ON THE COP NUMBER OF SUBDIVISIONS OF GRAPHS

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Abstract. The game of cops and robbers is a game played on graphs in which some number of cops move along a graph to try to catch a singular robber. In this paper, we examine the behavior of the cop number of graphs compared to their subdivision graphs, which we will refer to as k-tunnel graphs. We prove that for any graph, either every k-tunnel graph has the same cop number as the original, or there is some tail of k for which the cop number of the k-tunnel graphs increases by one.

1. Introduction

Throughout this paper, by a graph, we mean an irreflexive symmetric binary relation on a set (vertices), that is an undirected graph with no loops (we allow multiple edges between two vertices). Let G be a graph. For any point $g \in G$, $N(g) = \{h \in G \mid h = g \text{ or } E(g,h)\}$. $N(g)$ consists of g and the neighbors of g. Since G is undirected, g is a neighbor of h if and only if h is a neighbor of g. A *path* between two points x and y in a graph G is a sequence of points $x = g_0, g_1, \dots, g_{n-1}, g_n = y$ such that for all $0 \leq i \leq n-1$, $E(g_i, g_{i+1})$. The *length* of a path $g_0, g_1, \dots, g_n$ is n. For a graph G, the *graph distance* between two points g and h is equal to the length of the shortest path connecting them. We will denote this distance $\rho_G(g,h)$, and will omit the G when there is no danger of confusion.

The *game of cops and robbers* is a two-player game played on a graph in which player one controls some fixed number n of cops, while player two controls one robber. We call this the n-cop game on the graph. On the first turn of the game, player one places each cop on a vertex on the graph (we allow multiple cops to be on the same vertex). Player two responds by placing the robber on a vertex. The game proceeds over a countable sequence of turns. On each turn, player one moves each cop to a neighbor of the vertex they are currently occupying and then player two moves the robber to any neighbor of the vertex the robber is occupying. We note that in either case, a cop or the robber could chose to remain on the same vertex. Player one wins a run of the game if at some move, one of the cops is occupying the same vertex as the robber; in this case, we say the robber was *caught*. Otherwise player two wins. On a game with n cops, we will denote those cops by $c_1, c_2, \dots, c_n$, and denote the robber by r. For a more detailed introduction to the game of cops and robbers, see [1]

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A strategy for a player is a rule that tells the player how to move given the previous history of the game. As an abbreviation, if we have a strategy τ, we will let $\tau(v_r, v_1, \dots, v_n, t)$ denote what τ does on turn t when each cop c_i is on v_i, and the robber is on v_r. We view r as a symbol to denote the robber, not to be confused with an integer i. This strategy may depend on the previous history of the game, but we will suppress that in the notation. When we define a strategy we will just define the initial moves by saying, "place the cops down on vertices $v_1, v_2, \dots, v_n$ and robber on vertex v_r. To denote an initial move for the robber, we will use the notation $\tau(\cdot, v_1, \dots, v_n, 0)$, since the robber is not positioned on the graph on turn 0.

Consider a graph G, where the robber plays against n cops. A *game state s* for the game is a position in the game which we will denote as $s = (g_r, v_1, \dots, v_n, t, h) \in G^{n+1} \times \mathbb{N}$. Here g_r denotes the current position of the robber, v_1 through v_n denote the position of c_i, t denotes the current round of the game, and h is the history of the game, i.e., the sequence of prior moves. On turn $t = 0$, the cops must pick initial positions with no robber on the graph, and then the robber is allowed to pick any vertex. We will usually suppress h in our notation as it plays no role in our arguments. We say a player *follows* a strategy τ if for any game state s, the move that player makes is given by $\tau(s)$.

A strategy τ is *winning* for the cops if, when the cops follow τ, then at some finite turn t, there will be a game state $s = (v_r, v_1, \dots, v_n, t)$ such that $\tau(v_r, v_1, \dots, v_n, t) = (w_1, w_2, \dots, w_n)$, and for some i, $w_i = g_r$. A strategy τ is winning for the robber if when the robber follows τ he is never caught, i.e., $g_r \neq v_i$.

For general graphs G, this game is an open game. It is shown in [2] that any such game is determined, i.e., one of the players has a winning strategy.

Definition 1.1. The *cop number* of a graph G, denoted $C(G)$ is the minimum integer n such that player one has a winning strategy in the n-cop game on G. If there is no such number, then $C(G) = \infty$.

When G is finite, then $C(G)$ is necessarily finite with cop number at most $|G|$.

One fact about graphs in general is that the cops and robber game is always positional. A *positional strategy* is a strategy τ which only depends on the current round of the game. In our notation, a positional strategy is where $\tau(g_r, v_1, \dots, v_n, t, h) = \tau(g_r, v_1, \dots, v_n)$.

Lemma 1.2. *Whichever player has a winning strategy in a cops and robber game on a graph G has a positional winning strategy.*

Proof. (sketch) Say τ is a winning strategy for player one. The ordinal analysis of open games (cf reference) assigns to each winning position of player one an ordinal rank. Given a current position $(v_r, v_1, \dots, v_n)$ of the cops and robbers, let t and h such that $(v_r, v_1, \dots, t, h)$ minimizes the rank of the position in the open game. Let $\tau'(v_r, v_1 \dots, v_n) = \tau(v_r, v_1, \dots, v_n, t, h)$. It is easy to check τ' is a positional winning strategy for player one. The case where τ is winning for player two is similar. □

The main result of this paper concerns the relation between the cop number of a finite graph G, and the graph-theoretic subdivision of this graph. Informally recall that a subdivision of a graph means we insert some number of vertices between any two vertices of G. The k-tunnel graph associated to G is the subdivision of G

where we put $k-1$ vertices between any two vertices (defined formally in 2.1. We let G_k denote the k-tunnel graph associated with G. Our first result, Theorem 2.4, is that for any k, $C(G) \leq C(G_k) \leq C(G)+1$. The other major result, Theorem 4.5, is that the cop number is either constant with k, or increases by one on some tail of k-values for the tunnel graphs. We give an example where the cop number is constant, and also an example where the cop number increases.

2. The Tunnel Version

In this section we look at what happens when we add "tunnels" to graphs. The idea is we add a constant number of points between each vertex.

Definition 2.1. Given a graph G, and an integer $k > 1$ the *k-tunnel graph* of G, denoted G_k, is the graph constructed in the following way. If $x, y \in G$ are connected by an edge, then put vertices x and y in G_k, and insert points $g_1, g_2, \ldots, g_{k-1}$ such that $E(x, g_1), E(g_{k-1}, y)$, and $E(g_i, g_{i+1})$ for all i.

The vertices x and y in G_k will be referred to as *supervertices*, and the points $g_1, g_2, \ldots, g_{k-1}$ will be called *tunnel vertices*.

As an example, if we start with a graph G, and want to construct G_2, then we subdivide each edge of G, meaning that between any two points x and y of G, we delete the edge connecting x and y, and insert a vertex g which is a neighbor of both x and y. The tunnel graphs are denoted G_k because it takes k moves to get from one supervertex to another.

The cop number can change between a graph and its tunnel version; for example, consider the complete graph on 3 vertices. Then the 2-tunnel version of it is a 6-cycle. The first graph has cop number 1, while the latter has cop number 2. We will generalize this example later, but for now we can see this in figure 1.

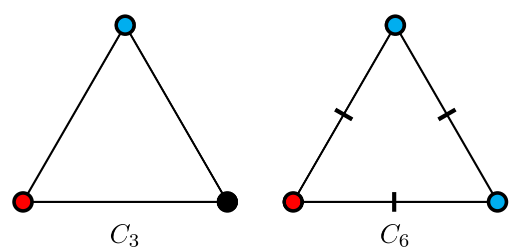


Figure 1. The cop number goes up in the 2-tunnel

There are some relationships between the cop number of a graph and its tunnel versions which we prove below. We will often have the G game and the G_k game going on at the same time. Depending on the situation we will want to associate the cops or robber in the G_k game with a supervertex so that we can place them in the G game. We will call this a *projection* of the cops or the robber. We will also sometimes want whichever side we are playing to commit a certain number of moves (usually k). We will refer to these as k-moves. These notions will be defined more precisely as they come up.

