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Baire Functions, Borel Sets, and Ordinary Function Systems

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1. INTRODUCTION

If Φ is a family of real-valued functions defined on a set X, then there is a smallest family, $B(\Phi)$, of real-valued functions defined on X which contains Φ and which is closed under the process of taking pointwise limits of sequences from $B(\Phi)$. This family is called the Baire system generated by Φ. One method of generating $B(\Phi)$ from Φ is by iteration of the operation of pointwise limits of sequences: Let Φ_0 be Φ and for each ordinal $\alpha > 0$, let Φ_α be the family of all pointwise limits of sequences from $\bigcup_{\gamma<\alpha} \Phi_\gamma$. Then $\Phi_{\omega_1} = \Phi_{\omega_1+1} = B(\Phi)$, where ω_1 is the first uncountable ordinal. This system was described by René Baire in this thése, published in 1899 [2].

This paper is meant to be an exposition of some of the main results concerning this process that have been obtained since then. The second section concerns itself with some properties of the classes Φ_α, under the assumption that the family Φ forms a lattice. In the third section, a development of the relationship between the classes Φ_α and Borel type sets which are inverse image sets of functions in Φ_α is given. That section concludes with Hausdorff's notion of an ordinary function system. These systems are completely characterized by their inverse image sets and yield appealing coextensive processes of generating the Baire system and of generating a certain Borel system (σ-algebra) of sets.

The Baire order of a family of functions Φ is the first ordinal α such that $\Phi_\alpha = \Phi_{\alpha+1}$. The Baire order problem for $C(X)$, the space of real-valued continuous functions on a topological space X, is studied in the fourth section. Two proofs, due to Lebesgue, are given to show that the Baire order of $C[0, 1]$ is ω_1. Later in this section it is shown that the Baire order of $C(X)$, with X compact and T_2, distinguishes those spaces which contain perfect sets from those which do not (dispersed spaces); the Baire order of the dispersed spaces being 0 or 1 and the others ω_1.

In the last section the Baire order of various families of functions which satisfy some relaxed continuity condition is investigated. The paper concludes with a return to the Baire order problem for an arbitrary complete function system Φ. It is shown that there is a completely regular space δX, an embedding of Φ in $C(\delta X)$ which extends to an embedding of the family Φ_α in $(C(\delta X))_\alpha$, for each α.

Because of the vast amount of work in this area, much has been left unsaid here. The author hopes that the reader will find the references, although uncomplete, useful. The journal *Fundamenta Mathematicae* contains much of the work in this area and the books on set theory and topology by Hausdorff [15] and Kuratowski [21] contain a wealth of information. Finally, the reader may find the recent papers of Lorch [26] and Frolik [11] helpful.

2. Lattices

In this section, it is assumed that the family Φ forms a lattice of real-valued functions defined on a set X. Consequently, for each ordinal α, the family Φ_α forms a lattice. In order to describe the families Φ_α in more detail, let us introduce the following notation. If H is a subfamily of R^X, let $USH(LSH)$ be the family of all pointwise limits of sequences $\{f_n\}_{n=1}^\infty$ from H such that for each integer n, $f_n \geqslant f_{n+1} (f_n \leqslant f_{n+1})$. It follows that the families $US\Phi$ and $LS\Phi$ are lattices.

The first theorem, due to Sierpinski [45], shows that the family Φ can be realized as the intersection of two families obtained from Φ by a twofold process of taking monotone limits.

THEOREM 2.1. *A function f belongs to Φ_1 if and only if f belongs to both $US(LS\Phi)$ and $LS(US\Phi)$.*

Proof. Suppose $f \in \Phi_1$. Let $\{f_n\}_{n=1}^\infty$ be a sequence from Φ converging pointwise to f. Let g_1 be the limit of the sequence $f_1, f_1 \vee f_2, f_1 \vee f_2 \vee f_3, \ldots$. The function g_1 belongs to $LS\Phi$. For each integer p, let

$$g_p = \lim_{n\to\infty} \left(\bigvee_{i=1}^{n} f_{i+p-1} \right).$$

For each p, g_p belongs to $LS\Phi$ and $g_p \geqslant g_{p+1} \geqslant f$. Also, it follows that $f = \lim_{p\to\infty} g_p$. Thus f belongs to $US(LS\Phi)$. Similarly, f belongs to $LS(US\Phi)$.

