Derived Models in PFA

Derek Levinson and Nam Trang

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Abstract

We discuss a conjecture of Wilson that under the proper forcing axiom, Θ_0 of the derived model at κ is below κ^+. We prove the conjecture holds for the old derived model. Assuming mouse capturing in the new derived model, the conjecture holds there as well. We also show $\Theta < \kappa^+$ in the case of the old derived model, and under additional hypotheses for the new derived model.

1 Introduction

In this paper, we prove several results which represent progress on a conjecture of Trevor Wilson relating large cardinals, derived models, and the proper forcing axiom (PFA).

Derived models were invented as a source of models of the axiom of determinacy (AD). Suppose κ is a limit of Woodin cardinals and G is $Col(\omega, <\kappa)$-generic. A relatively simple model of AD is $L(\mathbb{R}^*_G)$, where $\mathbb{R}^*_G$ represents the reals in the symmetric collapse of V induced by G. There may, however, be larger models of AD in $V[G]$. The "old derived model" at κ includes $\mathbb{R}^*_G$ as well as a large collection of set of reals Hom^*_G not contained in $L(\mathbb{R}^*_G)$ and, like $L(\mathbb{R}^*_G)$, can be shown to satisfy AD^+. The "new derived model" includes even more set of reals and is roughly the largest model of AD^+ contained in $V(\mathbb{R}^*_G)$.

Derived models have been extensively studied in the case that V is a mouse (see [Ste08]). Much less is known about the derived model if V lacks the fine-structural properties of mice. In the opposite direction, we are interested in the derived model in the case where V is very "wide," for example when forcing axioms hold in V. Recent progress was made here by Wilson:

Theorem 1.1 (Wilson)**.** *(PFA) If κ is a limit of Woodin cardinals of countable cofinality, then Θ_0 of the derived model at κ is below κ^+.*

In other words, the theorem says functions from $\mathbb{R}^*_G$ into κ^+ which are ordinal definable in the derived model are bounded in κ^+. On the other hand, $V[G] \models |\mathbb{R}^*_G| = \kappa$. So the theorem implies the ordinal definable part of the derived model is not close to being all of $V[G]$.

Wilson conjectured the assumption of countable cofinality in Theorem 1.1 is unnecessary:

Conjecture 1.2 (Wilson)**.** *(PFA) If κ is a limit of Woodin cardinals, then Θ_0 of the derived model at κ is below κ^+.*

We make substantial progress on Conjecture 1.2. First, we prove Conjecture 1.2 for the old derived model. In fact, we show something stronger:

Theorem 1.3. *(PFA) If κ is a limit of Woodin cardinals, then Θ of the old derived model at κ is below κ^+.*

We also prove the conjecture for the new derived model assuming mouse capturing:

Theorem 1.4. *(PFA) If κ is a limit of Woodin cardinals and the new derived model satisfies mouse capturing, then Θ_0 of the new derived model at κ is below κ^+.*

We also show Θ of the new derived model is below κ^+ under additional assumptions.

2 Background

2.1 The Derived Model

Definition 2.1. *Suppose κ is a limit of Woodin cardinals and G is $Col(\omega, <\kappa)$-generic over V. Let $\mathbb{R}^*_G = \bigcup_{\gamma<\kappa} \mathbb{R}^{V[G\restriction\gamma]}$ and $V(\mathbb{R}^*_G) = HOD^{V[G]}_{V\cup\mathbb{R}^*_G}$. Then*

1. *The "old" derived model is $olD(V,\kappa) = L(Hom^*_G, \mathbb{R}^*_G)$, where*

$$\begin{aligned} Hom^*_G = \{\rho[T]\cap\mathbb{R}^*_G : (\exists\gamma<\kappa)\, T \in V[G\restriction\gamma] \text{ and} \\ V[G\restriction\gamma] \models \text{“}T \text{ is } <\kappa-\text{absolutely complemented”}\}. \end{aligned}$$

2. *The "new" derived model is $D(V,\kappa) = L(\mathcal{A}, \mathbb{R}^*_G)$, where*

$$\mathcal{A} = \{A \subseteq \mathbb{R}^*_G : A \in V(\mathbb{R}^*_G) \wedge L(A, \mathbb{R}^*_G) \models AD^+\}.$$

Both $olD(V,\kappa)$ and $D(V,\kappa)$ satisfy AD^+.[1] $olD(V,\kappa)$ is our own notation to circumvent the unfortunate convention that the old and new derived models are typically not distinguished. Both $olD(V,\kappa)$ and $D(V,\kappa)$ technically depend on the generic G, but as their theories are independent of the generic, the notation makes no reference to G. We will also omit G as a subscript to $\mathbb{R}^*$, Hom^*, and $V(\mathbb{R}^*)$ where it is convenient to do so.

[1] See [Ste] for a proof in the case of the old derived model.

