$L = G \upharpoonright \lambda_G^*,$$

where λ_G^* is the largest proper cutpoint of G, this implies that

$$\lambda_L^{+,N} > \sup i_L``\kappa^{+,N}.$$

But E_j is the branch extender of $i_{\kappa,j(\kappa)}^{j(\mathcal{U})}$, and $j(\mathcal{U})$ is an ordinary plus tree in which all models satisfy the Jensen ISC. This implies that whenever L is a cutpoint initial segment of E_j, $\lambda_L^{+,N} = \sup i_L``\kappa^{+,N}$, contradiction. □

Corollary 3.11. *$d_i = \gamma_M$.*

Proof. If $d_i < \gamma_M$, then (Q, Σ^σ) has a strong failure of the ISC. This is preserved by nondropping iterations, so (Q, Σ^σ) cannot iterate to any $(M_{\nu,l}, \Omega_{\nu,l})$. It follows from Lemma 3.10 that (Q, Σ^σ) iterates strictly past $(M_{\eta_0,0}, \Omega_{\eta_0,0})$. Let $\mathcal{S}$ be the tree on (Q, Σ^σ) that witnesses this. The Dodd-Jensen Lemma now leads to a contradiction.

Let $\mathrm{lh}(\mathcal{S}) = \theta + 1$, and (P_ξ, Σ_ξ) be the pair at ξ in $\mathcal{S}$. let

$$\mathcal{T} = \sigma\mathcal{S}$$

be the copied tree, with pairs (P_ξ^*, Σ_ξ^*) and copy maps

$$\sigma_\xi \colon (P_\xi, \Sigma_\xi) \to (P_\xi^*, \Sigma_\xi^*).$$

Since either $[0,\theta)_S$ drops or $(M_{\eta_0,0}, \Omega_{\eta_0,0}) \lhd (P_\theta, \Sigma_\theta)$, we get that for $\bar{\sigma}_\theta = \sigma_\theta \upharpoonright M_{\eta_0,0}$, there is $(N, \Psi) \unlhd (P_\theta^*, \Sigma_\theta^*)$ such that

$$\bar{\sigma}_\theta \colon (M_{\eta_0,0}, \Omega_{\eta_0,0}) \to (N, \Psi)$$

is $\Sigma_1^{\mathcal{L}}$- elementary (not just $\Sigma_1^{\mathcal{L}^*}$). Moreover, $[0,\theta)_T$ drops or $(N,\Psi) \lhd (M_{\eta_0,0},\Omega_{\eta_0})$.

Let $\mathcal{U} = \mathcal{U}_{\eta_0,0}$ and $\beta + 1 = \mathrm{lh}(\mathcal{U})$; then

$$\bar{\sigma}_\theta \circ i^{\mathcal{U}}_{0,\beta} \colon (M,\Sigma) \to (N,\Psi)$$

is $\Sigma_1^{\mathcal{L}}$ elementary. Since the branch of $\mathcal{T}$ to (N,Ψ) drops, this contradicts the Dodd Jensen property of Σ. $\square$

3.4. Ruling out $d_i = \gamma_M$

We have now that $d_i = \gamma_M$. Thus $d_i = e_M(0)$, and $d_M \restriction i = p_M \restriction i \neq \emptyset$ by 3.3.

Our plan now is to use a phalanx comparison like the ones in the proofs of parameter solidity, closure under initial segment, condensation, and other similar results. See for example [4, Theorem 9.6.2] for the template.

Let us summarize the eventually contradictory properties of (M,Q,σ,d_i) that we have accumulated:

Definition 3.12. (N,P,ψ,α) is *problematic* iff

(1) N is an active, λ^*_N-sound lpm of type B and degree 0,

(2) $\alpha = \gamma_N = e_N(0)$ and $p_N - \alpha \neq \emptyset$,

(3) $P = \mathrm{cHull}_1^{*,N}(\lambda^*_N \cup (p_N - \alpha))$, $\psi \colon P \to N$ is the anticollapse map, and $\alpha = \mathrm{crt}(\psi)$, and

(4) $\dot{F}^P \notin N$.

$\dashv$

Lemma 3.13. *(M,Q,σ,d_i) is problematic.*

Proof. (1) is clear; in fact M is fully 1-sound. Lemma 3.3(a) and (c) imply the nontrivial part of (2) and (3). (4) is our assumption that Dodd solidity has failed. $\square$

Some observations:

Lemma 3.14. *If (N,P,ψ,α) is problematic, then for $F = \dot{F}^N$ and $G = \dot{F}^P$,*

(a) λ^*_N is inaccessible in N,

(b) λ^*_N is a limit of cutpoints of F and G,

(c) $\lambda^*_N = \lambda^*_P$ is the largest cutpoint of F and G, and

(d) $\alpha = (\lambda^*_P)^{+,P}$, so $E^P_\alpha = \emptyset$, and the Jensen ISC fails in P.

Proof. For (a): We have $\alpha = \text{crt}(\psi) = \gamma_N = (\lambda_N^*)^{+,P}$. Since λ_N^* is strongly inaccessible in $N||\gamma_N$, it is strongly inaccessible in P. But $\psi(\lambda_N^*) = \lambda_N^*$, so λ_N^* is strongly inaccessible in N.

For (b): Since $F \restriction \lambda_N^* \in N$, by (a), working in N we can find club many cutpoints of F below λ_N^*.

The proof of Lemma 3.3(b) yields (c). Part (d) is clear. □

We shall reach a contradiction by comparing the phalanx

$$\Phi_0 = ((M, \Sigma), (Q, \Sigma^\sigma), \lambda_M^*)$$

with (M, Σ), indirectly, by iterating it to or past the levels of $\mathbb{C}$. The definition of the coiterations is very similar to that in the proof of solidity for the standard parameter in [4, 9.6.2], and the proof that one of them succeeds is simpler than that in [4].[20] One difference is that our exchange ordinal is λ_M^* rather than d_i. This choice lets us avoid some anomalous cases that cause a fair amount of difficulty in the parameter solidity proof.[21]

We now define pseudo-iteration trees $\mathcal{S}_{\nu,l}$ on Φ_0 for certain $(\nu, l) \leq (\eta_0, 0)$. The definition is similar to the definition of $\mathcal{S}_{\nu,l}$ in [4, p. 420ff], so we'll go fast. Fix $(\nu, l) \leq (\eta_0, 0)$ for now, and assume $\mathcal{S}_{\nu',l'}$ is defined whenever $(\nu', l') <_{\text{lex}} (\nu, l)$. Let $\mathcal{U} = \mathcal{U}_{\nu,l}$, and for $\tau < \text{lh}(\mathcal{U})$, let

$$(Q_\theta, \Lambda_\theta) = (M_\theta^{\mathcal{U}}, \Sigma_{\mathcal{U}\restriction(\theta+1)})$$

be the mouse pair at θ in $\mathcal{U}$. As we define $\mathcal{S}$, we lift it to a padded tree $\mathcal{T}$ on (M, Σ) by copying. We write

$$(P_\theta^*, \Sigma_\theta^*) = (M_\theta^{\mathcal{T}}, \Sigma_{\mathcal{T}\restriction(\theta+1)})$$

for the mouse pair at θ in $\mathcal{T}$. For $\theta < \text{lh}(\mathcal{S})$, we have a nearly elementary copy map π_θ from $\mathcal{M}_\theta^{\mathcal{S}}$ into $\mathcal{M}_\theta^{\mathcal{T}}$. We attach the complete strategy $\Sigma_\theta = (\Sigma_\theta^*)^{\pi_\theta}$ to $\mathcal{M}_\theta^{\mathcal{S}}$, so that

$$(P_\theta, \Sigma_\theta) = (M_\theta^{\mathcal{S}}, (\Sigma_\theta^*)^{\pi_\theta})$$

is the (not quite mouse) pair at θ in $\mathcal{S}$. For the embeddings of $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ we write

$$\begin{aligned} i_{\alpha,\beta} &= i_{\alpha,\beta}^{\mathcal{S}}, \\ i_{\alpha,\beta}^* &= i_{\alpha,\beta}^{\mathcal{T}}, \\ j_{\alpha,\beta} &= i_{\alpha,\beta}^{\mathcal{T}}. \end{aligned}$$

We also define a non-decreasing sequence of ordinals $\varepsilon_\theta = \varepsilon_\theta^{\mathcal{S}}$ that measure agreement between models of $\mathcal{S}$, and tell us which model we should apply the next extender to.

[20] Mainly because we are relying on the fact that parameter solidity has already been proved, and M is 1-sound, so the reductions in the last subsection are available.

[21] See [4, 4.10.4] and [5] for a discussion of these cases.

The construction of $\mathcal{S}$ classifies nodes $\theta < \mathrm{lh}(\mathcal{S})$ as either stable or unstable, with the current last node always stable. If γ is our current last node, we shall have have $\sigma_\theta, \alpha_\theta$, for each unstable $\theta < \gamma$, that satisfy

$(\dagger)_\gamma$: If $\theta < \gamma$ is unstable, then

(i) $[0, \theta]_S$ does not drop, and all $\eta <_S \theta$ are unstable,

(ii) $e^{\mathcal{S}}_\theta = e^{\mathcal{U}}_\tau$ for some τ,

(iii) $(P_\theta, P_{\theta+1}, \sigma_\theta, \alpha_\theta)$ is problematic,

(iv) $\alpha_\theta = i_{0,\theta}(\alpha_0) = \sup i_{0,\theta}``\alpha_0$, and for all $\eta <_S \theta$, $\mathrm{crt}(i_{\eta,\theta}) < \alpha_\eta$,

(v) $\lambda^*_{P_\theta} = i_{0,\theta}(\lambda^*_{P_0}) = \sup i_{0,\theta}``\lambda^*_{P_\theta}$,

(vi) $\varepsilon_\theta = \inf(\lambda^*_{P_\theta}, \varepsilon_{\theta+1})$, or $\theta + 1 = \gamma$ and ε_θ is not yet defined.

