DERIVED MODELS AND SUPERCOMPACT MEASURES ON $\wp_{\omega_1}(\wp(\mathbb{R}))$

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Abstract

The main result of this paper is the proof of Theorem 0.1, which shows that it's possible for derived models to satisfy "$\mathsf{AD}_{\mathbb{R}} + \omega_1$ is $\wp(\mathbb{R})$-supercompact". Other constructions of models of this theory are also discussed; in particular, Theorem 3.1 constructs a normal fine measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$ and hence a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular $+ \omega_1$ is $\wp(\mathbb{R})$-supercompact" from a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable".

AD^+ models of the form $V = L(\wp(\mathbb{R}))$ have been studied extensively by Woodin and others. Woodin has shown that all models of AD^+ of the form $V = L(\wp(\mathbb{R}))$ arise as derived models (see [7] for a proof). It's natural then to consider AD^+ models of the form $V = L(\wp(\mathbb{R}))[X]$ where X codes some canonical information not coded by sets of reals in the model.

This paper gives various constructions of AD^+ models of the form $V = L(\wp(\mathbb{R}))[\mu]$ where μ is a normal fine measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$. The main result of the paper is a derived model construction of such a model. The notions used in the statement of Theorem 0.1 are spelled out in Sections 1 and 2 and its proof is given in Section 2.

Theorem 0.1. *Suppose there is a proper class of Woodin cardinals. Suppose δ_0 is a measurable cardinal which is a limit of Woodin and strong cardinals and $2^{\delta_0} = \delta_0^+$. Suppose $\langle \delta_i \mid 1 \leq i < \omega\rangle$ is an increasing sequence of good Woodin cardinals above δ_0 which are also strong cardinals. Let $G \subseteq Col(\omega, <\delta_0)$ be V-generic. Then in $V[G]$, there is a class model M containing $\mathbb{R}^{V[G]}$ such that $M \vDash$ "$\mathsf{AD}_{\mathbb{R}}+$ there is a normal fine measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$."*

In Section 3, we discuss models of the theory "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular $+ \omega_1$ is $\wp(\mathbb{R})$-supercompact".

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1. PRELIMINARIES

We start with the definition of Woodin's theory of AD^+. Recall the axiom of determinacy (AD) states that every game of length ω on integers is determined. In this paper, we identify $\mathbb{R}$ with ω^ω. We use Θ to denote the sup of ordinals α such that there is a surjection $\pi : \mathbb{R} \to \alpha$. Under

AC, Θ is just the successor cardinal of the continuum. In the context of AD, Θ is shown to be the supremum of $w(A)$[1] for $A \subseteq \mathbb{R}$. For $\alpha < \Theta$, we let $\wp_\alpha(\mathbb{R}) = \{A \subseteq \mathbb{R} \mid w(A) < \alpha\}$.

Definition 1.1. AD^+ *is the theory* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$ *and*

1. *for every set of reals A, there are a set of ordinals S and a formula φ such that $x \in A \Leftrightarrow L[S,x] \vDash \varphi[S,x]$. (S,φ) is called an ∞-Borel code for A;*
2. *for every $\lambda < \Theta$, for every continuous $\pi : \lambda^\omega \to \omega^\omega$, for every $A \subseteq \mathbb{R}$, the set $\pi^{-1}[A]$ is determined.*

AD^+ is equivalent to "AD + the set of Suslin cardinals is closed". If M is a model of $\mathsf{AD}^+ + V = L(\wp(\mathbb{R}))$ then in M, AD^+ is equivalent to the statement: every Σ_1 statement $\phi(A)$ about a Suslin co-Suslin set A is true in a model N (of a sufficient fragment of ZF), where $\mathbb{R} \cup \{A\} \subseteq N$ and N is coded by a Suslin co-Suslin set (see [7] for a proof).

Let $A \subseteq \mathbb{R}$, we let θ_A be the supremum of all α such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto α.

Definition 1.2 (AD^+)**.** *The* ***Solovay sequence*** *is the sequence $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ where*

1. *θ_0 is the supremum of ordinals β such that there is an OD surjection from $\mathbb{R}$ onto β;*
2. *if $\alpha > 0$ is limit, then $\theta_\alpha = \sup\{\theta_\beta \mid \beta < \alpha\}$;*
3. *if $\alpha = \beta + 1$ and $\theta_\beta < \Theta$ (i.e. $\beta < \Omega$), fixing a set $A \subseteq \mathbb{R}$ of Wadge rank θ_β, θ_α is the sup of ordinals γ such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto γ, i.e. $\theta_\alpha = \theta_A$.*

Note that the definition of θ_α for $\alpha = \beta + 1$ in Definition 1.2 does not depend on the choice of A.

The theory $\mathsf{AD}_\mathbb{R}$ is also a strengthening of AD; it states that every game of length ω where players play real numbers is determined. In this paper, by $\mathsf{AD}_\mathbb{R}$, we always mean the theory $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$. Using the derived model construction, Woodin has constructed (assuming large cardinals) models of $\mathsf{AD}_\mathbb{R} + V = L(\wp(\mathbb{R}))$. In a model of $\mathsf{AD}_\mathbb{R}$, the Solovay sequence always has limit length and every set of reals is Suslin.

Definition 1.3 (ZF + DC)**.** *Suppose X is an uncountable set. We say that ω_1* ***is X-supercompact*** *if there is a normal fine measure μ on $\wp_{\omega_1}(X) =_{def} \{\sigma \subseteq X \mid \sigma \text{ is countable}\}$, where μ is*

- *fine if whenever $x \in X$, the set $C_x =_{def} \{\sigma \in \wp_{\omega_1}(X) \mid x \in \sigma\} \in \mu$, and*
- *normal if whenever $F : \wp_{\omega_1}(X) \to \wp_{\omega_1}(X)$ is such that $\{\sigma \mid F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset\} \in \mu$ then there is some $x \in X$ such that $\{\sigma \mid x \in F(\sigma)\} \in \mu$.*

[1] $w(A)$ is the Wadge rank of A.

Definition 1.3 goes back to [5]. We will use the notation $\forall^*_\mu \sigma P(\sigma)$ for the statement "for μ-measure one many σ $P(\sigma)$". It's easy to see that whenever $\omega \subseteq X$ and μ is a normal fine measure on $\wp_{\omega_1}(X)$, then μ is in fact countably complete. The following lemma gives an alternative characterization of normality in terms of "diagonal intersection".