Remark 2.2. *The old derived model is contained in the new derived model. On the other hand, the Suslin-co-Suslin sets of the two are the same. For suppose $A \subseteq \mathbb{R}^*$ is Suslin-co-Suslin in $D(V,\kappa)$, so that $A = \rho[T] = \rho[S]^c$ for some $T, S \in V(\mathbb{R}^*)$. Then T, S are OD in $V[G]$ from s for some finite $s \subset ON \cup \mathbb{R}^*$. Pick γ large enough that $s \in V[G \restriction \gamma]$. Then $T, S \in V[G \restriction \gamma]$ and are $< \kappa$-absolutely complementing.*

Definition 2.3. *For $A \subseteq \mathbb{R}$, $\Theta(A) = sup\{\alpha \in On :$ exists surjection $f : \mathbb{R} \to \alpha$ which is $OD(A, x)$ for some $x \in \mathbb{R}\}$. We set*

1. $\Theta_0 = \Theta(\emptyset)$,
2. $\Theta_{\alpha+1} = \Theta(A)$ *for any A such that $w(A) = \theta_\alpha$, and*
3. $\Theta_\lambda = sup_{\alpha<\lambda}\Theta_\alpha$ *for λ a limit ordinal.*

Note $\Theta_{\alpha+1}$ is not defined if $\Theta_\alpha = \Theta$. The collection $\langle \Theta_\alpha : \alpha \leq \beta \rangle$ where β is least so that $\Theta_\beta = \Theta$ is called the Solovay hierarchy.

Remark 2.4. *Under AD^+ the following are equivalent:*

1. $AD_{\mathbb{R}}$
2. *Every set of reals is Suslin-co-Suslin.*
3. *Θ is a limit level of the Solovay hierarchy.*

By Remarks 2.2 and 2.4, if $D(V,\kappa) \models AD_{\mathbb{R}}$, then $D(V,\kappa) = olD(V,\kappa)$.

2.2 Wilson's Conjecture

Conjecture 2.5 (Wilson)**.** *Assume $PFA+$ "κ is a limit of Woodin cardinals." Then*

1. $\Theta_0^{D(V,\kappa)} < \kappa^+$ *and*
2. $\Theta_0^{D(V,\kappa)} < \Theta^{D(V,\kappa)}$.

Remark 2.6. *The 2nd part of Wilson's conjecture implies the 1st.*

Remark 2.7. *Assume $PFA+$ "κ is a limit of Woodin cardinals" is consistent. The assumption of PFA in Conjecture 2.5 in necessary.*

Proof. Let M be a model of $ZF + AD$. We may assume $M \models V = L(\mathbb{R})$, so that $\Theta_0^M = \Theta^M$. Let N be a Prikry-generic premouse over M and let δ_∞ be the supremum of the Woodin cardinals of N.[2] We will show $\Theta^{D(N,\delta_\infty)} = (\delta_\infty^+)^N$.

There exists H a $Col(\omega, <\delta_\infty)$-generic over N such that $\mathbb{R}^*_H = \mathbb{R}^M$.[3] In particular, $L(\mathbb{R}^*_H) = M$. We must show $(\delta_\infty^+)^N = \Theta_0^M$. Clearly, $(\delta_\infty^+)^N \geq \Theta^M$.

$N \subset M[H]$, where H is $\mathbb{P}$-generic for $\mathbb{P}$ the Prikry-forcing with the Martin measure. Since $\mathbb{P}$ is Θ^M-c.c, Θ^M is a cardinal of N. In particular, $(\delta_\infty^+)^N \leq \Theta^M$. □

[2]See Chapter 6.6 of [SW16].
[3]See Claim 6.46 of [SW16].

Wilson proved Theorem 1.1 by constructing a coherent covering matrix for κ^+ in V in the event that Θ_0 of the derived model is κ^+. But by a theorem of Viale, there is no coherent covering matrix for κ^+ assuming PFA. For κ of uncountable cofinality, we cannot use coherent covering matrices. Instead, we'll make use of the failure of square principles.

2.3 Square Sequences

Definition 2.8. *We call $\langle C_\alpha : \alpha < \lambda \rangle$ a coherent sequence iff for all limit $\alpha < \lambda$, C_α is a club subset of α and $C_\beta = \beta \cap C_\alpha$ whenever $\beta \in lim(C_\alpha)$.*

Definition 2.9. *We say $\square_\kappa$ holds if there is a coherent sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that $cof(\alpha) < \kappa \implies |C_\alpha| < \kappa$.*

Definition 2.10. *Suppose $S \subseteq \lambda$ is a club. We say a sequence $\vec{D} = \langle D_\alpha : \alpha \in S \rangle$ is almost coherent if for all $\alpha, \beta \in S$*