Given a graph G, and its k-tunnel graph G_k, we will define a projection function from G_k to G. For a supervertex v of G_k, we let v^* be the corresponding vertex in G. We define the projection $p(t, c_i)$ of a cop c_i at time t as follows. Suppose at time $t = 0$, c_i is on a supervertex v, or is on a tunnel vertex v_i that belongs to a tunnel whose corresponding supervertices are v and w, where $v < w$, where $<$ is a fixed linear order on the supervertices. We then define

$$p(0, c_i) = \begin{cases} v^* & \text{if } v_i \neq w \\ w^* & \text{otherwise.} \end{cases}$$

In other words, $p(0, c_i)$ is v^* if c_i is anywhere in the tunnel except at w; if c_i is at w, then $p(0, c_i) = w^*$. Suppose $t > 0$ and $p(t', c_i)$ have been defined for all $t' < t$. Suppose at time t, c_i is at vertex v_i which belongs to a tunnel whose endpoints are v and w and suppose that $p(t-1, c_i)$ is v^*. Then let

$$p(t, c_i) = \begin{cases} v^* & \text{if } v_i \neq w \\ w^* & \text{otherwise.} \end{cases}$$

In other words, we define $p(t, c_i)$ to the the projection of the cop c_i at time $t-1$, unless c_i is at the other supervertex w, then we switch the projection of c_i from v^* to w^*. We define the function $p(t, r)$ for the robber similarly.

Theorem 2.2. *If $C(G) = n$, then $C(G_k) \leq n + 1$, for $k \geq 1$.*

Proof. Suppose τ^* is a strategy for n cops to catch the robber in G, and we define a strategy τ for $n+1$ cops in G_k. At any time t, if the robber is on vertex v in G_k, let $v^* = p(v, t)$. To start, the strategy τ^* will place each cop c_i' on a vertex v_i of G. Have τ place n cops $c_1, \ldots, c_n$ at the corresponding supervertices of G_k, and place the $n+1$st cop (the "extra cop") c_{n+1} arbitrarily in G_k.

Throughout the game for G_k, the extra cop c_{n+1} will be dedicated to chasing the robber r in the following precise sense. Suppose at some time, the robber moves from vertex v_1 of G_k to an adjacent vertex v_2, and say c_{n+1} is currently at vertex u_1. Let $p = u_1, w_1, w_2, \ldots w_\ell = v_1$ be a path in G_k of minimal length. At the next move in G_k (the cop's turn), c_{n+1} will move from u_1 to $u_2 = w_1$. We clearly have that $\rho(u_2, v_2) \leq \rho(u_1, v_1)$. We will specify τ's moves for the other cops in a moment, but we will have that the distance from c_{n+1} to r is a monotonically decreasing function throughout the game. Roughly speaking, the role of c_{n+1} is to eventually prevent the robber from "back-tracking" inside the tunnels. For the rest of the game G_k, the extra cop c_{n+1} will always be following this rule. We will now define what each cop c_i does for $1 \leq i \leq n$ (so when we say c_i, we really mean c_i where $i \neq n+1$).

At any given time, if the robber is not on a supervertex, then we will call the *destination supervertex* the supervertex of the tunnel that the robber is in that does not correspond to his projection. We say a robber *backtracks* if he moves away from his destination vertex or passes.

We now define the strategy for the first n cops, who start on v_i. We say t is a *good time* if r and $c_1, \ldots, c_n$ are all on supervertices at time t. At a good time, τ will have each c_i pass. If t is not a good time, but $t-1$ is, then let w be the destination vertex for the robber at time t. We define the next k moves for the cops as follows. If $\tau^*(w^*, v_1^*, v_2^*, \ldots, v_n^*) = (w_1^*, w_2^*, \ldots, w_n^*)$, then have c_i move to w_i over the next k moves. At the end of the k moves, the robber is on a supervertex, we repeat the strategy. If he is not on a supervertex (which implies he has backtracked), then τ has c_i pass until the time (if any) the robber moves to a supervertex. Once the robber does move to a supervertex, the strategy repeats. By the definition of τ, at any time t, there will be some time $t < t' \leq t + k$ such that all of the n cops will be on supervertices.

We show τ is winning. We first observe that the robber can only backtrack finitely often, as each time he does, c_{n+1} will get closer to him. We may assume that there is some time T such that the robber will never backtrack after this time. Let $T' > T$ be a time such that all of the cops (besides possibly c_{n+1}) are on supervertices. Since there is no further backtracking, after at most k moves, the robber will be on a supervertex (and the cops will still be on supervertices). Thus we reach a time T'', such that the cops and robber will be on supervertices, and there is no further backtracking by the robber. Since there is no further backtracking, and by the definition of τ, every block of k moves in the G_k game corresponds to one move in the G game. Because τ^* is a winning strategy, after some number of these blocks, one of the c_i^* catches the robber in G. By the definition of τ, c_i will have caught the robber in the G_k game.

□

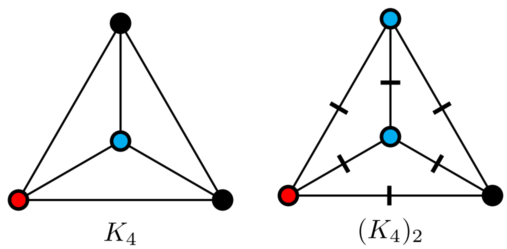


FIGURE 2. The cop number goes up in the 2-tunnel

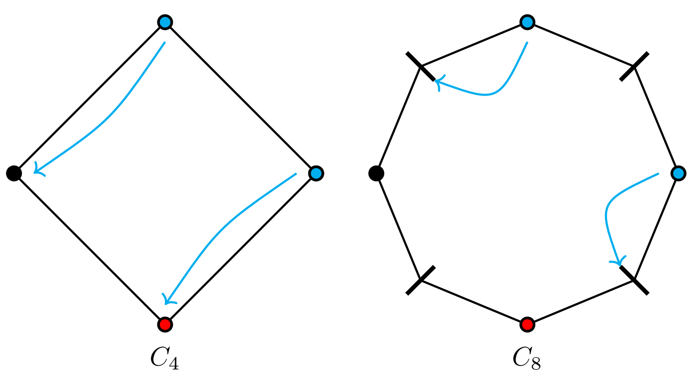


FIGURE 3. The cop number doesn't go up in the 2-tunnel

So adding in tunnels can increase the cop number by at most one. the example above shows that this bound is achievable. The next result shows that the cop number will never go down. i.e. adding tunnels can only help the robber.

Theorem 2.3. *If $C(G) = n$, then $C(G_k) \geq n$ for any $k > 0$.*

Proof. Suppose that τ^* is a winning strategy for the robber against $n-1$ cops in G (i.e., following τ^* will allow the robber to never get caught). We will translate τ^* to a strategy τ for the robber to avoid $n-1$ cops in G_k.

If v and w are supervertices in G_k, then we say that v and w are adjacent (supervertices) in G_k if $\rho_{G_k}(v,w)=k$. Then in G, $\rho_G(v^*,w^*)=1$.

First, at time $t=0$, cops $c_1,\dots,c_{n-1}$ position themselves at some vertices in G_k. For each $1\le i\le n-1$, suppose c_i is at vertex v_i. Let $v_i^*=p(0,c_i)$ for each i. Now $\tau^*(\cdot,v_1^*,\dots,v_{n-1}^*,0)=v_r^*$, which defines the vertex v_0 in G_k. Define $\tau(\cdot,v_1,\dots,v_{n-1},0)=v_r$. There are no cops next to the robber in G, so every cop is more than k away from the robber in G_k, i.e. $\rho_{G_k}(v_r,v_i)>k$ for each $1\le i\le n-1$.

