# The $\mathsf{AD}^+$ Conjecture and the Continuum Hypothesis *†‡

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**Abstract**

We show that Woodin's $\mathsf{AD}^+$ *Conjecture* follows from various hypotheses extending the Continuum Hypothesis ($\mathsf{CH}$). These results complement Woodin's original result that the $\mathsf{AD}^+$ *Conjecture* follows from $\mathsf{MM}(\mathfrak{c})$.

## 1 Introduction

This paper concerns Woodin's $\mathsf{AD}^+$ Conjecture, [Woo10, Definition 10.7.6]. The conjecture's original motivation was based on speculations from the Inner Model Program and has many important consequences, e.g. the definability of $\Omega$-logic. See [Woo10] for a more detailed discussion.

We identify elements of the Baire space $\omega^\omega$ with reals. Throughout the paper, by a "set of reals $A$", we mean $A \subseteq \omega^\omega$. Given a cardinal $\kappa$, we say $T \subseteq \bigcup_{n<\omega} \omega^n \times \kappa^n$ is a *tree on* $\omega \times \kappa$ if $T$ is closed under initial segments. Given a tree $T$ on $\omega \times \kappa$, we let $[T]$ be the set of its branches, i.e., $b \in [T]$ if $b \in \omega^\omega \times \kappa^\omega$ and letting $b = (b_0, b_1)$, for each $n \in \omega$, $(b_0 \restriction n, b_1 \restriction n) \in T$. We then let $p[T] = \{x \in \omega^\omega : \exists f((x, f) \in [T])\}$. A set $A$ is *Suslin* if it is $\kappa$-Suslin for some $\kappa$; $A$ is *co-Suslin* if its complement $\mathbb{R}\backslash A$ is Suslin. A set $A$ is *Suslin, co-Suslin* if both $A$ and its complement are Suslin. A cardinal $\kappa$ is a *Suslin cardinal* if there is a set of reals $A$ such that $A$ is $\kappa$-Suslin but

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$A$ is not $\lambda$-Suslin for any $\lambda < \kappa$. Suslin cardinals play an important role in the study of models of determinacy (see for example, various articles from the Cabal Volumes: [KMM83], [KMS88], [KLS08], [KLS12], [KLS16], [KMM81], [KM78]).

A set of reals $A$ is $\leq \gamma$-*universally Baire* if there are trees $T, U$ on $\omega \times \lambda$ for some $\lambda$ such that $A = p[T] = \mathbb{R} \backslash p[U]$ and whenever $g$ is $\leq \gamma$-generic (i.e. $g$ is $V$-generic for some forcing $\mathbb{P} \in V$ such that $|\mathbb{P}| \leq \gamma$), in $V[g]$, $p[T] = \mathbb{R} \backslash p[U]$. We write $A^g$ for $p[T]^{V[g]}$; this is the canonical interpretation of $A$ in $V[g]$.[1]

**Definition 1.1 ($\mathsf{AD}^+$ Conjecture, [Woo21])** *Suppose $A_0, A_1 \in \wp(\mathbb{R})$ are such that $L(A_i, \mathbb{R}) \vDash \mathsf{AD}^+$ for $i \in \{0,1\}$. Let $\Delta_i$ be the Suslin coSuslin sets of $L(A_i, \mathbb{R})$. Suppose that each $B \in \Delta_0 \cup \Delta_1$, $B$ is $\leq \omega_1$-universally Baire. Then*

$$L(\Delta_0, \Delta_1, \mathbb{R}) \vDash \mathsf{AD}^+.$$

It is consistent (relative to large cardinals) that there are divergent models of $\mathsf{AD}^+$, i.e. there are $A_0, A_1 \in \wp(\mathbb{R})$ such that $L(A_i, \mathbb{R}) \vDash \mathsf{AD}^+$ for $i \in \{0,1\}$ but $A_0 \notin L(A_1, \mathbb{R})$ and $A_1 \notin L(A_0, \mathbb{R})$. This is a theorem of Woodin (cf. [Far10]). That the hypothesis of the $\mathsf{AD}^+$ Conjecture is necessary follows from very deep analysis of divergent models of $\mathsf{AD}^+$. It is beyond the scope of this paper to discuss this any further.

Woodin, in [Woo21], has shown that $\mathsf{MM}(\mathfrak{c})$, the Martin's Maximum for partial orders of size at most the continuum, implies the $\mathsf{AD}^+$ Conjecture. This is the strongest known result regarding the conjecture in the context where the Continuum Hypothesis ($\mathsf{CH}$) fails.[2] The following two theorems show that the $\mathsf{AD}^+$ Conjecture can also hold with $\mathsf{CH}$.

Recall, for an infinite cardinal $\lambda$, the principle $\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for each $\alpha < \lambda^+$,

- $C_\alpha$ is club in $\alpha$;
- for each limit point $\beta$ of $C_\alpha$, $C_\beta = C_\alpha \cap \beta$;
- the order type of $C_\alpha$ is at most $\lambda$.