1. *$D_\alpha \subseteq S$ and is a closed subset of limit ordinals below α,*
2. *$cof(\alpha) > \omega \implies D_\alpha$ is unbounded in α, and*
3. *$\beta \in D_\alpha \implies D_\beta = \beta \cap D_\alpha$.*

Definition 2.11. *We say $\square'_\kappa$ holds if there is a sequence $\vec{C} = \langle C_\alpha : \alpha < \kappa^+ \rangle$ such that $\vec{C}$ is almost coherent and for all $\alpha < \kappa^+$, $ot(C_\alpha) \leq \kappa$.*

Remark 2.12. *Suppose $ZF + \square'_\kappa$ holds on a club — that is, there is a club $S \subseteq \kappa^+$ and an almost coherent sequence $\vec{C} = \langle C_\alpha : \alpha \in S \rangle$ so that $\alpha \in S \implies ot(C_\alpha) \leq \kappa$. Then $\square'_\kappa$ holds.*

Remark 2.13. *$ZFC + \square'_\kappa$ implies $\square_\kappa$.*[4]

For our main theorems, we will build $\square_\kappa$-sequences in the stages suggested by the definitions above. Inside a derived model, we will construct an almost coherent sequence. The particular way our almost coherent sequence is constructed will allow us to turn it into a $\square'_\kappa$-sequence on a club by a technique from [SZ04] and Remark 2.12. Then in a model of ZFC we can rearrange this as a $\square_\kappa$-sequence by Remark 2.13.

For one of our results, we shall only be able to construct a weaker version of square, which may not violate PFA.

Definition 2.14. *For cardinals $\nu \leq \kappa$, $\square_{\kappa,\nu}$ is the statement that there is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ s.t.*

1. *$\mathcal{C}_\alpha$ is nonempty and $|\mathcal{C}_\alpha| \leq \nu$, and each $C \in \mathcal{C}_\alpha \subset \alpha$ is a club.*

[4] See Lemma 5.1 of Chapter III of [Dev17].

2. *For all $C \in \mathcal{C}_\alpha$ and all $\beta \in lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.*

3. *If $cof(\alpha) < \kappa$ and $C \in C_\alpha$, then $|C| < \kappa$.*

Definition 2.15. *We write $\square^*_\kappa$ (called weak square) to mean $\square_{\kappa,\kappa}$.*

Remark 2.16. *$PFA \implies \neg\square_{\kappa,\omega_1}$ for all uncountable κ.*[5] *But by a result of Magidor, PFA is consistent with $\square_{\kappa,\omega_2}$ holds for all $\kappa \geq \omega_2$. In particular, PFA is consistent with weak square.*

3 Result in Old Derived Model

Theorem 3.1. *Suppose κ is a limit of Woodin cardinals and $\neg\square_\kappa$. Then $\Theta^{olD(V,\kappa)} < \kappa^+$.*

Proof. Let G be $Col(\omega, <\kappa)$-generic over V..

Claim 3.2. *There is a $Col(\omega, <\kappa)$-name π for $(Hom^*, \mathbb{R}^*)$ and a code $CODE$ for π such that $CODE \subset H^V_\kappa$.*

Proof. If $x \in \mathbb{R}^*$ then $x \in V[G \restriction \alpha]$ for some $\alpha < \kappa$, so x has a $Col(\omega, <\kappa)$-name in H^V_κ. Then there is a name π_2 for $\mathbb{R}^*$ contained in H^V_κ. And any $A^* \subseteq \mathbb{R}^*$ has a $Col(\omega, <\kappa)$-name $\dot{A}^*$ contained in H^V_κ (if σ is any name for A^*, we can take $\dot{A}^* = \{(\dot{x}, p) : \dot{x} \in H^V_\kappa \wedge p \Vdash^V_{Col(\omega,<\kappa)} \dot{x} \in \sigma\}$).

Since κ is a strong limit cardinal in $V[G \restriction \alpha]$, $|Hom^{V[G\restriction\alpha]}_{<\kappa}|^{V[G\restriction\alpha]} < \kappa$. Then let $\mathcal{A}_\alpha \in V$ be a $Col(\omega, \gamma)$-name for $Hom^{V[G\restriction\alpha]}_{<\kappa}$ such that $|\mathcal{A}_\alpha| < \kappa$. For each $A \in Hom^{V[G\restriction\alpha]}_{<\kappa}$, there is a unique $A^* \subseteq V[G]$ such that $A^* = \rho[T]$ for some tree T witnessing $A \in Hom^{V[G\restriction\alpha]}_{<\kappa}$. Let $\mathcal{A}^*_\alpha$ be a $Col(\omega, <\kappa)$-name (in V) such that

1. $|\mathcal{A}^*_\alpha| < \kappa$,

2. $\bigcup \mathcal{A}^*_\alpha \subset H^V_\kappa$, and

3. $\emptyset \Vdash (\mathcal{A}^*_\alpha)_G = \{(A_{G\restriction\alpha})^* : A \in \mathcal{A}_\alpha\}$.