The strategy τ for the robber r in G_k, derived from τ^*, is as follows: suppose r is at a supervertex v in G_k at time $t=mk+1$ for $m\ge 0$, so $v^*=p(t,r)$ is the vertex corresponding to v in G. Suppose $\tau^*(v^*,v_1^*,\dots,v_{n-1}^*,m+1)=w^*$. Let w be the corresponding supervertex in G_k and let the tunnel joining v,w consist of points $u_0=v,u_1,\dots,u_k=w$. In the next k moves in G_k, τ moves the robber through the tunnel from v to w, i.e. for each $i\in\{1,\dots,k\}$, $\tau(u_i,v_1,\dots,v_{n-1},mk+i)=u_i$. We note that if $\tau^*(v^*,v_1^*,\dots,v_{n-1}^*,m+1)=v^*$ then $\tau(v,v_1,v_2,\dots,v_{n-1},mk+i)=v$ for $1\le i<k$. At time $mk+k$, r is either at v or at w.

We now proceed by induction to show the robber is always safe in G_k. Assume (by induction) that the robber is on a supervertex v at time $t=mk$, and the projections of the cops and robbers at times $0,k,\dots,mk$ are consistent with τ^* (in the game G). Note that when $t=0$, equivalently $m=0$, the inductive hypothesis is satisfied. Suppose v^* is the vertex in G corresponding to v and $\tau^*(v^*,v_1^*,\dots,v_{n-1}^*,m+1)=w^*$. Let w be the supervertex in G_k that corresponds to w^*; as mentioned above, $\rho_{G_k}(w,v)$ is either k or 0. Now play according to τ, i.e. in the next k moves in G_k, move the robber from v to w precisely as described above. We claim the robber does not get caught. If he were caught, then that means there was a cop c_i whose position is v_i when the robber is at v at time t. Furthermore, $\rho_{G_k}(v_i,w)\le k$. Let $v_i^*=p(t,c_i)$. Therefore, in G, $\rho_G(v_i^*,w^*)\le 1$. But $\tau^*(v^*,v_1^*,\dots,v_n^*,m+1)=w^*$ and τ^* is a winning strategy for the robber in G, it is impossible for a cop to be at position v_i^*. Therefore, for each i, if cop c_i is at position v_i in G_k when the robber is at position v, and the robber moves from v to w in the next k moves according to τ', then $\rho_{G_k}(v_i,w)>k$. This implies that the robber can make such k moves and will be safe at w. Hence, he can evade the cops indefinitely.

□

Combining the previous two theorems, we get.

Theorem 2.4. *If $C(G)=n$, then $n\le C(G_k)\le n+1$ for all $k\ge 2$.*

Corollary 2.5. *If $C(G)=n$, and $C(G_k)=n+1$ for some $k>0$, then for any $m>0$, $C(G_{mk})=n+1$*

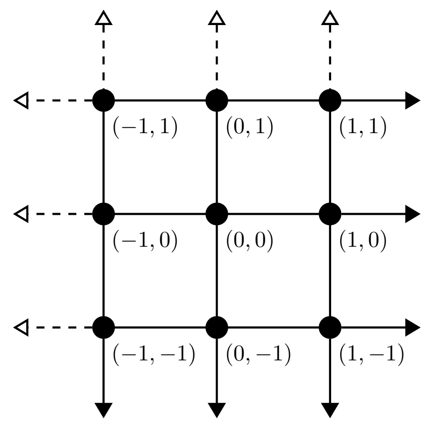
Proof. Suppose $C(G)=n$ and $C(G_k)=n+1$. Then G_{mk} is the m-tunnel version of G_k, so G_{mk} has cop number at least $n+1$. But G_{mk} is a tunnel graph of G, so it has cop number at most $n+1$ □

3. Examples of Graphs for which the Cop Number Increases

In this section, we will provide a class of graphs such that for each n, there is a graph that has cop number n, but the cop number of a tunnel graph is $n+1$.

Definition 3.1. We define the graph T^n as follows. The vertex set is the set $V = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \{-1, 0, 1\}\}$. The edge set is the set $E = \{(x, y) \mid$ there is a j such that $x_i = y_i$ if $i \neq j$, and $x_j \neq y_j\}$. In other words, two elements are connected by an edge if there is exactly one coordinate on which they are different.

Given an element $x \in T^n$, we will write x as $x = (x_1, x_2, \ldots, x_n)$. Since the possible value of any coordinate of an element $x \in T^n$ is in $\{-1, 0, 1\}$, $x_j \neq y_j$ implies that $x_j = y_j \pm 1 \mod 3$. Thus, x and y are neighbors if and only if there is exactly one $1 \leq i \leq n$ such that $x_i = y_i \pm 1$.



Lemma 3.2. *Suppose $C(G) = n$, and τ is a winning strategy for n cops in G. Then for any set of vertices $h_1, h_2, \ldots, h_n \in G$, there is a winning strategy σ where c_i starts on the vertex h_i.*

Proof. Define σ by first having σ place each cop c_i on h_i. Let τ be a winning strategy for the cops, and suppose that τ places c_i on a vertex g_i. Have σ move each c_i from h_i to g_i (possibly have c_i pass if the other cops have not yet gotten to their respective g_i). At this point, define σ to move the cops the same way τ would, starting at the robber's current position. □

Theorem 3.3. $C(T^n) = n$.

Proof. We note that the proof of the claim is simple for $n = 1$, so we will assume $n \geq 2$. We first show $C(T^n) > n - 1$. Start the game on the graph T^n with $n - 1$ cops, and suppose each cop c_i places down on some vertex v_i in T^n. We want to show that the robber has a vertex he can place down on which is at least two away from every cop. Any point $x \in T^n$ has $2n$ neighbors, so any cop can be within one of at most $2n+1$ points of T^n. Therefore, there are at most $(2n+1)(n-1) = 2n^2-n-1$ points that are within one vertex of the $n-1$ cops. T^n has 3^n vertices, so it suffices to show that $2n^2 - n - 1 < 3^n$ for $n \geq 2$. This however is easy to check. We will say a cop is *guarding* a point if she is within one of that point.

This shows that there are more vertices of T^n then there are guarded points. Therefore, the robber will have a vertex he can place down on to start the game which is at least two from every cop. Start his strategy τ by placing him down on any such vertex. Inductively, assume it is the robber's turn to move, and that the robber is on $\vec{0} = (0, 0, \dots, 0)$ by shifting the coordinates of the cops and robbers using the following method. If the robber is on the vertex $(r_1, r_2, \dots, r_n)$, then the new coordinates of a cop or robber will be $(v_1 - r_1, v_2 - r_2, \dots, v_n - r_n)$, where v_i is the ith coordinate of that cop or robber. Then the possible points the robber can move to, other than the vertex he is on, are the vertices that have exactly one nonzero coordinate. We will call this set N, and note that N has cardinality $2n$. We will show that there is always at least one point of N which has distance at least two from every cop, and then define his strategy τ to always move to such a point.

We claim that any one cop can guard at most two vertices of N. Since the robber is on $\vec{0}$, we always assume no cop is on $\vec{0}$. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, where the ith coordinate of e_i is 1. Suppose first that a cop is guarding two points that are nonzero on the same coordinate. Without loss of generality, we may assume the coordinates are e_1 and $-e_1 = (-1, 0, \dots, 0)$. We claim that for a cop to guard both of these points, they must either be on e_1 or $-e_1$.

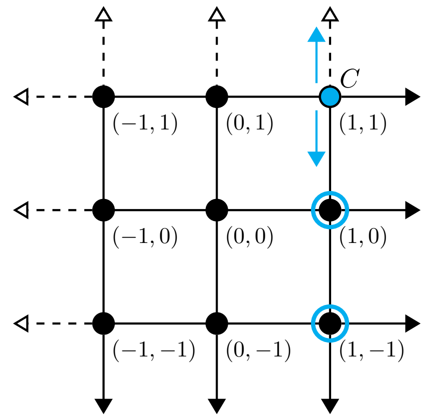
If a cop is guarding e_1, she must be on a point of the form $(\pm 1, 0, 0, \dots)$ or $(1, y_2 \dots, y_n)$, where $y_i \neq 0$ for exactly one $2 \leq i \leq n$. The first case is what we claimed, so suppose we are in the latter case. Then the shortest path from $(1, y_2, \dots, y_n)$ to $-e_1$ is of length 2. This is true, since she must move once in the first coordinate, and once in the ith coordinate. Thus, a cop on $(1, y_2, \dots, y_n)$ is not guarding $(-1, 0, \dots, 0)$. Therefore, for a cop to be guarding both $(1, 0, 0, \dots, 0)$ and $(-1, 0, 0, \dots, 0)$, she must be on $(\pm 1, 0, \dots, 0)$. Let $x \in N$ be such that the first coordinate of x is zero. Since the cop is on $(\pm 1, 0, 0, \dots, 0)$, the shortest path to x must be of length 2, using a similar argument as above. Therefore, the cop cannot guard $e_1, -e_1$, and another point of N.

