AD^+, Derived Models, and Σ_1-Reflection

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Let AD^+ be the theory $AD + DC_{\mathbb{R}} +$ "Every set of reals is ∞-Borel"$+$ "Ordinal Determinacy". For any $\Gamma \subseteq P(\mathbb{R})$, let $M_\Gamma = \cup\{m \mid m$ is transitive and $\exists E, F \subseteq \mathbb{R}\times\mathbb{R}$ ($E, F \in \Gamma$ and $(\mathbb{R}/E, F) \cong (m, \in))\}$. We'll prove the following theorems:

Theorem 1. *(Woodin) Assume $ZF + AD + V = L(P(\mathbb{R}))$. Then the following are equivalent:*

1. *AD^+*
2. *Letting $\mathcal{S} = \{B \subseteq \mathbb{R} \mid B$ is Suslin co-Suslin$\}$, $M_{\mathcal{S}} \prec_{\Sigma_1} V$.*

Let us call the statement in (2) above "Σ_1-reflection" to Suslin co-Suslin.

Theorem 2. *(Woodin) Assume $ZF + AD^+ + V = L(P(\mathbb{R}))$, then*

1. *Σ^2_1 has the scale property.*
2. *$M_{\mathbf{\Delta^2_1}} \prec_{\Sigma_1} V$.*

Proof. The theorem follows immediately from Theorem 1 and lemma 7.2 in [**?**], whose proof is essentially due to Woodin. $\square$

In the course of proving Theorem **??**, we shall prove part of the determinacy-to-large-cardinals direction of the Derived Model Theorem. Let λ be a limit of Woodin cardinals, and G be V-generic over $Col(\omega, <\lambda)$. We set

$$\mathbb{R}^*_G = \cup_{\alpha<\lambda}\mathbb{R}^{V[G|\alpha]},$$

$$Hom^*_G = \{p[T] \cap \mathbb{R}^*_G \mid \exists\alpha < \lambda(T \in V[G|\alpha], V[G|\alpha] \models T \textit{ is } \lambda\text{-absolutely complemented})\},$$

$$\mathcal{A}_G = \{A \subset \mathbb{R}^*_G \mid A \in V(\mathbb{R}^*_G) \textit{ and } L(A, \mathbb{R}^*_G) \models AD^+\}, \mathrm{where V}(\mathbb{R}^*_G) = \mathrm{HOD}^{\mathrm{V[G]}}_{\mathrm{V}\cup\mathbb{R}^*_G\cup\{\mathbb{R}^*_G\}}.$$

Theorem 3. *(Woodin) Assume $ZF + AD^+ + V = L(P(\mathbb{R}))$. Suppose also that if $AD_{\mathbb{R}}$ holds, then Θ is singular. Then there is a set X in some generic extension of V such that setting $M = L[X]$, then*

1. *for some λ, $M \models ZFC + \lambda$ is a limit of Woodins;*
2. *for some M-generic G over $Col(\omega, <\lambda)$:*
   - *$V = L(\mathcal{A}_G, \mathbb{R}^*_G)$, and*

- $Hom^*_G = \{B \subseteq \mathbb{R}^*_G \mid B$ *is Suslin co-Suslin in* $V\}$.

Remark 4.
- The model $L(\mathcal{A}_G, \mathbb{R}^*_G)$ as in 2 of the previous theorem is called the "new" derived model to distinguish it from the "old" derived model which is $L(Hom^*_G, \mathbb{R}^*_G)$.
- [**?**] shows that if $V \vDash AD^+ +$ "there is a largest Suslin cardinal", then we have the same conclusions as those of Theorem **??**. What we handle here is the case that $AD_{\mathbb{R}}+$"Θ is singular" holds in V.
- Characterization of derived models is one of the main themes in this paper. We want to answer the question: Is every model of AD^+ a derived model? Theorem **??** and the previous remark answer this question positively for the "no largest Suslin cardinal + Θ singular" and the "largest Suslin cardinal" cases. Woodin has shown that if $V \vDash AD_{\mathbb{R}} + \Theta$ is regular, then V is elementarily embeddable into a derived model of HOD. A proof of this fact can be found in [**?**]. It's not known whether V is actually a derived model in this case.

The proof of Theorem 3 is implicit in that of the direction (1) $\Rightarrow$ (2) of theorem 1. Before giving the proof of theorem 1, we'll state a couple of corollaries of the above theorems, and a key definition.

Corollary 5. *Let* $M \vDash ZFC + \lambda$ *is a limit of Woodins, and let D be a derived model of M below* λ*; then D satisfies:* Σ_1*-reflection (to Suslin co-Suslin),* Σ^2_1 *has the scale property, and every non-empty* Σ_1 *set* $\mathcal{A} \subseteq P(\mathbb{R})$ *has a* $\boldsymbol{\Delta^2_1}$ *member.*

Proof. Woodin has shown that $D \vDash AD^+$ (see [**?**] for a proof). Applying theorems 1 and 2 gives us the conclusions. □

Corollary 6. *Assume* AD^+*. Then Ult(V,*μ*) is well-founded where* μ *is the Martin measure on Turing degrees.*

Proof. If not, then by Theorem 1, there is $\alpha, \beta < \Theta$ such that $L_\alpha(\mathcal{P}_\beta(\mathbb{R})) \vDash$ "$Ult(V, \mu)$ is ill-founded." Since $DC_{\mathbb{R}}$ holds and there is a surjection from $\mathbb{R}$ onto $L_\alpha(\mathcal{P}_\beta(\mathbb{R}))$, $L_\alpha(\mathcal{P}_\beta(R)) \vDash DC$ and this is a contradiction. □

Definition 7. *(ZF + AD +* $DC_{\mathbb{R}}$*) Suppose X is a set. The* ***Solovay sequence*** *defined relative to X is the sequence* $\langle \Theta^X_\alpha : \alpha \leq \Upsilon_X \rangle$ *where*

(1) Θ^X_0 *is the supremum of the ordinals* ξ *such that there is a surjection* $\phi : \mathbb{R} \to \xi$ *such that* ϕ *is OD from X.*

(2) $\Theta^X_\alpha = sup\{\Theta^X_\beta \mid \beta < \alpha\}$ *if* $\alpha > 0$ *is limit.*

(3) If $\Theta^X_\alpha < \Theta$ *then* $\Theta^X_{\alpha+1}$ *is the supremum of the ordinals* ξ *such that there is a surjection* $\phi : \mathbb{R} \to \xi$ *such that* ϕ *is OD(X,A) where A is a set of reals of Wadge rank* Θ^X_α*.*

Remark 8. Suppose AD^+ holds. Let $\Theta^X_\alpha < \Theta$ be a member of the Solovay sequence and A be a set of reals with Wadge rank Θ^X_α. Let $\kappa = sup\{\boldsymbol{\delta^1_n}(A) \mid n < \omega\}$. Clearly $\kappa < \Theta^X_{\alpha+1}$. It's an AD^+ theorem that any B with Wadge rank Θ^X_α has an ∞-Borel code $C_B \subseteq \kappa$. Let $\xi < \Theta^X_{\alpha+1}$. We can define an OD_X surjection $\pi : P(\kappa) \to \xi$ as follows. Given $C \subseteq \kappa$, if C codes a tuple $\langle C_B, x, y \rangle$ where $x, y \in \mathbb{R}$, C_B is an $\infty - Borel$ code for a set B of Wadge rank Θ_α^X, and if there is a pre-wellordering of the reals of order type ξ that is $OD_X(B, x)$, then we let $\pi(C) = \pi_B(y)$ where $\pi_B : \mathbb{R} \to \xi$ is the surjection associated with the least such pre-wellordering; otherwise, $\pi(C) = 0$. So in fact, under AD^+, $\Theta_{\alpha+1}^X$ is the supremum of ordinals ξ such that there is an OD_X surjection from $P(\kappa)$ onto ξ.