Let $\pi_2 = \bigcup_{\alpha<\kappa} \mathcal{A}^*_\alpha$.

Let π be a natural[6] name for the pair (π_1, π_2). π is a name for $(Hom^*, \mathbb{R}^*)$ and since $|trcl(\pi)| = \kappa$, π can be coded by a set $CODE \subset H^V_\kappa$. □

Let $M = L(H^V_\kappa, CODE)$. Then $olD(V, \kappa) \subseteq M[G]$, since $\pi_1, \pi_2 \in M[G]$, $\pi_1[G] = \mathbb{R}^*$, and $\pi_2[G] = Hom^*$.

Suppose $\Theta^{olD(V,\kappa)} = \kappa^+$. Then $\Theta^{M[G]} = \kappa^+$, since $olD(V, \kappa) \subseteq M[G] \subseteq V[G]$. Since G does not collapse any cardinals above κ, it follows that $\Theta^M = \kappa^+$. Then in M, we have a club $S \subseteq \kappa^+$ and an almost coherent sequence $\vec{D} = \langle D_\alpha : \alpha \in S\rangle$.

[5] See p. 702 of [Jec02].

[6] I.e. defined in some reasonable way.

This is by the proof of [TZ]: Replace $Lp^{G\Sigma}(\mathbb{R})$ by $L(H_\kappa^V \cup CODE)$ in their argument. Then S and D_α are defined just as in [TZ], except we restrict S to ordinals greater than κ and only include ordinals above κ in D_α.

Note $M \subseteq V$, so $\vec{D} \in V$. Working in V, we will transform $\vec{D}$ into a sequence $\vec{C}$ of length κ^+ realizing $\square_\kappa$ holds. This is by now a standard argument, but we outline it below for the reader's convenience. The first step is to get a $\square'_\kappa$-sequence on a club.

Lemma 3.3. *There is a club $S \subset \kappa^+$ and a sequence $\vec{C'} = \langle C'_\alpha : \alpha \in S\rangle$ such that*

1. *C'_α is a closed set of ordinals below α,*
2. *$cof(\alpha) > \omega \implies C'_\alpha$ is unbounded in α,*
3. *$ot(C'_\alpha) \leq \kappa$, and*
4. *$\beta \in C'_\alpha \implies C'_\beta = \beta \cap C'_\alpha$.*

Proof. Recall our sequence $\vec{D}$ was obtained from [TZ]. We adopt the notation of that proof. In particular, for $\tau \in S$,

- N_τ is the least initial segment of $L(H_\kappa^V \cup CODE)$ such that there is $n_\tau < \omega$ satisfying $\rho_{n_\tau}^{N_\tau} \geq \tau$ and $\rho_{n_\tau+1}^{N_\tau} = H_\kappa^V \cup CODE$,
- $p_\tau = p_{n_\tau}^{N_\tau}$,
- $\tilde{h}_\tau$ is the $\Sigma_1^{(n_\tau)}$ Skolem function for N_τ,
- and if $\bar{\tau} \in D_\tau$, then $\sigma_{\bar{\tau}\tau} : N_{\bar{\tau}} \to N_\tau$ is a $\Sigma_0^{(n_\tau)}$-preserving map such that $crit(\sigma_{\bar{\tau}\tau}) = \bar{\tau}$, $\sigma_{\bar{\tau}\tau}(\bar{\tau}) = \tau$, and $\sigma_{\bar{\tau},\tau}(p_{\bar{\tau}}) = p_\tau$.

We will follow closely the proof of Lemma 3.6 of [SZ04].

Let $\langle x_\alpha : \alpha < \kappa\rangle$ be an enumeration of $H_\kappa^V \cup CODE$.[7] Let $X_\tau(\xi)$ be the $\Sigma_1^{(n_\tau)}$-hull of $\{x_\xi, p_\tau\}$ in N_τ. Define the sequence $\langle \tau_\iota, \xi_\iota\rangle$ as follows:

1. $\tau_0 = \min(D_\tau \cup \{\tau\})$.
2. $\xi_\iota^\tau =$ least $\xi < \kappa$ s.t. $X_\tau(\xi) \nsubseteq$ range$(\sigma_{\tau_\iota\tau})$.
3. $\tau_{\iota+1} =$ least $\bar{\tau} \in D_\tau \cup \{\tau\}$ s.t. $X_\tau(\xi_\iota^\tau) \subseteq$ range$(\sigma_{\bar{\tau}\tau})$.
4. For limit γ, $\tau_\gamma = \sup\{\tau_\iota : \iota < \gamma\}$.
5. $\iota_\tau =$ least ι s.t. $\tau_\iota = \tau$.