We start with
$$((P_0, \Sigma_0), (P_1, \Sigma_1), \sigma_0, \alpha_0) = ((M, \Sigma), (Q, \Sigma^\sigma), \sigma, d_i).$$
0 is unstable, and 1 is stable. Both are roots of $\mathcal{S}$. In $\mathcal{T}$, we let
$$(P^*_0, \Sigma^*_0) = (P^*_1, \Sigma^*_1) = (M, \Sigma).$$
The copy maps from $\mathcal{S}$ to $\mathcal{T}$ are
$$\pi_0 = \text{ id }, \pi_1 = \sigma.$$
0 and 1 are distinct roots of $\mathcal{S}$. We say that 0 is unstable, and 1 is stable. Clearly $(\dagger)_1$ holds.

The construction of $\mathcal{S}$ takes place in rounds in which we either add one stable θ, or one unstable θ and its stable successor $\theta + 1$. Thus the current last model is always stable, and all extenders used in $\mathcal{S}$ are taken from stable models. If γ is stable, then $\varepsilon_\gamma = \hat{\lambda}(E^{\mathcal{S}}_\gamma)$.

For a node γ of $\mathcal{S}$, we write $\mathrm{pd}_S(\gamma)$ for the immediate $\leq_{\mathcal{S}}$-predecessor of $\mathcal{S}$. We set
$$\mathrm{st}(\gamma) = \text{the least stable } \theta \text{ such that } \theta \leq_{\mathcal{S}} \gamma,$$
and [22]
$$\mathrm{rt}(\gamma) = \begin{cases} \mathrm{pd}_S(\mathrm{st}(\gamma))) & \text{if } \mathrm{pd}_S(\mathrm{st}(\gamma))) \text{ exists,} \\ \mathrm{st}(\gamma) & \text{otherwise.} \end{cases}$$
The construction of $\mathcal{S}$ ends when we reach a stable θ such that

(I) $(M_{\nu,l}, \Omega_{\nu,l}) \lhd (P_\theta, \Sigma_\theta)$, or $(P_\theta, \Sigma_\theta) = (M_{\nu,l}, \Omega_{\nu,l})$ and $[\mathrm{rt}(\theta), \theta]_S$ drops, or

(II) $(P_\theta, \Sigma_\theta) \trianglelefteq (M_{\nu,l}, \Omega_{\nu,l})$, and $[\mathrm{rt}(\theta), \theta]_S$ does not drop.

[22] Equivalently $\xi = \mathrm{rt}(\gamma)$ iff $\xi \leq_S \gamma$ and $e^{\mathcal{S}}_\xi$ is the longest initial segment of $e^{\mathcal{S}}_\gamma$ that belongs to $\mathcal{U}^{\mathrm{ext}}$. If $\xi = 0$ or $\xi = \theta + 1$ for θ unstable, then $e^{\mathcal{S}}_\xi = \emptyset$. Otherwise $e^{\mathcal{S}}_\xi \neq \emptyset$.

If case (I) occurs, then we go on to define $\mathcal{S}_{\nu,l+1}$. If case (II) occurs, we stop the construction.

We now describe how to extend $\mathcal{S}$ one more step. Suppose first that $\mathcal{S}$ has successor length $\gamma+1$, where γ is stable, and that $(\dagger)_\gamma$ holds. Suppose (I), (II) above do not hold for γ, so that there is a disagreement between $(P_\gamma, \Sigma_\gamma)$ and $(M_{\nu,l}, \Omega_{\nu,l})$.

It is convenient to isolate a certain special case.

Definition 3.15. $\gamma < \mathrm{lh}(\mathcal{S})$ is *special* iff $\gamma = \theta + 1$, where θ is unstable, and

(i) $P_{\theta+1}|(\alpha_\theta) = M_{\nu,l}||\alpha_\theta$, and

(ii) for some $\xi \geq \alpha_\theta$, $\mathrm{crt}(E_\xi^{P_{\theta+1}}) = \lambda^*_{P_\theta}$.

⊣

If γ is special and $\theta + 1 = \gamma$, then we set

$$\begin{aligned} E_\gamma^{\mathcal{S}} = &\text{ order 0 measure of } P_\gamma \text{ on } \lambda^*_{P_\theta} \\ = &\, E_\xi^{P_\gamma}, \text{ where } \xi \geq \alpha_\theta \text{ is least s.t. } \mathrm{crt}(E_\xi^{P_\gamma}) = \lambda^*_{P_\theta}, \\ \varepsilon_\gamma = &\,\alpha_\theta, \\ \varepsilon_\theta = &\,\lambda^*_{P_\theta}. \end{aligned}$$

As usual, $\mathrm{pd}_{\mathcal{S}}(\gamma+1)$ is the least ξ such that $\mathrm{crt}(E) < \varepsilon_\xi$, which in this case is γ. There is no dropping here. We let $E_\gamma^{\mathcal{T}} = \pi_\gamma(E_\gamma^{\mathcal{S}})$, so that

$$\begin{aligned} P_{\gamma+1} &= \mathrm{Ult}(P_\gamma, E_\gamma^{\mathcal{S}}), \\ P^*_{\gamma+1} &= \mathrm{Ult}(P^*_\gamma, \pi_\gamma(E_\gamma^{\mathcal{S}})), \end{aligned}$$

and $\pi_{\gamma+1}$ is given by the Shift Lemma. We declare $\gamma+1$ to be stable. $(\dagger)_{\gamma+1}$ follows vacuously from $(\dagger)_\gamma$.[23]

Suppose next that γ is not special, and that the least disagreement between $(P_\gamma, \Sigma_\gamma)$ and $(M_{\nu,l}, \Omega_{\nu,l})$ involves only an extender E on the sequence of P_γ. That is,

$$(M_{\nu,l}, \Omega_{\nu,l})|\mathrm{lh}(E) = (P_\gamma, \Sigma_\gamma)||\mathrm{lh}(E).$$

Later, we shall prove that this is the case.[24]

Set

$$\begin{aligned} E_\gamma^{\mathcal{S}} &= E^+, \\ \varepsilon_\gamma &= \lambda(E), \end{aligned}$$

[23] If γ is special, then $E_\gamma^{\mathcal{S}}$ may not be the least disagreement between P_γ and $M_{\nu,l}$, because we are using Jensen indexing. Also, setting $\varepsilon_\gamma = \alpha_\theta$ means we are using ms-rules at this point. We are defining $\mathcal{S}$ this way so that it will stay closer to $\mathcal{U}$, which (we shall see) must use $(E_{\alpha_\theta}^{P_\theta})^+$.

[24] Our convention is that $R||\xi$ is the passive version of $R|\xi$. The corresponding fact when γ is special is that $M_{\nu,l}|\alpha_{\gamma-1}$ is passive, and we shall also prove that. See Claim 3.21.

and if $\gamma = \theta + 1$ where θ is unstable[25], let

$$\varepsilon_\theta = \inf(\varepsilon_\gamma, \lambda^*_{P_\theta}).$$

Let ξ be least such that $\mathrm{crt}(E) < \varepsilon_\xi$. We let $\mathrm{pd}_S(\gamma+1) = \xi$.[26] Let (β, k) be lex least such that either $\rho(P_\xi|(\beta,k)) \leq \mathrm{crt}(E)$ or $(\beta,k) = (\hat{o}(P_\xi), k(\mathcal{M}^{\mathcal{S}}_\xi))$. Set

$$P_{\gamma+1} = \mathrm{Ult}(P_\xi|(\beta,k), E^+),\text{[27]}$$

and let $\hat{i}^{\mathcal{S}}_{\xi,\gamma+1}$ be the canonical embedding.

Now let

$$E^{\mathcal{T}}_\gamma = \pi_\gamma(E)^+,$$

so that $\mathcal{T} \restriction \gamma+2$ and $(P^*_{\gamma+1}, \Sigma^*_{\gamma+1})$ are now determined, and $\pi_{\gamma+1} \colon (P_{\gamma+1}, \Sigma_{\gamma+1}) \to (P^*_{\gamma+1}, \Sigma^*_{\gamma+1})$ is determined by the Shift Lemma.

If ξ is stable or $(\beta,k) < \langle \hat{o}(P_\xi), \deg(P_\xi)\rangle$, then we declare $\gamma+1$ to be stable. $(\dagger)_{\gamma+1}$ follows vacuously from $(\dagger)_\gamma$.

If ξ is unstable and E^+ is not used in $\mathcal{U}$, then again we declare $\gamma+1$ stable. Again, $(\dagger)_{\gamma+1}$ follows vacuously from $(\dagger)_\gamma$.

Finally, suppose ξ is unstable and E^+ is used in $\mathcal{U}$.[28] Say $E^+ = E^{\mathcal{U}}_\mu$. Let τ be such that

$$e^{\mathcal{S}}_\xi = e^{\mathcal{U}}_\tau,$$

so in particular,

$$(P_\xi, \Sigma_\xi) = (Q_\tau, \Lambda_\tau).$$

We have that $\mathrm{pd}_U(\mu+1) = \tau$ (see [4]), and

$$e^{\mathcal{S}}_{\gamma+1} = (e^{\mathcal{S}}_\xi)^\frown\langle E^+\rangle = (e^{\mathcal{U}}_\tau)^\frown\langle E^+\rangle = e^{\mathcal{U}}_{\mu+1}.$$

So we declare $\gamma+1$ to be unstable and $\gamma+2$ stable. Let

$$\begin{aligned}
\alpha_{\gamma+1} &= i_{\xi,\gamma+1}(\alpha_\xi),\\
P_{\gamma+2} &= \mathrm{Ult}(P_{\xi+1}, E^+)\\
\sigma_{\gamma+1} &= \text{copy map}.
\end{aligned}$$

We have the diagram

[25] So $\mathrm{lh}(E) < \alpha_\theta$ because γ is not special.