The principle $\square(\lambda)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda \rangle$ such that

1. for each $\alpha < \lambda$,
    - each $C_\alpha$ is club in $\alpha$;

[1] One can show $A^g$ does not depend on the choice of $T, U$.
[2] $\mathsf{MM}(\mathfrak{c})$ implies $\mathfrak{c} = \omega_2$.

- for each limit point $\beta$ of $C_\alpha$, $C_\beta = C_\alpha \cap \beta$; and

2. there is no thread through the sequence, i.e., there is no club $E \subseteq \lambda$ such that $C_\alpha = E \cap \alpha$ for each limit point $\alpha$ of $E$.

We remark that the hypothesis of Theorem 1.2 is consistent relative to large cardinals.

**Theorem 1.2** *Suppose* $\mathsf{CH}$ *holds and* $\neg\square(\omega_2) + \neg\square(\omega_3)$ *holds. Then the* $\mathsf{AD}^+$ *Conjecture holds.*

We say that an ideal $\mathcal{I}$ on $\omega_1$ is $\omega_1$-dense if the associated poset $\mathbb{P}_{\mathcal{I}} = \wp(\omega_1)/\mathcal{I}$ has a dense set of size $\omega_1$.[3]

**Theorem 1.3** *Suppose* $\mathsf{CH}$ *holds and there is an* $\omega_1$*-dense ideal on* $\omega_1$*. Then the* $\mathsf{AD}^+$ *Conjecture holds.*

As mentioned, the $\mathsf{AD}^+$ Conjecture was motivated by inner model theoretic considerations. One may attempt to prove the full conjecture (i.e. without any extra hypothesis) by extending the HOD analysis for all $\mathsf{AD}^+$ models. This is an active area of research in descriptive inner model theory and has many other applications (cf. [Sar14] for a treatment of the HOD analysis for all $\mathsf{AD}^+$ models below the minimal model of $\mathsf{AD}_{\mathbb{R}}$+"$\Theta$ is regular".)

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# 2 Preliminaries

## 2.1 AD and AD$^+$ Facts

We review basic facts about the Axiom of Determinacy ($\mathsf{AD}$). Suppose $A \subseteq {}^\omega\omega$. Let $G_A$ be the following game. Players I and II alternatively play natural numbers. Let $x_k$ be the natural number played at the $k$th move.

[3] The ideals considered in this paper are proper, normal, fine, and countably complete. Being $\omega_1$-dense is a very strong property; it implies for example that $\mathcal{I}$ is saturated.

| Player | 0 | | 1 | ... | n | | ... |
|---|---|---|---|---|---|---|---|
| I | $x_0$ | | $x_2$ | | $x_{2n}$ | | |
| II | | $x_1$ | | $x_3$ | | $x_{2n+1}$ | |

Let

$$x = (x_k)_{k<\omega}$$

Then $I$ wins iff $x \in A$.

$G_A$ is determined if one of the players has a winning strategy.

**Definition 2.1** ***Axiom of Determinacy,*** $\mathsf{AD}$***:*** *For every $A \subseteq {}^{\omega}\omega$, $G_A$ is determined.*

One particular game that is relevant to this paper is the Wadge game. We review it here. Let $A, B \subseteq \mathbb{R}$, the Wadge game $G_{A,B}$ for $A, B$ is defined as follows. Players I and II take turns to play integers $(n_i : i < \omega)$ and $(m_i : i < \omega)$ respectively. After $\omega$ many rounds (i.e. when the play is finished), letting $x = (n_i : i < \omega)$ and $y = (m_i : i < \omega)$, player $II$ wins the play if and only if

$$x \in A \Leftrightarrow y \in B.$$

$\mathsf{AD}$ implies that $G_{A,B}$ is determined and therefore $A, B$ are Wadge comparable. More precisely, if player II has a winning strategy $\tau$, then $\tau$ induces a continuous (in fact, Lipschitz) function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}[B] = A$; otherwise, there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g^{-1}[\mathbb{R} - A] = B$. In the first case, we say that $A$ is Wadge reducible to $B$ and we denote this by $A \leq_w B$; in the second case, we say $B$ is Wadge reducible to $\mathbb{R} - A$.

We continue with the definition of Woodin's theory of $\mathsf{AD}^+$. We use $\Theta$ to denote the sup of ordinals $\alpha$ such that there is a surjection $\pi : \mathbb{R} \to \alpha$. Under $\mathsf{AC}$, $\Theta$ is just the successor cardinal of the continuum. In the context of $\mathsf{AD}$, $\Theta$ is shown to be the supremum of $w(A)$[4] for $A \subseteq \mathbb{R}$ (cf. [Sol78]). The definition of $\Theta$ relativizes to any determined pointclass (with sufficient closure properties). We denote $\Theta^{\Gamma}$ for the supremum of ordinals $\alpha$ such that there is a surjection from $\mathbb{R}$ onto $\alpha$ coded by a set of reals in $\Gamma$.

