

# Derived Models in PFA

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## Abstract

We discuss a conjecture of Wilson that under the proper forcing axiom,  $\Theta_0$  of the derived model at  $\kappa$  is below  $\kappa^+$ . We prove the conjecture holds for the old derived model. Assuming mouse capturing in the new derived model, the conjecture holds there as well. We also show  $\Theta < \kappa^+$  in the case of the old derived model, and under additional hypotheses for the new derived model.

## 1 Introduction

In this paper, we prove several results which represent progress on a conjecture of Trevor Wilson relating large cardinals, derived models, and the proper forcing axiom (PFA).

Derived models were invented as a source of models of the axiom of determinacy (AD). Suppose  $\kappa$  is a limit of Woodin cardinals and  $G$  is  $Col(\omega, < \kappa)$ -generic. A relatively simple model of AD is  $L(\mathbb{R}_G^*)$ , where  $\mathbb{R}_G^*$  represents the reals in the symmetric collapse of  $V$  induced by  $G$ . There may, however, be larger models of AD in  $V[G]$ . The “old derived model” at  $\kappa$  includes  $\mathbb{R}_G^*$  as well as a large collection of set of reals  $Hom_G^*$  not contained in  $L(\mathbb{R}_G^*)$  and, like  $L(\mathbb{R}_G^*)$ , can be shown to satisfy  $AD^+$ . The “new derived model” includes even more set of reals and is roughly the largest model of  $AD^+$  contained in  $V(\mathbb{R}_G^*)$ .

Derived models have been extensively studied in the case that  $V$  is a mouse (see [Ste08]). Much less is known about the derived model if  $V$  lacks the fine-structural properties of mice. In the opposite direction, we are interested in the derived model in the case where  $V$  is very “wide,” for example when forcing axioms hold in  $V$ . Recent progress was made here by Wilson:

**Theorem 1.1** (Wilson). *(PFA) If  $\kappa$  is a limit of Woodin cardinals of countable cofinality, then  $\Theta_0$  of the derived model at  $\kappa$  is below  $\kappa^+$ .*

In other words, the theorem says functions from  $\mathbb{R}_G^*$  into  $\kappa^+$  which are ordinal definable in the derived model are bounded in  $\kappa^+$ . On the other hand,  $V[G] \models |\mathbb{R}_G^*| = \kappa$ . So the theorem implies the ordinal definable part of the derived model is not close to being all of  $V[G]$ .

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Wilson conjectured the assumption of countable cofinality in Theorem 1.1 is unnecessary:

**Conjecture 1.2** (Wilson). *(PFA) If  $\kappa$  is a limit of Woodin cardinals, then  $\Theta_0$  of the derived model at  $\kappa$  is below  $\kappa^+$ .*

We make substantial progress on Conjecture 1.2. First, we prove Conjecture 1.2 for the old derived model. In fact, we show something stronger:

**Theorem 1.3.** *(PFA) If  $\kappa$  is a limit of Woodin cardinals, then  $\Theta$  of the old derived model at  $\kappa$  is below  $\kappa^+$ .*

We also prove the conjecture for the new derived model assuming mouse capturing:

**Theorem 1.4.** *(PFA) If  $\kappa$  is a limit of Woodin cardinals and the new derived model satisfies mouse capturing, then  $\Theta_0$  of the new derived model at  $\kappa$  is below  $\kappa^+$ .*

We also show  $\Theta$  of the new derived model is below  $\kappa^+$  under additional assumptions.

## 2 Background

### 2.1 The Derived Model

**Definition 2.1.** *Suppose  $\kappa$  is a limit of Woodin cardinals and  $G$  is  $\text{Col}(\omega, < \kappa)$ -generic over  $V$ . Let  $\mathbb{R}_G^* = \bigcup_{\gamma < \kappa} \mathbb{R}^{V[G \upharpoonright \gamma]}$  and  $V(\mathbb{R}_G^*) = \text{HOD}_{V \cup \mathbb{R}_G^*}^{V[G]}$ . Then*

1. *The “old” derived model is  $olD(V, \kappa) = L(\text{Hom}_G^*, \mathbb{R}_G^*)$ , where  $\text{Hom}_G^* = \{\rho[T] \cap \mathbb{R}_G^* : (\exists \gamma < \kappa) T \in V[G \upharpoonright \gamma] \wedge V[G \upharpoonright \gamma] \models \text{“}T \text{ is } < \kappa\text{-absolutely complemented”}\}$*
2. *The “new” derived model is  $D(V, \kappa) = L(\mathcal{A}, \mathbb{R}_G^*)$ , where  $\mathcal{A} = \{A \subseteq \mathbb{R}_G^* : A \in V(\mathbb{R}_G^*) \wedge L(A, \mathbb{R}_G^*) \models AD^+\}$ .*

Both  $olD(V, \kappa)$  and  $D(V, \kappa)$  satisfy  $AD^+$ .<sup>1</sup>  $olD(V, \kappa)$  is our own notation to circumvent the unfortunate convention that the old and new derived models are typically not distinguished. Both  $olD(V, \kappa)$  and  $D(V, \kappa)$  technically depend on the generic  $G$ , but as their theories are independent of the generic, the notation makes no reference to  $G$ . We will also omit  $G$  as a subscript to  $\mathbb{R}^*$ ,  $\text{Hom}^*$ , and  $V(\mathbb{R}^*)$  where it is convenient to do so.