Conversely, suppose $f \in US(LS\Phi) \cap LS(US\Phi)$. Let $\{g_n\}_{n=1}^{\infty}$ be a nonincreasing sequence from $LS\Phi$ converging to f, and let $\{h_n\}_{n=1}^{\infty}$ be a nondecreasing sequence from $US\Phi$ converging to f. For each n, let $\{g_{np}\}_{p=1}^{\infty}$ be a nondecreasing sequence from Φ converging to g_n, and let $\{h_{np}\}_{p=1}^{\infty}$ be a nonincreasing sequence from Φ converging to h_n:

$$\begin{array}{lr}
g_{11} \leqq g_{12} \leqq g_{13} \leqq \cdots \leqq g_{1n} \leqslant \cdots & \to g_1 \\
 & \rotatebox{90}{$\leqq$} \\
g_{21} \leqslant g_{22} \leqslant g_{23} \leqslant \cdots & \to g_2 \\
 & \rotatebox{90}{$\leqq$} \\
g_{31} \leqslant g_{32} \leqslant g_{33} \leqslant \cdots & \to g_3 \\
 & \rotatebox{90}{$\leqq$} \\
 & \vdots \\
 & \downarrow \\
 & f \\
 & \uparrow \\
 & \vdots \\
 & \rotatebox{90}{$\leqq$} \\
h_{31} \geqq h_{32} \geqq h_{33} \geqq \cdots & \to h_3 \\
 & \rotatebox{90}{$\leqq$} \\
h_{21} \geqq h_{22} \geqq h_{23} \geqq \cdots & \to h_2 \\
 & \rotatebox{90}{$\leqq$} \\
h_{11} \geqq h_{12} \geqq h_{13} \geqq \cdots \geqq h_{1n} \geqq \cdots & \to h_1 .
\end{array}$$

For each pair np, let $k_{np} = h_{pn} \wedge g_{1n} \wedge g_{2n} \wedge \cdots \wedge g_{pn}$, and for each n, let $f_n = \bigvee_{p=1}^{n} k_{np}$. It will be shown that the sequence $\{f_n\}_{n=1}^{\infty}$ of functions from Φ converges to f pointwise. Let x be a member of the set X. First, it will be shown that the lower limit of $\{f_n(x)\}_{n=1}^{\infty}$ is not less than $f(x)$.

Let a be a number less than $f(x)$. Let w be an integer such that $h_w(x) > a$. Of course, $h_{wp}(x) > a$, for every integer p. Let r be an integer such that $g_{ml} > a$, for $m = 1, 2, \ldots, w$, and $l > r$.

Thus, if $n > r$, then $k_{nw}(x) = (h_{wn} \wedge g_{1n} \wedge g_{2n} \wedge \cdots \wedge g_{wn})(x) > a$. Also, if $n > r + w$, then $f_n(x) = (\bigvee_{p=1}^{n} k_{np})(x) \geqslant k_{nw}(x) > a$. Therefore, $\lim_{n-\infty} f_n(x) \geqslant f(x)$.

Next, it will be shown that the upper limit of $\{f_n(x)\}_{n=1}^{\infty}$ is not more than $f(x)$. Suppose $b > f(x)$. Let q be an integer such that $g_q(x) < b$. Of course, $g_{qn}(x) < b$, for every n. Let s be an integer such that $h_{mn}(x) < b$, for $m = 1, 2, \ldots, q$, and $n > s$.

Suppose $n > s$. If $m \leqslant q$, then $k_{nm}(x) \leqslant h_{mn}(x) < b$, and if $m \geqslant q$, then $k_{nm}(x) \leqslant g_{qn}(x) < b$. Therefore, $f_n(x) = (\bigvee_{p=1}^{n} k_{np})(x) < b$ and $\overline{\lim}_{n\to\infty} f_n(x) \leqslant f(x)$. This completes the argument for Theorem 2.1.