**Definition 2.2** $\mathsf{AD}^+$ *is the theory* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and*

[4] $w(A)$ is the Wadge rank of $A$. Under $\mathsf{AD}$ the Wadge reducibility relation is a prewellorder on $\wp(\mathbb{R})$.

1. *for every set of reals $A$, there are a set of ordinals $S$ and a formula $\varphi$ such that $x \in A \Leftrightarrow L[S, x] \vDash \varphi[S, x]$. $(S, \varphi)$ is called an $\infty$-Borel code for $A$;*

2. *for every $\lambda < \Theta$, for every continuous $\pi : \lambda^\omega \to \omega^\omega$, for every $A \subseteq \mathbb{R}$, the set $\pi^{-1}[A]$ is determined.*

$\mathsf{AD}^+$ is equivalent to "$\mathsf{AD}$ + the set of Suslin cardinals is closed". Another, perhaps more useful, characterization of $\mathsf{AD}^+$ is "$\mathsf{AD}$+$\Sigma_1$ statements reflect into the Suslin co-Suslin sets" (see [ST10] for the precise statement).

## 2.2 Term capturing under $\mathsf{AD}^+$

The definition of mice and iteration strategies used in this paper are standard, see [Ste10].

**Definition 2.3** *Let $A \subseteq \mathbb{R}$, $(M, \Sigma)$ is a (countable) mouse (pair), and $\delta$ a cardinal in $M$.*

1. *$(M, \Sigma)$ **term captures** $A$ **at** $\delta$ if there is a term $\tau \in M^{Col(\omega,\delta)}$ such that whenever $i : M \to N$ is according to $N$, and $g \subseteq Col(\omega, i(\delta))$ is $N$-generic, then $A \cap N[g] = i(\tau)_g$.*

2. *$(M, \Sigma)$ **Suslin captures** $A$ **at** $\delta$ if there is a pair of trees $(T, U) \in M$ such that whenever $i : M \to N$ is according to $N$, and $g \subseteq Col(\omega, i(\delta))$ is $N$-generic, then $A \cap N[g] = p[i(T)]^{N[g]} = \mathbb{R}^{N[g]} \backslash p[i(U)]$.*

*In the above, $\Sigma$ is a $\omega_1 + 1$-iteration strategy of $M$.*[5] *$M$ need not be fine structural.*

Clearly, Suslin capturing implies term capturing. The relationship between determinacy and term capturing is best expressed by the following theorem.

**Lemma 2.4 (Neeman, [Nee95a])** *Suppose $\delta$ is a Woodin cardinal in a countable mouse $M$ and $A \subseteq \mathbb{R}$ is Suslin captured by $(M, \Sigma)$ at $\delta$. Then $A$ is determined.*

We will call the triple $(M, \Sigma, \delta)$ in Lemma 2.4 a Woodin mouse pair or coarse Woodin mouse pair if $M$ is not fine structural. Under $\mathsf{AD}^+$, the following theorem, due to Woodin, gives the existence of coarse Woodin mouse pairs capturing Suslin co-Suslin sets of reals. See [Ste08, Section 10] for a more detailed version and its proof. In the following, we say that $\Sigma$ has condensation if whenever $\mathcal{T}$ is an iteration tree according to $\Sigma$ and $\mathcal{U}$ is a hull of $\mathcal{T}$ then $\mathcal{U}$ is according to $\Sigma$. As usual, $\mathcal{T}$ is equipped with a tree order $<_T$. $\alpha <_T \beta$ implies $\alpha < \beta$ and the interval $[\alpha, \beta]_T$ is the set of $\gamma$ such that $\alpha \leq_T \gamma \leq_T \beta$. See [SS, Ste08] for more details.

[5] Iteration trees according to $\Sigma$ are normal trees in the sense of [Ste10].

**Theorem 2.5** *Suppose* $\mathsf{AD}^+$ *holds and* $A$ *is a Suslin coSuslin set. Then there is a (coarse) Woodin mouse pair* $(M, \Sigma, \delta)$ *that Suslin captures* $A$*,* $\Sigma$ *has condensation (and hence has the Dodd-Jensen property), and* $\Sigma$ *is Suslin coSuslin.*[6]

## 2.3 Axiom of Strong Condensation

In this section, we briefly discuss the Axiom of Strong Condensation, isolated by Woodin. This axiom roughly abstracts essential condensation properties typically seen in canonical inner models (like $L$). For more details, see [Woo10].

**Definition 2.6 (Axiom of Strong Condensation)** *For each cardinal* $\kappa > \omega$*, there is a bijection* $F : \kappa \to H_\kappa$ *such that for all countable* $X \prec (H_\kappa, F)$*, letting* $F_X$ *be the transitive collapse of* $F \cap X$ *under the transitive collapse map,*

$$F_X \subset F.$$

*We say that* $F$ *witnesses Strong Condensation at* $\kappa$*.*

**Remark 2.7** *By absoluteness, the* $X$ *above can be taken to be in any outer model of* $V$*. Furthermore, [Woo21, Theorem 4.3] shows that the Axiom of Strong Condensation implies many consequences typically hold in* $L$*, like* $\mathsf{GCH}$ *and there are no measurable cardinals. [Woo21, Theorem 4.3] also shows that there is a "global"* $F$ *witnessing Strong Condensation, i.e. there is an* $F : ON \to V$ *that is* $\Sigma_2$*-definable from* $F|\omega_1$ *and for every cardinal* $\kappa > \omega$*,* $F|\kappa : \kappa \to H_\kappa$ *witnesses Strong Condensation at* $\kappa$*.*

## 2.4 $\omega_1$-dense ideals on $\omega_1$

Suppose $\mathcal{I}$ is an $\omega_1$-dense ideal on $\omega_1$. The following are standard facts; see [Woo10, Definition 6.19] and the discussion after it.