[26] $\mathcal{S}$ uses plus extenders, so letting $E^{\mathcal{S}}_\xi = G^+$, $\mathrm{crt}(E) \neq \lambda_G$. We have set $\varepsilon_\xi = \lambda(G)$, but in plus trees, $\mathrm{lh}(G)$ and λ_G lead to the same pd_S function.

[27] Recall E^+ is the plus-type extender derived from E.

[28] Since we are not dropping in $\mathcal{S}$ at $\gamma+1$, neither anomalous case applies.

$$\begin{tikzcd}
P_{\gamma+1} & P_{\gamma+2} \arrow[l, "\sigma_{\gamma+1}"'] \\
P_\xi \arrow[u, "i_{\xi,\gamma+1}"] & P_{\xi+1} \arrow[l, "\sigma_\xi"'] \arrow[u, "k"']
\end{tikzcd}$$

$i_{\xi,\gamma+1} = i^{P_\xi}_{E^+}$ and $k = i^{P_{\xi+1}}_{E^+}$, and $\sigma_{\gamma+1}$ is given by the Shift Lemma. In $\mathcal{T}$ we pad, that is, $P^*_{\gamma+2} = P^*_{\gamma+1}$.

Lemma 3.16. *If $\gamma+1$ is unstable, then $(\dagger)_{\gamma+2}$ holds.*

Proof. Items (i), (ii), and (vi) are immediate from the definitions.

Let $i = i_{\xi,\gamma+1} = i^{P_\xi}_{E^+}$. Since α_ξ has cofinality τ_{P_ξ} in P_ξ, i is continuous at α_ξ, moreover $\mathrm{crt}(i) < \lambda^*_{P_\xi}$. Thus (iv) holds.

i is $\Sigma_1^{\mathcal{L}}$-elementary, so $i(\lambda^*_{P_\xi}) = \lambda^*_{P_{\gamma+1}}$. Since $\lambda^*_{P_\xi}$ is inaccessible in P_ξ, i is continuous at $\lambda^*_{P_\xi}$. Thus (v) holds.

Finally, we must see that $(P_{\gamma+1}, P_{\gamma+2}, \sigma_{\gamma+1}, \alpha_{\gamma+1})$ is problematic. Let $p = p(P_\xi) - \alpha_\xi$; then $i(p) = p(P_{\gamma+1}) - \alpha_{\gamma+1}$ because i preserves the solidity witnesses.[29] But notice that for $\kappa = \mathrm{crt}(E^+)$,

$$^{\kappa}\alpha_\xi \cap P_\xi = {}^{\kappa}\alpha_\xi \cap P_{\xi+1}$$

by (iv) and (v) at ξ. Thus $\sigma_{\gamma+1} \restriction \alpha_{\gamma+1} = \mathrm{id}$, and

$$\begin{aligned}
P_{\gamma+2} &= \mathrm{cHull}_1^{*,P_{\gamma+1}}(\alpha_{\gamma+1} \cup \{p(P_{\gamma+1}) - \alpha_{\gamma+1}\}),\\
\sigma_{\gamma+1} &= \text{ the anticollapse map.}
\end{aligned}$$

We are left to show the Dodd solidity witness

$$H = \dot{F}^{P_{\gamma+1}} \restriction (\alpha_{\gamma+1} \cup \{p(P_{\gamma+1}) - \alpha_{\gamma+1}\})$$

is not in $P_{\gamma+1}$. But letting

$$G = \dot{F}^{P_\xi} \restriction (\alpha_\xi \cup \{p(P_\xi) - \alpha_\xi\}),$$

we have that $G \notin P_\xi$ by $(\dagger)_\xi$. G is coded by a set $\bar{G} \subseteq P_\xi || \alpha_\xi$ that is amenable to $P||\alpha_\xi$.[30] If $H \in P_{\gamma+1}$, then

$$\bigcup_{\beta < \alpha_\xi} i(\bar{G} \cap P||\beta) \in P_{\gamma+1}.$$

Since E^+ is weakly amenable to P_ξ, we can apply Schlutzenberg's Lemma (cf. [4, 9.6.1(a)]) to conclude that $\bar{G} \in P_\xi$, so $G \in P_\xi$, contradiction.

[29]The natural prewellorder on the $\Sigma_1^{\mathcal{L}}$ (equivalently $\Sigma_1^{\mathcal{L}^*}$) theories has cofinality τ_M, so i preserves the standard witnesses.

[30]Proof: Let $h\colon \tau_{P_\xi} \to \alpha_\xi$ witness that $\mathrm{cof}^{P_\xi}(\alpha_\xi) = \tau_\xi$. Put $\langle \eta, z\rangle \in \bar{G}$ iff ($\eta < \tau_{P_\xi}$ and $z \subseteq \lambda^*_{P_\xi}$ and z codes $G \cap ([h(\eta) \cup \{p(P_\xi) - \alpha_\xi\}])^{<\omega} \times P_\xi||\eta)$.

This finishes the proof of 3.16. $\square$

Now suppose we are given $\mathcal{S} \restriction \theta$, where θ is a limit ordinal. Let $b = \Sigma(\mathcal{T} \restriction \theta)$.

Case 1. There is a largest $\eta \in b$ such that η is unstable.

Fix η. There are two subcases.

(A) for all $\gamma \in b - (\eta + 1)$, $\mathrm{rt}(\gamma) = \eta + 1$. In this case, $b - (\eta + 1)$ is a branch of $\mathcal{S}$. Let $\mathcal{S}$ choose this branch,

$$[\eta + 1, \theta)_{\mathcal{S}} = b - (\eta + 1),$$

and let $\mathcal{M}_\theta^{\mathcal{S}}$ be the direct limit of the $\mathcal{M}_\gamma^{\mathcal{S}}$ for sufficiently large $\gamma \in b - (\eta + 1)$. We define the branch embedding $i_{\gamma,\theta}^{\mathcal{S}}$ a usual and $\pi_\theta : \mathcal{M}_\theta^{\mathcal{S}} \to \mathcal{M}_\theta^{\mathcal{T}}$ is given by the fact that the copy maps commute with the branch embeddings. We declare θ to be stable.

(B) for all $\gamma \in b - (\eta + 1)$, $\mathrm{rt}(\gamma) = \eta$. Let $\mathcal{S}$ choose

$$[0, \theta)_{\mathcal{S}} = (b - \eta) \cup [0, \eta]_{\mathcal{S}},$$

and let $\mathcal{M}_\theta^{\mathcal{S}}$ be the direct limit of the $\mathcal{M}_\gamma^{\mathcal{S}}$ for sufficiently large $\gamma \in b$. Branch embeddings $i_{\gamma,\theta}^{\mathcal{S}}$ for $\gamma \geq \eta$ are defined as usual. $\pi_\theta : \mathcal{M}_\theta^{\mathcal{S}} \to \mathcal{M}_\theta^{\mathcal{T}}$ is given by the fact that copy maps commute with branch embeddings. We declare θ to be stable.

Since θ is stable, $(\dagger)_\theta$ follows at once from $\forall \gamma < \theta\ (\dagger)_\gamma$.

Case 2. There are boundedly many unstable ordinals in b but no largest one.

We let η be the sup of the unstable ordinals in b. Let $\mathcal{S}$ choose

$$[0, \theta)_{\mathcal{S}} = (b - \eta) \cup [0, \eta]_{\mathcal{S}},$$

and define the corresponding objects as in case 1(B). We declare θ stable, and again $(\dagger)_\theta$ is immediate.

Case 3. There are arbitrarily large unstable ordinals in b. In this case, b is a disjoint union of pairs $\{\gamma, \gamma + 1\}$ such that γ is unstable and $\gamma + 1$ is stable. We set

$$[0, \theta)_{\mathcal{S}} = \{\xi \in b \mid \xi \text{ is unstable}\},$$

and let $\mathcal{M}_\theta^{\mathcal{S}}$ be the direct limit of the $\mathcal{M}_\xi^{\mathcal{S}}$'s for $\xi \in b$ unstable. There is no dropping in model or degree along $[0, \theta)_{\mathcal{S}}$. We define maps $i_{\xi,\theta}^{\mathcal{S}}, \pi_\theta$ as usual. If $e_\theta^{\mathcal{S}} \notin \mathcal{U}^{\mathrm{ext}}$, then we declare θ stable and $(\dagger)_\theta$ is immediate.

Suppose that $e_\theta^{\mathcal{S}} \in \mathcal{U}^{\mathrm{ext}}$. We declare θ unstable, and set

$$\begin{aligned}
\alpha_\theta &= i_{0,\theta}(\alpha_0),\\
P_{\theta+1} &= \mathrm{cHull}_1^{*,P_\theta}(\alpha_\theta \cup d_\theta \restriction i) \text{ where } d_\theta \restriction i = i_{0,\theta}^{\mathcal{S}}(d^M \restriction i),\\
\sigma_\theta &= \text{ anticollapse map.}
\end{aligned}$$

Note that by our assumptions, $d_\theta \restriction i = p_1(P_\theta) - \alpha_\theta$. $\theta+1$ is stable. As above, it is easy to check

Lemma 3.17. *$(\dagger)_{\theta+1}$ holds.*

Now we are back to the case that our current initial segment of $\mathcal{S}_{\nu,l}$ has a last stable node.