Now suppose a cop is guarding two points of N which do not share a nonzero coordinate, and the cop is not on $\vec{0}$. Without loss of generality, we may assume the cop guards e_1 and e_2. We claim a cop can guard these two points only if she is on $(1, 1, 0, \dots, 0)$. Suppose the cop is guarding e_1. Then she must be on a vertex of the form $(\pm 1, 0, 0, \dots)$ or $(1, y_2 \dots, y_n)$, where $y_i \neq 0$ for exactly one $2 \leq i \leq n$. Neither of the points in the first case neighbor $(0, 1, 0, \dots, 0)$, so a cop on $(\pm 1, 0, \dots, 0)$ is not guarding e_2. Now suppose a cop is on a point of the form $(1, y_2, \dots, y_n)$ as above. If $y_2 = 1$, then the cop is guarding both of the points as we claimed. If $y_2 = -1$, then the shortest path from $(1, -1, 0, \dots, 0)$ to e_2 is of length 2, contradicting that the cop was guarding $(0, 1, \dots, 0)$. Otherwise, $y_2 = 0$ and $y_i \neq 0$. The shortest path between $(1, y_2, y_3, \dots)$ and $(0, 1, 0, \dots, 0)$ has length at least 2, so she cannot be guarding e_2,

Thus, a cop can guard at most two points of N, meaning the $n - 1$ cops can guard at most $2n - 2$ points of N. Therefore, we define τ for the robber by having him move to any unguarded element of N on each of his turns.

We now show $C(G) \leq n$, which will imply $C(G) = n$. We proceed by induction. Suppose now that n cops have a winning strategy τ^* in T^n. We show that $n + 1$ cops can catch the robber in T^{n+1}. Let $T = \{(a_1, a_2, \dots, a_n, 0) \in T^{n+1} : a_j \in \{-1, 0, 1\}\}$. Define $\pi : T \to T^n$ by $\pi\left((a_1, a_2, \dots, a_n, 0)\right) = (a_1, a_2, \dots, a_n)$. For a point $x \in T$, let $x^* = \pi(x)$. We note x, y are neighbors in T if and only if x^* and y^* are neighbors in T^n.

Initially in T^n, suppose c_i starts on h_i^*. Have τ start c_i on h_i, where h_i is the point of T such that $\pi(h_i) = h_i^*$, and c_{n+1} on h_n. Now the robber starts the game



with $v_r = (v_1, \dots, v_n, v_{n+1})$. At each stage of the game, define the "fake robber" f to be on $v_f = (v_1, v_2, \dots, v_n, 0)$. Thus, v_f will always be in T. The fake robber moves according to the rules of the game since the robber does. Inductively, assume that at time t, f is on vertex v_f and each c_i is on some vertex v_i of T; furthermore, in T^n, the robber is on v_f^*, and each c_i is on v_i^*. If τ^* moves c_i from v_i^* to h_i^*, then have τ move c_i from v_i to h_i, and move c_{n+1} to h_n.

Since τ is winning, after some finite number of turns, a cop and robber will be on the same space in the T^n game. If, in the T^n game, the robber was caught by c_i on vertex z^*, then in T, c_i and f will be on z. Then the first n coordinates of the robber and c_i will be the same, and the last coordinate will differ by at most one. At this stage of the game, define τ to have c_i follow the robber in the following sense. If the robber moves from the vertex $(v_1, v_2, \dots, v_k, \dots v_{n+1})$ to $(v_1, v_2, \dots, v_k', \dots, v_{n+1})$, have c_i move from $(v_1, v_2, \dots, v_k, \dots, 0)$ to $(v_1, v_2, \dots, v_k', \dots, 0)$. This will inductively mean the first n coordinates of c_i and the robber's vertices will be the same. If the robber moves in the last coordinate, then c_i can catch him, since the robber either moved onto c_i, or moved onto a vertex neighboring c_i. Thus, the robber can never move on the last coordinate without getting caught.

There are now n cops who aren't following the robber. Relabel these cops to be $c_1, c_2, \dots, c_n$. The robber's coordinates are of the form $(v_1, v_2, \dots, v_{n+1})$, where $v_{n+1} \in \{-1, 1\}$. Define τ to move $c_1, c_2, \dots, c_n$ so that each cop's last coordinate is v_{n+1}. Let $T_r = \{(a_0, a_1, \dots, a_n, v_{n+1}) \in T^{n+1} : a_j \in \{-1, 0, 1\}\}$, and let the map $p : T_r \to T^n$ be defined by $p(a_1, a_2, \dots, a_n, r_{n+1}) = (a_1, a_2, \dots, a_n)$. Each cop c_i is on some vertex v_i of T_r. Using Lemma 3.2, we may assume that τ^* initially places c_i down on $p(v_i)$. Now we can repeat the algorithm above, but replacing π with p to define τ for $c_1, c_2, \dots, c_n$. We now argue that τ is a winning strategy. Since τ^* is winning, the cops will win in T^n, so there will be a cop c_i on the same vertex z^* as the robber in T^n. Since the robber was not allowed to move in the last coordinate, then he must have also been caught in T^{n+1} by c_i on the vertex z.

□

Theorem 3.4. $C(T_2^n) = n + 1$

Proof. By Theorems 2.2 and 2.3, $n \le C(T_2^n) \le n + 1$, so it is enough to show $C(T_2^n) > n$. We will adopt the convention that if v is a supervertex of T_2^n, and $v^* = (v_1, \dots, v_n)$ is the corresponding vertex of T^n, then v will have coordinates

$(v_1, \ldots, v_n)$. If n cops c_i place down on vertices v_i of T_2^n, then each cop can be within 1 of at most 2 supervertices, meaning there are at most $2n$ supervertices that the robber cannot place down on or else he will lose immediately. Since there are 3^n supervertices, the robber can place down safely on a supervertex. Define the robber's strategy τ by having the robber place down on any of these safe supervertices. We now define τ to not move the robber if there is no cop next to him.

We now show that if the robber is on a supervertex v, and there is a cop within distance one of him in T_2^n, then there is always a supervertex v' that is distance two from the robber in T_2^n which is not within distance two of a cop. Let N be the set of supervertices which are distance two from the robber. Then N has size $2n$.

If the robber is on a supervertex v, and a cop c_i is within one of him, then c_i must be in a tunnel, and the two endpoints of the tunnel are a vertex of N, and v. Thus, a cop next to a robber can only guard one vertex of N. If any other cop c_j is in a tunnel, she is next to two supervertices x and y, and any other supervertex has at least distance two from both x and y. Thus, she can only be within two moves of at most two points of N.

It remains to show that a cop on a supervertex can be within two moves at most two vertices of N. We will say a cop guards a supervertex w if she is within two moves of w. If a cop on a supervertex is guarding e_i and $-e_i = (0, 0, \ldots, -1, 0, \ldots, 0)$, then we can use a similar argument as in the proof of theorem 3.3 to show that it would take her more than two moves to move to any point of N other than e_i or $-e_i$, as it always takes one move to get through a tunnel, and then a second move to move to a supervertex. Similarly, a cop guarding e_i and e_j for $i \neq j$ cannot be guarding another point of N.