Lemma 1.4 (ZF + DC)**.** *Suppose μ is a fine measure on $\wp_{\omega_1}(X)$. The following are equivalent.*

1. *μ is normal.*
2. *Suppose we have $\langle A_x \mid x \in X \wedge A_x \in \mu\rangle$. Then $\triangle_{x\in X} A_x =_{def} \{\sigma \mid \sigma \in \bigcap_{x\in\sigma} A_x\} \in \mu$.*

Proof. Suppose μ is normal and we have a sequence $\langle A_x \mid x \in X \wedge A_x \in \mu\rangle$. We want to show $\triangle_{x\in X} A_x \in \mu$. Suppose not. Then

$$\forall^*_\mu \sigma \ \exists x \in \sigma \ \sigma \notin A_x.$$

Let then $F(\sigma) = \{x \in \sigma \mid \sigma \notin A_x\}$. Our assumption implies that $\forall^*_\mu \sigma \ F(\sigma) \neq \emptyset$. This means, by normality of μ, $\exists x \ \forall^*_\mu \sigma \ x \in F(\sigma)$ or equivalently there is some $x \in X$ such that $A_x \notin \mu$. Contradiction.

Now we show (2) $\Rightarrow$ (1). Let F be given and suppose there is no $x \in X$ such that $\forall^*_\mu \sigma \ x \in F(\sigma)$. Then for each $x \in X$,

$$A_x =_{\text{def}} \{\sigma \mid x \in F(\sigma)\} \notin \mu.$$

In other words, $\forall x \in X \ \neg A_x \in \mu$ and hence by (2), $\triangle_{x\in X} \neg A_x \in \mu$. This means $\forall^*_\mu \sigma \ \forall x \in \sigma \ \sigma \notin A_x$, i.e.

$$\forall^*_\mu \sigma \ F(\sigma) = 0.$$

This contradiction completes the proof of (2) $\Rightarrow$ (1). □

2. A PROOF OF THE MAIN THEOREM

We recall the notion of universal Baireness. We say that a tree T on $\omega \times \text{OR}$[2] is κ-absolutely complemented if there is a tree U on $\omega \times \text{OR}$ such that whenever g is $<$-κ generic over V[3],

$$V[g] \vDash p[T] = \mathbb{R} \backslash p[U].$$

A set $A \subseteq \mathbb{R}$ is *κ-universally Baire* if there is a κ-absolutely complemented tree T such that

$$A = p[T].$$

[2]Technically, T is a tree on $\omega \times \gamma$ for some ordinal γ but in this paper, it's not important what γ is.
[3]This means g is a generic filter over V for some forcing of size $< \kappa$.

We also say that $A \subseteq \mathbb{R}$ is *universally Baire* if A is κ-universally Baire for all κ.

Now we briefly recall the notion of homogeneously Suslin sets; a detailed discussion on this topic can be found in [6]. A countably complete measure μ on the set $Z^{<\omega}$ for some Z concentrates on Z^n for exactly one $n < \omega$ (that is, there is a unique n such that $\mu(Z^n) = 1$); we call this n $\dim(\mu)$. A *homogeneity system with support* Z is a function $\bar{\mu}$ from ${}^{<\omega}\omega$ into the set of countably complete measures on $Z^{<\omega}$, denoted $meas(Z)$, such that writing μ_s for $\bar{\mu}(s)$, for all $s, t \in \omega^{<\omega}$,

1. $\dim(\mu_t) = \mathrm{dom}(t)$;
2. $s \subseteq t \Rightarrow \mu_t$ projects to μ_s; that is, letting $m = \dim(\mu_s)$, for all $A \subseteq Z^m$,

$$A \in \mu_s \Leftrightarrow \{u \mid u \restriction m \in A\} \in \mu_t.$$

We say $\bar{\mu}$ is a *κ-complete homogeneity system* if every measure in $\bar{\mu}$ is κ-additive.

Suppose $\mu, \nu \in meas(Z)$ and μ projects to ν. Then there is a natural embedding $\pi_{\nu,\mu} : Ult(V, \nu) \to Ult(V, \mu)$ defined as: $\pi_{\nu,\mu}([f]_\nu) = [f^*]_\mu$, where $f^*(u) = f(u \restriction \dim(\nu))$. A *tower of measures* $\langle \mu_n \mid n < \omega \rangle$ then is a sequence of measures in $meas(Z)$ such that $n < m \Rightarrow \mu_m$ projects to μ_n. The tower of measures $\langle \mu_n \mid n < \omega \rangle$ is *countably complete* if the direct limit of the system $\{Ult(V, \mu_n), \pi_{\mu_n,\mu_m} \mid n < m < \omega\}$ is wellfounded; or equivalently, whenever $\mu_n(A_n) = 1$ for all n then there is an f such that for all n, $f \restriction n \in A_n$.

Let $\bar{\mu}$ be a homogeneity system with support Z as above. We define $S_{\bar{\mu}}$ to be the set of $x \in \mathbb{R}$ such that $\bar{\mu}_x =_{\mathrm{def}} \langle \mu_{x \restriction n} \mid n < \omega \rangle$ is countably complete. A set of reals A is *κ-homogenously Suslin* if $A = S_{\bar{\mu}}$ for some κ-complete homogeneity system with support Z for some set Z. We let Hom_κ denote the set of all κ-homogeneously Suslin sets. A is *homogeneous Suslin* if A is κ-homogeneous for all κ. We let Hom_∞ denote the set of homogenously Suslin sets. It's a basic fact that if there is a proper class of measurable cardinals then $Hom_\infty = Hom_\kappa$ for some measurable κ and Hom_∞ is determined.

Let λ be a limit of Woodin cardinals and let $G \subseteq Col(\omega, < \lambda)$. Let $\mathbb{R}^*_G = \bigcup_{\alpha<\delta} \mathbb{R}^{V[G|\alpha]}$ be the symmetric reals and

$$\begin{aligned} Hom^*_G = \{A \subseteq \mathbb{R}^* \mid \quad & A \in V(\mathbb{R}^*_G) \wedge \exists \alpha < \delta_0 \exists T \in V[G|\alpha]\ (V[G|\alpha] \models \\ & \text{“}T \text{ is } \delta_0\text{-absolutely complemented”} \wedge p[T] \cap \mathbb{R}^*_G = A)\}. \end{aligned}$$

Woodin has shown that the derived model $L(\mathbb{R}^*_G, Hom^*_G) \models \mathsf{AD}^+$. Additionally, if λ is a limit of $<$-λ-strong cardinals, then $L(\mathbb{R}^*_G, Hom^*_G) \models \mathsf{AD}_\mathbb{R}$ and $Hom^*_G = \wp(\mathbb{R})^{L(\mathbb{R}^*_G, Hom^*_G)}$. For more on Hom^* and derived models, see [6].

Recall that $\mathbb{Q}_{<\delta}$ is the “countable” stationary forcing, whose conditions are stationary sets $b \subseteq \wp_{\omega_1}(X)$ for some $X \in V_\delta$ (cf. [3]). The following definition comes from [3].

Definition 2.1. *Let Γ_{ub} be the collection of universally Baire sets and let δ be a Woodin cardinal. We say that δ is **good** if whenever g is a $< \delta$-generic over V and G is a stationary tower $\mathbb{Q}^{V[g]}_{<\delta}$*

generic over $V[g]$, then letting $j : V[g] \to M \subseteq V[g][G]$ be the associated embedding, $j(\Gamma_{ub}^{V[g]}) = \Gamma_{ub}^{V[g][G]}$.