Remark 9. It's worth pointing out that the Solovay sequence defined in Definition **??** is "globally defined" i.e. defined in V. On the other hand, one can define the notion of "locally defined" Solovay sequences, i.e. Solovay sequences defined in some $L(A, \mathbb{R})$, for $A \subseteq \mathbb{R}$. If $\Theta_{\alpha+1} < \Theta^{L(A,\mathbb{R})}$ then $\Theta_{\alpha+1}$ is a member of the "locally defined" Solovay sequence in $L(A, \mathbb{R})$. $\Theta_{\alpha+1}$ cannot be a limit of Suslin cardinals in $L(A, \mathbb{R})$ as otherwise, any $OD^V(A)$ relation would have an $OD^V(A)$ uniformization. Thus $\Theta_{\alpha+1} = (\Theta_{\gamma+1})^{L(A,\mathbb{R})}$, for some γ. Another key point is the following. Suppose $A \subseteq \Theta_{\alpha+1}$ is $OD^V(B)$ for some $B \subseteq \mathbb{R}$ such that $w(B) < \Theta_{\alpha+1}$. Let $\mathcal{C} = \langle C_\beta \mid \beta < \Theta_{\alpha+1} \rangle$, where $\mathcal{C}$ is an $OD^V(D)$ sequence such that each C_β is a pre-wellordering of $\mathbb{R}$ of length β, where $w(D) = \Theta_\alpha$. Then $\Theta_{\alpha+1}$ is regular in $L(\mathbb{R})[A, \mathcal{C}]$. This is important because it makes the Woodins' techniques for constructing measures under AD described in [**?**] relevant. We state here a theorem which will be used heavily.

Theorem 10. *(Woodin, see Theorem 5.6 of [**?**]) Assume ZF + DC + AD. Suppose X and Y are sets and let*

$$\Theta_{X,Y} = \sup\{\alpha \mid \text{there is an } \mathrm{OD_{X,Y}} \text{ surjection } \pi : \mathbb{R} \to \alpha\}.$$

Then

$$HOD_X \vDash ZFC + \Theta_{X,Y} \text{ is a Woodin cardinal.}$$

Proof of Theorem 1:

We deal with the easy direction (2) $\Rightarrow$ (1) first. Suppose there is a set of reals in V that has no ∞-Borel codes. One can show that A has an ∞-Borel code if and only if A has an ∞-Borel code which is coded by a set of reals projective in A. So our supposition is Σ_1^2. By (2), there is a Suslin co-Suslin set B that has no ∞-Borel codes; but this is absurd since any tree T such that p[T] = B is an ∞-Borel code of B.

For Ordinal Determinacy, again suppose there is a set B in V such that Ordinal Determinacy fails for B. The ordinal game associated to B and pre-wellordering $\leqslant$ of $\mathbb{R}$ has a winning strategy if and only if it has a winning strategy projective in $\leqslant$, by the Coding Lemma. So our supposition is Σ_1^2. By (2), there is a Suslin co-Suslin set B such that Ordinal Determinacy fails for B. This contradicts a theorem of Moschovakis and Woodin which states that Ordinal Determinacy holds for any Suslin co-Suslin set.

Finally, to see $DC_\mathbb{R}$ holds. Suppose not. Again, by our hypothesis, there is a Suslin co-Suslin relation $E \subseteq \mathbb{R} \times \mathbb{R}$ witnessing the failure of $DC_\mathbb{R}$. However, we can uniformize E using the scale associated with a tree T such that p[T]=E. This gives us an infinite E-chain, which is a contradiction. This completes the proof of (2) $\Rightarrow$ (1).

Remark 11. Our proof used that Σ_1^2 reflects to Suslin co-Suslin, rather than the full Σ_1-reflection in (2). Derived models satisfy Σ_1^2-reflection, hence they satisfy AD^+; see [**?**] and [**?**].

The rest of the paper is dedicated to the proof of (1) $\Rightarrow$ (2). First, assume there is a largest Suslin cardinal. This is the easier case.

Lemma 12. *If Θ is regular and $V = L(P(\mathbb{R})) \vDash \phi[x]$ where $x \in \mathbb{R}$ and ϕ is Σ_1, then there is a transitive M such that M is a surjective image of $\mathbb{R}$ and $(M, \in) \vDash \phi[x]$.*

Proof. By reflection, $L_\alpha(P(\mathbb{R})) \vDash \phi[x]$ for some ordinal α. We'll form a Skolem hull H of $L_\alpha(P(\mathbb{R}))$. First, fix a surjection $h : \alpha \times P(\mathbb{R}) \to L_\alpha(P(\mathbb{R}))$. Let $H_0 = \mathbb{R}$. Suppose we already have H_n and a surjection $\pi_n : \mathbb{R} \to H_n$. To build H_{n+1}, for any $a \in H_n$ and any formula φ such that $Ł_\alpha(P(\mathbb{R})) \vDash \exists y \varphi[y, a]$, pick the least β such that there is an $A \subseteq \mathbb{R}$ such that $L_\alpha(P(\mathbb{R})) \vDash \varphi[h(\beta, A), a]$. Then let γ be the least such that there is an $A \subseteq \mathbb{R}$ such that $w(A) = \gamma$ and $L_\alpha(P(\mathbb{R})) \vDash \varphi[h(\beta, A), a]$. Denote the (β, γ) above (β_a, γ_a). Now, let $H_{n+1} = H_n \cup \{h(\beta_a, A) \mid a \in H_n,\ w(A) = \gamma_a\}$. By regularity of Θ and the fact that $\pi_n : \mathbb{R} \to H_n$ is surjective, $\sup\{\gamma_a \mid a \in H_n\} < \Theta$. Hence, there is a surjection $\pi_{n+1} : \mathbb{R} \to H_{n+1}$. Finally, let $H = \cup_n H_n$. Hence H $\prec L_\alpha(P(\mathbb{R}))$ by construction. Since Θ is regular, $\mathbb{R} \subseteq H$, and $H \vDash V = L(P(\mathbb{R}))$, it is easy to see that H collapses to some $L_\delta(P_\gamma(\mathbb{R}))$ for some $\delta, \gamma < \Theta$. Since $L_\delta(P_\gamma(\mathbb{R})) \vDash \phi[x]$, $L_\delta(P_\gamma(\mathbb{R}))$ is the desired M. $\square$

Lemma 13. *Suppose there is a largest Suslin cardinal, then Θ is regular.*

Proof. Let κ be the largest Suslin cardinal and T be a tree on $\omega^2 \times \kappa$ such that p[T] is a universal Γ-set (where $\boldsymbol{\Gamma}$ is the boldface pointclass of κ-Suslin sets of reals).