This finishes our construction of $\mathcal{S} = \mathcal{S}_{\nu,l}$ and $\mathcal{T}$. Note that every extender used in $\mathcal{S}$ is taken from a stable node and every stable node, except the last model of $\mathcal{S}$ contributes exactly one extender to $\mathcal{S}$. The last model of $\mathcal{S}$ is stable.

Let us describe the failures of the ISC in $\mathcal{S}$.

Lemma 3.18. *The following are equivalent:*

(a) P_γ does not satisfy the Jensen ISC,

(b) there is a stable root $\xi+1$ of $\mathcal{S}$ such that $\xi+1 \leq_S \gamma$ and $[\xi+1,\gamma]_S$ does not drop.

Proof. All unstable P_γ are premice, and if there is a premouse on the branch of $\mathcal{S}$ to γ, then P_γ is a premouse. Thus (a) implies (b). Conversely, if ξ is unstable, then $P_{\xi+1}$ does not satisfy the Jensen ISC because $(P_\xi, P_{\xi+1}, \sigma_\xi, \alpha_\xi)$ is problematic. If $[\xi+1,\gamma]_S$ does not drop, then $i_{\xi+1,\gamma}$ propagates the failure of the ISC. Thus (b) implies (a). □

Remark 3.19. Suppose $\xi+1$ is a stable root of $\mathcal{S}$, $\xi+1 \leq_S \gamma$, and $[\xi+1,\gamma]_S$ does not drop. Let $\lambda^* = \lambda^*_{P_\xi} = \lambda(E^{P_\xi}_{\alpha_\xi})$; then $E^{P_\xi}_{\alpha_\xi} \restriction \lambda$ is the first missing-from-P_γ whole proper initial segment of $\dot{F}^{P_\gamma}$. It may not be the only one.

Our definitions imply that stable roots that are extended without a drop are special.

Proposition 3.20. *Suppose $\xi+1$ is a stable root of $\mathcal{S}$, $\xi+1 <_S \gamma$, and $[\xi+1,\gamma]_S$ does not drop; then $\xi+1$ is special and $\xi+2 \leq_S \gamma$.*

Proof. Let $\eta+1 \leq_S \gamma$ be such that $\mathrm{pd}_S(\eta+1) = \xi+1$, and $E = E^{\mathcal{S}}_\eta$. If $\mathrm{lh}(E) < \lambda^*_{P_\xi}$ then $\varepsilon_\xi = \varepsilon_{\xi+1}$, so $\mathrm{pd}_S(\eta+1) \leq \xi$, contradiction. Thus $\mathrm{lh}(E) > \lambda^*_{P_\xi}$ and $\varepsilon_\xi = \lambda^*_{P_\xi}$. Since E is total over $P_{\xi+1}$, $P_{\xi+1}|\alpha_\xi = P_\eta|\alpha_\xi = M_{\nu,l}|\alpha_\xi$. This implies that $\xi+1$ is special.

Since $\xi+1$ is special, $\mathrm{crt}(E) < \varepsilon_{\xi+1} = \alpha_\xi$. If $\eta > \xi+1$ then $\mathrm{crt}(E) \neq \lambda^*_{P_\xi}$ because $\lambda^*_{P_\xi}$ is not measurable in P_η (since $E^{\mathcal{S}}_{\xi+1}$ has order 0). Thus $crt(E) < \lambda^*_{P_\xi} = \varepsilon_\xi$, so $\mathrm{pd}_S(\eta+1) \leq \xi$, contradiction.

Thus $\eta+1 = \xi+2$, as desired. □

Combining the arguments in [4, Lemma 9.6.5] with an additional element that deals with the fact that extenders used in $\mathcal{S}$ may fail the Jensen ISC, we show

Lemma 3.21. *Let γ be stable and suppose neither (I) nor (II) holds at γ; then the least disagreement between $(P_\gamma, \Sigma_\gamma)$ and $(M_{\nu,l}, \Omega_{\nu,l})$ is an extender disagreement, and it is passive on the $M_{\nu,l}$-side.*

Proof. The proof of [4, Lemma 9.6.5] shows that it is an extender disagreement. We shall not go into the many details of that proof, but rather focus on the proof that the disagreement is passive on the $M_{\nu,l}$ side. This is where the failure of the Jensen ISC in Q adds a complication.

So suppose toward contradiction that $(P_\gamma, \Sigma_\gamma)||\eta = (M_{\nu,l}, \Omega_{\nu,l})||\eta$ and $P_\gamma|\eta \neq M_{\nu,l}|\eta$, and $M_{\nu,l}|\eta$ is active. Let ν, l be the lex least counterexample. As in [4], we get that $l = 0, \eta = \nu$. Again, to simplify the notation, we write $\mathcal{S}$ for $\mathcal{S}_{\nu,0}$, $\mathcal{U}$ for $\mathcal{U}_{\nu,0}$ etc.

Let $F = \dot{F}^{M_{\nu,0}}$ and let F^* be the background extender for F. Let $j = i^V_{F^*}$ and $\kappa = \mathrm{crt}(F) = \mathrm{crt}(j)$. We have

(a) F^* backgrounds F^+.

(b) $j(M_{\nu,0})|\langle \nu, 0\rangle = M_{\nu,0}||\nu$ and $\mathrm{lh}(F) = o(M_{\nu,0})$ is a cardinal in $j(M_{\nu,0})$.[31]

(c) $\mathcal{S} \restriction \gamma + 1 = j(\mathcal{S}) \restriction \gamma + 1$.

(d) $j \restriction \mathcal{M}^{\mathcal{S}}_\kappa = i^{j(\mathcal{S})}_{\kappa, j(\kappa)}$.

(e) $j \restriction \mathcal{M}^{\mathcal{U}}_\kappa = i^{j(\mathcal{U})}_{\kappa, j(\kappa)}$.

(f) $\mathcal{M}^{\mathcal{S}}_\kappa|(\kappa^+)^{\mathcal{M}^{\mathcal{S}}_\kappa} = \mathcal{M}^{\mathcal{U}}_\kappa|(\kappa^+)^{\mathcal{M}^{\mathcal{U}}_\kappa}$.

Since our claim holds when $\mathcal{S}$ is replaced by $\mathcal{U}$ (by [4]), $M_{\nu,0} \trianglelefteq \mathcal{M}^{\mathcal{U}}_\tau$ for $\tau + 1 = \mathrm{lh}(\mathcal{U})$. It follows that $\mathcal{U} = j(\mathcal{U}) \restriction \tau + 1$, so $Q_\tau = \mathcal{M}^{j(\mathcal{U})}_\tau$, and

$$E^{j(\mathcal{U})}_\tau = F^+.$$

Let

$$\begin{aligned} E_j &= \{(a, X) \mid a \in [j(\kappa)]^{<\omega} \wedge X \in \mathcal{M}^{\mathcal{U}}_\kappa \wedge a \in j(X)\}, \\ N &= \mathcal{M}^{j(\mathcal{U})}_{j(\kappa)}||j(\kappa) \\ &= \mathcal{M}^{j(\mathcal{S})}_{j(\kappa)}||j(\kappa), \end{aligned}$$

That is, E_j is the the common (short) $(\kappa, j(\kappa))$ extender of $i^{j(\mathcal{S})}_{\kappa,j(\kappa)}$ and $i^{j(\mathcal{U})}_{\kappa,j(\kappa)}$. Let H be the first whole initial segment K of E_j such that $K \notin N$. The Jensen ISC holds for the models of $\mathcal{U}$ and $j(\mathcal{U})$, so H^+ must be used in $j(\mathcal{U})$. But F^+ is used in $j(\mathcal{U})$ and $F^+ \triangleleft E_j$, so $F \restriction \lambda_F = H$, i.e.

$$F \restriction \lambda_F = \text{ first whole initial segment } K \text{ of } E_j \text{ such that } K \notin N.$$

Let

$$G^+ = E^{j(\mathcal{S})}_\theta,$$

where $\theta + 1 <_{j(S)} j(\kappa)$ and $\mathrm{pd}_{j(S)}(\theta + 1) = \kappa$. We have that $G \restriction \lambda_G \triangleleft E_j$, and $G \notin N$, so by the ISC for F, $F \trianglelefteq G$. If F is on the sequence of $\mathcal{M}^{j(\mathcal{S})}_\theta$ then since $P_\gamma = \mathcal{M}^{j(\mathcal{S})}_\gamma$ and $\gamma \leq \theta$ and

[31]See [4, Theorem 10.4.1].

$\mathrm{lh}(E_\gamma^{j(\mathcal{S})}) > \mathrm{lh}(F)$, F is on the sequence of P_γ. Thus F is not on the sequence of $\mathcal{M}_\theta^{j(\mathcal{S})}$, and

$$F \upharpoonright \lambda_F \lhd G \upharpoonright \lambda_G \lhd E_j,$$

and F witnesses that $\mathcal{M}_\theta|\mathrm{lh}(G)$ does not satisfy the Jensen ISC.