There is another property that the families Φ_α, $\alpha > 0$ enjoy. It will be formulated in the next definition.

DEFINITION. A subfamily H of R^X will be said to have the interposition property provided that if $g \in LSH$, $h \in USH$, and $g \geqslant h$, then there is a function f in H such that $g \geqslant f \geqslant h$.

Note that if the lattice Φ has the interposition property, then the proof of the second part of Theorem 2.1 becomes very simple: For each p, let f_p be a function from Φ such that $g_p \geqslant f_p \geqslant h_p$. The sequence $\{f_p\}_{p=1}^{\infty}$ converges pointwise to f.

The next theorem yields as a corollary a characterization of those lattices which have the interposition property.

THEOREM 2.2. *Suppose $g \in LS\Phi$, $h \in US\Phi$, and $g \geqslant h$. There is a function f in $US\Phi \cap LS\Phi$ such that $g \geqslant f \geqslant h$* [49].

Proof. Let $\{g_p\}_{p=1}^{\infty}$ be a nondecreasing sequence Φ converging to g, and let $\{h_p\}_{p=1}^{\infty}$ be a nonincreasing sequence from Φ converging to h. Let $u_1 = g_1$, and for each n, let $v_n = u_n \vee h_n$ and $u_{n+1} = v_n \wedge g_{n+1}$. It can be shown that (1) $u_n \leqslant u_{n+1}$, $v_{n+1} \leqslant v_n$, and $v_n \geqslant u_n$, for each n, and (2) there is a function f which is the pointwise limit of the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$. Finally, this function f, which is in $US\Phi \cap LS\Phi$, is interposed between g and h, $g \geqslant f \geqslant h$.

Remark. It should be noted that if the sequences $\{g_p(x)\}_{p=1}^{\infty}$ and $\{h_p(x)\}_{p=1}^{\infty}$ are strictly monotone for some x in X, and $g(x) > h(x)$, then $g(x) < f(x) < h(x)$. Also, sequences $\{g_p\}_{p=1}^{\infty}$ can be constructed which are strictly monotone for every x, if Φ is a linear lattice containing constants [30].

As an obvious corollary of Theorem 2.2, we have:

COROLLARY 2.2a. *The lattice Φ has the interposition property if and only if $US\Phi \cap LS\Phi = \Phi$.*

Also we obtain:

COROLLARY 2.2b. *The lattices Φ_α, $\alpha > 0$ have the interposition property.*

Proof. It is enough to show that Φ_1 has the interposition property. If $f \in US\Phi_1$, then $f \in US(US(LS\Phi))$. But for any lattice H,

$US(USH) = USH$. So $f \in US(LS\Phi)$. Similarly, if $f \in LS\Phi_1$, then $f \in LS(US\Phi)$ and by Theorem 2.1, $(US\Phi_1) \cap (LS\Phi_1) = \Phi_1$.

It has been demonstrated that if Φ is a lattice of functions, then not only are the families Φ_α, $\alpha > 0$, lattices, but they also have the interposition property.

Note that Φ_1 is a lattice containing Φ having the interposition property and

COROLLARY 2.2c. *If $H = US\Phi \cap LS\Phi$, then H is the smallest lattice containing Φ having the interposition property, and $H_1 = \Phi_1$.*

Proof. As before, $USH = US\Phi$ and $LSH = LS\Phi$. From these facts it follows that H has the interposition property. Moreover, $US(LSH) = US(LS\Phi)$ and $LS(USH) = LS(US\Phi)$. Employing Theorem 2.1, one has $\Phi_1 = H_1$.

Thus, in the study of the families Φ_α, $\alpha > 0$, there is no loss in assuming that the original lattice Φ has the interposition property.