**Remark 2.2.** *The old derived model is contained in the new derived model. On the other hand, the Suslin-co-Suslin sets of the two are the same. For suppose  $A \subseteq \mathbb{R}^*$  is Suslin in  $D(V, \kappa)$ , so that  $A = \rho[T] = \rho[S]^c$  for some  $T, S \in V(\mathbb{R}^*)$ . Then  $T, S$  are OD in  $V[G]$  from  $s$  for some finite  $s \subset ON \cup \mathbb{R}^*$ . Pick  $\gamma$  large enough that  $s \in V[G \upharpoonright \gamma]$ . Then  $T, S \in V[G \upharpoonright \gamma]$  and are  $< \kappa$ -absolutely complementing.*

<sup>1</sup>See [Ste] for a proof in the case of the old derived model.

**Definition 2.3.** For  $A \subseteq \mathbb{R}$ ,  $\Theta(A) = \sup\{\alpha \in \text{On} : \text{exists surjection } f : \mathbb{R} \rightarrow \alpha \text{ which is } OD(A, x) \text{ for some } x \in \mathbb{R}\}$ . We set

1.  $\Theta_0 = \Theta(\emptyset)$ ,
2.  $\Theta_{\alpha+1} = \Theta(A)$  for any  $A$  such that  $w(A) = \theta_\alpha$ , and
3.  $\Theta_\lambda = \sup_{\alpha < \lambda} \Theta_\alpha$  for  $\lambda$  a limit ordinal.

Note  $\Theta_{\alpha+1}$  is not defined if  $\Theta_\alpha = \Theta$ . The collection  $\langle \Theta_\alpha : \alpha \leq \beta \rangle$  where  $\beta$  is least so that  $\Theta_\beta = \Theta$  is called the Solovay hierarchy.

**Remark 2.4.** Under  $AD^+$  the following are equivalent:

1.  $AD_{\mathbb{R}}$
2. Every set of reals is Suslin-co-Suslin.
3.  $\Theta$  is a limit level of the Solovay hierarchy.

By Remarks 2.2 and 2.4, if  $D(V, \kappa) \models AD_{\mathbb{R}}$ , then  $D(V, \kappa) = olD(V, \kappa)$ .

## 2.2 Wilson's Conjecture

**Conjecture 2.5** (Wilson). Assume  $PFA + \text{“}\kappa \text{ is a limit of Woodin cardinals.”}$  Then

1.  $\Theta_0^{D(V, \kappa)} < \kappa^+$ ,
2.  $\Theta_0^{D(V, \kappa)} < \Theta^{D(V, \kappa)}$ , and
3.  $V^{Col(\omega, \omega_1)} \models (\Sigma_1^2)^{uB_\kappa} \subset uB_\kappa$ .

**Remark 2.6.** Conjectures 1, 2, and 3 are of increasing strength.

**Remark 2.7.** Assume  $PFA + \text{“}\kappa \text{ is a limit of Woodin cardinals”}$  is consistent. The assumption of  $PFA$  in Conjecture 2.5 is necessary.

*Proof.* Let  $M$  be a model of  $ZF + AD$ . We may assume  $M \models V = L(\mathbb{R})$ , so that  $\Theta_0^M = \Theta^M$ . Let  $N$  be a Prikry-generic premouse over  $M$  and let  $\delta_\infty$  be the supremum of the Woodin cardinals of  $N$ .<sup>2</sup> We will show  $\Theta^{D(N, \delta_\infty)} = (\delta_\infty^+)^N$ .

There exists  $H$  a  $Col(\omega, \delta_\infty)$ -generic over  $N$  such that  $\mathbb{R}_H^* = \mathbb{R}^M$ .<sup>3</sup> In particular,  $L(\mathbb{R}_H^*) = M$ . We must show  $(\delta_\infty^+)^N = \Theta_0^M$ . Clearly,  $(\delta_\infty^+)^N \geq \Theta^M$ .

$N \subset M[H]$ , where  $H$  is  $\mathbb{P}$ -generic for  $\mathbb{P}$  the Prikry-forcing with the Martin measure. Since  $\mathbb{P}$  is  $\Theta^M$ -c.c.,  $\Theta^M$  is a cardinal of  $N$ . In particular,  $(\delta_\infty^+)^N \leq \Theta^M$ .  $\square$

Wilson proved Theorem 1.1 by constructing a coherent covering matrix for  $\kappa^+$  in  $V$  in the event that  $\Theta_0$  of the derived model is  $\kappa^+$ . But by a theorem of Viale, there is no coherent covering matrix for  $\kappa^+$  assuming  $PFA$ . For  $\kappa$  of uncountable cofinality, we cannot use coherent covering matrices. Instead, we'll make use of the failure of square principles.

<sup>2</sup>See Chapter 6.6 of [SW16].

<sup>3</sup>See Claim 6.46 of [SW16].

## 2.3 Square Sequences

**Definition 2.8.** We call  $\langle C_\alpha : \alpha < \lambda \rangle$  a coherent sequence iff for all limit  $\alpha < \lambda$ ,  $C_\alpha$  is a club subset of  $\alpha$  and  $C_\beta = \beta \cap C_\alpha$  whenever  $\beta \in \text{lim}(C_\alpha)$ .

**Definition 2.9.** Suppose  $S \subseteq \lambda$  is a club. We say a sequence  $\vec{D} = \langle D_\alpha : \alpha \in S \rangle$  is almost coherent if for all  $\alpha, \beta \in S$

1.  $D_\alpha \subseteq S$  and is a closed subset of limit ordinals below  $\alpha$ ,
2.  $\text{cof}(\alpha) > \omega \implies D_\alpha$  is unbounded in  $\alpha$ , and
3.  $\beta \in D_\alpha \implies D_\beta = \beta \cap D_\alpha$ .