We have shown that if there is a cop next to the robber, she is guarding at most one point of N, and any other of the $n-1$ cops can guard at most two points of N. Thus, if there is some i such that w_i is a neighbor of v, then the cops can guard at most $1+2(n-1) = 2n-1$ vertices of N. This means there will be a vertex v' that is within distance two of the robber, and there is no cop within distance two of v'. v and v' are adjacent supervertices We now define τ to move the robber to from v to v' in two moves

τ is winning, because in order for the cops to win, one cop must move to a neighbor of the robber, but whenever a cop moves to a neighbor of the robber, the robber can commit two moves to moving to a safe supervertex. □

4. Monotonicity

In this section, we will prove that if G is a graph, then the cop number is constant for some tail of k-tunnel graphs of G.

Definition 4.1. Suppose G is a graph, and k, m are positive integers. then G_k has $k-1$ points between tunnel vertices, and $G_{1+m(k-1)}$ has $m(k-1)$ tunnel points between supervertices. Suppose x, y and x^*, y^* are two corresponding neighboring supervertices of $G_{1+m(k-1)}$ and G_k respectively, i.e. they came from the same vertices of G. Let $v_1, v_2, \ldots, v_{m(k-1)}$ be the tunnel vertices between x and y, and let $u_1, u_2, \ldots, u_{k-1}$ be the tunnel vertices between x^* and y^*, where v_1 neighbors x, and u_1 neighbors x^*.

We define a map $p : G_{1+m(k-1)} \to G_k$ as follows:

$$p(v) = \begin{cases} x^* & \text{if } v = x \\ y^* & \text{if } v = y \\ u_i & \text{if } v = v_{m(i-1)+j}, \text{ where } 1 \leq j \leq m \end{cases}$$

We can now define an equivalence relation P in $G_{1+m(k-1)}$ by x and y in $G_{1+m(k-1)}$ are in the same P equivalence class if $p(x) = p(y)$.

FIGURE 4. A 3-tunnel shown on the left, and a 5-tunnel on the right

The following lemma is a special case of Lemma 4.4, but the argument is quite a bit simpler, and omits a fair number of the factors that make lemma 4.4 complicated.

Lemma 4.2. *If* $C(G) = n$*, and* $C(G_k) = n+1$*, then* $C(G_j) = n+1$ *for any* $j \geq k$ *with* $j \cong 1 \mod k-1$

Proof. Suppose $j = m(k-1)+1$, where $m \geq 1$. Given a point x in G_j, let $x^* = p(x)$. We say a point x in a P equivalence class is *extreme* if it shares an edge with a point in a different equivalence class. Suppose τ^* is a strategy for the robber to evade n cops in G_k. We will use it to show the robber can evade n cops in G_j, which will show $C(G_j) = n+1$

We will define a strategy for the robber in which the robber can commit to making m moves at a time. If the robber makes an m-move, it is possible for a cop to move from a P class next to a supervertex to a supervertex, and then over to a different P class. This means that If we use p to map the cops' coordinates, then a cop can move over two vertices in G_k, which would be an illegal move. We therefore define a map π for each cop that will "fix" this issue. We will have a partial map defined for specific values of t, which we will refer to as *good* times, $t \to t^*$ mapping specific times in the G_j game to times in the G_k game. Suppose t is in the domain of the function. Then $t+1$ or $t+m$ will be in the domain of the function depending on whether the robber makes a 1-move or an m-move. We accordingly set $(t+1)^*$ or $(t+m)*$ to be t^*+1.

Let $\pi_i : G_j \times \mathbb{N} \to G_k$ be defined recursively. $\pi_i(v, 0) = v^*$, where v is the initial position of c_i. Now suppose c_i is on v at some good time s and moves to w when the robber commits to a move. Then if the cop completes this (1 or m) move at time t, we let:

$$\pi_i(w,t) = \begin{cases} w^* & \text{if } w \text{ is a supervertex corresponding to a point in } G. \\ w^* & \text{if } \pi_i(v,s) = v^* \text{ or } v \text{ and } w \text{ are in the same equivalence class.} \\ v^* & \text{if } \pi_i(v,s) \neq v^* \text{ and } v \text{ and } w \text{ lie in different equivalence classes.} \end{cases}$$

We will define a cop on vertex v to be *ahead* if at stage t, $\pi(v,t) \neq v$. i.e. we are projecting a cop to be on a point of G_k which does not correspond to the equivalence class of vertices she is in. We note here that at the initial step of the game, no cop is ahead.

On the initial turn, the cops place themselves down on vertices $v_1, v_2, \ldots, v_n$ in G_j. Place the cops down on $v_1^*, v_2^*, \ldots, v_n^*$ in G_k. Let $v^* = \tau^*(r, v_1^*, \ldots, v_n^*, 0)$. We let $\tau(r, v_1, \ldots, v_n) = v$, where v is an extreme point such that $p(v) = v^*$. When the cops make their first move, no cop can catch the robber, or else there was a cop next to the robber in G_j. This would imply there was a cop next to the robber in G_k, violating τ^* being winning.

We now define the robber's strategy recursively. Suppose at turn t, the cops are on $v_1^*, v_2^*, \ldots, v_n^*$ in G_k, the robber is at an extreme vertex r of an equivalence class in G_j, the robber in G_k is on r^*. Let $v^* = \tau^*(r^*, \pi_1(v_1, t), \ldots, \pi_n(v_n, t), t^*)$. Let v be the extreme vertex in $p^{-1}(v^*)$ which is closest to the robber. Then either v and r are neighbors, or there are $m-1$ tunnel vertices $x_1, x_2, \ldots, x_{m-1}$ between v and r. If v and r are neighbors, let $\tau(r, v_1, \ldots, v_n, t) = v$. In the other case, have the robber move from r to v. More precisely, suppose that x_1 is the neighbor of r and we let:

$$
\begin{aligned}
&\tau(r, v_1, \ldots, v_n, t) = x_1 \\
&\tau(x_1, v_1^1, \ldots, v_n^1, t+1) = x_2, \text{ for all neighbors } v_i^1 \text{ of each } v_i. \\
&\ldots \\
&\tau(x_{m-1}, v_1^{m-1}, \ldots v_n^{m-1}, t+m-1) = v, \text{ for all neighbors } v_i^{m-1} \text{ of } v_i^{m-2}.
\end{aligned}
$$

Using the above strategy, we note that the robber either makes a 1-move or an m-move before he consults τ^* for his next move. We will refer to either of these moves as the robber *committing* to a move. Furthermore, he will make at most $k-1$ m-moves in a row before making a 1-move.

At any time good time t, let $R_i(t)$ be the number of good times c_i was ahead since the last good time that c_i was not ahead. If c_i is in a tunnel, let x and y be the supervertices corresponding to that tunnel, and assume that c_i crossed over x to get into the tunnel. We claim that the following hold for all good values of t.

(1) If c_i on vertex v is ahead, then $\pi_i(v, t)$ is a neighbor of v^*.
(2) If c_i on vertex v is ahead, then she is within $m-1$ of the class corresponding to $\pi_i(v, t)$.
(3) If c_i is ahead, then she is at least $(k-1-R_(t))m+2$ away from y.
(4) If c_i is ahead on vertex v, then v^* is closer to y^* than $\pi_i(v, t)$.
(5) If the robber makes a 1-move, then $R_i(t+1) = 0$.
(6) $0 \leq R_(t) \leq k-1$

We prove the claim by induction on the value of $R_i(t)$. Since the robber can only make $k-1$ m-moves in a row, (5) being true will always imply (6). If $R_i(t) = 0$, 1-4 and 6 are trivially satisfied, and (5) is easy to check. First suppose that c_i gets ahead, making $R_i(t) = 1$. Then she must have crossed a supervertex which would have taken at least one move, and then had at most $m-1$ moves left to head into the tunnel class. She therefore must be at least $m(k-2)+2$ away from y, and at most $m-1$ from x, verifying (2) and (3). By definition of π_i, $\pi_i(v, t) = x^*$, where v is the vertex the cop ended on, showing (1) and (4). Since a cop just got ahead, (5) does not apply. If c_i does not get ahead, then the hypotheses are vacuously true.