By Lemma 3.18 we have a stable root $\mu+1$ such that $\mu+1 \leq_{j(S)} \theta$ and $(\mathcal{M}_\mu, \mathcal{M}_{\mu+1}, \sigma_\mu, \alpha_\mu)^{j(\mathcal{S})}$ is problematic. (In $j(N^*)$.) Letting $R = \mathcal{M}_\mu^{j(\mathcal{S})}$ and $W = \mathcal{M}_{\mu+1}^{j(\mathcal{S})}$, we have $\lambda_F = \lambda_R^* = \lambda_W^*$ by Remark 3.19. Moreover $F = E_{\alpha_\mu}^R$. So $\alpha_\mu = o(M_{\nu,0})$. It follows that

$$\gamma = \mu + 1,$$
$$(P_\mu, P_{\mu+1}) = (\mathcal{M}_\mu^{j(\mathcal{S})}, \mathcal{M}_{\mu+1}^{j(\mathcal{S})}),$$
$$F = E_{\alpha_\mu}^{P_\mu},$$

and

$$e_\mu^{j(\mathcal{S})} = e_\mu^{\mathcal{S}} = e_\tau^{\mathcal{U}} = e_\tau^{j(\mathcal{U})}.$$

Claim 3.22. *$\kappa \leq_S \mu$. In particular, κ is unstable in $\mathcal{S}$ and $j(\mathcal{S})$.*

Proof. $\kappa = \mathrm{crt}(\dot{F}^{P_\mu})$, so $\kappa = i_{0,\mu}(\kappa_0)$ where $\kappa_0 = \mathrm{crt}(\dot{F}^{P_0})$. For $\alpha \leq_S \mu$ let

$$\kappa_\alpha = i_{0,\alpha}(\kappa_0),$$

and let $\xi \leq_S \mu$ be least such that $\kappa = \kappa_\xi$. Since κ is inaccessible, $\xi \geq \kappa$.

Suppose toward contradiction that $\xi > \kappa$, and let $\eta + 1 \leq_S \xi$ with $\eta \geq \kappa$, and

$$\mathrm{pd}_S(\eta + 1) = \gamma < \kappa.$$

So γ and $\eta + 1$ are unstable. Let $E^+ = E_\eta^{\mathcal{S}}$; then $\kappa \leq \lambda(E)$ because the $\lambda(E_\alpha)$ increase strictly with α. Also, $\mathrm{crt}(E) \leq \kappa_\gamma$ since otherwise $\kappa_\gamma = \kappa_\xi = \kappa$. But then

$$\kappa_{\eta+1} = i_{E^+}(\kappa_\gamma) > \lambda(E) \geq \kappa,$$

so $\kappa_\xi > \kappa$, contradiction.

Thus $\xi = \kappa$, so $\kappa \leq_S \mu$. □

By the claim, $e_\kappa^{\mathcal{S}} = e_\kappa^{\mathcal{U}}$ and $e_\kappa^{\mathcal{S}} \trianglelefteq e_\mu^{\mathcal{S}}$. It follows that $e_\kappa^{\mathcal{U}} \trianglelefteq e_\tau^{\mathcal{U}}$; that is,

$$\kappa \leq_U \tau.$$

Also, $P_\kappa = Q_\kappa$, which improves the agreement given in (f) above. Finally, $\mathrm{crt}(i_{\kappa,\mu}) > \kappa$ or $\kappa = \mu$, so P_κ agrees with P_μ up to some inaccessible of P_κ that is $> \kappa$. The relevant diagram is Figure 1.

Notice that $\mu+1$ is special in $j(\mathcal{S})$, because $(P_{\mu+1}, \Sigma_{\mu+1})||\alpha_\mu = j((M_{\nu,0}, \Omega_{\nu,0}))|\alpha_\mu = (M_{\nu,0}, \Omega_{\nu,0})||\alpha_\mu$.

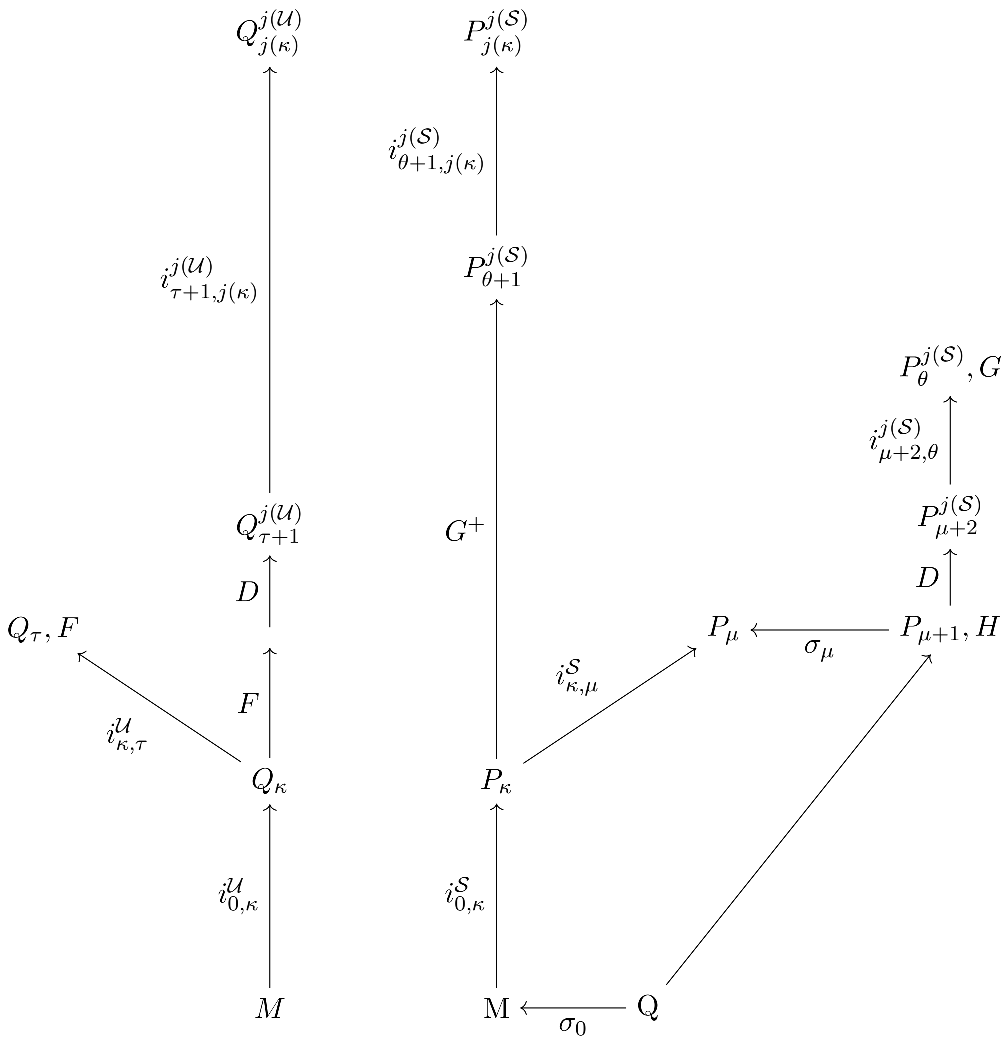

Figure 1: Diagram if $F = \dot{F}^{M_{\nu,0}}$ is an extender disagreement in the phalanx comparison. $P_\kappa = Q_\kappa$, $P_\mu = Q_\tau$, and $P^{j(\mathcal{S})}_{j(\kappa)} = Q^{j(\mathcal{U})}_{j(\kappa)}$. $F^+ = F^\frown D$ and $F \restriction \lambda_F \lhd G \restriction \lambda_G$. $i^{\mathcal{S}}_{0,\kappa} = i^{\mathcal{U}}_{0,\kappa}$ and $i^{\mathcal{S}}_{\kappa,\mu} = i^{\mathcal{U}}_{\kappa,\tau}$.

Let

$$\begin{aligned}D &= E^{j(\mathcal{S})}_{\mu+1}\\ &= \text{ order 0 measure of } P_{\mu+1} \text{ on } \lambda_F.\end{aligned}$$

Claim 3.23. *Let $\beta_\mu = \alpha_\mu^{+,P_{\mu+1}}$; then $P_{\mu+1}||\beta_\mu \trianglelefteq \mathrm{Ult}(P_\mu, F)$.*

Proof. Let N be a level of $P_{\mu+1}$ projecting to α_μ. We can apply the condensation theorem of [3] to $\sigma_\mu \restriction N$. Since $\mathrm{crt}(\sigma_\mu) = \alpha_\mu = \mathrm{lh}(F)$, the "ultrapower away" conclusion of the theorem holds, as desired. In other words, $N \trianglelefteq \mathrm{Ult}(P_\mu, F)$. Since N is arbitrary, we have $P_{\mu+1}||\beta_\mu \trianglelefteq \mathrm{Ult}(P_\mu, F)$ as claimed. □

Claim 3.24. *$F^+ = F$-then-D.*

Proof. By definition, $F^+ = F$-then-E, where E is the order 0 measure of $\mathrm{Ult}(Q_\kappa, F)$ on λ_F. But $\mathrm{Ult}(Q_\kappa, F) = \mathrm{Ult}(P_\kappa, F)$, and $\mathrm{Ult}(P_\kappa, F)$ agrees with $\mathrm{Ult}(P_\mu, F)$ below $i_F(\rho)$, where ρ is the first inaccessible of P_κ strictly above κ. Since $i_F(\rho) > \beta_\mu$, we get $E = D$ from Claim 3.23. □

Let

$$H = \dot{F}^{P_{\mu+1}}.$$

Since $(P_\mu, P_{\mu+1}, \sigma_\mu, \alpha_\mu)$ is problematic, $H \notin P_\mu$. Letting $\sigma_\mu(q) = p(P_\mu) - \alpha_\mu$, we have that H is equivalent to $H \restriction \lambda_F \cup q = G \restriction \lambda_F \cup t$, where $t = i^{j(\mathcal{S})}_{\mu+1,\theta}(q)$. But $Q_\tau = P_\mu$, so it will be enough for a contradiction to show:

Claim 3.25. . *For any finite $t \subseteq \lambda_G$, $G \restriction \lambda_F \cup t \in Q_\tau$.*

Proof. We have that $F^+ = E^{j(\mathcal{U})}_\tau$ and $F^+ \triangleleft E_j$, so $\kappa = \mathrm{pd}_{j(U)}(\tau+1)$ and $\tau + 1 <_{j(U)} j(\kappa)$. Let

$$\begin{aligned}\lambda^* &= i^{j(\mathcal{S})}_{\mu+1,\theta}(\lambda_F),\\ \rho &= \text{ least } \beta \in [\mu+1, \theta]_{j(S)} \text{ s.t. } \lambda^* < \mathrm{crt}(i^{j(\mathcal{S})}_{\beta,\theta}),\\ \xi &= \text{ least } \beta \in [\tau+1, j(\kappa)]_{j(U)} \text{ s.t. } \lambda^* \leq \mathrm{crt}(i^{j(\mathcal{U})}_{\beta,j(\kappa)}).\end{aligned}$$

Subclaim 1. $G \restriction \lambda^*$ is the (κ, λ^*)-extender derived from $i^{j(\mathcal{S})}_{\mu+1,\rho} \circ i_F$.