Remark 3.4. $\tau_\iota < \tau \implies \xi_\iota^\tau$ *exists. In particular, the construction does not stop until it reaches ι_τ.*

[7] This exists (in V) because κ is a limit of Woodin cardinals and $CODE \subset H_\kappa^V$. We need to work in V for this so that we can use AC, which is why we defined our sequence $\vec{D}$ from $L(H_\kappa^V \cup CODE)$ instead of from $L(Hom^*, \mathbb{R}^*)$.

Proof. Let $\xi < \kappa$ be s.t. $\tau_\iota = \tilde{h}_\tau(x_\xi, p_\tau)$. Then $X_\tau(\xi) \not\subseteq \text{range}(\sigma_{\tau_\iota\tau})$ since $\tau_\iota = crit(\sigma_{\tau_\iota\tau})$. □

Let $C'_\tau = \{\tau_\iota : \iota < \iota_\tau\}$. One can show $\vec{C}' = \langle C'_\tau : \tau \in S\rangle$ realizes the lemma just as in [SZ04].

□

By Lemma 3.3, together with Remarks 2.12 and 2.13, $V \models \square_\kappa$, a contradiction. □

Theorem 1.3 is immediate from Theorem 3.1 and that $PFA \implies \neg\square_\kappa$.

4 Results from PFA in New Derived Model

Our results in the new derived model rely upon mouse capturing and other similar principles.

Definition 4.1. *Assume AD^+. Mouse capturing (MC) holds if for every $x \in \mathbb{R}$ and every $y \in OD(x) \cap \mathbb{R}$, there is an x-mouse M such that $y \in M$.*

The mouse set conjecture says MC holds in every model of $AD^+ + V = L(P(\mathbb{R}))$. In particular, we expect MC is true in any derived model. We use only the following consequence of mouse capturing. Let

$$Lp(A) = \bigcup\{M : M \text{ is a sound } A\text{-mouse projecting to } A\}.$$

$Lp(A)$ can be reorganized as an A-premouse. Assuming $AD^+ + V = L(P_{\Theta_0}(\mathbb{R})) + MC$, $V = L(Lp(\mathbb{R}))$.[8]

Our first result of this section gives Theorem 1.4.

Theorem 4.2. *Suppose $V \models \neg\square_\kappa$, κ is a limit of Woodin cardinals and $D(V,\kappa) \models MC$. Then $\Theta_0^{D(V,\kappa)} < \kappa^+$.*

Proof. Let $M = D(V,\kappa)$ and suppose for contradiction $\Theta_0^M = \kappa^+$. Let G be the $Col(\omega, <\kappa)$-generic used in constructing M. Of course $\Theta^M \leq (\kappa^+)^{V[G]} \leq (\kappa^+)^V$, so our assumption implies $\Theta^M = \Theta_0^M$. Any derived model satisfies $V = L(P(\mathbb{R}^*))$, so in this case $M \models V = L(P_{\Theta_0}(\mathbb{R}^*))$. Then by the assumption of mouse capturing, $M = L(Lp(\mathbb{R}^*))$. In particular, the height of $Lp(\mathbb{R}^*)^M$ is $(\kappa^+)^V$.

Let π be a $Col(\omega, <\kappa)$-name for $\mathbb{R}^*$ (in V) such that $\pi \subset H_\kappa^V$. Let S be the output of the S-construction[9] in $Lp(\mathbb{R}^*)$ over $\pi \cup H_\kappa^V$. Then $S \in V$ and $S \trianglelefteq Lp(\pi \cup H_\kappa^V)$. In particular, $Lp(\pi \cup H_\kappa^V)$ has height κ^+.

By the argument of [TZ], there is (in V) a club $C \subseteq \kappa^+$ and an almost coherent sequence $\vec{D} = \langle D_\alpha : \alpha \in C\rangle$.[10]

[8] This is a special case of the result of [SS15].

[9] First defined under the name P-construction in [SS09].

[10] We can apply the argument of [TZ] because none of the extenders on the extender sequence of S have critical point less than κ.

$\vec{D}$ can be transformed into a $\Box_\kappa$-sequence as in the proof of Theorem 3.1, contradicting that $V \models \neg\Box_\kappa$.

$\Box$

We next adapt the technique above to show $\Theta_{\alpha+1} < \kappa^+$ if in place of our mouse capturing assumption from the previous theorem, we assume $D(V,\kappa) = L(Lp^\Sigma(\mathbb{R}))$ for a sufficiently nice iteration strategy Σ.