**Definition 2.10.** We say  $\square_\kappa$  holds if there is a coherent sequence  $\langle C_\alpha : \alpha < \kappa^+ \rangle$  such that  $\text{cof}(\alpha) < \kappa \implies |C_\alpha| < \kappa$ .

**Definition 2.11.** We say  $\square'_\kappa$  holds if there is a sequence  $\langle C_\alpha : \alpha < \kappa^+ \rangle$  such that

1.  $C_\alpha$  is a closed subset of limit ordinals below  $\alpha$ ,
2.  $\text{cof}(\alpha) > \omega \implies C_\alpha$  is unbounded in  $\alpha$ ,
3.  $\text{ot}(C_\alpha) \leq \kappa$ , and
4.  $\beta \in C_\alpha \implies C_\beta = \beta \cap C_\alpha$ .

**Remark 2.12.** Suppose  $ZF + \square'_\kappa$  holds on a club — that is, there is a club  $S \subseteq \kappa^+$  and a sequence  $\vec{C} = \langle C_\alpha : \alpha \in S \rangle$  satisfying the four properties of Definition 2.11. Then  $\square'_\kappa$  holds (an order-preserving surjection between  $S$  and limit ordinals below  $\kappa^+$  moves  $\vec{C}$  to a  $\square'_\kappa$ -sequence).

**Remark 2.13.**  $ZFC + \square'_\kappa$  implies  $\square_\kappa$ .<sup>4</sup>

For our main theorems, we will build  $\square_\kappa$ -sequences in the stages suggested by the definitions above. Inside a derived model, we will construct an almost coherent sequence. The particular way our almost coherent sequence is constructed will allow us to turn it into a  $\square_\kappa$ -sequence on a club by a technique from [SZ04] and Remark 2.12. Then in a model of  $ZFC$  we can rearrange this as a  $\square_\kappa$ -sequence by Remark 2.13.

For one of our results, we shall only be able to construct a weaker version of square, which may not violate  $PFA$ .

**Definition 2.14.** For cardinals  $\nu \leq \kappa$ ,  $\square_{\kappa, \nu}$  is the statement that there is a sequence  $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$  s.t.

1.  $\mathcal{C}_\alpha$  is nonempty and  $|\mathcal{C}_\alpha| \leq \nu$ , and each  $C \in \mathcal{C}_\alpha \subset \alpha$  is a club.

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<sup>4</sup>See Lemma 5.1 of Chapter III of [Dev17].

2. For all  $C \in \mathcal{C}_\alpha$  and all  $\beta \in \text{lim}(C)$ ,  $C \cap \beta \in \mathcal{C}_\beta$ .
3. If  $\text{cof}(\alpha) < \kappa$  and  $C \in \mathcal{C}_\alpha$ , then  $|C| < \kappa$ .

**Definition 2.15.** We write  $\square_\kappa^*$  (called weak square) to mean  $\square_{\kappa, \kappa}$ .

**Remark 2.16.**  $\text{PFA} \implies \neg \square_{\kappa, \omega_1}$  for all uncountable  $\kappa$ .<sup>5</sup> But by a result of Magidor,  $\text{PFA}$  is consistent with  $\square_{\kappa, \omega_2}$  holds for all  $\kappa \geq \omega_2$ . In particular,  $\text{PFA}$  is consistent with weak square.

### 3 Result in Old Derived Model

**Theorem 3.1.** Suppose  $\kappa$  is a limit of Woodin cardinals and  $\neg \square_\kappa$ . Then  $\Theta^{\text{old}(V, \kappa)} < \kappa^+$ .

*Proof.* Let  $G$  be  $\text{Col}(\omega, < \kappa)$ -generic over  $V$ .

**Claim 3.2.** There is a  $\text{Col}(\omega, < \kappa)$ -name  $\pi$  for  $(\text{Hom}^*, \mathbb{R}^*)$  and a code  $\text{CODE}$  for  $\pi$  such that  $\text{CODE} \subset H_\kappa^V$ .

*Proof.* If  $x \in \mathbb{R}^*$  then  $x \in V[G \upharpoonright \alpha]$  for some  $\alpha < \kappa$ , so  $x$  has a  $\text{Col}(\omega, < \kappa)$ -name in  $H_\kappa^V$ . Then there is a name  $\pi_2$  for  $\mathbb{R}^*$  contained in  $H_\kappa^V$ . And any  $A^* \subseteq \mathbb{R}^*$  has a  $\text{Col}(\omega, < \kappa)$ -name  $\dot{A}^*$  contained in  $H_\kappa^V$  (if  $\sigma$  is any name for  $A^*$ , we can take  $\dot{A}^* = \{(\dot{x}, p) : \dot{x} \in H_\kappa^V \wedge p \Vdash_{\text{Col}(\omega, < \kappa)}^V \dot{x} \in \sigma\}$ ).

Since  $\kappa$  is a strong limit cardinal in  $V[G \upharpoonright \alpha]$ ,  $|\text{Hom}_{< \kappa}^{V[G \upharpoonright \alpha]}|^{V[G \upharpoonright \alpha]} < \kappa$ . Then let  $\mathcal{A}_\alpha \in V$  be a  $\text{Col}(\omega, \gamma)$ -name for  $\text{Hom}_{< \kappa}^{V[G \upharpoonright \alpha]}$  such that  $|\mathcal{A}_\alpha| < \kappa$ . For each  $A \in \text{Hom}_{< \kappa}^{V[G \upharpoonright \alpha]}$ , there is a unique  $A^* \subseteq V[G]$  such that  $A^* = \rho[T]$  for some tree  $T$  witnessing  $A \in \text{Hom}_{< \kappa}^{V[G \upharpoonright \alpha]}$ . Let  $\mathcal{A}_\alpha^*$  be a  $\text{Col}(\omega, < \kappa)$ -name (in  $V$ ) such that

1.  $|\mathcal{A}_\alpha^*| < \kappa$ ,
2.  $\bigcup \mathcal{A}_\alpha^* \subset H_\kappa^V$ , and
3.  $\emptyset \Vdash (\mathcal{A}_\alpha^*)_G = \{(A_{G \upharpoonright \alpha})^* : A \in \mathcal{A}_\alpha\}$ .