Proof. Recall $H = \dot{F}^{P_{\mu+1}}$; then for $A \subseteq \kappa$,

$$i_F(A) = i_H(A) \cap \lambda_F,$$

so for $k = i^{j(\mathcal{S})}_{\mu+1,\rho}$,

$$\begin{aligned} k \circ i_F(A) &= k(i_H(A) \cap \lambda_F) \\ &= i_{k(H)}(A) \cap k(\lambda_F) \\ &= i_G(A) \cap \lambda^*, \end{aligned}$$

as desired. □

Subclaim 2. $e^{j(\mathcal{U})}_{\tau+1,\xi} = e^{j(\mathcal{S})}_{\mu+2,\rho}$.

Proof. Let $s = e^{j(\mathcal{U})}_{\kappa,\xi}$, so that

$$s = \langle F^+\rangle^\frown e^{j(\mathcal{U})}_{\tau+1,\xi}.$$

We can recover s from $E_j \restriction \lambda^*$ by looking at missing-from-N initial segments of tail factors. More precisely, let

$$\begin{aligned} R_0 &= Q_\kappa | \kappa^{+,Q_\kappa}, \\ k_0 &= i^{j(\mathcal{U})}_{\kappa,\xi} \restriction R_0, \\ W &= k_0(R_0), \\ E_0 &= E_{k_0} = E_j \restriction \lambda^*. \end{aligned}$$

Then $s(0) = E^+$, where E is the shortest whole initial segment of E_0 such that $E \notin N$. (That is, $s(0) = F^+$.) Let $R_1 = \mathrm{Ult}(R_0, s(0))$ and $k_1 \colon R_1 \to W$ be the factor map; then $R_1 \lhd \mathcal{M}^{j(\mathcal{U})}_{\tau+1}$ and $k_1 = i^{j(\mathcal{U})}_{\tau+1,\xi} \restriction R_1$. Letting $E_1 = E_{k_1}$, we let $s(1) = E^+$, where E is the shortest whole initial segment of E_1 such that $E \notin N$. And so on.

But $E_j \restriction \lambda^* = G \restriction \lambda^*$, and so by Subclaim 1 $E_j \restriction \lambda^*$ is the extender (over the same R_0) of $i^{j(\mathcal{S})}_{\mu+1,\rho} \circ i_F$. Further,

$$i^{j(\mathcal{S})}_{\mu+1,\rho} \circ i_F = i^{j(\mathcal{S})}_{\mu+2,\rho} \circ i_D \circ i_F = i^{j(\mathcal{S})}_{\mu+2,\rho} \circ i_{F^+}.$$

Thus the recovery process above must yield

$$s = \langle F^+\rangle^\frown e^{j(\mathcal{S})}_{\mu+2,\rho}.$$

This proves Subclaim 2. □

Subclaim 3. In $j(\mathcal{S})$, every extender in $\mathrm{ran}(e^{j(\mathcal{S})}_{\mu+2,\rho})$ is very close[32] to the model to which it is applied.

Proof. Let $r = e^{j(\mathcal{S})}_{\mu+2,\rho}$. By the proof of [4, 4.5.7] it is enough to show that all extenders used in r are very close to the models to which they are applied.[33] Suppose that $E = E^{j(\mathcal{S})}_\alpha$ is used in r, and let

$$\mathrm{pd}_{j(\mathcal{S})}(\alpha+1) = \beta \in [\mu+2, \rho)_{j(\mathcal{S})}.$$

[32] E is very close to N iff $E_a \in N$ for all finite $a \subset \mathrm{lh}(E)$.

[33] [4, 4.5.7] is stated for plus trees, whereas $j(\mathcal{S})$ is a pseudo-tree, but the difference is not relevant to the proof.

By [4, 4.5.3(1)] we may assume that $\beta <_{j(\mathcal{S})} \alpha$. Let $\eta+1$ be least in $(\beta, \alpha]_{j(\mathcal{S})}$. By [4, 4.5.7(2)(iv)], we are done unless $\mathcal{M}^{*,j(\mathcal{S})}_{\eta+1} = \mathcal{M}^{*,j(\mathcal{S})}_{\alpha+1}$ and $E^- = k(\dot{F}^{\mathcal{M}^{*,j(\mathcal{S})}_{\alpha+1}})$, where $k = (i_{\eta+1,\alpha} \circ i^*_{\eta+1})^{j(\mathcal{S})}$. But the branch $[\mu+1, \rho)_{j(S)}$ does not drop, so $\mathcal{M}^{*,j(\mathcal{S})}_{\alpha+1} = \mathcal{M}^{j(\mathcal{S})}_{\beta}$, and $E = k(i^{j(\mathcal{U})}_{\mu+1,\beta}(H))$. But then $\mathrm{crt}(E) < \varepsilon_\mu$, so $\beta \leq \mu$, contradiction. $\square$

Subclaim 4. In $j(\mathcal{U})$, every extender in $\mathrm{ran}(e^{j(\mathcal{U})}_{\tau+1,\xi})$ is very close to the model to which it is applied.

Proof. Let $r = e^{j(\mathcal{U})}_{\tau+1,\xi} = e^{j(\mathcal{S})}_{\mu+2,\rho}$. Let W_i be the model to which $r(i)$ is applied in $j(\mathcal{S})$ and Z_i the model to which $r(i)$ is applied in $j(\mathcal{U})$. Thus

$$\begin{aligned} W_0 &= \mathcal{M}^{j(\mathcal{S})}_{\mu+2} \\ &= \mathrm{Ult}(P_{\mu+1}, D), \\ Z_0 &= \mathcal{M}^{j(\mathcal{U})}_{\tau+1} = \mathrm{Ult}(Q_\kappa, F^+) \\ &= \mathrm{Ult}(\mathrm{Ult}(Q_\kappa, F), D). \end{aligned}$$

Let $\beta = \lambda_F^{++,P_{\mu+1}}$; then by Claim 3.23, $P_{\mu+1}||\beta \trianglelefteq \mathrm{Ult}(P_\mu, F)$, and hence $P_{\mu+1}||\beta \trianglelefteq \mathrm{Ult}(Q_\kappa, F)$. But then i_D propagates this agreement to $i_D(\beta) = \lambda_D^{++,W_0}$. That is

$$W_0||\lambda_D^{++,W_0} \trianglelefteq Z_0.$$

This agreement propagate under the $r \restriction k$ ultrapowers, so

$$W_k||i_{r\restriction k}(\lambda_D)^{++,W_k} \trianglelefteq Z_k$$

for all $k \leq \mathrm{dom}(r)$. But for $k \in \mathrm{dom}(r)$,

$$\mathrm{crt}(r(k)) \leq i_{r\restriction k}(\lambda_D)$$

by our definition of ρ. Since $r(k)$ is very close to W_k, every $r(k)_a$ belongs to $W_k||i_{r\restriction k}(\lambda_D)^{++,W_k}$, and hence belongs to Z_k, as desired. $\square$

We have not quite reached λ_G on the $\mathcal{U}$ side yet. There is one more extender to go. Let

$$\xi_1 = \text{ least } \alpha <_{j(U)} j(\kappa) \text{ s.t. } \lambda_G \leq \mathrm{crt}(i^{j(\mathcal{U})}_{\alpha,j(\kappa)}).$$

Subclaim 5. $\mathrm{pd}_{j(\mathcal{U})}(\xi_1) = \xi$; moreover, letting $E^+ = E^{j(\mathcal{U})}_{\xi_1 - 1}$, we have

(a) $\mathrm{crt}(E) = \lambda^*$, $\lambda_E = \lambda_G$, and E has type A, and

(b) E is very close to $\mathcal{M}^{j(\mathcal{U})}_{\xi}$.

Proof. For (a): λ_G and λ^* are cutpoints of E_j. Moreover, $\lambda^* = i^{j(\mathcal{S})}_{\mu+1,\rho}(\lambda_F)$, $\lambda_G = i^{j(\mathcal{S})}_{\mu+1,\rho}(\lambda_H)$, and λ_F is the largest proper cutpoint of H. It follows that λ^* is the largest proper cutpoint of G.

Also

$$\begin{aligned}\lambda^* &= i^{j(\mathcal{S})}_{\mu+2,\rho} \circ i_D \circ i_F(\kappa)\\ &= i^{j(\mathcal{U})}_{\tau+1,\xi} \circ i_{F^+}(\kappa)\\ &= i^{j(\mathcal{U})}_{\kappa,\xi}(\kappa),\end{aligned}$$

so $\operatorname{crt}(i^{j(\mathcal{U})}_{\xi,j(\kappa)}) \leq \lambda^*$, so then $\operatorname{crt}(i^{j(\mathcal{U})}_{\xi,j(\kappa)}) = \lambda^*$ by our choice of ξ.