For each $A \subseteq \mathbb{R}$, we have $L(T, A, \mathbb{R}) \vDash DC$ because $V \vDash DC_{\mathbb{R}}$. Let T_A be the image of T under the Martin measure ultrapower map where the ultrapower is computed with respect to functions in $L(T, A, \mathbb{R})$. Because $L(T, A, \mathbb{R}) \vDash DC$, $Ult(L(T, A, \mathbb{R}), \mu_T)$ is wellfounded. By relativizing the proof that $P(\mathbb{R}) \subseteq L(T^*, \mathbb{R})$ to the universe $L(T, A, \mathbb{R})$ (see [**?**]), we get that $A \in L(T_A, \mathbb{R})$. Notice that T_A only depends on w(A) but not A itself. So we in fact have an enumeration $\langle T_\alpha \mid \alpha < \Theta \rangle$ where for each $\alpha < \Theta$, $T_\alpha = T_A$ for any A with Wadge rank α. Now let $\gamma = sup\{supT_\alpha \mid \alpha < \Theta\}$ and $C \subseteq \Theta \times \gamma$ is such that $(\alpha, \beta) \in C \Leftrightarrow \beta \in T_\alpha$. Then $T_A \in L[C]$ for any $A \subseteq \mathbb{R}$. So $P(\mathbb{R}) \subseteq L(C, \mathbb{R})$. So $V = L(C, \mathbb{R})$. The following claim supplies an important step toward proving Θ is regular.

Claim 14. *Θ is regular if and only if Collection holds, where Collection is the following statement: " $(\forall x \in \mathbb{R})(\exists A \subseteq \mathbb{R})\ (x, A) \in U \to (\exists B \subseteq \mathbb{R})(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\ (x, B_{x,y}) \in U$, where $B_{x,y} = \{z \mid \langle x, y, z \rangle \in B\}$. "*

Proof. ($\Leftarrow$) Suppose Θ is singular. Let $f : \mathbb{R} \to \Theta$ be cofinal. So $(\forall x \in \mathbb{R})(\exists A \subseteq \mathbb{R})$ (A is a pre-wellordering of $\mathbb{R}$ of length f(x)). By Collection, $(\exists B \subseteq \mathbb{R})(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})$ ($B_{x,y}$ is a pre-wellordering of length f(x)). Define $g : \mathbb{R} \to \Theta$ as follows: for any $x \in \mathbb{R}$, if $x = (x_0, x_1, x_2)$ and B_{x_0,x_1} is a pre-wellordering of $\mathbb{R}$ of order type $f(x_0)$, then let g(x) = rank of x_2 in the pre-wellordering B_{x_0,x_1}; otherwise, let $g(x) = 0$. Clearly, g is onto. This is a contradiction.

($\Rightarrow$) Suppose Θ is regular. Let U be as in the hypothesis of Collection. For $x \in \mathbb{R}$, let f(x) be the least ξ such that there is $A \subseteq \mathbb{R}$ with Wadge rank ξ and $(x, A) \in U$. Since Θ is regular, f is bounded in Θ. Fix an $\alpha < \Theta$ such that $\alpha \geq sup(rng(f))$. Let $B = \{(x, y, B_{x,y}) \mid x, y \in \mathbb{R},$ y Wadge reduces $B_{x,y}$ to $A\}$. Clearly, B satisfies the conclusion of Collection. $\square$

By claim **??**, it suffices to prove that $L(C,\mathbb{R}) \vDash \mathit{Collection}$. So let U be as in the hypothesis of Collection. Let $B = \{(x, y, B_{x,y}) \mid B_{x,y}$ is the least $OD_C(y)$ set such that $(x, B_{x,y}) \in U\}$. This B clearly works because every set of reals in $V = L(C,\mathbb{R})$ is $OD_C(y)$ for some real y. □

The proof of Lemma **??** also implies that DC holds, hence the Martin measure ultrapower is well-founded. This fact is used to show that there is an M in a generic extension of V such that V is a derived model of M. See [**?**] for a proof of this. The conclusion 2 of Theorem **??** then follows from Lemma **??** above and Lemma 7 of [**?**].

Now, we're on to the "no largest Suslin cardinal" case. So we have $AD_{\mathbb{R}}$. First, assume Θ is regular. By Lemma **??**, $M_{P(\mathbb{R})} \prec_{\Sigma_1} V$. Since all sets of reals are Suslin co-Suslin, we're done.

From now on, we may assume that Θ is singular. We have that every set of reals is Suslin co-Suslin. Our strategy is to Prikry-force a universe M such that V is a derived model of M. This guarantees that Σ^2_1-reflection holds in V, but with a little more argument, we'll be able to show Σ_1-reflection holds in V. Most of what we are doing here, then, is proving Theorem **??** in the case $AD_{\mathbb{R}} + \Theta$ is singular.

Case 1: cof$(\Theta) = \omega$.

Let $\langle \Theta_\alpha \mid \alpha < \Upsilon \rangle$ be the Solovay sequence of V. Notice that cof$(\Upsilon) = \omega$. Hence, there is a sequence $\langle \alpha_i \mid i < \omega \rangle$ cofinal in Υ. We can and do take the sequence $\langle \alpha_i \mid i < \omega \rangle$ to be definable from a set of reals and from no ordinal parameters. The hypothesis implies that every set of reals is Suslin, so given an $\alpha < \Upsilon$, let κ be the largest Suslin cardinal below $\Theta_{\alpha+1}$. Set $HOD_{P(\kappa)} = \{A \mid \forall C \in TC(A \cup \{A\})$ C is OD from some B $\in P(\kappa)\}$, then the following hold:

(1.1) $\Theta_{\alpha+1}$ is the supremum of the ordinals ξ for which there is a surjection $\phi : P(\kappa) \to \xi$ such that ϕ is OD.

(1.2) $\Theta_{\alpha+1} = \Theta^{HOD_{P(\kappa)}}$.

(1.3) $HOD_{P(\kappa)} = HOD_X$, where $X = \{B \subseteq \mathbb{R} \mid w(B) < \Theta_{\alpha+1}\}$.

(1.1) follows from Remark 8. Both (1.2) and (1.3) are immediate consequences of (1.1). By (1.3), every bounded subset of $\Theta_{\alpha+1}$ belongs to $HOD_{P(\kappa)}$. Now, for each i $< \omega$, let κ_i be the largest Suslin cardinal below Θ_{α_i+1} and μ_i be the supercompact (nonprincipal, fine, and normal) measure on $P_{\omega_1}(P(\kappa_i))$. Notice here that by $AD_{\mathbb{R}}$, Solovay's super-compactness measure on $P_{\omega_1}(\mathbb{R})$ exists and is unique. Since $P(\kappa_i)$ is the surjective image of $\mathbb{R}$, μ_i exists and is unique. Because it is unique, μ_i is OD. Also, let X_i be the set of all $\sigma \in P_{\omega_1}(P(\kappa_i))$ such that

(2.1) $\text{HOD}_{\sigma\cup\{\sigma\}} \vDash AD^+$

(2.2) $\text{HOD}_{\sigma\cup\{\sigma\}} \nvDash AD_{\mathbb{R}}$

(2.3) the transitive collapse of σ is $P(\kappa_i^\sigma) \cap$ $\text{HOD}_{\sigma\cup\{\sigma\}}$ where κ_i^σ is the largest Suslin cardinal in $HOD_{\sigma\cup\{\sigma\}}$.

Lemma 15. $\mu_i(X_i) = 1$

Proof. Let

$$\prod_\sigma HOD_{\sigma\cup\{\sigma\}}/\mu_i = M,$$

where the ultraproduct is formed in the universe $HOD_{P(\kappa_i)}$. The reason we do this is that we do not have DC in V, and thus the ultraproduct formed in V might be illfounded. On the other hand, $HOD_{P(\kappa_i)} \models DC$, so M is well-founded, and we take it to be transitive. Let σ^∞ be the element of M represented by the identity function. By Łos, for all formulas ϕ,

$$M \models \phi[\sigma^\infty] \Leftrightarrow \mu_i(\{\sigma \in P_{\omega_1}(P(\kappa_i)) \mid HOD_{\sigma\cup\{\sigma\}} \models \phi[\sigma]\}) = 1.$$

We should remark here that even though we don't have AC, Łos theorem still goes through because of normality (closure under diagonal intersections) of μ_i. The following claim will complete the proof of the lemma.