Let  $\pi_2 = \bigcup_{\alpha < \kappa} \mathcal{A}_\alpha^*$ .

Let  $\pi$  be a natural<sup>6</sup> name for the pair  $(\pi_1, \pi_2)$ .  $\pi$  is a name for  $(\text{Hom}^*, \mathbb{R}^*)$  and since  $|\text{trcl}(\pi)| = \kappa$ ,  $\pi$  can be coded by a set  $\text{CODE} \subset H_\kappa^V$ .  $\square$

Let  $M = L(H_\kappa^V, \text{CODE})$ . Then  $\text{old}(V, \kappa) \subseteq M[G]$ , since  $\pi_1, \pi_2 \in M[G]$ ,  $\pi_1[G] = \mathbb{R}^*$ , and  $\pi_2[G] = \text{Hom}^*$ .

Suppose  $\Theta^{\text{old}(V, \kappa)} = \kappa^+$ . Then  $\Theta^{M[G]} = \kappa^+$ , since  $\text{old}(V, \kappa) \subseteq M[G] \subseteq V[G]$ . Since  $G$  does not collapse any cardinals above  $\kappa$ , it follows that  $\Theta^M = \kappa^+$ . Then in  $M$ , we have a club  $S \subseteq \kappa^+$  and an almost coherent sequence  $\vec{D} = \langle D_\alpha : \alpha \in S \rangle$ . This

<sup>5</sup>See p. 702 of [Jec02].

<sup>6</sup>I.e. defined in some reasonable way.

is by the proof of [TZ]: Replace  $\mathbb{R}$  by  $H_\kappa^V \cup CODE$  in their argument. Then  $S$  and  $D_\alpha$  are defined just as in [TZ], except we restrict  $S$  to ordinals greater than  $\kappa$  and only include ordinals above  $\kappa$  in  $D_\alpha$ .

Note  $M \subseteq V$ , so  $\vec{D} \in V$ . Working in  $V$ , we will transform  $\vec{D}$  into a sequence  $\vec{C}$  of length  $\kappa^+$  realizing  $\square_\kappa$  holds. This is by now a standard argument, but we outline it below for the reader's convenience. The first step is to get a  $\square'_\kappa$ -sequence on a club.

**Lemma 3.3.** *There is a club  $S \subset \kappa^+$  and a sequence  $\vec{C}' = \langle C'_\alpha : \alpha \in S \rangle$  such that*

1.  $C'_\alpha$  is a closed set of ordinals below  $\alpha$ ,
2.  $\text{cof}(\alpha) > \omega \implies C'_\alpha$  is unbounded in  $\alpha$ ,
3.  $\text{ot}(C'_\alpha) \leq \kappa$ , and
4.  $\beta \in C'_\alpha \implies C'_\beta = \beta \cap C'_\alpha$ .

*Proof.* Recall our sequence  $\vec{D}$  was obtained from [TZ]. We adopt the notation of that proof. In particular, for  $\tau \in S$ ,

- $N_\tau$  is the least initial segment of  $Lp(H_\kappa^V \cup CODE)$  such that there is  $n_\tau < \omega$  satisfying  $\rho_{n_\tau}^{N_\tau} \geq \tau$  and  $\rho_{n_\tau+1}^{N_\tau} = H_\kappa^V \cup CODE$ ,
- $p_\tau = p_{n_\tau}^{N_\tau}$ ,
- $\tilde{h}_\tau$  is the  $\Sigma_1^{(n_\tau)}$  Skolem function for  $N_\tau$ ,
- and if  $\bar{\tau} \in D_\tau$ , then  $\sigma_{\bar{\tau}\tau} : N_{\bar{\tau}} \rightarrow N_\tau$  is a  $\Sigma_0^{(n_\tau)}$ -preserving map such that  $\text{crit}(\sigma_{\bar{\tau}\tau}) = \bar{\tau}$ ,  $\sigma_{\bar{\tau}\tau}(\bar{\tau}) = \tau$ , and  $\sigma_{\bar{\tau},\tau}(p_{\bar{\tau}}) = p_\tau$ .

We will follow closely the proof of Lemma 3.6 of [SZ04].

Let  $\langle x_\alpha : \alpha < \kappa \rangle$  be an enumeration of  $H_\kappa^V \cup CODE$ .<sup>7</sup> Let  $X_\tau(\xi)$  be the  $\Sigma_1^{(n_\tau)}$ -hull of  $\{x_\xi, p_\tau\}$  in  $N_\tau$ . Define the sequence  $\langle \tau_\iota, \xi_\iota \rangle$  as follows:

1.  $\tau_0 = \min(D_\tau \cup \{\tau\})$ .
2.  $\xi_\iota^\tau =$  least  $\xi < \kappa$  s.t.  $X_\tau(\xi) \not\subseteq \text{range}(\sigma_{\tau_\iota\tau})$ .
3.  $\tau_{\iota+1} =$  least  $\bar{\tau} \in D_\tau \cup \{\tau\}$  s.t.  $X_\tau(\xi_\iota^\tau) \subseteq \text{range}(\sigma_{\bar{\tau}\tau})$ .
4. For limit  $\gamma$ ,  $\tau_\gamma = \sup\{\tau_\iota : \iota < \gamma\}$ .
5.  $\iota_\tau =$  least  $\iota$  s.t.  $\tau_\iota = \tau$ .