Inductively suppose the claims are satisfied for the current nonzero value of $R_i(t)$, and that $R_i(t) < k-1$. If the robber makes a 1-move, then by (2), (5) is true, and (1)-(4) become vacuously true. So we assume at time t the robber makes an m-move. If c_i is ahead at time t, then by (3), c_i is at least $(k-1-R_i(t))m+2$ away from y, so in m-moves, the cop can be at least $(k-1-R_i(t))m+2-m =$

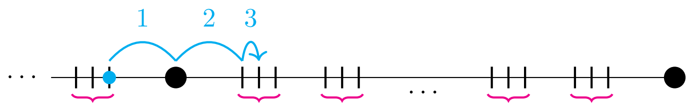


FIGURE 5. Cop over supervertex

$(k-1-(R_i(t)+1))m+2 = (k-1-R_i(t+m))m+2$ away from y, verifying (3) at time $t+m$.

Every P class is of size m, so c_i cannot travel more than one P class in one m-move, so (1) holds; furthermore, since c_i was within $m-1$ of $\pi_i(v,t)$, after m moves, she takes at least one move to get to the extreme vertex of the class she is in, and then $m-1$ moves into the next class, implying (2). We can use similar reasoning to show that (4) is true. If a cop was not ahead at time t, and then got ahead, we can repeat the arguments of the case $R_i(t) = 1$. Finally if c_i was not ahead at times t or $t+m$, then (1)-(4) are vacuously true.

If $R_i(t) = k-1$, then the robber must make a 1-move. By (2), no cops will be ahead. Thus, the new value of $R_i(t)$ will be 0, and (1)-(4) vacuously hold again.

We now show the robber can evade the cops indefinitely by showing that he will not be caught whenever he commits to moving at some good time t. To do this, we will show that for every value of $R_i(t)$, the robber can safely commit to making a move without being caught by c_i.

Suppose that $R_i(t) = 0$; then if c_i is on v_i, $\pi_i(v_i,t) = v_i^*$. Suppose the robber commits a move from vertex v to w, and c_i is on v_i. Whether the robber makes a 1-move or an m-move, it must have been the case that $\tau^*(v^*, v_1^*, \ldots, v_n^*, z) = w^*$. If the robber made a 1-move, c_i cannot catch him or else $v_i^* = \pi_i(v_i,t)$ must have been w^* or a neighbor of w^*. Thus, in G_k, the robber was caught on w^* by c_i, violating τ^* being winning. Next suppose the robber makes an m-move. We first show that the robber cannot be caught by a cop who doesn't get ahead. Towards a contradiction, suppose the robber gets caught by c_i who does get ahead. Then c_i must have been in a P class, crossed over a supervertex x, and moved at most $m-1$ vertices into a different P class. If the robber was caught on x, then he started in one of the P classes adjacent to x. In either case, he moved to x^*, which was adjacent to v_i^*, contradicting τ^* being winning; therefore, the robber must have been caught in the P class that c_i moved to. Since the robber made an m-move, and always moves to the closest vertex of a P class, he is m away from x. Since c_i must be within $m-1$ of x after her move, she could not have caught the robber.

Suppose that $1 \le R_i(t) \le k-2$ and that if c_i is ahead, then they crossed over the supervertex x_i, and the supervertex on the other side of the tunnel is y_i. Then c_i is within $m-1$ of the class corresponding to $\pi_i(v_i,t)$, and is at least $(k,-1-r)m+2$ away from the y_i. If the robber makes a 1-move or an m-move, he will not get caught by a cop who is not ahead using the same argument as in the $R_i(t) = 0$ case, so it suffices to show that the robber will never be caught by an ahead cop.

If the robber makes a 1-move, then for any cop c_i, $v_i^* = \pi_i(v_i, t+1)$, i.e. the cop is projected to the correct place. Thus, τ^* would not move the robber to v_i^*, as that would violate τ^* being winning. Therefore, the robber does not get caught if he makes a 1-move. Assume next that the robber makes an m-move. Towards a contradiction, suppose he is caught by an ahead cop c_i. There are two cases; either the robber moved to a supervertex, or he moved to a tunnel class of P.

First assume the robber moved into a tunnel class to vertex w. If he was caught in the class that he started in, then c_i must have been in an adjacent class, say on vertex v_i. By the claims above, $\pi(v_i, t)$ must be a neighbor of v_i^*, implying that in G_k, the robber and c_i were at most distance two apart, and τ^* moved the robber to v_i^*. This contradicts τ^* being winning; therefore, we may conclude that the robber was caught on w. Using similar reasoning as above, c_i did not start in the class the robber moved to. By (2), c_i was not on the extreme vertex distance m away from w, so she could not have caught the robber in m moves. Therefore, if the robber moves to a tunnel class of P, he will not be caught.

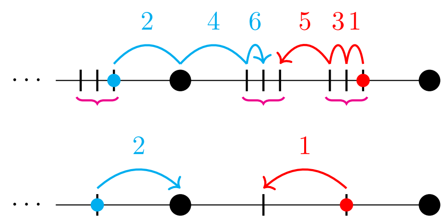


FIGURE 6. We see the projected game on the bottom is not violated

Finally suppose w is a supervertex. The robber cannot be caught by c_i, as if w is opposite c_i, then the cop must be at least $(k-1-R_i(t))m+2 \geq (k-1-(k-2))m+2 = m+2$ away from w. If c_i used w to get ahead, then she can't be within m of the supervertex, or else she would be on w^* in G_k, once again violating τ^* being winning.

If $R_i(t) = k-1$, then the robber is forced to make a 1-move, in which case we have already showed he will be safe. □

We now work towards proving a similar lemma which will allow us to prove our monotonicity result. Suppose that k is an integer. Then G_{2k} has $2k-1$ tunnel vertices. If a and b are positive integers with $a > b$, then we will enlarge each of the classes of G_{2k} by starting with the first tunnel point, making it an equivalence class of size a, moving to the next tunnel point, and making it size b.

Definition 4.3. Suppose G is a graph, and a, b and k are positive integers with $a > b$.Suppose x,y and x^*, y^* are two corresponding neighboring supervertices of $G_{ak+b(k-1)}$ and G_{2k} respectively. Let $v_1, v_2, \ldots, v_{ak+b(k-1)-1}$ be the tunnel vertices between x and y, and let $u_1, u_2, \ldots, u_{2k-1}$ be the tunnel vertices between x^* and y^*, where v_1 neightbors x and u_1 neighbors x^*. Let $f(x) : G_{ak+b(k-1)} \to G_{2k}$ be defined by:

$$p(v) = \begin{cases} x^* & \text{if } v = x \\ y^* & \text{if } v = y \\ u_i & \text{if } i = 2m+1 \text{ and } v = v_{ma+mb+j}, \text{ where } 1 \leq j \leq a \\ u_i & \text{if } i = 2m \text{ and } v = v_{ma+(m-1)b+j}, \text{ where } 1 \leq j \leq b \end{cases}$$

Lemma 4.4. *If $C(G) = n$ and $C(G_{2k}) = n+1$, then $C(G_{ak+b(k-1)+1}) = n+1$ for any $a, b \in \mathbb{N}^+$ such that $a > b$.*

Proof. Assume τ^* is a winning strategy for the robber to evade n cops in G_{2k}. Let $j = ak + b(k-1) + 1$, and for any vertex v of G_j, let $v^* = p(v)$. Suppose each cop

c_i places down on v_i; then in G_{2k}, place c_i on v_i^*. Let $v^* = \tau^*(r, v_1^*, v_2^*, \ldots, v_n^*, 0)$. In G_j, let $\tau(r, v_1, v_2, \ldots, v_n, 0) = v$, where v is an extreme vertex corresponding to v^*. v is not in a class next to a class a cop is in, or else τ^* would have placed the robber directly next to the corresponding cop in G_{2k}, so no cop can catch him on the first move.

Our strategy for the robber will allow him to commit to a 1-move, an a-move, or a b-move. To do this, we will map certain special values of times t in G_j to times t^* in G_k. 0 will be called a *good time.* If t is a good time and the robber makes a z-move, where $z \in \{1, a, b\}$. We will then say $(t+z)$ is also a good time and let $(t+z)^* = t^* + 1$.