In the presence of a proper class of Woodin cardinals, $\Gamma_{ub} = Hom_\infty$ (see [6] or [3] for a proof). For the reader's convenience, we state the tree production lemma (cf. Lemma 4.2 of [6]), which features in a key argument of the proof of Theorem 0.1.

Theorem 2.2 (Tree production lemma, Woodin)**.** *Let $\varphi(v_0, v_1)$ be a formula; let a be a parameter; and let δ be a Woodin cardinal. Suppose the following hold.*

1. *(Generic absoluteness) If G is $<$-δ generic over V, and H is $<$-δ^+ generic over $V[G]$ then for all $x \in V[G] \cap \mathbb{R}$,*

$$V[G] \vDash \varphi[x, a] \Leftrightarrow V[G][H] \vDash \varphi[x, a].$$

2. *(Stationary tower correctness) If G is $\mathbb{Q}_{<\delta}$-generic and $j : V \to M \subseteq V[G]$ is the associated generic embedding, then for all $x \in \mathbb{R} \cap V[G]$[4],*

$$V[G] \vDash \varphi[x, a] \Leftrightarrow M \vDash \varphi[x, j(a)].$$

Then the set $\{x \mid \varphi(x, a)\}$ is δ-universally Baire.

We are ready to give a proof of Theorem 0.1, which is inspired by Woodin's construction of a model of "$\mathsf{AD}^+ + \omega_1$ is $\mathbb{R}$-supercompact" from ω^2 Woodin cardinals. But first let us remark that the hypothesis of the theorem is consistent relative to, for example, the existence of a proper class of Woodin cardinals and a huge cardinal plus a supercompact cardinal above (the proof is basically an easy modification of the proof of Theorem 3.4.17 in [3]).

Proof of Theorem 0.1. Again, let Γ_{ub} denote the collection of universally Baire sets. The hypothesis of the theorem implies $\Gamma_{ub} = Hom_\infty$. Let $G \subseteq Col(\omega, < \delta_0)$ be V-generic. In $V[G]$, let $\mathbb{R}^* = \mathbb{R}^*_G = \mathbb{R}^{V[G]}$ (the second equality follows from the fact that δ_0 is inaccessible) and $Hom^* = Hom^*_G$. By results of Woodin, $Hom^* = \wp(\mathbb{R})^{L(Hom^*, \mathbb{R}^*)}$ and $L(Hom^*, \mathbb{R}^*) \vDash$ "$\mathsf{AD}_\mathbb{R} + \mathsf{DC}$".

Lemma 2.3. *In $V[G]$, $Hom^* = \Gamma_{ub}$.*

Proof. Since δ_0 is a limit of strong cardinals, it's easy to see that $Hom^* \subseteq \Gamma_{ub}$. To see the reverse inclusion, let $A \in \Gamma_{ub} = Hom_\infty$. Let $\bar{\mu}$ be a (countable) homogeneity system witnessing this . We may assume the measures in $\bar{\mu}$ have additivity κ for some $\kappa >> \delta_0$ and $Hom_\infty = Hom_\kappa$. Any $\mu \in \bar{\mu}$ is the canonical extension of some $\nu \in V$ ($A \in \mu \Leftrightarrow \exists B \in \nu\ B \subseteq A$) (see [6, Proposition 4.4]). Since $\bar{\mu}$ is countable, there is an $\alpha < \delta_0$ such that $\bar{\mu} \cap V[G|\alpha] \in V[G|\alpha]$. We may also pick κ sufficiently large so that in $V[G|\alpha]$, $\bar{\mu}$ witnesses that $A \cap V[G|\alpha]$ is in $Hom_\infty^{V[G|\alpha]}$ and hence in $\Gamma_{ub}^{V[G|\alpha]}$. This gives $A \in Hom^*$. □

[4] x is also in M because $M^{<\delta} \subseteq M$ in $V[G]$. Furthermore, $j(\omega_1) = \delta$.

Now note that $|Hom^*| = \omega_1$ in $V[G]$. This is because Hom^* is determined (in $V[G]$) by V_{δ_0} and the sequence $\langle G|\alpha \mid \alpha < \delta_0\rangle$.

Let $j : V \to M$ witness δ_0 is measurable, i.e. j is the ultrapower map by a normal measure U on δ_0. We define a filter $\mathcal{F}$ on $\wp_{\omega_1}(Hom^*)$ as follows.

$$A \in \mathcal{F} \Leftrightarrow V[G] \vDash \text{“}\emptyset \Vdash_{Col(\omega,<j(\delta_0))} j^+[Hom^*] \in j^+(A)\text{”}^5.$$

It's clear that $\mathcal{F} \in V[G]$; in fact, $\mathcal{F}$ is definable over $V[G]$ from parameters $\{Hom^*, U, G\}$. Note also that since $L(Hom^*, \mathbb{R}^*) \vDash \mathsf{DC}$,

$$\wp_{\omega_1}(Hom^*)^{V[G]} = \wp_{\omega_1}(Hom^*)^{L(Hom^*,\mathbb{R}^*)} \in \mathcal{F}.$$

Lemma 2.4. $L(Hom^*)[\mathcal{F}] \vDash$ *“$\mathcal{F}$ is a normal fine measure on $\wp_{\omega_1}(Hom^*)$”.*

Proof. First we show that $\mathcal{F}$ is a normal fine filter. We verify fineness. Let $A \in Hom^*$, we show $X_A =_{\text{def}} \{\sigma \in \wp_{\omega_1}(Hom^*) \mid A \in \sigma\} \in \mathcal{F}$. Using the notation introduced before the lemma, since $A \in Hom^*$, $j^+(A) \in j^+[Hom^*]$ and hence

$$j^+[Hom^*] \in j^+(X_A).$$

This shows $X_A \in \mathcal{F}$. To show normality, let $F \in V[G]$ be such that

$$A_F =_{\text{def}} \{\sigma \in \wp_{\omega_1}(Hom^*) \mid F(\sigma) \neq \emptyset \wedge F(\sigma) \subseteq \sigma\} \in \mathcal{F}.$$

By the definition of $\mathcal{F}$, $j^+[Hom^*] \in j^+(A_F)$. This means there is some $A \in Hom^*$ such that

$$j^+(A) \in j^+(F)(j^+[Hom^*]).$$

This implies that

$$\{\sigma \in A_F \mid A \in F(\sigma)\} \in \mathcal{F}.$$

This is what we want.