**Remark 3.4.**  $\tau_\iota < \tau \implies \xi_\iota^\tau$  exists. In particular, the construction does not stop until it reaches  $\iota_\tau$ .

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<sup>7</sup>This exists (in  $V$ ) because  $\kappa$  is a limit of Woodin cardinals and  $CODE \subset H_\kappa^V$ . We need to work in  $V$  for this so that we can use  $AC$ , which is why we built our sequence  $\vec{D}$  over  $CODE$  instead of over  $\mathbb{R}^*$ .

*Proof.* Let  $\xi < \kappa$  be s.t.  $\tau_\iota = \tilde{h}_\tau(x_\xi, p_\tau)$ . Then  $X_\tau(\xi) \not\subseteq \text{range}(\sigma_{\tau_\iota, \tau})$  since  $\tau_\iota = \text{crit}(\sigma_{\tau_\iota, \tau})$ .  $\square$

Let  $C'_\tau = \{\tau_\iota : \iota < \iota_\tau\}$ . One can show  $\vec{C}' = \langle C'_\tau : \tau \in S \rangle$  realizes the lemma just as in [SZ04].  $\square$

By Lemma 3.3, together with Remarks 2.12 and 2.13,  $V \models \square_\kappa$ , a contradiction.  $\square$

Theorem 1.3 is immediate from Theorem 3.1 and that  $PFA \implies \neg \square_\kappa$ .

## 4 Results from PFA in New Derived Model

Our results in the new derived model rely upon mouse capturing and other similar principles.

**Definition 4.1.** *Assume  $AD^+$ . Mouse capturing (MC) holds if for every  $x \in \mathbb{R}$  and every  $y \in OD(x) \cap \mathbb{R}$ , there is an  $x$ -mouse  $M$  such that  $y \in M$ .*

The mouse set conjecture says  $MC$  holds in every model of  $AD^+ + V = L(P(\mathbb{R}))$ . In particular, we expect  $MC$  is true in any derived model. We use only the following consequence of mouse capturing. Let

$$Lp(A) = \bigcup \{M : M \text{ is a sound } A\text{-mouse projecting to } A\}$$

$Lp(A)$  can be reorganized as an  $A$ -premouse. Assuming  $AD^+ + V = L(P(\mathbb{R})) + MC$ ,  $V = L(Lp(\mathbb{R}))$ .

Our first result of this section gives Theorem 1.4.

**Theorem 4.2.** *Suppose  $V \models \neg \square_\kappa$ ,  $\kappa$  is a limit of Woodin cardinals and  $D(V, \kappa) \models MC$ . Then  $\Theta_0^{D(V, \kappa)} < \kappa^+$ .*

*Proof.* Let  $M = D(V, \kappa)$  and suppose for contradiction  $\Theta_0^M = \kappa^+$ . Let  $G$  be the  $Col(\omega, < \kappa)$ -generic used in constructing  $M$ . Of course  $\Theta^M \leq (\kappa^+)^{V[G]} \leq (\kappa^+)^V$ , so our assumption implies  $\Theta^M = \Theta_0^M$ . Any derived model satisfies  $V = L(P(\mathbb{R}^*))$ , so in this case  $M \models V = L(P_{\Theta_0}(\mathbb{R}^*))$ . Then by the assumption of mouse capturing,  $M = L(Lp(\mathbb{R}^*))$ . In particular, the height of  $Lp(\mathbb{R}^*)^M$  is  $(\kappa^+)^V$ .

Let  $\pi$  be a  $Col(\omega, < \kappa)$ -name for  $\mathbb{R}^*$  (in  $V$ ) such that  $\pi \subset H_\kappa^V$ . Let  $S$  be the output of the  $S$ -construction<sup>8</sup> in  $Lp(\mathbb{R}^*)$  over  $\pi \cup H_\kappa^V$ . Then  $S \in V$  and  $S \trianglelefteq Lp(\pi \cup H_\kappa^V)$ . In particular,  $Lp(\pi \cup H_\kappa^V)$  has height  $\kappa^+$ .

By the argument of [TZ], there is (in  $V$ ) a club  $S \subseteq \kappa^+$  and an almost coherent sequence  $\vec{D} = \langle D_\alpha : \alpha \in S \rangle$ .<sup>9</sup>

<sup>8</sup>First defined under the name  $P$ -construction in [SS09]

<sup>9</sup>We can apply the argument of [TZ] because none of the extenders on the extender sequence of  $S$  have critical point less than  $\kappa$ .

$\vec{D}$  can be transformed into a  $\square_\kappa$ -sequence as in the proof of Theorem 3.1, contradicting that  $V \models \neg \square_\kappa$ . □

We next adapt the technique above to show  $\Theta_{\alpha+1} < \kappa^+$  if in place of our mouse capturing assumption from the previous theorem, we assume  $D(V, \kappa) = L(Lp^\Sigma(\mathbb{R}))$  for a sufficiently nice iteration strategy  $\Sigma$ .