**Fact 2.8**

1. $\mathbb{P}_\mathcal{I}$ *is a homogeneous forcing.*[7]
2. *There is a boolean isomorphism* $\pi : \mathbb{P}_\mathcal{I} \to RO(Coll(\omega, \omega_1))$[8]*. In particular,* $\mathbb{P}_\mathcal{I}$ *is forcing equivalent to* $Coll(\omega, \omega_1)$*.*

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[6] In fact, we can find $\Sigma \in \Sigma^1_\omega(A)$.

[7] A forcing $\mathbb{P}$ is homogeneous if whenever $p, q \in \mathbb{P}$, there is an automorphism $\sigma : \mathbb{P} \to \mathbb{P}$ such that $\sigma(p)$ is compatible with $q$.

[8] RO(Coll($\omega, \omega_1$)) is the regular open algebra of Coll($\omega, \omega_1$).

3. *For any $V$-generic filter $G \subset Coll(\omega, \omega_1)$, $\pi$ induces a $V$-generic filter $H \subset \mathbb{P}_{\mathcal{I}}$, and letting $j : V \to M =_{def} Ult(V, H) \subset V[H]$ be the associated generic ultrapower map, we have:*

   (a) *$j(f)(\omega_1^V) = G$ for some $f : \omega_1 \to H_{\omega_1}$; in particular, $V[H] = V[G]$.*

   (b) *$j(\omega_1^V) = \omega_2^V$.*

   (c) *$M$ is well-founded and $M^\omega \subset M$ in $V[H]$.*

# 3 Proof of Theorem 1.2

Suppose $A_0, A_1$ are as in the hypothesis of the $\mathsf{AD}^+$ Conjecture. By Theorem 2.5, there are coarse Woodin pairs $(M_i, \Sigma_i, \delta_i)$ that Suslin captures $A_i$ for each $i \in \{0, 1\}$. Let $(T_i, U_i)$ witness $\Sigma_i$ is $\leq \omega_1$-universally Baire. In the following, $\mathcal{M}_1^{(\Sigma_0,\Sigma_1),\sharp}$ is the minimal active mouse with a Woodin cardinal that is closed under $\Sigma_0$ and $\Sigma_1$. This is a kind of strategy mice and its general theory has been fully developed in for example [ST16]. We fix a canonical coding $Code : H_{\omega_1} \to \mathbb{R}$ as in [Woo10, Chapter 2]. This coding is simply definable and generically absolute.

**Lemma 3.1** *$\mathcal{M}_1^{(\Sigma_0,\Sigma_1),\sharp}$ exists.*

*Proof.* For each $i$, since $\Sigma_i$ is $\leq \omega_1$-universally Baire, $\Sigma_i$ can be uniquely extended to a strategy (which we will also call $\Sigma_i$) on $H_{\omega_2}$. For a tree $\mathcal{T} \in H_{\omega_2}^V$, according to $\Sigma_i$,

$$\Sigma_i(\mathcal{T}) = b \text{ iff } \emptyset \Vdash_{Coll(\omega,\omega_1)} (Code(\mathcal{T}), Code(b)) \in p[T_i].$$

Fix $i$ and let $\Lambda = \Sigma_i$. We let $\Lambda^g = p[T_i] \cap V[g]$ for any generic $g \subseteq Coll(\omega, \omega_1)$. It is easy to see that the definition of $b$ is independent of generics and therefore $b \in V$. To see this, suppose there is an ordinal $\gamma$ and conditions $p, q$ such that $p \Vdash_{Coll(\omega,\omega_1)} \check{\gamma} \in \dot{\Lambda}^g(\check{T})$ iff $q \Vdash_{Coll(\omega,\omega_1)} \check{\gamma} \notin \dot{\Lambda}^g(\check{T})$. But then, by the homogeneity of $Coll(\omega, \omega_1)$, we can find generics $g_0, g_1$ such that

- $V[g_0] = V[g_1]$.
- $p \in g_0$ and $q \in g_1$.

Since $p[T_i] \cap V[g_0] = p[T_i] \cap V[g_1]$, $\Lambda^{g_0}(\mathcal{T}) = \Lambda^{g_1}(\mathcal{T})$. This contradicts what $p, q$ force.