Claim 16. *The following hold:*

1. *The transitive collapse of σ^∞ is $P(\kappa_i)$.*
2. $\mathbb{R} \cap M = \mathbb{R}$.
3. $P(\mathbb{R}) \cap M = \{B \mid w(B) < \Theta_{\alpha_i+1}\} = P(\mathbb{R}) \cap HOD_{P(\kappa_i)}$.

Proof. (1) and (2) are easy consequences of normality, so we leave them to the reader. To prove (3), suppose first that $w(B) < \Theta_{\alpha_i+1}$. So $B \in HOD_{P(\kappa_i)}$. Let $f(\sigma) = B \cap \sigma$ for $\sigma \in P_{\omega_1}(P(\kappa_i))$. Then $f \in HOD_{P(\kappa_i)}$ and $[f]_{\mu_i} = B$. On the other hand, $M \subseteq HOD_{P(\kappa_i)}$ as the ultraproduct is formed in $HOD_{P(\kappa_i)}$. □

□

Let

$$T_0 = \{\langle\sigma_0, ..., \sigma_n\rangle \mid \sigma_i \in P_{\omega_1}(P(\kappa_i)) \text{ for all i}\}.$$

Let T be the set of all $s = \langle\sigma_0, ..., \sigma_n\rangle \in T_0$ such that for all i $\leq$ n

(3.1) $P(\mathbb{R})^{HOD_{\{s\}}} = P(\mathbb{R})^{HOD}$,

(3.2) $\sigma_i \in X_i$,

(3.3) $\sigma_k \subset \sigma_i$ and $\sigma_k \in HOD_{\sigma_i\cup\{\sigma_i\}}$ for all k $\leq$ i,

(3.4) σ_k is countable in $HOD_{\sigma_i\cup\{\sigma_i\}}$ for all k $<$ i,

(3.5) θ^{σ_i} is Woodin in $HOD_{\{s|(i+1)\}}$ and $P(\theta^{\sigma_i}) \cap HOD_{\{s|(i+1)\}} = P(\theta^{\sigma_i}) \cap HOD_{\{s\}}$, where $\theta^{\sigma_i} = \Theta^{HOD_{\sigma_i\cup\{\sigma_i\}}}$. Note here that θ^{σ_i} is a successor in the Solovay sequence of $HOD_{\sigma_i\cup\{\sigma_i\}}$

Remark 17. For any $s = \langle \sigma_0, ..., \sigma_n \rangle \in T_0$, $HOD_{\{s\}} = HOD_s$. From now on, we'll write HOD_s for $HOD_{\{s\}}$

Lemma 18. *Let $t = \langle \sigma_0, ..., \sigma_n \rangle$ be such that (3.1)-(3.4) hold. Let $\sigma = \sigma_n$, and set $H = HOD_t$. Then*

$$H = HOD_H^{HOD_{\sigma \cup \{\sigma\}}}.$$

Proof. Here HOD_H consists of all sets HOD from members of H. Notice here that $H \subseteq HOD_{\sigma \cup \{\sigma\}}$; hence the right hand side of the equation makes sense and $H \subseteq HOD_H^{HOD_{\sigma \cup \{\sigma\}}}$. The $\supseteq$ direction follows from the fact that σ is OD from t. □

Lemma 19. *Let $s \in T$ and dom(s) = i; then $\forall^*_{\mu_i} \sigma$ $(s + \langle \sigma \rangle \in T)$.*

Proof. Fix $s \in T$ with dom(s) = i. It is easy to see that $\forall^*_{\mu_i} \sigma, s + \langle \sigma \rangle$ satisfies (3.1)-(3.4), so we address (3.5). We want to show $\forall^*_{\mu_i} \sigma, HOD_{s+\langle \sigma \rangle} \vDash \theta^\sigma$ is Woodin. Let $H = HOD_{s+\langle \sigma \rangle}$. Let us work now in $HOD_{\sigma \cup \{\sigma\}}$, where AD^+ holds and $AD_{\mathbb{R}}$ fails. This implies that $\Theta = \Theta_Y$ for some Y. Also, Θ is regular and DC holds. We have then from Theorem 5.6 of [**?**] that

$$HOD_H \vDash \Theta \text{ is Woodin.}$$

By the previous theorem, $H = HOD_H$, hence we're done.

Let $s = \langle \sigma_0, ..., \sigma_{i-1} \rangle$. Without loss of generality, it is enough to see that $(\forall^*_{\mu_i} \sigma)$ $P(\theta^{\sigma_{i-1}}) \cap HOD_s = P(\theta^{\sigma_{i-1}}) \cap HOD_{s+\langle \sigma \rangle}$. It is clearly enough to show $(\forall^*_{\mu_i} \sigma)$ $P(\theta^{\sigma_{i-1}}) \cap HOD_s \supseteq P(\theta^{\sigma_{i-1}}) \cap HOD_{s+\langle \sigma \rangle}$. Suppose not. We have $(\forall^*_{\mu_i} \sigma)(\exists A_\sigma \subseteq \theta^{\sigma_i - 1})$ $(A_\sigma \in HOD_{s+\langle \sigma \rangle} \backslash HOD_s)$. Here we take A_σ to be the least such set. Since $\theta^{\sigma_i - 1}$ is a fixed countable ordinal, we have $(\exists A \subseteq \theta^{\sigma_i - 1})(\forall^*_{\mu_i} \sigma)$ $(A = A_\sigma)$. But this A is in fact HOD_s since the supercompactness measures are OD. Contradiction. □

Lemma 20. *Let $s \in T$ with dom(s) = i. Let $\sigma = s(dom(s) - 1)$. Then there is a partial order $\mathbb{P}$ such that*

1. *$HOD_s \vDash \mathbb{P}$ is a θ^σ-c.c. complete boolean algebra of cardinality θ^σ, and*
2. *for any $A \subseteq \kappa^\sigma$ such that $A \in HOD_{\sigma \cup \{\sigma\}}$, there is a filter G_A on $\mathbb{P}$ such that*
    - *G_A is HOD_s-generic over $\mathbb{P}$, and*
    - *$HOD_{\{s,A\}} = HOD_s[G_A]$.*

Proof. Let $H = HOD_s$. Working in $HOD_{\sigma \cup \{\sigma\}}$, where $H = HOD_H$ by Lemma **??**, let $\mathbb{P}$ be the Vopenka algebra for adding subsets of κ^σ to HOD_H. So $\mathbb{P}$ is isomorphic to $(\mathcal{O}, \subseteq)$, where $\mathcal{O}$ is the collection of all OD_H subsets of $P(\kappa^\sigma)$. Then (1) and (2) are standard properties of the Vopenka algebra, where the filter G_A in (2) is the filter generated by A. □

Now we're ready to define our Prikry forcing $\mathbb{P}$. Conditions in $\mathbb{P}$ are pairs (s,F) such that $s \in T$ and $F : T \to V$, $F(\varnothing) \in \mu_0$, and for all $\langle \sigma_0, ..., \sigma_n \rangle \in T$, $F(\langle \sigma_0, ..., \sigma_n \rangle) \in \mu_{n+1}$. The ordering is defined by

$$(s_0, F_0) \preceq (s_1, F_1) \Leftrightarrow s_1 \subseteq s_0, (\forall s \in T)(F_0(t) \subseteq F_1(t)), (\forall i \in dom(s_0) - dom(s_1))(s_0(i) \in F_1(s_0|i)).$$

Lemma 21. *Suppose $Z \subset V^{\mathbb{P}}$ is countable, ϕ is a formula, and $(s_0, F_0) \in \mathbb{P}$. Then there is a condition $(s_0, G) \in \mathbb{P}$ deciding $\phi[\tau]$ for all $\tau \in Z$.*

Proof. Since the usual proof requires DC, which we don't have, we'll give here a DC-free proof. Fix $\tau \in Z$. We'll show that there is an (s_0, G) deciding $\phi[\tau]$ such that G is OD from s_0, F, and τ. Let us say that $u \in T$ is **positive** if and only if $(\exists G)$ $((u, G) \Vdash \phi[\tau])$, **negative** if and only if $(\exists G)$ $((u, G) \Vdash -\phi[\tau])$, and **ambiguous** if and only if it is neither positive nor negative. Notice that u cannot be both positive and negative.