Let $K^+ = E^{j(\mathcal{U})}_{\eta+1}$ where $\eta+1 <_{j(U)} j(\kappa)$ and $\mathrm{pd}_{j(U)}(\eta+1) = \xi$. Since λ_K is a cutpoint of E_j, we have $\lambda_G \leq \lambda_K$. But $\lambda_K \leq \lambda_G$, since otherwise $K \restriction \lambda_G \in N$ by the Jensen ISC, which is impossible because $i_{K\restriction\lambda_G}$ maps $(\lambda^*)^{+,N}$ cofinally into $\lambda_G^{+,N}$, and hence collapses it. Thus $\lambda_K = \lambda_G$. It follows that $K = E$ and $\eta+1 = \xi_1$.

If γ is a proper cutpoint of E, then γ is a cutpoint of G strictly between λ^* and λ_G. So there are no such γ. This finishes the proof of (a)

For (b), we apply [4, 4.5.3(1)]. We may assume that $\xi <_{j(U)} \alpha$, where $\xi_1 = \alpha+1$. Let $\eta+1$ be least in $(\xi,\alpha]_{j(U)}$. By [4, 4.5.7(2)(iv)], we are done unless $\mathcal{M}^{*,j(U)}_{\eta+1} = \mathcal{M}^{*,j(U)}_{\alpha+1}$ and $E = (i_{\eta+1,\alpha} \circ i^*_{\eta+1})^{j(U)}(\dot{F}^{\mathcal{M}^{*,j(U)}_{\alpha+1}})$. But the branch $[0,\alpha+1]_{j(U)}$ does not drop, so $M^*_{\alpha+1} = M^*_{\eta+1} = \mathcal{M}_\xi$ in $j(U)$. Thus $E = i^{j(U)}_{\xi,\alpha}(K)$, where $K = i^{j(U)}_{0,\xi}(\dot{F}^M)$. But E has type A and $\dot{F}^M$ has type B, so this impossible. This proves (b). □

We can now finish the proof of Claim 3.25. Let I be the branch extender of $i^{j(\mathcal{U})}_{\tau+1,\xi_1}$, that is,

$$I = E_k, \text{ where } k = i^{j(\mathcal{U})}_{\tau+1,\xi_1}.$$

By [4, 4.5.7], I is very close to $\mathcal{M}^{j(\mathcal{U})}_{\tau+1}$. Thus

$$I_t \in \mathcal{M}^{j(\mathcal{U})}_{\tau+1}.$$

Since $\mathcal{M}^{j(\mathcal{U})}_{\tau+1}$ agrees with $\mathrm{Ult}(\mathcal{M}^{j(\mathcal{U})}_\tau, F^+)$ below $i_{F^+}(\kappa)$, we have $\mathcal{M}^{j(\mathcal{U})}_{\tau+1}||i_{F^+}(\kappa) \in \mathcal{M}^{j(\mathcal{U})}_\tau = \mathcal{M}^{\mathcal{U}}_\tau$, so

$$I_t \in \mathcal{M}^{\mathcal{U}}_\tau.$$

But $F^+ \in \mathcal{M}^{\mathcal{U}}_\tau$, and from F^+ and I_t one can recover $G \restriction \lambda_F \cup t$. (Let K be the extender of F^+-then-I_t; then $K \restriction \lambda_F \cup t = G \restriction \lambda_F \cup t$.) This proves Claim 3.25. □

That in turn completes the proof of Lemma 3.21. □

Lemma 3.26. *For some $(\nu,l) \leq (\eta_0,0)$, the construction of $\mathcal{S}_{\nu,l}$ stops for reason (II), that is, letting $\theta+1 = \mathrm{lh}(\mathcal{S}_{\nu,l})$,*

$$(P_\theta, \Sigma_\theta) \trianglelefteq (M_{\nu,l}, \Omega_{\nu,l}),$$

and $[rt(\theta),\theta]_{\mathcal{S}}$ does not drop.

Proof. We use the Dodd-Jensen argument in the proof of 3.11.

Suppose not, and let $\mathcal{S} = \mathcal{S}_{\eta_0,0}$ and $\theta + 1 = \text{lh}(\mathcal{S})$. We adopt the notation above, so that (P_ξ, Σ_ξ) is the pair at ξ in $\mathcal{S}$, $\mathcal{T} = (\text{ id}, \sigma)\mathcal{S}$, and (P^*_ξ, Σ^*_ξ) is the pair at ξ in $\mathcal{T}$. The copy maps are $\pi_\xi \colon (P_\xi, \Sigma_\xi) \to (P^*_\xi, \Sigma^*_\xi)$.

$\mathcal{U} = \mathcal{U}_{\eta_0,0}$ iterates (M, Σ) to $(M_{\eta_0,0}, \Omega_{\eta_0,0})$, while $\mathcal{S}$ iterates it strictly past. More precisely, let $\gamma + 1 = \text{lh}(\mathcal{U})$ and (Q_ξ, Λ_ξ) be the pair at ξ in $\mathcal{U}$; then

$$(Q_\gamma, \Lambda_\gamma) = (M_{\eta_0,0}, \Omega_{\eta_0,0}) \trianglelefteq (P_\theta, \Sigma_\theta).$$

Let $i_{\alpha,\beta}, i^*_{\alpha,\beta}$, and $j_{\alpha,\beta}$ be the branch embeddings of $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$,

Since either $[0, \theta)_S$ drops or $(M_{\eta_0,0}, \Omega_{\eta_0,0}) \triangleleft (P_\theta, \Sigma_\theta)$, we get that for $\bar{\sigma}_\theta = \sigma_\theta \restriction M_{\eta_0,0}$, there is $(N, \Psi) \trianglelefteq (P^*_\theta, \Sigma^*_\theta)$ such that

$$\bar{\sigma}_\theta \colon (M_{\eta_0,0}, \Omega_{\eta_0,0}) \to (N, \Psi)$$

is $\Sigma_1^{\mathcal{L}}$- elementary. Moreover, $[0, \theta)_T$ drops or $(N, \Psi) \triangleleft (P^*_\theta, \Sigma^*_\theta)$. But then

$$\bar{\sigma}_\theta \circ j_{0,\gamma} \colon (M, \Sigma) \to (N, \Psi)$$

is $\Sigma_1^{\mathcal{L}}$ elementary, and maps (M, Σ) to a iterate along a branch that has dropped, contradiction. □

Now fix $(\nu, l) \le (\eta_0, k_0)$ as in the 3.26. Let $\mathcal{S} = \mathcal{S}_{\nu,l}$, $\mathcal{U} = \mathcal{U}_{\nu,l}$, and $\mathcal{T} = (\text{ id }, \sigma_0)\mathcal{S}$ be the lift of $\mathcal{S}$ defined above. Let us adopt the rest of the notation above for the nodes and branch embeddings of these trees, the copy maps, the problematic tuples, and so on. Let $\gamma + 1 = \text{lh}(\mathcal{U})$, so that

$$(P_\theta, \Sigma_\theta) \trianglelefteq (M_{\nu,l}, \Omega_{\nu,l}) \trianglelefteq (Q_\gamma, \Lambda_\gamma).$$

Lemma 3.27. *For some unstable ξ, $rt(\theta) = \xi + 1$.*

Proof. Suppose the claim is false. Then

(i) either $\text{rt}(\theta)$ is unstable,

(ii) or $\text{rt}(\theta)$ is stable, and $\text{rt}(\theta)$ is a limit of unstable $\xi <_S \text{rt}(\theta)$.

In either case, $0 \le_S \text{rt}(\theta) \le_S \theta$ and $[0, \theta]_S$ does not drop. Let $i = i_{0,\theta}$, and $i^* = i^*_{0,\theta}$ be the branch embeddings of $\mathcal{S}$ and $\mathcal{T}$. The relevant diagram is

$$\begin{tikzcd} P^*_\theta & P_\theta \arrow[l, "\pi_\theta"'] \\ M \arrow[u, "i^*"] & M \arrow[l, "\text{id}"] \arrow[u, "i"'] \end{tikzcd}$$

Note first that i is elementary in the category of mouse pairs, because

$$\begin{aligned}\Sigma &= (\Sigma_\theta^*)^{i^*} \\ &= (\Sigma_\theta^*)^{\pi_\theta \circ i} \\ &= ((\Sigma_\theta^*)^{\pi_\theta})^{i} \\ &= \Sigma_\theta^i.\end{aligned}$$

Thus we can apply Dodd-Jensen:

Subclaim 1. $(P_\theta, \Sigma_\theta) = (Q_\gamma, \Lambda_\gamma)$ and $[0, \gamma]_U$ does not drop.

Proof. We have that $i\colon (M, \Sigma) \to (P_\theta, \Sigma_\theta)$ is elementary, and $(P_\theta, \Sigma_\theta) \trianglelefteq (Q_\gamma, \Lambda_\gamma)$. Since (M, Σ) is a mouse pair, Dodd-Jensen gives the desired conclusion.

□

It follows that $\langle \nu, l \rangle = \langle \eta_0, 0 \rangle$. Let $j = j_{0,\gamma}$.

Subclaim 2. $i = j$

Proof. j is an iteration map, so $i(\eta) \leq j(\eta)$ for all η. Since i^* is an iteration map

$$\pi_\theta \circ i(\eta) = i^*(\eta) \leq \pi_\theta \circ j(\eta)$$

for all η. Applying π_θ^{-1}, we get that $i(\eta) \leq j(\eta)$ for all η.

□

Subclaim 3. $\mathrm{rt}(\theta)$ is is not a limit ordinal.