We have shown that $\Lambda$ can be extended to a (necessarily) unique strategy, also called $\Lambda$, acting on trees in $H_{\omega_2}$. Now we extend $\Lambda$ to $H_{\omega_3}$. Suppose $\mathcal{T}$ is a normal tree of length $\geq \omega_2$ in $H_{\omega_3}$. If $\text{cof}(lh(\mathcal{T})) = \omega_2$, using $\neg\Box(\omega_2)$, we can easily find a cofinal branch $b$ through $\mathcal{T}$.[9] This branch is necessarily unique and well founded. We define $\Lambda(\mathcal{T}) = b$.

Suppose $\text{cof}(lh(\mathcal{T})) \leq \omega_1$. We define $X$ to be *good* if

$$X \prec (H_{\omega_3}, \in), |X| = \omega_1, \text{ and } X^\omega \subseteq X.$$

Note that such good hulls exist by $\mathsf{CH}$. For a good $X$ such that $\mathcal{T} \in X$, we let $\pi_X : M_X \to X$ be the uncollapse map, $\mathcal{T}_X = \pi_X^{-1}(\mathcal{T})$ and $b_X \in \Lambda(\mathcal{T}_X)$. We claim.

**Claim 3.2** *There is a good $X$ such that for any good $X \prec Y$, letting $\pi_{X,Y} = \pi_Y^{-1} \circ \pi_X$, $c_{X,Y} = \pi_{X,Y}[b_X]$, then $c_{X,Y} \subseteq b_Y$.*

*Proof.* Since $\text{cof}(lh(\mathcal{T})) \leq \omega_1$, for any good $X$, $X \cap \mathcal{T}$ is cofinal in $\mathcal{T}$. Suppose $\text{cof}(lh(\mathcal{T})) = \omega_1$, then it is easy to see that the claim holds for any good $X$ such that $\mathcal{T} \in X$.

Now suppose $\text{cof}(lh(\mathcal{T}) = \omega$, then note that for any good $X$, since $X^\omega \subset X$, $b_X \in M_X$. The argument is as in [Ste05]. Suppose there is no such $X$ as in the claim, we can form an elementary chain $(X_\nu : \nu \in \omega_2)$ such that:

1. If $\nu$ is a limit ordinal then $X_\nu = \bigcup_{\alpha<\nu} X_\alpha$.

2. If $\nu$ is a successor ordinal, then $X_\nu$ is good.

3. For each successor $\nu$ or for each limit $\nu$ such that $\text{cof}(\nu) > \omega$, $c_{\nu,\nu+1} =_{def} c_{X_\nu, X_{\nu+1}} \neq b_{\nu+1} =_{def} b_{X_{\nu+1}}$.

We also write $\pi_{\alpha,\beta}$ for $\pi_{X_\alpha, X_\beta}$ etc. An easy argument (using that for each $\nu < \omega_2$ with $\text{cof}(\nu) > \omega$, $b_\nu \in M_{X_\nu}$) gives a stationary $S \subset \omega_2$ such that

- $\nu \in S \Rightarrow \text{cof}(\nu) > \omega$, and

- $\nu, \gamma \in S \Rightarrow \pi_{\nu,\gamma}(b_\nu) = b_\gamma$.[10]

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[9]This is a standard argument. The set $\vec{C} = \{[0, \alpha]_T : \alpha < lh(\mathcal{T})\}$ is a coherent sequence on $lh(\mathcal{T})$. Fix a continuous, increasing function $f : \omega_2 \to lh(\mathcal{T})$, we can use $f$ to pullback $\vec{C}$ into a coherent sequence $\vec{D}$ in $\omega_2$. Now apply $\neg\Box(\omega_2)$ to get a thread $E$ for $\vec{D}$. Then $f[E]$ is a thread through $\vec{C}$ and gives a cofinal branch through $lh(\mathcal{T})$.

[10]Suppose the set $S$ as defined is not stationary. So there is a club $C$ such that $C \cap S = \emptyset$. For any $\nu < \gamma \in C$ with uncountable cofinality, $\pi_{\nu,\gamma}(b_\nu) \neq b_\gamma$. Let $\gamma \in lim(C)$ be of uncountable cofinality and is a limit of points in $C$ of uncountable cofinality. Since $C$ is club below $\gamma$ and $\text{cof}(lh(\mathcal{T}_\gamma)) = \omega$, we can easily find $\nu \in C \cap \gamma$ such that $\text{rng}(\pi_{\nu,\gamma}) \cap b_\gamma$ is cofinal in $b_\gamma$ and $b_\gamma \in \text{rng}(\pi_{\nu,\gamma})$, but then $\pi_{\nu,\gamma}^{-1}[b_\gamma] = b_\nu$ by condensation of $\Sigma$, i.e. $\pi_{\nu,\gamma}(b_\nu) = b_\gamma$. Contradiction.