Similar to Lemma 4.2, we will define a partial function $\pi_i : G_j \times \mathbb{N} \to G_{2k}.\pi_i$ will compromise between projecting c_i to G_{2k}, and making sure that every move in the smaller game is a legal one. Let $\pi_i(v, 0) = v^*$, where v is the initial position of c_i. Inductively, assume that π_i has been defined for a good time s and vertex v. Then if at the next good time t, c_i is on w, we define π_i as follows:

$$\pi_i(w,t) = \begin{cases} w^* & \text{if } w \text{ is a supervertex corresponding to a point in } G. \\ w^* & \text{if } v \text{ and } w \text{ are in the same equivalence class.} \\ w^* & \text{if } \pi_i(v,s) = v^*\text{, and } w^* \text{ is a neighbor of } v^* \\ w^* & \text{if } \pi_i(v,s) = w^* \text{ and } w^* \neq v^* \\ x^* & \text{if } \pi_i(v,s) = v^*\text{, and } x^* \text{ is the unique class between } w^* \text{ and } v^* \\ v^* & \text{if } \pi_i(v,s) \neq v^*,\ \pi_i(v,s) \neq w^* \text{ and } v \text{ and } w \text{ lie in different equivalence classes.} \end{cases}$$

We define a a cop c_i on vertex v_i as being *poorly projected* at a good time t if $\pi_i(v_i, t) \neq v_i^*$. We will say she is *ahead* if she started on an a-class, moved over a supervertex, and then moved into a different a-class to become poorly projected. A cop can get ahead on either an a-move or a b-move. A cop c_i is *pseudoahead* if she started on an a-class, moved over a b-class, and then moved into a different a-class to become poorly projected. We note that a cop can only get pseudoahead on an a-move, meaning she would be at most $a - b$ away from the b-class she crossed. Since the next move is at most a b-move, she will no longer be pseudoahead at the end of the next move.

We will now define the robber's strategy recursively. Suppose at a good time t, the cops are on $v_1^*, v_2^* \ldots, v_n^*$ in G_{2k}, the robber is at an extreme vertex r of an equivalence class in G_j, and the robber in G_{2k} is on r^*. Let $v^* = \tau^*(r^*, \pi_1(v_1, t), \ldots, \pi_n(v_n, t), t^*)$. Let v be the extreme vertex in $p^{-1}(v^*)$ which is closest to the robber. Then there are three possibilities. v and r are neighbors, there are $a-1$ tunnel vertices between v and r, or there are $b-1$ tunnel vertices between v and r. If v and r are neighbors, let $\tau(r, v_1, \ldots, v_n, t) = v$. Otherwise, have the robber move from r to v. We will define τ in the case that the robber makes an a-move, and the case for a b-move will be similar. Suppose that $x_1, \ldots, x_{a-1}$ are the tunnel vertices between r and v, and that x_1 is a neighbor of r. Then we define τ by:

$$\tau(r, v_1, \dots, v_n, t) = x_1$$
$$\tau(x_1, v_1^1, \dots, v_n^1, t+1) = x_2, \text{ for all neighbors } v_i^1 \text{ of each } v_i.$$
$$\dots$$
$$\tau(x_{a-1}, v_1^{a-1}, \dots v_n^{a-1}, t+a-1) = v, \text{ for all neighbors } v_i^{a-1} \text{ of } v_i^{a-2}.$$

We note that the robber will make at most $2k-1$ non-1-moves in a row, and will never make two consecutive a or b moves in a row.

For any good time t, let $R_i(t)$ be the number of good times c_i since the last good time that c_i was not poorly projected. Let $F_i : \mathbb{N} \to \mathbb{N}$ be defined by $F_i(t) = t - k$, where k is the greatest good time with $k \leq t$ such that $R_i(k) = 0$. If c_i is on vertex v in a tunnel, let x_i and y_i be the supervertices corresponding to that tunnel, and assume that c_i crossed over x_i to get into the tunnel. Then we claim the following hold for all good times t.

(1) $\pi_i(v,t)$ is a neighbor of v^*
(2) If the move the robber made at the good time previous to t was an a-move, then c_i is within $a-1$ of the class corresponding to $\pi_i(v,t)$.
(3) If the move the robber made at the good time previous to t was b-move, then c_i is within $b-1$ of the class corresponding to $\pi_i(v,t)$.
(4) If c_i is ahead, then she is at least $ak + b(k-1) - F_i(t) + 2$ away from y_i.
(5) If c_i on vertex v is ahead, then v^* is closer to y^* than $\pi_i(v,t)$.
(6) If the previous move was a 1-move, then $R_i(t) = 0$.
(7) $0 \leq R_i(t) \leq 2k-1$

We prove the claim by induction based on the value of $R_i(t)$. We will assume the induction assumptions hold at a good time t and show they hold at the next good time s. At any stage, if $(6)_t$ is true, then $(7)_t$ will also be true, since the robber can make at most $2k-1$ non 1-moves in a row. Similarly, if $(1)_t, (2)_t$, and $(3)_t$ are true at a good time t by induction, then if the next move is a 1-move, $(6)_{t+1}$ will be true.

If $R_i(t) = 0$, then $(1)_t - (5)_t$ are vacuously true, and $(6)_t$ is simple to check. If $R_i(t) = 2k+1$, then we know the robber is forced to make a 1-move, so by $(6)_t$, $R_i(t+1)$ will be zero.

We next consider the case where c_i is not ahead and gets ahead, i.e., $R_i(t) = 0$ to $R_i(s) = 1$. Since $R_i(t) = 0$, if the robber makes an a-move, then if c_i gets ahead, she started on an a-class, made at least one move to the neighboring supervertex, and had at most $a-1$ moves left to move into the next class. If the previous move was a b-move, then she could only move $b-1$ times in that class, meaning $(2)_s$ and $(3)_s$ are satisfied. In either case, the distance from x_i to y_i is $ak+b(k-1)+1$, and she moved at most $a-1$ steps away from x_i. She is therefore at most $ak+b(k-1)-(a-1)+1 > ak + b(k-1) - F_i(1) + 2$ moves away from y_i, satisfying $(4)_s$. $(5)_s$ is true, since $\pi_i(v_i, s)$ is x_i.

Assume $1 \leq R_i(t) < 2k-1$. If the move the robber made at the last good time before t was an a-move, then by $(2)_t$ and $(4)_t$, c_i is within $a-1$ of the class corresponding to $\pi_i(v_i,t)$, and is at least $ak + b(k-1) - F(t) + 2$ away from y_i. If the robber makes a b-move, then $(2)_{t+b}$ will be vacuously true. By $(4)_t$, c_i won't move to y_i, as $ak + b(k-1) - F(t) + 2 > b$; therefore, she must be moving to a class of size at least b. It will take c_i at least one move to get to the extreme vertex of the class she is in, and then she can move at most $b-1$ into the next class, so $(1)_{t+b}, (3)_{t+b}$ and $(5)_{t+b}$ will be true. Since the cop was allowed at most b moves, she is at least $ak + b(k-1) - F(t) + 2 - b = ak + b(k-1) - F(t+b) + 2$ away from y_i, implying $(4)_{t+b}$.

Now suppose the move that was made at the last good time before t was a b-move. By $(3)_t$, c_i is within $b-1$ of the class corresponding to $\pi_i(v,t)$. If the robber makes an a-move, then $(3)_{t+a}$ will be vacuously true. If c_i is on an a-class, it will take her at least $a-(b-1) = a-b+1$ moves to get to the extreme vertex of her class, giving her at most $b-1$ moves left to move into the next class. This implies she can at most move into the next b-class, so $(1)_{t+a}$, $(2)_{t+a}$ and $(5)_{t+a}$ will be true. If c_i was in a b-class, then it will take her at least one move to get to the end of the b-class, and then she has at most $a-1$ moves remaining. In either case, $(4)_{t+a}$ is true, as c_i can be at most $ak + b(k-1) - F(t) + 2 - a = ak + b(k-1) - F(t+a) + 2$ away from the supervertex.