Proof. Suppose $\xi = \mathrm{rt}(\theta)$ is a limit ordinal. Since ξ is a limit of unstables, $\mathrm{ran}(e_\xi^{\mathcal{S}})$ consists of extenders used in $\mathcal{U}$, and hence extenders of the form E^+ where E is on the sequence of a premouse with the ISC. It follows that the $e_\xi^{\mathcal{S}}$ can be recovered from $i = j$ by looking at E^+ for E the first missing whole initial segment of the current tail factor. Thus $e_\xi^{\mathcal{S}} \trianglelefteq e_\gamma^{\mathcal{U}}$, so $e_\xi^{\mathcal{S}} = e_\tau^{\mathcal{U}}$ for some τ. But then ξ is unstable, contrary to $\xi = \mathrm{rt}(\theta)$.

□

By Subclaim 3 $\mathrm{rt}(\theta) = \eta + 1$ where $\beta = \mathrm{pd}_{\mathcal{S}}(\eta + 1)$ is unstable. Let τ be such that $e_\beta^{\mathcal{S}} = e_\tau^{\mathcal{U}}$. Since $e_\beta^{\mathcal{S}} \triangleleft e_\theta^{\mathcal{S}} = e_\gamma^{\mathcal{U}}$, we have that $\tau <_U \gamma$ and $e_{\tau,\gamma}^{\mathcal{U}} = e_{\beta,\theta}^{\mathcal{S}}$. In particular, $e_{\eta+1}^{\mathcal{S}} \in \mathcal{U}^{\mathrm{ext}}$, so $\eta + 1$ is unstable, contradiction.

This proves Lemma 3.27. □

By Lemma 3.27 and Lemma 3.18, P_θ does not satisfy the Jensen ISC. On the other hand, $P_\theta \trianglelefteq M_{\nu,l}$, so P_θ does satisfy the Jensen ISC. This contradiction completes the proof of Theorem 1.8 in the case that $d_i = \gamma_M$. □

4. Dodd solidity for unsound structures

In this section, we work with a possibly unsound mouse pair (M, Σ) such that M is of type B and $\deg(M) = 0$. As before, we let $F = \dot{F}^M$, $\mu = \mathrm{crt}(F)$, and $\tau = (\mu)^{+,M}$. Recall Definition 1.10:

- $\hat{\rho}_M$ is the least $\alpha \geq \tau$ such that there is a finite d such that $F \restriction (\alpha \cup d) \notin M$.
- $\hat{d}_M$ is the $<^*$-least d such that $F \restriction (\hat{\rho}_M \cup d) \notin M$.

Lemma 4.1. *Suppose M is strongly 1-sound; then $d_M = \hat{d}_M$ and $\rho^*_M = \hat{\rho}_M$.*

Proof. $\hat{\rho}_M \leq \rho^*_M$ because $h^*_M(\rho^*_M \cup d_M) = M$, so $F \restriction (\rho^*_M \cup d_M) \notin M$. For the reverse inequality, let $\eta < \rho^*_M$ and $c \in [o(M)]^{<\omega}$; we must see that $F \restriction (\eta \cup c) \in M$. Let $a \subset \rho^*_M$ be finite such that $c = h^*_M(a, d_M)$. Let $\gamma = \max(\eta, \max(a) + 1)$. By Proposition 2.12(a)[34], $F \restriction (\gamma \cup d_M) \in M$. Thus $F \restriction (\eta \cup c) \in M$, as desired.

$\hat{d}_M \leq^* d_M$ by definition and the fact that $\rho^*_M = \hat{\rho}_M$. If $\hat{d}_M <^* d_M$, by Theorem 1.8, $F \restriction (\rho^*_M \cup d) \in M$, contradiction. So $\hat{d}_M = d_M$. □

Definition 4.2. For $i \in \mathrm{dom}(\hat{d}_M)$,

$$W^i_M = \mathrm{Th}_1^{*,M}(\hat{d}_M(i) \cup \{\hat{d}_M(0), ..., \hat{d}_M(i-1)\}).$$

We say that M is *Dodd solid at i* iff $W^i_M \in M$. We say M is *Dodd solid* iff M is Dodd solid at all $i \in \mathrm{dom}(\hat{d}_M)$. ⊣

Equivalently, we could take W^i_M to be $F \restriction (\hat{d}_M(i) \cup \{\hat{d}_M(0), ..., \hat{d}_M(i-1)\})$. The W^i_M are the (standard) Dodd solidity witnesses for $\hat{d}_M$. We don't need generalized witnesses, because the standard ones are preserved by Σ_0 ultrapowers.

Theorem 4.3. *Suppose (M, Σ) is a mouse pair of type B and degree 0; then M is Dodd solid.*

Proof. Let N be the strong core of M; that is, $N = \bar{\mathfrak{C}}(M)^-$. Since N is strongly 1-sound, $\hat{d}_N = d_N$ and $\rho^*_N = \hat{\rho}_N$. Let $\sigma : N \to M$ be the anticore map and $(\tau_N, \gamma_N) = \sigma^{-1}(\tau, \gamma_M)$. We have $crt(\sigma) \geq \rho_1(N) = \rho_1(M)$.

Claim 1. $\mathrm{ran}(\sigma)$ is cofinal in τ, $\tau^{+,M}$, and $o(M)$.

Proof. $M = \mathrm{Ult}_0(N, s)$, where s is a sequence of extenders that are close to the models to which they are applied, moreover σ is the ultrapower map.The claim follows. □

The natural prewellorder of W^i_N has cofinality τ_N in N, so we easily get

Claim 2. Suppose that $\mathrm{Th}_1^{*,N}(\beta \cup \{c\}) \in N$; then $\sigma(\mathrm{Th}_1^{*,N}(\beta \cup \{c\})) = \mathrm{Th}_1^{*,M}(\sigma(\beta) \cup \{\sigma(c)\})$, so $\mathrm{Th}_1^{*,M}(\sigma(\beta) \cup \{\sigma(c)\}) \in M$.

[34] Here we use 1-soundness of M.

It follows that for $i \in \mathrm{dom}(d_N)$, $\sigma(W_N^i) = \mathrm{Th}_1^{*,M}(\sigma(d_N(i) \cup \{\sigma(d_N(0), ..., \sigma(d_N(i-1))\}) \in M$. This leads at once to

Claim 3.

(a) $\hat{\rho}_M = \sup \sigma``\rho_N^*$, and

(b) $\hat{d}_M = \sigma(d_N)$.

Proof. For (a): We have that $\hat{\rho}_N = \rho_N^* = \max(\rho_1(N), \tau_M)$. Suppose first $\rho_N^* = \rho_1(N) \geq \tau_N$; then

$$\rho_N^* = \sup \sigma``\rho_N^* = \rho_1(M) \geq \tau_M = \tau_N.$$

Clearly $\rho_1(M) \leq \hat{\rho}_M$. On the other hand, $\mathrm{Th}_1^{*,N}(\rho_N^* \cup d_N) \notin N$ and $\sigma \restriction \rho_N^* = \mathrm{\ id\ }$, so $\mathrm{Th}_1^{*,M}(\rho_N^* \cup \sigma(d_N)) \notin M$. This implies $\hat{\rho}_M \leq \rho_1(M)$. Thus $\hat{\rho}_M = \sup \sigma``\rho_N^*$.

Suppose next that $\rho_N^* = \tau_N > \rho_1(N)$. Thus $\sup \sigma``\rho_N^* = \tau_M$. Since $\mathrm{Th}_1^{*,N}(\tau_N \cup d_N) \notin N$, by Schlutzenberg's lemma $\mathrm{Th}_1^{*,M}(\tau_M \cup \sigma(d_N) \notin M$. (Note that the theories are amenable to N and M respectively.) Thus $\tau_M \leq \hat{\rho}_M$, so $\tau_M = \hat{\rho}_M$, as desired.

For (b), we have $\mathrm{Th}_1^{*,M}(\hat{\rho}_M \cup \sigma(d_N)) \notin M$ by (a) and Schlutzenberg's lemma. Thus $\hat{d}_M \leq^* \sigma(d_N)$. Suppose toward contradiction that i is least such that $\hat{d_M}(i) < \sigma(d_N(i))$, and let $\alpha = d_N(i)$. By Claim 2, and induction, $\sigma(W_N^i) = \mathrm{Th}_1^{*,M}(\sigma(\alpha) \cup \{\hat{d}_M(0), ..., \hat{d}_M(i-1))\} \in M$, so $\mathrm{Th}_1^{*,M}(\hat{d}_M(i) + 1 \cup \{\hat{d}_M(0), ..., \hat{d}_M(i-1))\} \in M$, contradiction.

□

Combining Claims 2 and 3, we get that $\sigma(W_N^i) = W_M^i \in M$, for all $i \in \mathrm{dom}(d_N) = \mathrm{dom}(\hat{d}_M)$. That finishes the proof of Theorem 4.3. □

We also get a version of the Zeman Exchange Lemma for unsound M.

Lemma 4.4. *Suppose (M, Σ) is a mouse pair of type B and degree 0; then $\bar{d}_M = (p_M \cup e_M) - \tau_M$ and $p_M \cap e_M = \emptyset$.*

Proof. Let $N = \bar{\mathfrak{C}}_1(M)^-$, and let $\sigma : N \to M$ be the anticore map. Thus $\sigma(\tau_N) = \tau_M$. Since N is strongly 1-sound, Theorem 2.18 gives us that $d_N = \bar{d}_N = (p_N \cup e_N) - \tau_N$ and $p_N \cap e_N = \emptyset$. We showed in the proof of 4.3 that $\sigma(d_N) = \hat{d}_M$. Clearly $\sigma(p_N) = p_M$, so it is enough to show that $\sigma(e_N) = e_M$. This is a straightforward calculation, based on the fact that $\sigma(\gamma_N) = \gamma_M$. We leave it to the reader. □

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