We now show $\mathcal{F} \cap L(Hom^*)[\mathcal{F}]$ is a measure. Suppose $A \subseteq \wp_{\omega_1}(Hom^*)$ in $L(Hom^*)[\mathcal{F}]$ is a counterexample. Every set in $L(Hom^*)[\mathcal{F}]$ is ordinal definable in $V[G]$ from elements of Hom^* and $\{Hom^*, U, G\}$. Let $\varphi(v_0, v_1, v_2, v_3)$ be a formula, $B \in Hom^*$, $s \in \mathrm{OR}^{<\omega}$ be such that

$$\sigma \in A \Leftrightarrow V[G] \vDash \varphi[\sigma, B, s, \{U, G, Hom^*\}].$$

By minimizing the the parameter s that goes into the definition of a counterexample, we may choose a counterexample A such that there is a formula $\varphi(v_0, v_1, v_2)$, a $B \in Hom^*$ such that

$$\sigma \in A \Leftrightarrow V[G] \vDash \varphi[\sigma, B, \{U, G, Hom^*\}].$$

[5] $j^+ : V[G] \to M[G][H]$ for $H \subseteq Col(\omega, < j(\delta_0))$ being $V[G]$-generic is the canonical extension of j. j^+ is defined as: $j^+(\tau_G) = j(\tau)_{G*H}$ for any $Col(\omega, < \delta_0)$-name τ in V. We also note that $j^+[Hom^*] \in M[G][H]$ since in $V[G]$, Hom^* has cardinality ω_1 and Hom^*, in turns, can be represented by a set of names of cardinality δ_0 in V.

Let $\alpha < \delta_0$ be such that there is a δ_0-absolutely complemented tree $T \in V[G|\alpha]$ such that $p[T] \cap V[G] = B$ and $\emptyset$ forces over $V[G|\alpha]$ all relevant facts above. In $V[G|\alpha]$, let U^* be the canonical extension of U; note that j naturally lifts to a map from $V[G|\alpha]$ to $M[G|\alpha]$, which we also call j. For $\gamma < \delta_0$, a limit of Woodin and strong cardinals, let $\dot{\mathbb{R}}_\gamma$ be the canonical (symmetric) $Col(\omega, < \gamma)$-name for $\mathbb{R}^*_\gamma$, and $\dot{Hom}^*_\gamma$ be the canonical (symmetric) name for Hom^*_γ, where $\mathbb{R}^*_\gamma$ and Hom^*_γ are defined similarly to $\mathbb{R}^*$, Hom^* above but at γ instead of at δ_0 (so $\mathbb{R}^* = \mathbb{R}^*_{\delta_0}$ and $Hom^* = Hom^*_{\delta_0}$). Let $\dot{Hom}^*_{\gamma,\delta_0}$ be the canonical name for Hom^*_{γ,δ_0}, where

$$Hom^*_{\gamma,\delta_0} = \{A \in Hom^* \mid \exists\alpha < \gamma \exists T \in V[G|\alpha]\ (A = p[T] \wedge p[T] \cap V(\mathbb{R}^*_\gamma) \in Hom^*_\gamma)\}.$$

Since U^* is a measure in $V[G \restriction \alpha]$, either

$$\forall^*_{U^*}\gamma\ \emptyset \Vdash_{Col(\omega,<\gamma)} \Vdash_{Col(\omega,<\delta_0)} \varphi[\dot{Hom}^*_{\gamma,\delta_0}, p[T], \{\check{U}, \dot{G}, \dot{Hom}^*_{\delta_0}\}]$$

or

$$\forall^*_{U^*}\gamma\ \emptyset \Vdash_{Col(\omega,<\gamma)} \Vdash_{Col(\omega,<\delta_0)} \neg\varphi[\dot{Hom}^*_{\gamma,\delta_0}, p[T], \{\check{U}, \dot{G}, \dot{Hom}^*_{\delta_0}\}].$$

This implies in $M[G \restriction \alpha]$, either

$$\emptyset \Vdash_{Col(\omega,<\delta_0)} \Vdash_{Col(\omega,<j(\delta_0))} \varphi[j^+[\dot{Hom}^*], p[j(T)], \{j(\check{U}), \dot{H}, \dot{Hom}^*_{j(\delta_0)}\}]^6 \qquad (\dagger)$$

or

$$\emptyset \Vdash_{Col(\omega,<\delta_0)} \Vdash_{Col(\omega,<j(\delta_0))} \neg\varphi[j^+[\dot{Hom}^*], p[j(T)], \{j(\check{U}), \dot{H}, \dot{Hom}^*_{j(\delta_0)}\}] \qquad (\dagger\dagger).$$

In the above, note that if $H \subseteq Col(\omega, < j(\delta_0))$ is $V[G]$-generic and $j^+ : V[G] \to M[G][H]$ is the canonical extension of j, then $j^+[Hom^*] = (Hom^*_{\delta_0, j(\delta_0)})^{M[G][H]}$.

($\dagger$) and ($\dagger\dagger$) easily give that A is measured by $\mathcal{F}$, hence a contradiction. This completes the proof of the lemma. $\square$

Since $|\wp_{\omega_1}(Hom^*)| = \omega_1$ in $V[G]$, we can use the club-shooting construction $\mathbb{P}$ described in Section 17.2 of [1] to shoot a club through each $A \in \mathcal{F}$[7]. The forcing $\mathbb{P}$ is ω-distributive. Let $G' \subseteq \mathbb{P}$ be $V[G]$-generic.

Lemma 2.5. *In $V[G][G']$, the following hold.*

(a) $(\mathrm{OR}^\omega)^{\mathrm{V[G]}} = (\mathrm{OR}^\omega)^{\mathrm{V[G][G']}}$, *hence in particular,* $\mathbb{R}^* = \mathbb{R}^{V[G][G']}$.

(b) $Hom^* = \Gamma^{V[G]}_{ub} = \Gamma^{V[G][G']}_{ub}$.

(c) In $V[G][G']$, $L(Hom^*)[\mathcal{F}] \models$ "$\mathcal{F}$ is a normal fine measure on $\wp_{\omega_1}(Hom^*)$" and $\mathcal{F} \subseteq \mathcal{C}_{Hom^*}$, where $\mathcal{C}_{Hom^*}$ is the club filter on $\wp_{\omega_1}(Hom^*)$ in $V[G][G']$.

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[6] $\dot{G}$ is the canonical name for a generic $G \subseteq Col(\omega, < \delta_0)$ and $\dot{H}$ is the canonical name for a generic $H \subseteq Col(\omega, < j(\delta_0))$.

[7] Very roughly, this is a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \omega_2^{V[G]}, \beta < \omega_2^{V[G]} \rangle$, where $\dot{\mathbb{Q}}_\beta$ is the $\mathbb{P}_\beta$-name for $CU(\omega_1, S)$, where S is a stationary subset of ω_1. Conditions of $CU(\omega_1, S)$ are countable closed bounded subsets of S ordered by end-extension. Recall also that $\omega_2^{V[G]} = (2^{\delta_0})^{V[G]} = (2^{\omega_1})^{V[G]}$. By fixing a bijection $\pi : \omega_1 \to \wp_{\omega_1}(Hom^*)$ in advance, we can identify stationary sets in $\wp_{\omega_1}(Hom^*)$ with stationary sets in ω_1. Since $\mathcal{F}$ is a normal fine filter as shown in the proof of Lemma 2.4, if $A \in \mathcal{F}$ then A is stationary.