For $\nu < \gamma \in S$, $\pi_{\nu+1,\gamma}$ witnesses $(\mathcal{T}_{\nu+1})^\frown \pi_{\nu,\nu+1}(b_\nu)$ is a hull of $\mathcal{T}_\gamma^\frown b_\gamma$, and so $\pi_{\nu,\nu+1}(b_\nu)$ is according to $\Sigma$, i.e. $c_{\nu,\nu+1} = b_{\nu+1}$. Contradiction. $\square$

Using the claim, we can define $\Lambda(\mathcal{T})$ to be the downward closure of $\pi_X[b_X]$, where $X$ is as in the claim.

Now we show $\mathcal{M}_1^{(\Sigma_0,\Sigma_1),\sharp}$ exists and is $\omega_3$-iterable. The first step is to show $H_{\omega_3}^V$ is closed under $(\Sigma_1, \Sigma_2)^\sharp$. More precisely, for any $A \in H_{\omega_3}$, $A^{(\Sigma_0,\Sigma_1),\sharp}$ exists.[11] If not, then by covering, letting $M = L_{\omega_3}^{(\Sigma_0,\Sigma_1)}[A]$,[12] where $A$ is some subset of $\omega_2$, letting $\gamma = (\omega_2^V)^{+,M}$,

$$\mathrm{cof}(\gamma) \geq \omega_2.$$

But $\neg\square(\omega_3) \Rightarrow \gamma < \omega_3^V$. This is because if $\gamma = \omega_3^V$, letting $\vec{C}$ be the canonical $\square_{\omega_2^V}$-sequence defined over $M$, then $\neg\square(\omega_3)$ implies there is a thread $D$. The thread $D$, as usual, gives a collapsing structure for $\gamma$, i.e. some sound model $N$ such that $\gamma \in N$ and $\rho_\omega(N) = \omega_2^V$ (this means $N$ projects to $\omega_2$). This is a contradiction as $\gamma$ was assumed to be a cardinal in $V$. So $\gamma < \omega_3^V$. Then $\neg\square(\omega_2) \Rightarrow \mathrm{cof}(\gamma) < \omega_2$ by a similar argument. Contradiction.

Similarly, we can then show $\mathcal{M}_1^{(\Sigma_0,\Sigma_1),\sharp}$ exists. Otherwise, the core model $K = K^{(\Sigma_0,\Sigma_1)}$ exists.[13] Let $\gamma = (\omega_2^V)^{+,K}$. By covering, cf. [JS13], $\mathrm{cof}(\gamma) \geq \omega_2$, but as before $\neg\square(\omega_3) + \neg\square(\omega_2) \Rightarrow \mathrm{cof}(\gamma) < \omega_2$. Contradiction.

Now to finish the proof of the theorem, it is enough to show that the Wadge game $G_{A_0,A_1}$ is determined. Let $H = \mathcal{M}_1^{(\Sigma_0,\Sigma_1),\sharp}$, $\delta^H$ be the Woodin cardinal of $H$, $\Sigma$ be $H$'s canonical strategy, and let $\tau_i \in H$ be $Coll(\omega, \delta^H)$-terms for $A_i$. In particular, for any $g \subseteq Coll(\omega, \delta^H)$ in $V$, $(\tau_i)_h = A_i \cap H[g]$.[14]. First, we note that the Wadge game $G_{A_0,A_1}$ is determined in $H[g]$ for any $H$-generic $g \subseteq Coll(\omega, \delta^H)$ in $V$ (via a strategy in $H$). This follows from the fact that the corresponding Neeman's game $\hat{G}_{A_0,A_1}$ is determined in $H[g]$ (see [Nee95b]).

Now we let $(M_i, \Sigma_i)_{i<\omega}$ enumerate the coarse Woodin pairs that Suslin captures all sets in $\Sigma^1_\omega(A_0) \cup \Sigma^1_\omega(A_1)$. By a similar proof, $H = \mathcal{M}_1^{(\Sigma_i:i<\omega),\sharp}$ exists and $G_{A_0,A_1}$

[11] This is the theory of the indiscernibles for the model $L_{\omega_3}^{(\Sigma_0,\Sigma_1)}[A]$. Here the language is the language of set theory augmented by the following predicate symbols: a unary predicate symbol $\dot{A}$ and two binary predicate symbols $\dot{\Sigma}_1, \dot{\Sigma}_2$.

[12] Again, $L_{\omega_3}^{(\Sigma_0,\Sigma_1)}[A]$ is the minimal model over $A$ of height $\omega_3$ and is closed under strategies $\Sigma_0$ and $\Sigma_1$. [ST16] gives a detailed treatment of how to feed strategy information of $\Sigma_0$ and $\Sigma_1$ into the model. This model is $L$-like in that it satisfies all condensation properties $L$ satisfies. The key here, of course, is because $\Sigma_0$ and $\Sigma_1$ have condensation properties.

[13] Here we construct the Jensen-Steel core model as in [JS13] up to $\omega_3^V$. Again, the core model and the corresponding $K^c$-construction are hybrid, relativized to $(\Sigma_0, \Sigma_1)$.