For notational convenience, for $u \in T$ with dom(u) = n+1, we write $\forall^*_u \sigma P(\sigma)$ to mean $\{\sigma \mid P(\sigma)\} \in \mu_{n+1}$. Now define $G = G_\tau$ by: for $v \in T$, G(v) = $\{\sigma \mid v + \langle\sigma\rangle$ is positive$\} \cap F_0(v)$ if $(\forall^*_v \sigma)$ $(v + \langle\sigma\rangle$ is positive); G(v) = $\{\sigma \mid v + \langle\sigma\rangle$ is negative$\} \cap F_0(v)$ if $(\forall^*_v \sigma)$ $(v + \langle\sigma\rangle$ is negative); G(v) = $\{\sigma \mid v + \langle\sigma\rangle$ is ambiguous$\} \cap F_0(v)$ if $(\forall^*_v \sigma)$ $(v + \langle\sigma\rangle$ is ambiguous). Clearly G is OD from s_0, τ, F_0 and $(s_0, G) \preceq (s_0, F_0)$. If remains to see that (s_0, G) decides $\phi[\tau]$.

Claim 22. *Let $u \in T$ with dom(u) = n+1. Then*

1. *u is positive $\Rightarrow \forall^*_u \sigma$ $(u + \langle\sigma\rangle$ is positive);*
2. *u is negative $\Rightarrow \forall^*_u \sigma$ $(u + \langle\sigma\rangle$ is negative)*
3. *u is ambiguous $\Rightarrow \forall^*_u \sigma$ $(u + \langle\sigma\rangle$ is ambiguous)*

Proof. If u is positive, then there is an H such that $(u, H) \Vdash \phi[\tau]$. But then whenever $\sigma \in H(u)$, $(u + \langle\sigma\rangle, H) \Vdash \phi[\tau]$. Since $H(u) \in \mu_{n+1}$, we're done. The proof is the same for u being negative.

Suppose u is ambiguous and the conclusion of (3) is false. Without loss of generality, we may assume $\forall^*_u \sigma$ $(u + \langle\sigma\rangle$ is positive). Let $G = G_\tau$ be as above. Then $(u, G) \Vdash \phi[\tau]$ since if $(v, H) \preceq (u, G)$, then v is positive by and easy induction using part (1), and thus $(v, H) \nVdash -\phi[\tau]$. Hence u is in fact positive. Contradiction. ☐

Claim 23. *No $u \in T$ is ambiguous.*

Proof. Suppose u is ambiguous. Let $G = G_\tau$ be as in the previous claim. Let $(v, H) \preceq (u, G)$ and (v,H) decide $\phi[\tau]$. Then v is not ambiguous. On the other hand, by induction using Claim 18 part (3), v is ambiguous. Contradiction. ☐

By the previous claim, we may assume without loss of generality that s_0 is positive. But then $(s_0, G_\tau) \Vdash \phi[\tau]$, for otherwise, we have $(v, H) \preceq (s_0, G_\tau)$ forcing $-\phi[\tau]$. This implies that v is negative. However, an induction using Claim 18 part (1) shows that v is positive.

Finally, let $H(v) = \cap_{\tau \in Z} G_\tau(v)$. We get that (s_0, H) decides $\phi[\tau]$ for all $\tau \in Z$. ☐

Let $G \subset \mathbb{P}$ is V-generic and $s_G = \cup\{s \mid (s, F) \in G\}$. Now we use Lemma **??** to prove the following:

Lemma 24. *For all $i < \omega$, $P(\theta_i) \cap HOD^V_{s_G|(i+1)} = P(\theta_i) \cap HOD^{(V[G],V)}_{\{s_G\}}$, where $\theta_i = \Theta^{HOD^V_{s_G(i) \cup \{s_G(i)\}}}$.*

Proof. The $\subseteq$ direction is evident because we use V as a predicate in the definition of $HOD_{\{s_G\}}^{(V[G],V)}$. Suppose the converse direction fails for some i. Then there is a formula $\varphi(x_0, x_1, x_2)$, an ordinal ξ, an $n > i$, an F such that $(s_G|n, F) \in G$, and

$$(s_G|n, F) \Vdash \{\beta < \theta_i \mid (V[G], V) \vDash \varphi[\beta, \xi, s_G]\} \notin HOD^V_{\{s_G|(i+1)\}}.$$

By Lemma ??, given any $(s_G|n, F)$ as above, there is $(s_G|n, F^*) \preceq (s_G|n, F)$ such that for all $\beta < \theta_i$, either $(s_G|n, F^*) \Vdash (V[G], V) \vDash \varphi[\beta, \xi, s_G]$, or $(s_G|n, F^*) \Vdash (V[G], V) \vDash -\varphi[\beta, \xi, s_G]$. Hence we can find such a $(s_G|n, F^*)$ in G. So $\{\beta < \theta_i \mid (s_G|n, F^*) \Vdash (V[G], V) \vDash \varphi[\beta, \xi, s_G]\} = \{\beta < \theta_i \mid \exists F^* \ (s_G|n, F^*) \Vdash (V[G], V) \vDash \varphi[\beta, \xi, s_G]\} \in HOD^V_{\{s_G|n\}}$. But $s_G|n \in T$ and $n > i$, so by (3.5) $\{\beta < \theta_i \mid (s_G|n, F^*) \Vdash (V[G], V) \vDash \varphi[\beta, \xi, s_G]\} \in HOD^V_{\{s_G|(i+1)\}}$. This is a contradiction. $\square$

Fix a $G \subset \mathbb{P}$ such that G is V-generic. Let

$$N = HOD_{\{s_G\}}^{(V[G],V)}.$$

It's easy to see that ω_1^V is a limit of Woodin cardinals in N, $N \vDash ZFC$. Here is the key lemma.

Lemma 25. *V is a derived model of N.*

Proof. To simplify the notation, let $N_i = HOD^V_{s_G|(i+1)}$ and $\theta_i = \Theta^{HOD^V_{s_G(i) \cup \{s_G(i)\}}}$ for each $i < n$. Then θ_i is Woodin in N_i and $P(\theta_i) \cap N_i = P(\theta_i) \cap N_j = P(\theta_i) \cap N$ for all $j \geq i$. As mentioned above, $\omega_1^V = sup\{\theta_i \mid i < \omega\}$.

Now, let K be a $Col(\omega, < \omega_1^V)$-generic over N such that $\mathbb{R}^*_K = \mathbb{R}^V$. To see that there is such a K, it suffices to show that any $x \in \mathbb{R}^V$ is generic over N for some poset $\mathbb{P} \in N|sup_i(\theta_i)$. Fix such an x and pick i such that $x \in s_G(i)$. By Lemma ??, x is $\mathbb{P}$-generic over N_i, where $\mathbb{P}$ is the Vopenka algebra of $HOD_{s_G(i) \cup \{s_G(i)\}}$ for adding a subset of $\kappa^{s_G(i)}$ to $HOD_{s_G|(i+1)} = N_i$. But $P(\theta_i)^{N_i} = P(\theta_i)^N$, so x is $\mathbb{P}$-generic over N.