**Theorem 4.3.** *Suppose  $V \models \neg \square_\kappa$ ,  $\kappa$  is a limit of Woodin cardinals and  $D(V, \kappa) \models$  “ $\Theta_{\alpha+1}$  exists” + “there is a hod pair  $(P, \Sigma)$  such that  $P_{\Theta_{\alpha+1}}(\mathbb{R}^*) = Lp^\Sigma(\mathbb{R}^*) \cap P(\mathbb{R}^*)$ ,  $\Sigma$  is fullness-preserving, and has branch condensation.”<sup>10</sup> Then  $\Theta_{\alpha+1}^{D(V, \kappa)} < \kappa^+$ .*

The assumptions on  $D(V, \kappa)$  are known to hold in any model of  $AD^+$  in which  $\Theta_{\alpha+1}$  exists and a sufficient smallness condition on  $D(V, \kappa)$  holds (see e.g. [Sar15]).

*Proof of Theorem 4.3.* Let  $M = D(V, \kappa)$  and suppose  $\Theta_{\alpha+1}^M = \kappa^+$ . Then  $M \models V = L(P_{\Theta_{\alpha+1}}(\mathbb{R}^*))$ . Let  $(P, \Sigma)$  be as in the statement of the theorem. So  $M = L(Lp^\Sigma(\mathbb{R}^*))$ .

Let  $G$  be the  $Col(\omega, < \kappa)$ -generic used to construct  $M$ , so that  $\mathbb{R}^* = \mathbb{R}^M = \bigcup_{\gamma < \kappa} \mathbb{R}^{V[G \upharpoonright \gamma]}$ .

Note  $P$  is coded by a real and  $\Sigma$  is Suslin-co-Suslin in  $M$ , since  $\Sigma$  is an iteration strategy. There there is  $\gamma < \kappa$  and a tree  $T \in V[G \upharpoonright \gamma]$  such that  $(\rho[T])^M$  codes  $(P, \Sigma)$ .<sup>11</sup>

**Claim 4.4.**  $\Sigma \upharpoonright V[G \upharpoonright \gamma] \in V[G \upharpoonright \gamma]$

*Proof.* Let  $S$  be an iteration tree (according to  $\Sigma$ ) of limit length on  $P$ , with  $S \in V[G \upharpoonright \gamma]$ . Let  $b$  be the branch through  $S$  in  $V[G]$  selected by  $\rho[T]$ . Suppose  $H$  is another  $Col(\omega, < \kappa)$ -generic such that  $H \upharpoonright \gamma = G \upharpoonright \gamma$ . Then there is a branch  $b' \in V[H]$  through  $S$  chosen by  $\rho[T]$ . But we can build a third  $Col(\omega, < \kappa)$  generic  $H'$  such that  $H' \upharpoonright \gamma = G \upharpoonright \gamma$  and  $G, H \in V[H']$ . Then  $b$  and  $b'$  are both branches through  $S$  according to  $\rho[T]$ , so  $b = b'$ . It follows that  $b \in V[G \upharpoonright \gamma]$ . □

**Remark 4.5.** *If  $(P, \Sigma) = \rho[T]$  for some  $T \in V$ , then we could proceed as in the proof of Theorem 4.2 to obtain a  $\square_\kappa$ -sequence in  $V$ . Instead, we first have to modify  $(P, \Sigma)$  to get a pair  $(Q, \Lambda)$  which is sufficiently definable in  $V$ , so that the coherent sequence induced by  $Lp^\Lambda$  is definable in  $V$ . To do this, we'll use a boolean-valued comparison argument similar to the one in [SS].*

Let  $G = H \times L$ , where  $H$  is  $Col(\omega, \gamma)$ -generic and  $L$  is  $Col(\omega, < \nu)$ -generic for some  $\nu$  such that  $\gamma + \nu = \kappa$ . Let  $\tau$  be a  $Col(\omega, \gamma)$ -name for  $T$  such that

$$\emptyset \Vdash_{Col(\omega, \gamma)}^V \text{ “} \Vdash_{Col(\omega, < \kappa)}^{V[\dot{G} \upharpoonright \gamma]} \rho[\tau] \text{ satisfies all properties of } (P, \Sigma) \text{ mentioned above.”}$$

Let  $\tau_1$  and  $\tau_2$  be names such that  $\emptyset \Vdash \text{ “} \rho[\tau] \cap \mathbb{R}^* \text{ codes } (\tau_1, \tau_2) \text{”}$ . So  $\tau_1$  is a name for a hod mouse with strategy  $\tau_2$ . For  $q \in Col(\omega, \gamma)$ , let  $H_q = q \cup H \upharpoonright \omega \setminus \text{domain}(q)$ . Note

<sup>10</sup>In the sense of Definitions 1.36, 0.20, and 2.14 of [Sar15].

<sup>11</sup> $T$  can be taken to be  $< \kappa$ -absolutely complemented, but we don't need this property.



$V[H_q] = V[H]$  for any  $q \in \text{Col}(\omega, \gamma)$ . Let  $P_q = \tau_1^{H_q}$  and  $\Sigma_q = \tau_2^{H_q}$ . Note  $P_q \in V[H]$  and while  $\Sigma_q \in M$ ,  $\Sigma_q \upharpoonright V[H] \in V[H]$ .