Proof. (a) follows from the ω-distributivity of $\mathbb{P}$; the details of this are given in [1, Section 17.2].

We verify (b). The first equality of (b) is just the statement of Lemma 2.3. For the second equality, let $A \in \Gamma_{ub}^{V[G][G']} = Hom_{\infty}^{V[G][G']}$ and let $\bar{\mu}$ be a homogeneity system witnessing this. Again, by [6, Proposition 4.4], each measure $\mu \in \bar{\mu}$ is the canonical extension of a measure $\nu \in V[G]$ (this is because the forcing $\mathbb{P}$ is small). We write $\mu = \nu^*$ to mean μ is the canonical extension of ν. By the ω-distributivity of $\mathbb{P}$ and the fact that $\bar{\mu}$ is countable, the set $\{\nu \mid \nu^* \in \bar{\mu}\} \in V[G]$ and witnesses that $A \in Hom_{\infty}^{V[G]}$.

(c) follows from the fact whenever $A \in \mathcal{F}$ (in $V[G]$), then in $V[G][G']$, A contains a club. □

To ease the notation, we rename $V[G][G']$ to $V[G]$. It remains to prove the following

Lemma 2.6. $\wp(\mathbb{R})^{L(Hom^*)[\mathcal{F}]} = Hom^*$.

Proof. Let δ be the limit of the δ_i's. Let $K \subseteq Col(\omega, <\delta)$ be $V[G]$-generic. Let $\mathbb{R}^{**} = \bigcup_{\alpha<\delta} \mathbb{R}^{V[G][K|\alpha]}$ and Hom^{**} be defined in $V[G](\mathbb{R}^{**})$ the same way Hom^* is defined in $V(\mathbb{R}^*)$. By Lemma 6.6 of [6], there is an $H \subseteq \mathbb{Q}_{<\delta}^{V[G]}$ generic over $V[G]$ such that for all $1 \leq n < \omega$, $H \cap \mathbb{Q}_{\delta_n} =_{\text{def}} H_n$ is $V[G]-$generic. Let

$$j : V[G] \to M \subseteq V[G][H]$$

be the generic embedding associated to H (the embedding j before is behind us now); Lemma 6.6 of [6] also allows us to choose H so that $\mathbb{R}^{**} = \mathbb{R} \cap M$[8]. Let $j_n : V[G] \to M_n$ be the generic embedding given by H_n, hence M is the direct limit of the M_n's. Note that the j_n's factor into j via map k_n (i.e. $j = k_n \circ j_n$). Also for $n \leq k$, let $j_{n,k} : M_n \to M_k$ be the natural embedding so that k_n is the limit of the $j_{n,k}$'s. By our assumption on the δ_i's,

$$j_n(\Gamma_{ub}^{V[G]}) = j_n(Hom^*) = \Gamma_{ub}^{V[G][H_n]}.$$

Hence

$$j(\Gamma_{ub}^{V[G]}) = j(Hom^*) = Hom^{**}.$$[9]

Let $\mathcal{F}^*$ be the "tail filter" defined in $V[G][H]$ as follows: for $A \subseteq \wp_{\omega_1}(Hom^{**})$

$$A \in \mathcal{F}^* \Leftrightarrow \exists n \forall m \geq n \ k_m[\sigma_m] \in A.$$

Claim 1: Let $\mathcal{C}_{Hom^*}$ be the club filter on $\wp_{\omega_1}(Hom^*)$ in $V[G]$, then

$$L(Hom^{**})[j(\mathcal{C}_{Hom^*})] = L(Hom^{**})[\mathcal{F}^*] \vDash \text{“}j(\mathcal{C}_{Hom^*}) = \mathcal{F}^* \text{ is a normal fine measure on } \wp_{\omega_1}(Hom^{**}).\text{”}$$

[8]Also, we can pick in advance an $\alpha >> \delta$ and have α in the wellfounded part of M. We suppress this α and pretend that $\alpha = \text{OR}$.

[9]The $\subseteq$-direction of the second equality needs that δ is a limit of good Woodins.

Proof. For each $i < \omega$, let $\sigma_i = (Hom^*)^{M_i} = \Gamma_{ub}^{V[G][H|\delta_i]}$. We claim that if $A \in j(\mathcal{C}_{Hom^*})$ then $A \in \mathcal{F}^*$. To see this, let $n < \omega$ such that M_n contains the preimage of A, say $k_n(A_n) = A$. Then A_n is a club in M_n. We claim that $\forall m \geq n\ k_m[\sigma_m] \in A$. We prove this for the case $m = n$. The other cases are similar. Since $k_n = k_{n+1} \circ j_{n,n+1}$, it suffices to show $j_{n,n+1}[\sigma_n] \in j_{n,n+1}(A_n)$. We have that in M_n, $\sigma_n = \bigcup_{\alpha<\omega_1} \tau_\alpha$ where $\tau_\alpha \in A_n$ for each $\alpha < \omega_1$; this is because A_n is club. In M_{n+1}, $\{j_{n,n+1}(\tau_\alpha) \mid \alpha < \omega_1^{M_n}\}$ is a countable, directed subset of $j_{n,n+1}(A_n)$ whose union is $j''_{n,n+1}\sigma_n$. Since $j_{n,n+1}(A_n)$ is a club in M_{n+1}, $j_{n,n+1}[\sigma_n] \in j_{n,n+1}(A_n)$. Hence we're done with the claim.

The argument also gives: $L(Hom^*)[\mathcal{C}_{Hom^*}] = L(Hom^*)[\mathcal{F}]$ is embeddable into $L(Hom^{**})[\mathcal{F}^*]$. The above argument and the fact that $j(Hom^*) = Hom^{**}$ prove that

$$L(Hom^{**})[j(\mathcal{C}_{Hom^*})] = L(Hom^{**})[\mathcal{F}^*] \vDash \text{“}j(\mathcal{C}_{Hom^*}) = \mathcal{F}^* \text{ is a normal fine measure on } \wp_{\omega_1}(Hom^{**})\text{”}.$$

This completes the proof of Claim 1. □

Let $\mathcal{G}$ be the "tail filter" defined in $V[G][K]$ from the sequence $\langle \sigma_i \mid i < \omega \rangle$. More precisely, for $A \subseteq \wp_{\omega_1}(Hom^{**})$:

$$A \in \mathcal{G} \Leftrightarrow \exists n \forall m \geq n\ \sigma_m^* \in A,$$

where $\sigma_m^* = \{A^* \mid A \in \sigma_m\}$ and $A^* = p[T] \cap V[G][K]$ where $T \in V[G][K|\delta_m]$ witnesses that A is δ-universally Baire.

Claim 2: $L(Hom^{**})[\mathcal{G}] \vDash$ "$\mathcal{G}$ is a normal fine measure on $\wp_{\omega_1}(Hom^{**})$".