To finish the proof, we need to see that $P(\mathbb{R})^V = Hom^*_K$. It suffices to show that $P(\mathbb{R})^V \subseteq Hom^*_K$. Because then if $P(\mathbb{R})^V \subsetneq Hom^*_K$, we get a sharp for V in a generic extension of V. This is impossible.

So let $B \in P(\mathbb{R})^V$. B is Suslin co-Suslin. By Martin's theorem, B and $\mathbb{R}\backslash B$ are homogeneously Suslin as witnessed by homogeneous trees on $\omega \times \kappa$ for some $\kappa < \Theta$. So we can find a countable sequence of ordinals f such that sup(range(f)) $< \Theta$ from which we can define a pair of trees (T,U) over V such that p[T] = B = $\mathbb{R}\backslash$p[U]. The sequence f comes from the measures of the homogeneity systems from which T and U are defined. Pick k large enough so that ran(f) $\subseteq s_G(k)$. Also $s_G(k) \cap Ord \in N$. $s_G(k)$ is made countable in $N(\mathbb{R}^V)$ and some real coding ran(f) is added. Hence, for some $i < \omega$ and $g \in V$ generic over N_i for the collapse of an ordinal $< \theta_i$, we have $f \in N_i[g]$. So, for any $j \geq i$, $N_j[g]$ can decode f to get the pair (T,U). Moreover, $p[T]^{N_j[g]} = B \cap \mathbb{R}^{N_j[g]} = \mathbb{R}^{N_j[g]} - p[U]^{N_j[g]}$. Hence, $B \in Hom^*_K$ as desired. $\square$

Now let ϕ be a Σ_1 formula such that $V \vDash \phi[\mathbb{R}]$. We want to show that there are $\alpha, \beta < \Theta$ such that $L_\alpha(P_\beta(\mathbb{R})) \vDash \phi[\mathbb{R}]$.

Lemma 26. *There is an $A \in (Hom_{<\omega_1^V})^N$ such that $L(A, \mathbb{R}^N) \vDash \phi[\mathbb{R}^N]$.*

Proof. Let γ be the least such that $L_\gamma(P(\mathbb{R})) \vDash \phi[\mathbb{R}]$ and $\langle \alpha_i \mid i < \omega \rangle$ is definable $L_\gamma(P(\mathbb{R}))$ from a set of reals and no ordinal parameters. Since V is the derived model of N at ω_1^V, the ($\mathbb{Q}$ version of) stationary tower forcing gives an elementary embedding $j : N \to (M, E)$ such that

(10.1) crt(j) $= \omega_1^N$ and $j(\omega_1^N) = \omega_1^V$;

(10.2) $\mathbb{R}^{(M,E)} = \mathbb{R}^V$;

(10.3) $P(\mathbb{R})^V = (Hom^N_{<\omega_1^V})^* \subseteq j((Hom_{<\omega_1^V})^N)$

(10.4) $j(A) = A^*$ for each $A \in (Hom_{<\omega_1^V})^N$, where $A^* = p[T] \cap \mathbb{R}^V$ for T a homogeneous tree in N such that $p[T] \cap \mathbb{R}^N = A$;

(10.5) γ is in the well-founded part of (M,E).

If $(P(\mathbb{R}))^V \neq j((Hom_{<\omega_1^V})^N)$, then there is an $A \in j((Hom_{<\omega_1^V})^N) \backslash (P(\mathbb{R}))^V$. Since ϕ is Σ_1 and by (10.2), (M,E) $\vDash L(A, \mathbb{R}^{(M,E)}) \vDash \phi[\mathbb{R}^{(M,E)}]$. By elementarity, there is an $A \in (Hom_{<\omega_1^V})^N$ such that $L(A, \mathbb{R}^N) \vDash \phi[\mathbb{R}^N]$. Hence, we may assume $(P(\mathbb{R}))^V = j((Hom_{<\omega_1^V})^N)$. Since $\langle \alpha_i \mid i < \omega \rangle$ is definable in $L_\gamma(P(\mathbb{R}))$, from some $B \in P(\mathbb{R})^V = (Hom^N_{<\omega_1^V})^*$, let $\beta < \omega_1^V$ such that there is a $D \in N[K|\beta]$ such that $B = D^*$. Replacing N by $N[K|\beta]$ if neccessary where K is as in the previous lemma, we can assume $\langle \alpha_i \mid i < \omega \rangle$ is in the range of j, say $j(\langle \alpha_i^* \mid i < \omega \rangle) = \langle \alpha_i \mid i < \omega \rangle$. Since N is a model of choice, we can choose (using $\langle \alpha_i^* \mid i < \omega \rangle$) a sequence $\langle A_i \mid i < \omega \rangle \in N$ cofinal in $(Hom_{<\omega_1^V})^N$. Let $A \in (Hom_{<\omega_1^V})^N$ code the A_i's, say $A = \{\langle i, x(0), x(1)... \rangle \mid x = \langle x(0), x(1)... \rangle \in A_i\}$. Then A is in $Hom^N_{<\omega_1^V}$ but not Wadge reducible to any A_i. Contradiction. □

Lemma **??** and the elementarity of the map j defined there finish the proof of the theorem in the case $cof(\Theta) = \omega$.

Case 2: cof(Θ) $> \omega$

By a result of Solovay, DC holds in this case (see [**?**]). Let μ be a measure on $\{\alpha \mid cof(\alpha) = \omega\}$ induced by the measure on $cof(\Theta) < \Theta$ which in turn is induced by the Martin measure on Turing degrees.

For each $\alpha < \Upsilon$ such that $cof(\alpha) = \omega$, let $I_\alpha = \{A \subset \Theta_\alpha \mid sup(A) < \Theta_\alpha\}$. Therefore,

(11.1) $HOD_{I_\alpha} \vDash AD^+ + AD_\mathbb{R}$

(11.2) $\Theta^{HOD_{I_\alpha}} = \Theta^V_\alpha$

(11.3) for each $X \in HOD_{I_\alpha}$, $\Theta^{HOD_{I_\alpha}}$ is a limit of Woodin cardinals in $HOD_{\{X\}}$.

We'll use a slightly different Prikry forcing to add an inner model N like before. The only difference in this case is that we want ω_1^V to be a limit of limits of Woodin cardinals in N.

For each $\alpha < \Upsilon$ such that $cof(\alpha) = \omega$, let μ_α be the supercompact measure on $P_{\omega_1}(I_\alpha)$ induced by the Solovay measure on $P_{\omega_1}(\mathbb{R})$.