[14] In fact, we can take $\tau_i$ to be some tree $T_i \in \omega \times \lambda$ for some $\lambda \in H$.

is determined in $H^{Coll(\omega,\delta)}$ via a strategy in $H$, where $\delta$ is the Woodin cardinal of $H$. We note that $H$ has $Coll(\omega,\delta)$-terms that capture $A_0, A_1$ as well as scales on $A_0$, $A_1$. Therefore,

$$(H[g] \cap V_{\omega+1}, H[g] \cap A_0, H[g] \cap A_1) \prec (V_{\omega+1}, A_0, A_1),$$

for any $H$-generic $g \subset Coll(\omega,\delta)$ such that $g \in V$. This implies that $G_{A_0,A_1}$ is determined in $V$. This completes the proof of the theorem.

□

# 4 Proof of Theorem 1.3

Let $\mathcal{I}$ be an $\omega_1$-dense ideal on $\omega_1$. Let $A_0, A_1$ be as in the hypothesis of the $\mathsf{AD}^+$ Conjecture. We note that by our hypothesis, for any $C \in \Sigma^1_\omega(A_0) \cup \Sigma^1_\omega(A_1)$, for any generic $g$ for a poset $\mathbb{P} \in H_{\omega_2}$,

$$(V_{\omega+1}, C) \prec (V[g]_{\omega+1}, C_g), \tag{1}$$

where $C_g$ is the canonical interpretation of $C$ in $V[g]$.

Let $P$ be the term relation for $\Sigma^1_\omega(A_0) \cup \Sigma^1_\omega(A_1)$. More precisely, $P$ consists of tuples $(i, \varphi, \mathbb{P}, \sigma, q)$ such that

- $i \in \{0, 1\}$.
- $\mathbb{P} \in H_{\omega_2}$ is a poset.
- $\sigma \in V^{\mathbb{P}} \cap H_{\omega_2}$ is a term for a real.
- For a closed unbounded set of countable $X \prec H_{\omega_2}$, for a comeager set of $X$-generic filter $g \subset X \cap \mathbb{P}$: if $i = 0$ and $q \in g$ then $(V_{\omega+1}, A_0) \vDash \varphi[\sigma_g]$; and if $i = 1$ and $q \in g$ then $(V_{\omega+1}, A_1) \vDash \varphi[\sigma_g]$.

By [Woo21] and 1, for all generic $g$ for a poset in $H_{\omega_2}$, for all bounded $Z \subset \omega_2^V$ in $V[g]$,

$$L_{\omega_2^V}[Z, P]^{Coll(\omega,\sup(Z))} \vDash \mathsf{ZFC}+ \text{Axiom of Strong Condensation}.$$

Let $G \subset Coll(\omega,\omega_1)$ be $V$-generic. Note that $Coll(\omega,\omega_1) \in H^V_{\omega_2}$ and $G$ induces a generic $g \subseteq \wp(\omega_1)/\mathcal{I}$ and a generic elementary embedding $j : V \to M \subseteq V[g]$. Similarly, over $M$, we let $k : M \to N$ be the generic embedding induced by an

$M$-generic $h \subseteq Coll(\omega, \omega_1^M)$. We note that by Sectiion 2.4, $M^\omega \subset M$ in $V[G]$; in particular, $\mathbb{R}^M = \mathbb{R}^{V[G]}$.

We then have by strong condensation of $P$ and the fact that $M^\omega \subset M$ in $V[G]$:

$$j(P) = P_G \cap M.\text{[15]} \tag{2}$$

**Claim 4.1** *For all bounded $Z \subset \omega_2^V$, there is a closed and unbounded set $C \subset \omega_2$ of indiscernibles for the structure $(N_Z =_{def} L_{\omega_2^V}[Z, P], P \cap N_Z)$.*

*Proof.* First let $Z$ be a bounded subset of $\omega_1$. We note the following:

1. $N_Z = L_{\omega_2^V}[Z, j(P)]$.
2. $\omega_1^V$ and $\omega_2^V = j(\omega_1) = \omega_1^M$ are strongly inaccessible in $N_Z$.

Item (1) is proved in [Woo21] using the fact that $N_Z$ satisfies the Axiom of Strong Condensation. Here is a quick sketch. Let $F : ON^{N_Z} \to N_Z$ be a function witnessing Strong Condensation of $N_Z$. For all uncountable cardinal $\kappa < \omega_2^V$ of $N$, $j(F)|\kappa$ witnesses Strong Condensation. By Remark 2.7, $\pi(F)|\kappa = F|\kappa$ because

$$(j[H_\kappa^{N_Z}], j[F|\kappa]) \prec (j(H_\kappa^{N_Z}), j(F|\kappa)) \prec (H_{j(\kappa)}^{j(N_Z)}, j(F)|j(\kappa)).$$

This gives (1).