**Claim 4.6.** *If  $q_1, q_2 \in \text{Col}(\omega, \gamma)$ , then there are countable stacks  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $V[H]$  such that (for  $i \in \{1, 2\}$ )  $U_i$  is a stack on  $P_{q_i}$  according to  $\Sigma_{q_i}$  and letting  $Q_i$  be the last model of  $U_i$  and  $\Lambda_i = (\Sigma_{q_i})_{Q_i, U_i}$ ,<sup>12</sup> either*

1.  $Q_1 \trianglelefteq Q_2$  and  $\Lambda_1 = (\Lambda_2)_{Q_1}$ <sup>13</sup> or
2.  $Q_2 \trianglelefteq Q_1$  and  $\Lambda_2 = (\Lambda_1)_{Q_2}$ .<sup>14</sup>

*Proof.* Since  $\Sigma_{q_1}$  and  $\Sigma_{q_2}$  are both  $\Gamma$ -fullness preserving for  $\Gamma = D(V, \kappa) \cap P(P(\mathbb{R}))$  and have branch condensation,  $(P_{q_1}, \Sigma_{q_1})$  and  $(P_{q_2}, \Sigma_{q_2})$  are of the same kind.<sup>15</sup> Then  $V[H]$  satisfies  $(P_{q_1}, \Sigma_{q_1} \upharpoonright V[H])$  and  $(P_{q_2}, \Sigma_{q_2} \upharpoonright V[H])$  are of the same kind, so we may apply Theorem 2.47 of [Sar15] in  $V[H]$  to  $(P_{q_1}, \Sigma_{q_1} \upharpoonright V[H])$  and  $(P_{q_2}, \Sigma_{q_2} \upharpoonright V[H])$ . Let  $U_1$  and  $U_2$  be the stacks on  $P_{q_1}$  and  $P_{q_2}$  obtained from applying this theorem, and define  $(Q_i, \Lambda_i)$  from  $U_i$  as in the statement of the claim. The theorem gives (for  $i \in \{1, 2\}$ )  $U_i$  is countable,  $U_i \in V[H]$ , and, without loss of generality,  $Q_1 \trianglelefteq Q_2$  and  $\Lambda_1 \upharpoonright V[H] = (\Lambda_2)_{Q_1} \upharpoonright V[H]$ . It remains to show  $\Lambda_1 = (\Lambda_2)_{Q_1}$ . Since  $\Lambda_i$  is a tail strategy of  $\Sigma_{q_i}$ , there is a tree  $T_i \in V[H]$  such that  $\Lambda_i = (\rho[T_i])^M$ . We know  $(\rho[T_1])^{V[H]} = (\rho[T_2])^{V[H]}$ . Then standard absoluteness arguments imply  $(\rho[T_1])^M = (\rho[T_2])^M$ .  $\square$

By Claim 4.6, we may define a hod pair  $(Q, \Lambda)$  to be the direct limit of all countable iterates in  $V[H]$  of every  $(P_q, \Sigma_q)$  for  $q \in \text{Col}(\omega, \gamma)$ .

**Claim 4.7.**  $Lp^\Sigma(\mathbb{R}^*) \subset Lp^\Lambda(\mathbb{R}^*)$

*Proof.*

**Subclaim 4.8.** *The iteration strategy for  $M_1^{\Sigma, \#}$  (restricted to trees in  $Lp^\Lambda(\mathbb{R}^*)$ ) is definable in  $Lp^\Lambda(\mathbb{R}^*)$ .*

*Proof.* Suppose  $T$  is a tree on  $P$  according to  $\Sigma$  of limit length and  $T \in Lp^\Lambda(\mathbb{R}^*)$ . Lift  $T$  to a tree  $T'$  on  $Q$  (this can be done in  $Lp^\Lambda(\mathbb{R}^*)$  since the iteration from  $P$  to  $Q$  is countable in  $V[G]$  and thus coded by a real). Since  $\Lambda$  is a tail strategy of  $\Sigma$  and  $\Sigma$  has branch condensation,  $\Sigma(T)$  is definable from  $\Lambda(T')$ . Since  $\Lambda(T')$  is definable in  $Lp^\Lambda(\mathbb{R}^*)$  from  $T$ ,  $\Sigma(T)$  is as well.

It follows that the operation  $a \mapsto M_0^{\Sigma, \#}(a)$  (for transitive  $a \in Lp^\Lambda(\mathbb{R}^*)$ ) is definable in  $Lp^\Lambda(\mathbb{R}^*)$ .  $M_0^{\Sigma, \#}(a)$  is the minimal active, sound,  $\Sigma$ -premouse over  $a$  projecting to  $a$  such that any iterate of  $M_0^{\Sigma, \#}$  by hitting its top extender is a  $\Sigma$ -premouse. This is definable in  $Lp^\Lambda(\mathbb{R}^*)$  from  $a$  since  $\Sigma \upharpoonright Lp^\Lambda(\mathbb{R}^*)$  is definable in  $Lp^\Lambda(\mathbb{R}^*)$ .

Then we have an iteration strategy for  $M_1^{\Sigma, \#}$  in  $Lp^\Lambda(\mathbb{R}^*)$  using  $Q$ -structures.  $\square$

<sup>12</sup>I.e.  $\Lambda_i$  is the tail strategy.

<sup>13</sup>I.e.  $\Lambda_1$  is the restriction of  $\Lambda_2$  to trees on  $Q_1$ .

<sup>14</sup>Don't need this, but can we easily say they are actually equal?

<sup>15</sup>Is this true? See Lemma 3.32 of [Sar15] - that is only for limit case and not even assuming full branch condensation, but maybe our case is easier - does it even require branch condensation?

The claim is immediate from Subclaim 4.8 and inspecting the definition of  $Lp^\Lambda$ .  $\square$

It follows from the claim that  $L(Lp^\Lambda(\mathbb{R}^*)) = M$ . In particular,  $Lp^\Lambda(\mathbb{R}^*)$  has height  $\kappa^+$ .