Lemma 27. *For each $\alpha < \Upsilon$ such that $cof(\alpha) = \omega$, there are μ_α-measure 1 many σ such that*

(12.1) $HOD_{\sigma\ \cup\{\sigma\}} \models AD_{\mathbb{R}}$

(12.2) The transitive collapse of σ is the set $\{A \subset \Theta \mid sup(A) < \Theta\}$ as computed in $HOD_{\sigma\cup\{\sigma\}}$

Proof. Notice that because of DC, the ultraproduct $\prod_\sigma HOD_{\sigma\cup\{\sigma\}}/\mu_\alpha$ is wellfounded. So identifying it with its transitive collapse, we get $I_\alpha \subset \prod_\sigma HOD_{\sigma\cup\{\sigma\}}/\mu_\alpha \subset HOD_{I_\alpha}$. Also $\Theta_\alpha = \Theta^{HOD_{I_\alpha}} = \Theta^{\prod_\sigma HOD_{\sigma\cup\{\sigma\}}/\mu_\alpha}$. This proves the claim. $\square$

Now like before, let T_0 be the set of all finite sequences $\langle \sigma_i \mid i \leq n\rangle$ such that for all $i \leq n$, there is an $\alpha < \Upsilon$ such that

(13.1) $cof(\alpha) = \omega$

(13.2) $\Theta_\alpha = sup\{\gamma \mid \gamma \in \sigma_i\}$

(13.3) $\sigma_i \in P_{\omega_1}(I_\alpha)$

(13.4) $HOD_{\sigma_i\cup\{\sigma_i\}} \models AD_{\mathbb{R}}$

(13.5) The transitive collapse of σ_i is $\{A \subset \Theta \mid sup(A) < \Theta\}$ as computed in $HOD_{\sigma_i\cup\{\sigma_i\}}$

For each $\langle \sigma_i \mid i \leq n\rangle \in T_0$, let $\alpha_{\sigma_i} = sup\{\gamma \mid \gamma \in \sigma_i\}$. Now let T be the set of all $s = \langle \sigma_i \mid i \leq n\rangle \in T_0$ such that for all $i \leq n$,

(14.1) $P(\mathbb{R})^{HOD_{\{s\}}} = P(\mathbb{R})^{HOD}$

(14.2) $\alpha_{\sigma_i} < \alpha_{\sigma_{i+1}}$

(14.3) $\sigma_k \subset \sigma_i$, $\sigma_k \in HOD_{\sigma_i\cup\{\sigma_{i+1}\}}$ for all $k \leq i$, and σ_k is countable in $HOD_{\sigma_i\cup\{\sigma_i\}}$ for all $k < i$,

(14.4) $P(\theta^{\sigma_i}) \cap HOD_{\{s|(i+1)\}} = P(\theta^{\sigma_i}) \cap HOD_{\{s\}}$, where $\theta^{\sigma_i} = \Theta^{HOD_{\sigma_i\cup\{\sigma_i\}}}$.

From the definition of T and a similar proof to that of Lemma **??**, if $s \in T$ then for μ-almost all $\alpha < \Upsilon$, for μ_α-almost all $\sigma \in P_{\omega_1}(I_\alpha)$, $s + \langle\sigma\rangle \in T$. Now we're ready to define the Prikry forcing $\mathbb{P}$. Conditions in $\mathbb{P}$ are pairs (s,F) such that $s \in T$ and $F : T \to V$ such that for all $t \in T$, $t + \langle\sigma\rangle \in T$ for all $\sigma \in F(t)$ and for μ-almost all $\alpha < \Upsilon$, for μ_α-almost all $\sigma \in P_{\omega_1}(I_\alpha)$, $\sigma \in F(t)$. The ordering on $\mathbb{P}$ is defined by:

$$(s_1, F_1) \preceq (s_0, F_0) \Leftrightarrow s_0 \subset s_1, \forall i \in dom(s_1) - dom(s_0), s_1(i) \in F_0(s_1|i), \text{ and } F_1 \subset F_0 \text{ pointwise.}$$

Lemma 28. *Suppose $Z \subset V^{\mathbb{P}}$ is countable, ϕ is a formula, and $(s_0, F_0) \in \mathbb{P}$. Then there is a condition $(s_0, F_1) \in \mathbb{P}$ that decides $\phi[\tau]$ for every $\tau \in Z$.*

Proof. Same as that of Lemma **??**. $\square$

Let $G \subset \mathbb{P}$ be V-generic and let $s_G = \{s \mid \exists F(s,F) \in G\} = \langle \sigma_i \mid i < \omega\rangle$.

Lemma 29. *(a) For all $i < \omega$, $P(\theta^{\sigma_i}) \cap HOD^V_{s_G|(i+1)} = P(\theta^{\sigma_i}) \cap HOD^{(V[G],V)}_{\{s_G\}}$, where $\theta^{\sigma_i} = \Theta^{HOD^V_{\sigma_i \cup \{\sigma_i\}}}$.*

(b) For all $i < \omega$, for all A bounded subset of θ^{σ_i} and $A \in HOD_{\sigma_i \cup \{\sigma_i\}}$, there is a partial order $\mathbb{P}$ such that $|\mathbb{P}| < \theta^{\sigma_i}$ and $\mathbb{P}$ is θ^{σ_i}-c.c. as computed in $HOD^V_{s_G|(i+1)}$, and $HOD^V_{\{s_G|(i+1),A\}} = HOD_{s_G|(i+1)}[G_A]$ for some $HOD^V_{s_G|(i+1)}$-generic filter $G_A \subset \mathbb{P}$ in V.

(c) θ^{σ_i} is a limit of Woodin cardinals in $HOD^{(V[G],V)}_{\{s_G\}}$

Proof. (a),(b) have the same proofs as those of Lemma **??** and **??**. It remains to prove (c). By (a), it suffices to prove

$$HOD^V_{s_G|(i+1)} \vDash \theta^{\sigma_i} \text{ is Woodin.}$$

We know $HOD^V_{\sigma_i \cup \{\sigma_i\}} \vDash AD_{\mathbb{R}}$, and in $HOD^V_{\sigma_i \cup \{\sigma_i\}}$, $HOD_{s_G|(i+1)} = HOD_{HOD_{s_G|(i+1)}}$, so by Theorem 5.6 of [**?**], θ^{σ_i} is a limit of Woodin cardinals in $HOD^V_{s_G|(i+1)}$. Hence we're done. $\square$

Now, fix some $G \subset \mathbb{P}$ such that G is V-generic, and let

$$N = HOD^{(V[G],V)}_{\{s_G\}}.$$

As before, for any $x \in \mathbb{R}^V$, $N[x] \vDash ZFC$, and V is the derived model of N[x]. By part (c) of the previous lemma, ω_1^V is a limit of limits of Woodin cardinals in N[x]. Before stating the next lemma, we need the following:

Definition 30. *Suppose δ is a limit of Woodin cardinals, then $Hom_{<\delta}$ is weakly sealed if the following hold.*

(1) Suppose $\kappa < \delta$ is a Woodin cardinal and $G \subset \mathbb{Q}_{<\kappa}$ is V-generic. Let $j : V \to M \subset V[G]$ be the associated generic embedding. Then $j(Hom_{<\delta}) = (Hom_{<\delta})^{V[G]}$.

(2) Suppose that $G \subset \mathbb{P}$ is V-generic and $\mathbb{P} \in V_\delta$. Then (1) holds in V[G].

Lemma 31. *One of the following must hold.*

(a) There is an $x \in \mathbb{R}^V$ and $A \in (Hom_{<\omega_1^V})^{N[x]}$ such that $L(A, \mathbb{R}^{N[x]}) \vDash \phi[\mathbb{R}^{N[x]}]$.

(b) $Hom^N_{<\omega_1^V}$ is weakly sealed in N.