For item (2), first note that $j(N_Z)$ has the form $L_{j(\omega_2^V)}[Z, j(P)] = L_{j(\omega_2^V)}[Z, P_G]$. Suppose there is a $\kappa < \omega_1^V$ such that in $N_Z$, $\wp(\kappa) \geq \omega_1^V$, we may assume $\kappa^+ = \omega_1^V$ in $N_Z$. But this means in $j(N_Z)$,

$$\kappa^+ = j(\omega_1^V) = \omega_2^V.$$

This contradicts the agreement of $N_Z$ and $j(N_Z)$ in item (1) and the fact that $\omega_1^V, \omega_2^V$ are cardinals of $N_Z, j(N_Z)$. Now use the fact that $\omega_2^V = j(\omega_1^V)$ and apply the above argument to $j(N_Z)$ and $k$, we conclude that $\omega_2^V$ is strongly inaccessible in $j(N_Z)$. By item (1), $\omega_2^V$ is also strongly inaccessible in $N_Z$.

The proof of [Woo21, Theorem 4.4] then shows the conclusion of the claim for this particular choice of $Z$.[16]

[15] Here $P_G$ is $P$ as interpreted in $V[G]$. Note here that $j(A_i) = (A_i)_G$, the canonical interpretation of $A_i$ in $V[G]$.

[16] In fact, one gets that $(N_Z, P)^\sharp$ exists. Roughly, we can show that letting $\mu$ be the measure over $N_Z$ derived from $j$ and let $M_Z = Ult(N_Z, \mu)$. Let $i_\mu : N_Z \to M_Z$ be the ultrapower map. Then there is a canonical factor map $\sigma : M_Z \to j(N_Z)$ such that $j \restriction N_Z = \sigma \circ i_\mu$. We also have $\mathrm{crt}(\sigma) = i_\mu(\omega_1^V)$. In fact by condensation, $M_Z = N_Z$. We can then derive a measure over $M_Z$ from $\sigma$. We continue this process, showing that $(N_Z, \mu)$ is iterable.

Now assume $Z \subset \omega_2^V$ is bounded. Note then that in $M$, $Z$ is a bounded subset of $\omega_1$. We apply the above proof to $L_{\omega_2^M}[Z, j(P)] =_{def} N_Z^*$ and use $k$. We conclude that in $k(N_Z^*)$, $\omega_2^M$ is a limit of a club of indiscernibles. By elementarity, in $N_Z^*$, $\omega_1^M$ is a limit of a club of indiscernibles. But again by strong condensation, $N_Z^* \restriction \omega_1^M = N_Z$. This completes the proof of the claim. $\square$

Now let $R = L_{\omega_2^V}(\mathbb{R})[P]$. Let $(M_0, \Sigma_0, \delta_0)$, $(M_1, \Sigma_1, \delta_1) \in R$ be coarse Woodin pairs that capture $A_0, A_1$ respectively. As in the proof of Theorem 1.2, it suffices to show $\mathcal{M}_1^{(\Sigma_0,\Sigma_1),\sharp}$ exists. Suppose not. Then for any $H \subset Coll(\omega_1, \mathbb{R})$ in $V$, $K = K^{(\Sigma_0,\Sigma_1)}$ exists in $R[H]$.[17] We note that:

- $R[H]$ has the form $L[Z, P]$, where $Z$ is a bounded subset of $\omega_2^V$ coding $(\mathbb{R}^V, H)$.
- $K \in R$ by homogeneity of $Coll(\omega_1, \mathbb{R})$.
- $j(K) \in V$ by homogeneity of $Coll(\omega, \omega_1)$ and $j(R)$ being definable in $V[g]$ from parameters in $V$.[18]

From the above, the proof of [SS, Section 2.11] goes through and show that $\omega_1^V$ must be Shelah in $j(K)$. Contradiction. Hence $\mathcal{M}_1^{(\Sigma_0,\Sigma_1),\sharp}$ exists. As before, we let $(M_i, \Sigma_i)_{i<\omega}$ enumerate the coarse Woodin pairs that Suslin captures all sets in $\Sigma^1_\omega(A_0) \cup \Sigma^1_\omega(A_1)$. By a similar proof, $H = \mathcal{M}_1^{(\Sigma_i:i<\omega),\sharp}$ exists and $G_{A_0,A_1}$ is determined in $H^{Coll(\omega,\delta)}$ via a strategy in $H$, where $\delta$ is the Woodin cardinal of $H$. This implies $G_{A_0,A_1}$ is determined in $V$ and completes the proof of the theorem.

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[17] That we can find such a $H$ in $V$ follows from CH. $K$ is the Jensen-Steel core model, cf [JS13]. Since $R[H]$ contains all reals in $V$ and is closed under $(\Sigma_0, \Sigma_1)$, whether $\mathcal{M}_1^{(\Sigma_0,\Sigma_1),\sharp}$ exists is absolute between $V$ and $R[H]$.

[18] The parameters are the collection of trees $\{(T_C, S_C) : C \in \Sigma^1_\omega(A_0) \cup \Sigma^1_\omega(A_1)\}$ witnessing each $C \in \Sigma^1_\omega(A_0) \cup \Sigma^1_\omega(A_1)$ is $\leq \omega_1$-universally Baire. Here, we also use equation 2 to get that $j(R)$ has the form $L[j(Z), P_G]$ to be able to use homogeneity of $Coll(\omega, \omega_1)$.

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