Proof. To see this, note that for each $A \subseteq \wp_{\omega_1}(Hom^{**})$ in $V[G][H] \cap V[G][K]$,

$$A \in \mathcal{F}^* \Leftrightarrow A \in \mathcal{G} \qquad (\dagger).$$

This is because for each m, $k_m[\sigma_m] = \sigma_m^*$.

Now we prove the claim. First recall that $(\wp_{\omega_1}(Hom^*))^{V(\mathbb{R}^*)} = (\wp_{\omega_1}(Hom^*))^{V[G]}$. Now note that since $L(Hom^*)[\mathcal{C}_{Hom^*}] = L(Hom^*)[\mathcal{F}]$ is definable in $V(\mathbb{R}^*)$ as the model constructed from Hom^* and the club filter on $\wp_{\omega_1}(Hom^*)$ [10], for any $A \subseteq \wp_{\omega_1}(Hom^{**})$ in $L(Hom^{**})[\mathcal{F}^*] = L(Hom^{**})[j(\mathcal{C}_{Hom^*})]$ (definable over $V[G](\mathbb{R}^{**})$ as the model constructed from Hom^{**} and the club filter on $\wp_{\omega_1}(Hom^{**})$), $A \in V[G][H] \cap V[G][K]$. This fact and (†) imply

$$L((Hom^{**})[\mathcal{G}] = L(Hom^{**})[\mathcal{F}^*] \vDash \text{“}\mathcal{F}^* = \mathcal{G} \text{ is a normal fine measure on } \wp_{\omega_1}(Hom^{**})\text{”}.$$

This finishes the proof of Claim 2. □

Claim 3: $L(Hom^{**})[\mathcal{F}^*] \vDash \mathsf{AD}^+$.

[10]This is true because letting $\mathcal{C}'$ be the club filter on $\wp_{\omega_1}(Hom^*)^{V(\mathbb{R}^*)}$ in $V(\mathbb{R}^*)$, then $\mathcal{C}' \subseteq \mathcal{C}_{Hom^*}$. Since $L(Hom^*)[\mathcal{C}_{Hom^*}] \vDash$ "$\mathcal{C}_{Hom^*}$ is a measure", it's easy to see that $L(Hom^*)[\mathcal{C}'] = L(Hom^*)[\mathcal{C}_{Hom^*}]$ and $L(Hom^*)[\mathcal{C}'] \cap \mathcal{C}' = L(Hom^*)[\mathcal{C}_{Hom^*}] \cap \mathcal{C}_{Hom^*}$.

Proof. To show $L(Hom^{**})[\mathcal{F}^*] \vDash \mathsf{AD}^+$, we use the tree production lemma, Theorem 2.2. Suppose not. Let $x \in \mathbb{R}^{**}$, $T \in V[G][K|\alpha]$ for some $\alpha < \delta$ be a δ-complemented tree, γ be least such that there is a counter-example of AD^+ $B \in L(Hom^{**})[\mathcal{F}^*]$ definable over $L_\gamma(Hom^{**})[\mathcal{F}^*]$ from $(\varphi, x, p[T] \cap \mathbb{R}^{**})$ i.e.

$$y \in B \Leftrightarrow L_\gamma(Hom^{**})[\mathcal{F}^*] \vDash \varphi[y, p[T] \cap \mathbb{R}^{**}, x].$$

Let $\theta(u,v)$ be the natural formula defining B (where $\mathcal{C}$ is the club filter and the parameter v can be construed as a pair (v_0, v_1)):

$$\begin{aligned}\theta(u,v) \quad = \quad & \text{“}L(\Gamma_{ub})[\mathcal{C}] \vDash \mathcal{C} \text{ is a normal fine measure on } \wp_{\omega_1}(\Gamma_{ub}) \text{ and } L(\Gamma_{ub})[\mathcal{C}] \vDash \exists B(\mathsf{AD}^+ \\ & \text{fails for } B) \text{ and if } \gamma_0 \text{ is the least } \gamma \text{ such that } L_\gamma(\Gamma_{ub})[\mathcal{C}] \vDash \exists B(\mathsf{AD}^+ \text{ fails for } B) \\ & \text{then } L_{\gamma_0}(\Gamma_{ub})[\mathcal{C}] \vDash \varphi[u, p[v_0] \cap \mathbb{R}, v_1]\text{”}.\end{aligned}$$

We verify that the tree production lemma holds for $\theta(-,(T,x))$. This gives $B \in Hom^{**}$. Without loss of generality, let $g \in HC^{V[G][K]}$ be such that $(G, K|\alpha, x, T) \in V[g]$ and $(Hom^*)^{V[G][g]} = \Gamma_{ub}^{V[G][g]}$ and

$$L((Hom^*)^{V[G][g]})[\mathcal{C}] \vDash \mathcal{C} \text{ is a normal fine measure on } \wp_{\omega_1}((Hom^*)^{V[G][g]})$$

where $\mathcal{C}$ is the club filter in $V[G][g]$. We can make this assumption about g because δ is a limit of measurable cardinals which are limits of Woodin and strong cardinals (again, we note that the argument that $L(Hom^*)[\mathcal{C}^{V[G][G']}] \vDash$ “$\mathcal{C}^{V[G][G']}$ is a normal fine measure” only uses that δ_0 is a measurable limit of Woodin and strong cardinals, where $\mathcal{C}^{V[G][G']}$ is the club filter on $\wp_{\omega_1}(Hom^*)$ in $V[G][G']$). δ is still a limit of good Woodin cardinals in $V[G][g]$. Let $\xi < \delta$ be a good Woodin cardinal in $V[G][g]$.

We first verify stationary correctness. Let $K' \subseteq \mathbb{Q}_{<\xi}^{V[G][g]}$ be $V[G][g]$-generic, and

$$k : V[G][g] \to N \subseteq V[G][g][K']$$

be the associated embedding. By the property of ξ, $k(\Gamma_{ub}^{V[G][g]}) = \Gamma_{ub}^N = \Gamma_{ub}^{V[G][g][K']}$. Furthermore, $\mathcal{C}^N \subseteq \mathcal{C}^{V[G][g][K']}$ (here $\mathcal{C}$ denotes the club filter in the relevant universe) and by elementarity,

$$L(\Gamma_{ub}^N)[\mathcal{C}^N] \vDash \mathcal{C}^N \text{ is a normal fine measure on } \wp_{\omega_1}(\Gamma_{ub}^N).$$

This implies $L(\Gamma_{ub})[\mathcal{C}]^N = L(\Gamma_{ub})[\mathcal{C}]^{V[G][g][K']}$. Furthermore, $p[T] \cap V[G][g][K'] = p[k(T)] \cap N$. This easily implies stationary correctness.