Proof. Let γ be large enough that $L_\gamma(P(\mathbb{R}^V)) \vDash \phi[\mathbb{R}^V]$. For any $x \in \mathbb{R}^V$, there is a generic elementary embedding $j_x : N[x] \to (M_x, E_x)$ induced by a $\mathbb{Q}^{N[x]}_{<\omega_1^V}$-generic such that

(15.1) $crt(j_x) = \omega_1^{N[x]}$ and $j_x(\omega_1^{N[x]}) = \omega_1^V$,

(15.2) $\mathbb{R}^{(M_x,E_x)} = \mathbb{R}^V$,

(15.3) $(P(\mathbb{R})^V \subseteq j_x(Hom^{N[x]}_{<\omega_1^V})$, (15.4) $\forall A \in Hom^{N[x]}_{<\omega_1^V}, j_x(A) = A^*$,

(15.5) for all successor Woodin cardinals $\kappa < \omega_1^V$ in $N[x]$, there is an $N[x]$-generic $H \subset \mathbb{Q}^{N[x]}_{<\kappa}$ inducing a generic elementary embedding $j_H : N[x] \to Ult(N[x], E_H)$, and an elementary embedding $k_H : Ult(N[x], E_H) \to (M_x, E_x)$ such that $j_x = k_H \circ j_H$.

(15.6) γ is in the well-founded part of (M_x, E_x).

If overspill occurs, i.e. if there is some $x \in \mathbb{R}^V$ such that $P(\mathbb{R})^V \neq j_x(Hom^{N[x]}_{<\omega_1^V})$ then (a) holds by the same argument as in Lemma **??**. So suppose $P(\mathbb{R})^V = j_x(Hom^{N[x]}_{<\omega_1^V})$ for all $x \in \mathbb{R}^V$. Then $j_H(Hom^{N[x]}_{<\omega_1^V}) = Hom^{N[x][H]}_{<\omega_1^V}$ for all H in (15.5) because $k_H(Hom^{N[x][H]}_{<\omega_1^V}) \supseteq P(\mathbb{R})^V$ and $j_H(Hom^{N[x]}_{<\omega_1^V}) \supseteq Hom^{N[x][H]}_{<\omega_1^V}$. By varying j_x and (M_x, E_x) to ensure the filters H contain any specified condition, we get (b). $\square$

If (a) holds in the previous lemma, we're done with the proof of case 2. So suppose (b) holds.

Lemma 32. $Hom^N_{<\omega_1^V} = L(Hom^N_{<\omega_1^V}) \cap P(\mathbb{R}^N)$

Proof. We first show:

(16.1) If $\mathbb{P} \in V^N_{\omega_1^V}$ and $G \subset \mathbb{P}$ is N-generic then in N[G], there is an elementary embedding $j_G : L(Hom^N_{<\omega_1^V}) \to (L(Hom_{<\omega_1^V}))^{N[G]}$ such that $j_G(Hom^N_{<\omega_1^V}) = (Hom_{<\omega_1^V})^{N[G]}$.

To show (16.1), fix $\mathbb{P} \in V^N_{\omega_1^V}$ and an N-generic $G \subset \mathbb{P}$. Fix an increasing sequence $\langle \delta_i \mid i < \omega \rangle$ of Woodin cardinals in N bounded below ω_1^V and let $\kappa = sup\{\delta_i \mid i < \omega\} > |\mathbb{P}|^N$. Let $\delta_\omega < \omega_1^V$ be a Woodin cardinal in N larger than κ.

Let σ be the symmetric reals for a $Col(\omega, < \kappa)$-generic over N. Let $G_\omega \subset \mathbb{Q}_{<\delta_\omega}$ be N-generic such that for all i, $G_i = G_\omega \cap \mathbb{Q}_{<\delta_i}$ is N-generic and $\sigma = \cup\{\mathbb{R}^{N[G_i]} \mid i < \omega\}$.

Let, for each $i \leq \omega$, $j_i : N \to M_i \subset N[G_i]$ be the generic elementary embedding given by G_i. Let $j_{i_1,i_2} : M_{i_1} \to M_{i_2}$ be the induced embeddings for pairs $i_1 < i_2$ and M^* be the corresponding direct limit with associated embedding $j^* : N \to M^*$. M^* can be embedded into M_ω hence is well-founded. Also, since $Hom^N_{<\omega_1^V}$ is weakly-sealed, $j_i(Hom^N_{<\omega_1^V}) = Hom^{N[G_i]}_{<\omega_1^V}$, hence $j^*(Hom^N_{<\omega_1^V}) = Hom^{N(\sigma)}_{\omega_1^V}$. Using this, we'll show (16.1).

Using the notation of (16.1), let N[G](τ) be a symmetric extension of N[G] for $Col(\omega, < \kappa)$ such that $N(\sigma) = N[G](\tau)$. Now, j^* induces an elementary embedding $j_\sigma : L(Hom^N_{<\omega_1^V}) \to L(Hom_{<\omega_1^V})^{N(\sigma)}$ such that $j_\sigma(Hom^N_{<\omega_1^V}) = Hom^{N(\sigma)}_{<\omega_1^V}$. Similarly, there is an elementary embedding $j_\tau : (L(Hom_{<\omega_1^V})^{N[G]} \to (L(Hom_{<\omega_1^V}))^{N[G](\tau)}$ such that $j_\tau(Hom^{N[G]}_{<\omega_1^V}) = Hom^{N[G](\tau)}_{<\omega_1^V}$. But $N[G](\tau) = N(\sigma)$ so this induces an elementary embedding $j_G : L(Hom^N_{<\omega_1^V}) \to (L(Hom_{<\omega_1^V}))^{N[G]}$ such that $j_G(Hom^N_{<\omega_1^V}) = Hom^{N[G]}_{<\omega_1^V}$. This proves (16.1)

Now to see that (16.1) implies the lemma, we need to use Woodin's tree production lemma. Suppose for contradiction that $Hom^N_{<\omega_1^V} \neq L(Hom^N_{<\omega_1^V}) \cap P(\mathbb{R}^N)$. Let α be least such that $Hom^N_{<\omega_1^V} \neq L_\alpha(Hom^N_{<\omega_1^V}) \cap P(\mathbb{R}^N)$. Then there is an $A \in L_\alpha(Hom^N_{<\omega_1^V}) \cap P(\mathbb{R}^N) \backslash Hom^N_{<\omega_1^V}$ such that N can define A by a formula ϕ with parameters a pair of trees (T, S) representing a $Hom^N_{<\omega_1^V}$ set. It is then easy to check the hypotheses of the tree production lemma hold true for N and ϕ, i.e.

(a) (Generic Absoluteness) Let $\delta < \omega_1^V$ be Woodin in N, G be $< \delta$-generic over N, and H be $< \delta^+$-generic over N[G]. For all x $\in \mathbb{R} \cap N[G]$, $N[G] \vDash \phi[x, T, S] \Leftrightarrow N[G][H] \vDash \phi[x, T, S]$.

(b) (Stationary Tower Correctness) Let $\delta < \omega_1^V$ be Woodin in N, G be $\mathbb{Q}_{<\delta}$-generic over N, and $j : N \to M \subseteq N[G]$ be the induced embedding. Then for all $x \in \mathbb{R} \cap N[G]$, $N[G] \vDash \phi[x, T, S] \Leftrightarrow M \vDash \phi[x, j(T), j(S)]$

The tree production lemma then implies that $A \in Hom^N_{<\omega_1^V}$. This is a contradiction. $\square$

This implies that $L(Hom^N_{<\omega_1^V})$ is a counterexample to the theorem in the sense that $L(Hom^N_{<\omega_1^V}) \vDash AD^+ + \phi[\mathbb{R}^N]$ but no $A \in (P(\mathbb{R}))^{L(Hom^N_{<\omega_1^V})}$ satisfies that $L(A, \mathbb{R}^N) \vDash \phi[\mathbb{R}^N]$. By induction on Θ of AD^+ models and the fact that $\Theta^{L(Hom^N_{<\omega_1^V})} < \Theta^V$, we have a contradiction. So (b) of Lemma **??** can't hold; hence, (a) is the only possibility. (Theorem 1)