Theorem 4.3. *Suppose* $V \models \neg\Box_\kappa$*,* κ *is a limit of Woodin cardinals and* $D(V,\kappa) \models$ *"*$\Theta_{\alpha+1}$ *exists" + "there is a hod pair* (P,Σ) *such that* $P_{\Theta_{\alpha+1}}(\mathbb{R}^*) = Lp^\Sigma(\mathbb{R}^*) \cap P(\mathbb{R}^*)$*,* Σ *is fullness-preserving, and has branch condensation.*[11] *Then* $\Theta_{\alpha+1}^{D(V,\kappa)} < \kappa^+$*.*

The assumptions on $D(V,\kappa)$ are known to hold in any model of AD^+ in which $\Theta_{\alpha+1}$ exists and a sufficient smallness condition on $D(V,\kappa)$ holds (see e.g. [Sar15]).

Proof of Theorem 4.3. Let $M = D(V,\kappa)$ and suppose $\Theta_{\alpha+1}^M = \kappa^+$. Then $M \models V = L(P_{\Theta_{\alpha+1}}(\mathbb{R}^*))$. Let (P,Σ) be as in the statement of the theorem. So $M = L(Lp^\Sigma(\mathbb{R}^*))$.

Let G be the $Col(\omega,<\kappa)$-generic used to construct M, so that $\mathbb{R}^* = \mathbb{R}^M = \bigcup_{\gamma<\kappa} \mathbb{R}^{V[G\restriction\gamma]}$.

Note P is coded by a real and Σ is Suslin-co-Suslin in M, since Σ is an iteration strategy. Then there is $\gamma < \kappa$ and a tree $T \in V[G \restriction \gamma]$ such that $(\rho[T])^M$ codes (P,Σ).[12]

Claim 4.4. $\Sigma \restriction V[G\restriction\gamma] \in V[G\restriction\gamma]$

Proof. Let S be an iteration tree (according to Σ) of limit length on P, with $S \in V[G \restriction \gamma]$. Let b be the branch through S in $V[G]$ selected by $\rho[T]$. Suppose H is another $Col(\omega,<\kappa)$-generic such that $H \restriction \gamma = G \restriction \gamma$. Then there is a branch $b' \in V[H]$ through S chosen by $\rho[T]$. But we can build a third $Col(\omega,<\kappa)$ generic H' such that $H' \restriction \gamma = G \restriction \gamma$ and G, $H \in V[H']$. Then b and b' are both branches through S according to $\rho[T]$, so $b = b'$. It follows that $b \in V[G \restriction \gamma]$. $\Box$

Remark 4.5. *If* $(P,\Sigma) = \rho[T]$ *for some* $T \in V$*, then we could proceed as in the proof of Theorem 4.2 to obtain a* $\Box_\kappa$*-sequence in* V*. Instead, we first have to modify* (P,Σ) *to get a pair* (Q,Λ) *which is sufficiently definable in* V*, so that the coherent sequence induced by* Lp^Λ *is definable in* V*. To do this, we'll use a boolean-valued comparison argument similar to the one in [SS].*

Claim 4.6. *Suppose* U *is a non-dropping iteration tree on* P *according to* Σ *with last model* Q *and* $\Lambda = \Sigma_{Q,U}$*.*[13] *Then* $Lp^\Sigma(\mathbb{R}^*) \subseteq Lp^\Lambda(\mathbb{R}^*)$*.*

Proof.

Subclaim 4.7. *The iteration strategy for* $M_1^{\Sigma,\#}$ *(restricted to trees in* $Lp^\Lambda(\mathbb{R}^*)$*) is definable in* $Lp^\Lambda(\mathbb{R}^*)$*.*

[11] In the sense of Definitions 1.36, 0.20, and 2.14 of [Sar15].

[12] T can be taken to be $<\kappa$-absolutely complemented, but we don't need this property.

[13] I.e. Λ is the tail strategy.

Proof. Suppose T is a tree on P according to Σ of limit length and $T \in Lp^{\Lambda}(\mathbb{R}^*)$. Lift T to a tree T' on Q (this can be done in $Lp^{\Lambda}(\mathbb{R}^*)$ since the iteration from P to Q is countable in $V[G]$ and thus coded by a real). Since Λ is a tail strategy of Σ and Σ has branch condensation, $\Sigma(T)$ is definable from $\Lambda(T')$. Since $\Lambda(T')$ is definable in $Lp^{\Lambda}(\mathbb{R}^*)$ from T, $\Sigma(T)$ is as well.

It follows that the operation $a \mapsto M_0^{\Sigma,\#}(a)$ (for transitive $a \in Lp^{\Lambda}(\mathbb{R}^*)$) is definable in $Lp^{\Lambda}(\mathbb{R}^*)$. $M_0^{\Sigma,\#}(a)$ is the minimal active, sound, Σ-premouse over a projecting to a such that any iterate of $M_0^{\Sigma,\#}$ by hitting its top extender is a Σ-premouse. This is definable in $Lp^{\Lambda}(\mathbb{R}^*)$ from a since $\Sigma \restriction Lp^{\Lambda}(\mathbb{R}^*)$ is definable in $Lp^{\Lambda}(\mathbb{R}^*)$.