To verify generic absoluteness at ξ. We rename $V[G][g]$ to V to save space. Let g be $<\xi$-generic over V (the old g is behind us now) and h be $<\xi^+$-generic over $V[g]$. Let $y \in \mathbb{R}^{V[g]}$. We want to show

$$V[g] \vDash \theta[y,(T,x)] \Leftrightarrow V[g][h] \vDash \theta[y,(T,x)].$$

There are $G_0, G_1 \subseteq Col(\omega, <\delta)$ such that G_0 is generic over $V[g]$ and G_1 is generic over $V[g][h]$ with the property that $\mathbb{R}^{V[G_0|\delta_i]} = \mathbb{R}^{V[G_1|\delta_i]}$ for all $\delta_i > \xi$. Also, $(Hom^*)^{V[G_0|\delta_i]} = (Hom^*)^{V[G_1|\delta_i]} = \Gamma_{ub}^{V[G_0|\delta_i]} = \Gamma_{ub}^{V[G_1|\delta_i]}$. Let us denote this σ_i. Such G_0 and G_1 exist since h is generic over $V[g]$ and

$\xi < \delta$. So we get that $(Hom^*)^{V[g][G_0]} = (Hom^*)^{V[g][h][G_1]}$. The proofs of Claims 1 and 2 imply $L(Hom^*)[\mathcal{C}]^{V[g]}$ is embeddable into $L(Hom^*)[\mathcal{G}]^{V[g][G_0]}$, and $L(Hom^*)[\mathcal{C}]^{V[g][h]}$ is embeddable into $L(Hom^*)[\mathcal{G}]^{V[g][h][G_1]}$, and $L(Hom^*)[\mathcal{G}]^{V[g][G_0]} = L(Hom^*)[\mathcal{G}]^{V[g][h][G_1]}$, where $\mathcal{G}$ is the "tail filter"[11] defined from the sequence $\langle \sigma_i \mid i < \omega \rangle$ (this also uses the homogeneity of $Col(\omega, < \delta)$ and that's why we proved Claim 2). This implies generic absoluteness. □

Claim 3 implies

$$\wp(\mathbb{R})^{L(Hom^{**})[\mathcal{F}^*]} = Hom^{**}$$

since otherwise, let $A \in \wp(\mathbb{R})^{L(Hom^{**})[\mathcal{F}^*]} \backslash Hom^{**}$. Then $L(A, \mathbb{R}^{**}) \vDash \mathsf{AD}^+$. By the choice of δ and a theorem of Woodin (Theorem 8.3 of [6]), $Hom^{**} = \{A \subseteq \mathbb{R}^{**} \mid A \in V[G](\mathbb{R}^{**}) \wedge L(A, \mathbb{R}^{**}) \vDash \mathsf{AD}^+\}$[12]. This is a contradiction. By elementarity of j and the fact that $L(Hom^*)[\mathcal{F}]$ embeds into $L(Hom^{**})[\mathcal{F}^*]$ (see the argument in Claim 1), $\wp(\mathbb{R})^{L(Hom^*)[\mathcal{F}]} = Hom^*$ and hence Lemma 2.6 follows.

□

Lemma 2.6 completes the proof of the theorem since by the derived model theorem (cf. [6]), $L(Hom^*, \mathbb{R}^*) \vDash \mathsf{AD}_{\mathbb{R}}$ and $Hom^* = \wp(\mathbb{R})^{L(Hom^*, \mathbb{R}^*)}$, hence $M = L(Hom^*)[\mathcal{F}] \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \mathcal{F}$ is a normal fine measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$". □

The hypothesis of Theorem 0.1 is very strong. This is because we want to show the derived model satisfies "$\mathsf{AD}_{\mathbb{R}} + \omega_1$ is $\wp(\mathbb{R})$-supercompact."[13] In terms of consistency strength, the theory "$\mathsf{AD}_{\mathbb{R}} + \omega_1$ is $\wp(\mathbb{R})$-supercompact" is much weaker as demonstrated by Theorem 2.7.

Theorem 2.7. *Assume $\mathsf{AD}_{\mathbb{R}} + \Theta = \theta_{\alpha+\omega}$ where α is a limit ordinal and $cf(\theta_\alpha)$ is uncountable. Let $\Gamma = \{A \subseteq \mathbb{R} \mid w(A) < \theta_\alpha\}$. Let μ be the measure on $\wp_{\omega_1}(\Gamma)$ induced by the Solovay measure on $\wp_{\omega_1}(\mathbb{R})$. Let $M = \mathrm{HOD}_\Gamma$. Then $\wp(\mathbb{R})^M = \Gamma$ and $M \vDash \mathsf{AD}_{\mathbb{R}} + \mu$ is a normal fine measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$.*

Proof. First, it's easy to see that $\wp(\mathbb{R})^M = \Gamma$; hence $M \vDash \mathrm{AD}_{\mathbb{R}} + \Theta = \theta_\alpha$. By [10], μ is unique and hence OD and hence $\mu \cap M \in M$. Now the key point is $\wp_{\omega_1}(\Gamma)^M = \wp_{\omega_1}(\Gamma)$. This is because $\mathrm{cf}(\theta_\alpha)$ is uncountable, Γ is closed under ω-sequences. This means μ concentrates on $\wp_{\omega_1}(\wp(\mathbb{R}))^M$ and hence $M \vDash$ "μ is a normal fine measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$". □

We remark that the hypothesis of the theorem is consistent, e.g. relative to "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". The exact consistency strength of the theory "$\mathsf{AD}_{\mathbb{R}} + \omega_1$ is $\wp(\mathbb{R})$-supercompact" is still unknown. Theorem 2.7 implies that an upper bound is "$\mathsf{AD}_{\mathbb{R}} + \Theta = \theta_{\omega_1+\omega}$", which is slightly stronger than "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$". On the other hand, suppose $M \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \omega_1$ is $\wp(\mathbb{R})$-supercompact" then $M \vDash$ "$\mathrm{cf}(\Theta) > \omega$" hence $L(\wp(\mathbb{R}))^M \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$". To see that $M \vDash$ "$\mathrm{cf}(\Theta) > \omega$", suppose

[11] This piece of the proof was pointed out by John Steel. The author would like to thank him for this.

[12] In fact, this equality holds for δ being limit of Woodin and $< -\delta$-strong cardinals.

[13] One can show the model M in Theorem 0.1 satisfies Θ is regular. This uses significantly the supercompactness of δ_0. Again the proof of Theorem 0.1 only uses that δ_0 is a measurable limit of Woodin and strong cardinals.

not. Working in M, suppose μ is a normal fine measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$ and let $f : \omega \to \Theta$ be cofinal. For each $n < \omega$, let

$$A_n = \{\sigma \mid \sup_{A\in\sigma} w(A) \geq f(n)\}.$$

By fineness of μ, it's easy to see that each $A_n \in \mu$. By countable completeness of μ, $\bigcap_{n<\omega} A_n \neq \emptyset$. Let $\sigma \in \bigcap_{n<\omega} A_n$. Then σ is Wadge-cofinal in $\wp(\mathbb{R})$. Say $\sigma = \{B_n \mid n < \omega\}$; let $B = \{(x, n) \mid x \in B_n\}$. B clearly has Wadge rank above that of each B_n. This contradicts the fact that σ is Wadge confinal.