We have symmetric terms for  $Q$  and  $\Lambda$ , so  $Q \in V$  and  $\Lambda \upharpoonright V \in V$ . Let  $\pi$  be a  $Col(\omega, < \kappa)$ -name (in  $V$ ) for  $\mathbb{R}^*$  such that  $\pi \subset H_\kappa^V$  and  $|\pi| = \kappa$ . Then  $Lp^\Lambda(\pi \cup H_\kappa^V) \in V$ .

Let  $S$  be the result of the  $S$ -construction<sup>16</sup> in  $Lp^\Lambda(\mathbb{R}^*)$  over  $\pi \cup H_\kappa^V$ . Standard properties of the  $S$ -construction imply  $S \leq Lp^\Lambda(\pi \cup H_\kappa^V)$ . In particular,  $Lp^\Lambda(\pi \cup H_\kappa^V)$  has height  $(\kappa^+)^V$ .

From here the argument is the same as in the proof of Theorem 4.2.  $\square$

What happens if  $\Theta^{D(V,\kappa)}$  is a limit level of the Solovay hierarchy? In this case, Remarks 2.2 and 2.4 imply  $D(V, \kappa) = olD(V, \kappa)$ . So as a corollary to Theorem 3.1, if  $\square_\kappa$  fails, then  $\Theta^{D(V,\kappa)} < \kappa^+$ . Here is another, more elementary, proof which does not use any assumption on the failure of square principles:

**Theorem 4.9.** *If  $\kappa$  is a limit of Woodin cardinals and  $D(V, \kappa) \models AD_{\mathbb{R}}$ , then  $\Theta^{D(V,\kappa)} < \kappa^+$ .*

*Proof.* By Remarks 2.2 and 2.4,  $P(\mathbb{R}^*) = Hom^*$ . The proof of Claim 3.2 shows that  $|Hom^*|^{V[G]} = \kappa$ . If  $\Theta^{D(V,\kappa)} = \kappa^+$ , then the map  $A \mapsto w(A)$  is a surjection of  $Hom^*$  onto  $\kappa^+$  in  $V[G]$ , contradicting the claim.  $\square$

Probably the reader by now also expects the techniques above will go as far up the Solovay hierarchy as progress on studying hod mice allows and shall not be surprised by the following conjecture.

**Conjecture 4.10.** *(PFA) If  $\kappa$  is a limit of Woodin cardinals, then  $\Theta^{D(V,\kappa)} < \kappa^+$ .*

## 5 Result from failure of Weak Square in New Derived Model

In this section, we prove the conclusion of Conjecture 4.10 at regular  $\kappa$  from a version of mouse capturing and the failure of weak square at  $\kappa$ . In exchange for strengthening our assumption on the failure of square, we will not need so many assumptions about the hod pair we use in the proof. In the proofs of Section 4, we used a boolean-valued comparison argument to pull back a hod pair in the derived model to a hod pair in  $V$ . For the following theorem, we instead simply pull back a  $\square_\kappa$ -sequence in  $V[G]$  to a  $\square_{\kappa,\kappa}$ -sequence in  $V$ .

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<sup>16</sup>[ST12] develops  $S$ -constructions for “ $\Theta$ -g-organized hod mice” — the  $S$  construction for  $Lp^\Sigma$  is defined similarly.

**Theorem 5.1.** *Suppose  $V \models \neg \square_\kappa^*$ ,  $\kappa$  is a regular limit of Woodin cardinals, and there is a least branch hod pair  $(P, \Sigma)$  in  $D(V, \kappa)$  such that  $D(V, \kappa) = L(Lp^\Sigma(\mathbb{R}^*))$ . Then  $\Theta^{D(V, \kappa)} < \kappa^+$ .*

See [Ste22] for the definition of a least branch hod pair. In [ST12],  $Lp^\Sigma$  is defined with the hod pairs of [Sar15] in mind, but the definition is the same for least branch hod pairs. In Theorem 4.3, we worked with the rigidly layered hod pairs of [Sar15] to facilitate pulling back the strategy of a hod mouse to  $V$ , layer by layer. We don't see how to replicate that argument with least branch hod pairs. We can use least branch hod pairs in Theorem 5.1 because failure of weak square will allow us to sidestep this part of the argument.

*Proof of Theorem 5.1.* Let  $\Theta = \Theta^{D(V, \kappa)}$  and suppose  $\Theta = \kappa^+$ . Then by [TZ], there is a club  $S \subset \kappa^+$  and an almost coherent sequence  $\vec{C}' = \langle C'_\alpha : \alpha \in S \rangle$  in  $D(V, \kappa)$ .

Inspecting the proof of [TZ], we have an explicit construction of the almost coherent sequence  $\vec{C}'$  which is definable (in  $D(V, \kappa)$ ) from  $(P, \Sigma)$ . In particular,  $\vec{C}' \in V[G \upharpoonright \gamma]$  for some  $\gamma < \kappa$ . Working in  $V[G \upharpoonright \gamma]$ , transform  $\vec{C}'$  into a  $\square_\kappa$ -sequence  $\vec{C} = \langle C_\xi : \xi < \kappa^+ \rangle$  exactly as we did in the proof of Theorem 3.1. Then since  $\kappa$  is regular, the proof of Theorem 8.2 of [CFM01] gives  $V \models \square_\kappa^*$ , a contradiction.  $\square$

**Conjecture 5.2.** *“PFA +  $\kappa$  is a regular limit of Woodin cardinals +  $\square_\kappa^*$ ” is consistent.<sup>17</sup>*

If Conjecture 5.2 is false, then Theorem 5.1 implies Theorem 4.3.

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<sup>17</sup>See Remark 2.16.

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