Then we have an iteration strategy for $M_1^{\Sigma,\#}$ in $Lp^{\Lambda}(\mathbb{R}^*)$ using Q-structures. □

The claim is immediate from Subclaim 4.7 and inspecting the definition of Lp^{Λ}. □

Let $G = H \times L$, where H is $Col(\omega, \gamma)$-generic and L is $Col(\omega, <\nu)$-generic for some ν such that $\gamma + \nu = \kappa$. Let τ be a $Col(\omega,\gamma)$-name for T such that

$$\emptyset \Vdash^{V}_{Col(\omega,\gamma)} \text{“} \Vdash^{V[\dot{G}\restriction\gamma]}_{Col(\omega,<\kappa)} \rho[\tau] \text{ satisfies all properties of } (P,\Sigma) \text{ mentioned above.”}$$

Let τ_1 and τ_2 be names such that $\emptyset \Vdash$ “$\rho[\tau] \cap \mathbb{R}^*$ codes (τ_1, τ_2).” So τ_1 is a name for a hod mouse with strategy τ_2. For $q \in Col(\omega,\gamma)$, let $H_q = q \cup H \restriction \omega \backslash domain(q)$. Note $V[H_q] = V[H]$ for any $q \in Col(\omega,\gamma)$. Let $P_q = \tau_1^{H_q}$ and $\Sigma_q = \tau_2^{H_q}$. Note $P_q \in V[H]$ and while $\Sigma_q \in M$, $\Sigma_q \restriction V[H] \in V[H]$.

Claim 4.8. *If $q_1, q_2 \in Col(\omega,\gamma)$, then there are countable stacks $\mathcal{U}_1$ and $\mathcal{U}_2$ in $V[H]$ such that (for $i \in \{1,2\}$) U_i is a stack on P_{q_i} according to Σ_{q_i} and letting Q_i be the last model of U_i and $\Lambda_i = (\Sigma_{q_i})_{Q_i,U_i}$, $Q_1 = Q_2$ and $\Lambda_1 = \Lambda_2$.*

Proof. Since Σ_{q_1} and Σ_{q_2} are both Γ-fullness preserving for $\Gamma = D(V,\kappa) \cap P(\mathbb{R})$ and have branch condensation, (P_{q_1}, Σ_{q_1}) and (P_{q_2}, Σ_{q_2}) are of the same kind.[14] Then $V[H]$ satisfies $(P_{q_1}, \Sigma_{q_1} \restriction V[H])$ and $(P_2, \Sigma_{q_2} \restriction V[H])$ are of the same kind, so we may apply Theorem 2.47 of [Sar15] in $V[H]$ to $(P_{q_1}, \Sigma_{q_1} \restriction V[H])$ and $(P_2, \Sigma_{q_2} \restriction V[H])$. Let U_1 and U_2 be the stacks on P_{q_1} and P_{q_2} obtained from applying this theorem, and define (Q_i, Λ_i) from U_i as in the statement of the claim. The theorem gives (for $i \in \{1,2\}$) U_i is countable, $U_i \in V[H]$, and, without loss of generality, $Q_1 \trianglelefteq Q_2$ and $\Lambda_1 \restriction V[H] = (\Lambda_2)_{Q_1} \restriction V[H]$.[15] By Claim 4.6, $\Theta(\Lambda_1) = \Theta = \Theta(\Lambda_2)$, and therefore $Q_1 = Q_2$ and $\Lambda_1 \restriction V[H] = \Lambda_2 \restriction V[H]$. It remains to show $\Lambda_1 = \Lambda_2$. Since Λ_i is a tail strategy of Σ_{q_i}, there is a tree $T_i \in V[H]$ such that $\Lambda_i = (\rho[T_i])^M$. We know $(\rho[T_1])^{V[H]} = (\rho[T_2])^{V[H]}$. Then standard absoluteness arguments imply $(\rho[T_1])^M = (\rho[T_2])^M$. □

[14] Lemma 3.32 of [Sar15] proves this in the (harder) case that the order type of the Woodin cardinals in P_{q_1} and P_{q_2} are limit ordinals.

[15] I.e. Λ_1 is the restriction of Λ_2 to trees on Q_1.

By Claim 4.8, we may define a hod pair (Q, Λ) to be the direct limit of all countable iterates in $V[H]$ of every (P_q, Σ_q) for $q \in Col(\omega, \gamma)$. It follows from Claim 4.6 that $L(Lp^{\Lambda}(\mathbb{R}^*)) = M$. In particular, $Lp^{\Lambda}(\mathbb{R}^*)$ has height κ^+.

We have symmetric terms for Q and Λ, so $Q \in V$ and $\Lambda \restriction V \in V$. Let π be a $Col(\omega, < \kappa)$-name (in V) for $\mathbb{R}^*$ such that $\pi \subset H^V_\kappa$. Then $Lp^{\Lambda}(\pi \cup H^V_\kappa) \in V$.