T. Wilson and the author have shown in [9] that "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\wp(\mathbb{R})$-supercompact" implies that there are models of "$\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \Theta = \theta_\alpha$" for all countable limit ordinal α; this means that the best known lower bound consistency strength of theory "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\wp(\mathbb{R})$-supercompact" is very close to "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$". We conjecture that

Conjecture 2.8. *The following theories are equiconsistent.*

1. $\mathsf{ZF} + \mathsf{DC} + \omega_1$ *is* $\wp(\mathbb{R})$*-supercompact.*
2. $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC} + \omega_1$ *is* $\wp(\mathbb{R})$*-supercompact.*

If we add "Θ is regular" to the clauses in the conjecture, then the corresponding conjecture in fact has a positive answer (cf. Theorem 3.2).

3. Θ IS REGULAR

Woodin had conjectured that the theory "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" has consistency strength on the order of a supercompact cardinal. He's shown that assuming there is a supercompact cardinal below the cardinal δ_0 in the hypothesis of Theorem 0.1, then the derived model $L(Hom^*, \mathbb{R}^*)$ at δ_0 satisfies "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular." In fact, since the model $L(Hom^*)[\mathcal{F}]$ in Theorem 0.1 is very close to $L(Hom^*, \mathbb{R}^*)$, Woodin's proof can be used to show $L(Hom^*)[\mathcal{F}] \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular + ω_1 is $\wp(\mathbb{R})$-supercompact." Part of the reason for Woodin's conjecture is that it's not clear how to show derived models satisfy "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" without assuming the existence of supercompact cardinals.

However, G. Sargsyan in [4] has reduced the consistency strength of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" to below that of "ZFC + there exists a Woodin limit of Woodin cardinals." G. Sargsyan and Y. Zhu have subsequently computed the exact strength of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular." We'd like to do the same for the theory "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular + ω_1 is $\wp(\mathbb{R})$-supercompact." The following theorem is a first step toward that goal. It belongs to the folklore but whose proof seems to be unpublished.

Theorem 3.1. *Suppose* $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ *holds and there is a* $\mathbb{R}$*-complete measure on* Θ[14]*. Then there is a normal fine measure on* $\wp_{\omega_1}(\wp(\mathbb{R}))$*.*

[14] A measure ν on Θ is $\mathbb{R}$-complete if whenever $\langle A_x \mid x \in \mathbb{R}\rangle$ is a sequence of ν-measure one sets then $\cap_{x\in\mathbb{R}} A_x \in \mu$.

Proof. The hypothesis implies there is a $\mathbb{R}$-complete and normal measure on Θ by a standard argument (see Theorem 10.20 of [2] and note that DC is enough for the proof of the theorem). Let ν be such a measure. For each $\alpha < \Theta$, let μ_α be the normal fine measure on $\wp_{\omega_1}(\wp_\alpha(\mathbb{R}))$ derived from the Solovay measure μ_0 on $\wp_{\omega_1}(\mathbb{R})$ (i.e. we first fix a surjection $\pi : \mathbb{R} \to \wp_\alpha(\mathbb{R})$; then we let $\pi^* : \wp_{\omega_1}(\mathbb{R}) \to \wp_{\omega_1}(\wp_\alpha(\mathbb{R}))$ be the surjection induced from π and let $A \in \mu_\alpha \Leftrightarrow (\pi^*)^{-1}[A] \in \mu_0$). It's worth noting that by [10], μ_α are unique for all $\alpha < \Theta$. We derive from ν a measure μ on $\wp_{\omega_1}(\wp(\mathbb{R}))$ as follows. Let $A \subseteq \wp_{\omega_1}(\wp(\mathbb{R}))$, then

$$A \in \mu \Leftrightarrow \forall^*_\nu \alpha \ A \restriction \wp_\alpha(\mathbb{R}) =_{def} \{\sigma \in A \mid \sigma \in \wp_{\omega_1}(\wp_\alpha(\mathbb{R}))\} \in \mu_\alpha.$$

It's clear that μ is a measure. It's also clear that μ is fine since the measures μ_α's are fine. It remains to show normality of μ.

We use the alternative characterization of normality in Lemma 1.4. Suppose μ is not normal. By Lemma 1.4, there is a sequence $\langle A_x \mid x \in \wp(\mathbb{R}) \wedge A_x \in \mu \rangle$ but $\triangle_{x\in\wp(\mathbb{R})} A_x \notin \mu$. This means

$$\forall^*_\nu \alpha \forall^*_{\mu_\alpha} \sigma \exists x \in \sigma \ \sigma \notin A_x.$$

By normality of μ_α, we then have

$$\forall^*_\nu \alpha \exists x \forall^*_{\mu_\alpha} \sigma \ x \in \sigma \wedge \sigma \notin A_x. \tag{3.1}$$

We now define a regressive function $F : \Theta \to \Theta$ as follows. Let $F(\alpha)$ be the least $\beta < \alpha$ such that there is an $x \in \wp(\mathbb{R})$ such that $w(x) = \beta$ and $\forall^*_{\mu_\alpha} \sigma \ \sigma \notin A_x$; otherwise, let $F(\alpha) = 0$. By 3.1, $\forall^*_\nu \alpha \ 0 < F(\alpha) < \alpha$. By normality of ν, there is a β such that $\forall^*_\nu \alpha \ F(\alpha) = \beta$.

For each x such that $w(x) = \beta$, let

$$B_x = \{\alpha < \Theta \mid \forall^*_{\mu_\alpha} \sigma \ \sigma \notin A_x\}.$$

Note that $\cup_x B_x \in \nu$. Since there are only $\mathbb{R}$-many such x, by $\mathbb{R}$-completeness of ν, there is an x such that $B_x \in \nu$. Fix such an x. We then have

$$\forall^*_\nu \alpha \forall^*_{\mu_\alpha} \sigma \ \sigma \notin A_x. \tag{3.2}$$

The above equation implies $A_x \notin \mu$. Contradiction. □

We call the theory in the hypothesis of Theorem 3.1 "$\mathsf{AD}_\mathbb{R} + \Theta$ is measurable". Theorem 3.1 proves one direction of the following theorem, whose proof is beyond the scope of this paper (cf. [8]).

Theorem 3.2. *The following theories are equiconsistent:*

1. $\mathsf{AD}_\mathbb{R} + \Theta$ *is measurable.*

2. $\mathsf{ZF} + \mathsf{DC} + \Theta$ *is regular* $+ \omega_1$ *is* $\wp(\mathbb{R})$*-supercompact.*

As a consequence, the above theories are equiconsistent with

3. $\mathsf{AD}_\mathbb{R} + \mathsf{DC} + \Theta$ *is regular* $+ \omega_1$ *is* $\wp(\mathbb{R})$*-supercompact.*

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