We now show that τ is a winning strategy for the robber. We will do this by showing that if the robber has not yet been caught at a good time t, and for all i, c_i's position in G_{2k} is consistent with τ, then if he makes a 1-move, an a-move, or a b-move, he will be safe for all times between t and the next good time. To do this, we will show he is not caught by c_i based on the value of $R_i(t)$. ***

We first claim that at any good time t, it will never be the case that the robber is in an a or b class C_1, commits to making an a or b-move into a different class C_2, and there is a cop c_i in C_1. The robber must make a 1-move to enter the tunnel, so there was some good time t such that for each c_i, $\pi_i(v_i,t) = v_i^*$. At this time, a cop cannot be in the same class as the robber, or else v_i^* is in the same class as the robber in G_{2k}. From this position, if the robber makes an a-move or b-move into another class, he will be moving towards a supervertex y and away from a supervertex x. If c_i is between the robber and x before the robber makes his a or b move, then $\pi_i(v_i,t)$ will be between the robber's position and x^* in G_{2k}. Similarly, if c_i is between the robber and y when $R_i(v_i,t) = 0$, then $\pi_i(v_i,t)$ is between the robber and y^* in G_{2k}. π_i and v^* in G_{2k} can move at most one class between successive good times, and if v^* is adjacent to $\pi_i(v_i,t)$ for a good time t, then τ will not move the robber into the class corresponding to $\pi_i(v_i,t)$. Thus, for the next good time s, $\pi_i(v_i,s)$ can be in the same class as the robber, but c_i would catch the robber before he could make it to the next class.

If $R_i(t) = 0$, then, $\pi_i(v_i,t) = v_i^*$. If the robber makes a 1-move, c_i cannot catch him, else the robber's position in G_{2k} is the same as c_i, contradicting τ^* being winning. If the robber makes an a-move or a b-move, he will never be caught by a cop who does not get ahead or pseudoahead, or else the robber would have been caught by c_i on $\pi_i(v_i,t) = v_i^*$, meaning that the robber lost while following τ^*. If the robber makes a b-move, then he must be moving to the extreme vertex v of an a-class; furthermore, he will be at least a moves away from a supervertex. If c_i does gets ahead on the robber's b-move, then she will be at most $b-1$ steps into the a-class she moved into to, so she could not have caught him. If the robber makes an a-move to a vertex x, then the class corresponding to x neighbors two a-classes. For c_i to catch the robber, she must either be on x itself, or within a moves of x. In either case, $\pi_i(v_i,t)$ neighbors x^*, so since τ^* is winning, the robber would not have moved to x.

Assume $1 \leq R_i(t) < 2k-1$. We note that for this to be possible, the robber must have previously made an a-move or a b-move at the last good time. If the robber makes a b-move, he must move from a b-class to an extreme vertex x of an a-class. If c_i is ahead at time t, and is in the a-class the robber is moving to, then by $(1)_t$, the robber would be moving to a class that neighbors c_i in G_{2k}, a contradiction. The only other class within b moves of the vertex the robber is moving to is the b-class he is in. If an ahead cop is in this class, then by $(1)_t$, $\pi_i(v_i, t)$ neighbors that class, so τ^* will move the robber away from $\pi_i(v_i, t)$. It is possible that a cop c_i is pseudoahead, but if that is the case, $\pi_i(v_i, t)$ is a neighbor of the a-class she is in, and c_i is at most $b-1$ moves into that a-class. Since the robber is on a b-class, he won't move to the a-class c_i is on, as in G_{2k}, the corresponding class neighbors $\pi_i(v_i, t)$. Thus, the robber can never be caught by a pseudoahead cop.

Suppose now that the robber makes an a-move. Then he is either moving towards a b-class or a supervertex. If he's moving towards a b-class B then for c_i to catch him, she must be within a of the point he is moving to. She will not be in B, or else by $(1)_t$, $\pi_i(v_i, t)$, will be a neighboring class, meaning τ^* moved the robber into the cop in G_{2k}. If she is in the a-class A neighboring B, and $\pi_i(v_i, t)$ is not the point corresponding to B, then by $(3)_t$, she is within $b-1$ of $\pi_i(v_i, t)$. It will take her at least $a-(b-1)$ moves to leave A, so she will have $b-1$ moves left to move into B. This means she cannot catch the robber. If the robber is moving to a supervertex x, then by $(4)_t$, any ahead cop whose projection is not x is at least

$$\begin{aligned} ak + b(k-1) - F(t) + 2 &>= ak + b(k-1) - (a(k-1) + b(k-1)) + 2 \\ &= a + 2 \end{aligned}$$

moves away from x, so she cannot catch the robber in a moves.

If $R_i(t) = 2k-1$, then the robber must make a 1-move, so he will not get caught using the same argument as in the $0 \leq R_i(t) \leq 2k-1$ case.

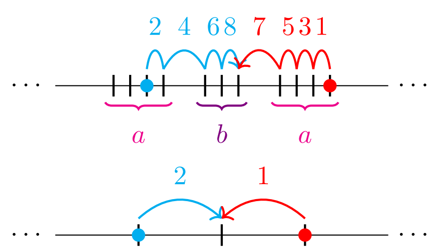


FIGURE 7. We see the projected game strategy on the bottom is violated

□

With the previous two proofs, we can prove that the cop number of tunnel graphs is eventually monotonic in the sense that if $C(G) = n$, then either for all $k \in \mathbb{N}, C(G_k) = n$, or there is some $m \in \mathbb{N}$ such that for all $k > m, C(G_k) = n+1$. We will call a number m *good* if $C(G) = n$, but $C(G_m) = n+1$.

Lemma 4.5. *If $C(G) = n$, and $C(G_{2k}) = n+1$, for some integer k then $C(G_z) = n+1$ for $z \geq 2k^2 - 2k + 2$*

Proof. By lemma 4.2 and lemma 4.4, we have that $C(G_{ak+b(k-1)+1}) = n+1$ for any $a \geq b > 0$. Lemma 4.4 allows us to substitute $a > b$, while lemma 4.2 allows us to let $a = b$.

We claim any number greater than or equal to $2k^2 - 2k + 2$ is good. To do this, we will first argue that every number in the interval $[2k^2 - 2k + 2, 2k^2 - k + 1]$ equals $ak + b(k-1) + 1$ for some a and b, where $a \geq b > 0$. Since this interval is a complete set of representatives mod k, by increasing a it is clear we can represent any integer $z \geq 2k^2 - 2k + 2$ as $ak + b(k-1) + 1$, where $a \geq b > 0$. If $a = b = k$, then we get that $2k^2 - k + 1$ is of the desired form. If $b \in \{1, 2, \ldots, k-1\}$, let $a = 2k - b - 1 > b$. Then

$$(2k - b - 1)k + b(k-1) + 1 = 2k^2 - k - b + 1$$

Since $0 \leq b \leq k - 1$, we get that

$$2k^2 - 2k + 2 \leq 2k^2 - k - b + 1 \leq 2k^2 - k + 1$$

Therefore, as b ranges over the interval $[1, k]$, all of the numbers in $[2k^2 - 2k + 2, 2k^2 - k + 1]$ are represented.

□

If the cop number first goes up at an even number, we can just use the above bound. If the cop number is odd, we can double it first, and then apply the above theorem.

Theorem 4.6. *If $C(G) = n$, and $C(G_k) = n+1$. Then $C(G_m) = n+1$ for each $m \geq 2k^2 - 2k + 2$.*

Proof. Since $C(G_k) = n+1$, $C(G_{2k}) = n+1$ by corollary 2.5; thus we may apply Lemma 4.5. □

The bound in 4.6 is not always optimal. For example, if $C(G_3) = n+1$, then Theorem 4.6 states that we are guaranteed all numbers $m \geq 14$ are good. But in fact, we can show all $m \geq 9$ are good. To see this, lemma 4.2 states that $C(G_{1+2k}) = n+1$ for any natural number k. Thus, any odd number after 3 is good, so in particular, every odd prime is good. Thus, any multiple of an odd prime is good by Corollary 2.5. On the other hand, we claim 16 is good. Since 3 is good, 6 is also good. But then, $6 + 2(6-1) = 16$ is good. Therefore, we see that any number bigger than 8 must be good, because it is either divisible by an odd prime, or is divisible by 16.

References

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