Let S be the result of the S-construction[16] in $Lp^{\Lambda}(\mathbb{R}^*)$ over $\pi \cup H^V_\kappa$. Standard properties of the S-construction imply $S \trianglelefteq Lp^{\Lambda}(\pi \cup H^V_\kappa)$. In particular, $Lp^{\Lambda}(\pi \cup H^V_\kappa)$ has height $(\kappa^+)^V$.

From here the argument is the same as in the proof of Theorem 4.2. $\square$

What happens if $\Theta^{D(V,\kappa)}$ is a limit level of the Solovay hierarchy? In this case, Remarks 2.2 and 2.4 imply $D(V, \kappa) = olD(V, \kappa)$. So as a corollary to Theorem 3.1, if $\square_\kappa$ fails, then $\Theta^{D(V,\kappa)} < \kappa^+$. Here is another, more elementary, proof which does not use any assumption on the failure of square principles:

Theorem 4.9. *Suppose κ is a limit of Woodin cardinals and $D(V, \kappa) \models AD_{\mathbb{R}}$. Then $\Theta^{D(V,\kappa)} < \kappa^+$.*

Proof. By Remarks 2.2 and 2.4, $P(\mathbb{R}^*) = Hom^*$. The proof of Claim 3.2 shows that $|Hom^*|^{V[G]} = \kappa$. If $\Theta^{D(V,\kappa)} = \kappa^+$, then the map $A \mapsto w(A)$ is a surjection of Hom^* onto κ^+ in $V[G]$, contradicting the claim. $\square$

Probably the reader by now also expects the techniques above will go as far up the Solovay hierarchy as progress on studying hod mice allows and shall not be surprised by the following conjecture.

Conjecture 4.10. *(PFA) If κ is a limit of Woodin cardinals, then $\Theta^{D(V,\kappa)} < \kappa^+$.*

5 Result from failure of Weak Square in New Derived Model

In this section, we prove the conclusion of Conjecture 4.10 at regular κ from a version of mouse capturing and the failure of.weak square at κ. In exchange for strengthening our assumption on the failure of square, we will not require as many assumptions about the hod pair we use as we needed for Theorem 4.3. In the proof of Theorem 4.3, we used a boolean-valued comparison argument to pull back a hod pair in the derived model to a hod pair in V. For the following theorem, we instead simply pull back a $\square_\kappa$-sequence in $V[G]$ to a $\square_{\kappa,\kappa}$-sequence in V.

Theorem 5.1. *Suppose $V \models \neg\square^*_\kappa$, κ is a regular limit of Woodin cardinals, and there is a least branch hod pair (P, Σ) in $D(V, \kappa)$ such that $D(V, \kappa) = L(Lp^{\Sigma}(\mathbb{R}^*))$. Then $\Theta^{D(V,\kappa)} < \kappa^+$.*

[16] [ST12] develops S-constructions for "Θ-g-organized hod mice" — the S construction for Lp^{Σ} is defined similarly.

See [Ste22] for the definition of a least branch hod pair. In [ST12], Lp^Σ is defined with the hod pairs of [Sar15] in mind, but the definition is the same for least branch hod pairs. In Theorem 4.3, we worked with the rigidly layered hod pairs of [Sar15] to facilitate pulling back the strategy of a hod mouse to V, layer by layer. We don't see how to replicate that argument with least branch hod pairs. We can use least branch hod pairs in Theorem 5.1 because failure of weak square will allow us to sidestep this part of the argument.

Proof of Theorem 5.1. Let $\Theta = \Theta^{D(V,\kappa)}$ and suppose $\Theta = \kappa^+$. Then by [TZ], there is a club $S \subset \kappa^+$ and an almost coherent sequence $\vec{D} = \langle D_\alpha : \alpha \in S \rangle$ in $D(V,\kappa)$.

Inspecting the proof of [TZ], we have an explicit construction of the almost coherent sequence $\vec{D}$ which is definable (in $D(V,\kappa)$) from (P,Σ). In particular, $\vec{D} \in V[G \restriction \gamma]$ for some $\gamma < \kappa$. Working in $V[G \restriction \gamma]$, transform $\vec{D}$ into a $\square_\kappa$-sequence $\vec{C} = \langle C_\xi : \xi < \kappa^+ \rangle$ exactly as we did in the proof of Theorem 3.1. Then, since κ is regular, the proof of Theorem 8.2 of [CFM01] gives $V \models \square^*_\kappa$, a contradiction. □

Conjecture 5.2. *"PFA + κ is a regular limit of Woodin cardinals + $\square^*_\kappa$" is consistent.*[17]

If Conjecture 5.2 is false, then Theorem 5.1 implies Theorem 4.3.

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