3. Borel Sets and Ordinary Function Systems

Throughout this section it is assumed that Φ is a linear (vector) lattice containing the constant functions. Under this assumption, the families Φ_α, $\alpha > 0$, have a rich algebraic structure. These families form uniformly closed algebras containing the constant functions which are also closed under inversion [Theorem 3.1]. Such families were termed complete ordinary function systems by Hausdorff and were thoroughly investigated by him [15]. Some of his principal results are that such systems are in exact correspondence with a certain family of sets [Corollary 3.2a] and that the Baire system generated by such a family is coextensive with a certain Borel system or σ-algebra of sets [Theorem 3.5]. The presentation given here strengthens these results of Hausdorff in that in [15], it is assumed that the family Φ forms an ordinary system (see Definition), whereas here it is only assumed that Φ is a vector lattice containing the constant functions.

THEOREM 3.1. *The family Φ_1 is an algebra and lattice containing the constant functions which is closed under both uniform limits and inversion.*

Proof. Since Φ is a vector lattice containing the constant functions, it is clear that Φ_1 is a linear lattice containing the constant functions.

Suppose that for each p, $|g_p(x) - f(x)| < 1/p$, for each x in X and $g_p \in \Phi_1$. Let $h_n = \wedge_{p=1}^{n} (g_p + (1/p))$. It follows that $\{h_n\}_{n=1}^{\infty}$ is a non-increasing sequence from Φ_1 converging pointwise to f. Thus, f is in $US\Phi_1$. Similarly, f is in $LS\Phi_1$. Since Φ_1 has the interposition property, $f \in \Phi_1$. Therefore, Φ_1 is closed under uniform limits.

In order to show that Φ_1 is an algebra, we show that if $f \in \Phi_1$, then $f^2 \in \Phi_1$. First, it is easy to see that $\Phi_1 = (\Phi_u)_1$, where Φ_u is the space of all uniform limits of sequences from Φ. If $f \in \Phi_1$, then f is the pointwise limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of bounded functions from Φ. Since the bounded functions in Φ_u form an algebra [30], $f_n^2 \in \Phi_u$ and thus $f^2 \in (\Phi_u)_1 = \Phi_1$.

If f is a nonnegative function in Φ_1, then there is a sequence $\{f_n\}_{n=1}^{\infty}$ of positive bounded functions in Φ, each bounded away from 0, converging pointwise to f. Since Φ_u is an algebra and is closed under uniform limits, we have $1/f_n \in \Phi_u$ [30], for each n. Hence, $1/f \in (\Phi_u)_1 = \Phi_1$. If f is an arbitrary function in Φ_1, then $1/f^2 \in \Phi_1$ and $1/f = f \cdot 1/f^2 \in \Phi_1$. Thus, Φ_1 is closed under inversion.

We now give some definitions made by Hausdorff [15].

DEFINITION. The statement that $\mathfrak{A}$ is on ordinary funtion system means that $\mathfrak{A}$ *is* both a vector lattice and algebra containing the constant functions and $\mathfrak{A}$ is closed under inversion. An ordinary function system is said to be complete if it is also closed under uniform limits.

Of course, a complete ordinary function system is an algebra containing the constant functions which is closed under both uniform limits and inversion. These systems have been the object of some recent studies of Hager [16].

We can restate Theorem 3.1 as:

THEOREM 3.1. *If Φ is a vector lattice containing the constant functions, then Φ_1 is a complete ordinary function system.*

We shall now develop two descriptions of the smallest complete ordinary function system containing Φ. This development will be needed in the sequel.

Let $O(\Phi)$ be the family of all inverse images of right open rays by functions in Φ : $O(\Phi) = \{f^{-1}(a, \infty): f \in \Phi, a \in R\}$, and let $F(\Phi)$ be the family of all inverse images of right closed rays: $F(\Phi) = \{f^{-1}[a, \infty): f \in \Phi, a \in R\}$. Let $O_\sigma(\Phi)$ be the family of all countable unions of sets in $O(\Phi)$. Let $Z(\Phi)$ be the family of all zero sets of functions in Φ. We will need the following technicalities:

Remark 3.1. A set K is in $F(\Phi)$ if and only if $K = f^{-1}(O)$, for some nonnegative function f in Φ. In other words, $Z(\Phi)$, the family of zero sets of functions in Φ, is $F(\Phi)$.

Remark 3.2. The family $O(\Phi)$ is the memberwise complement of the family $F(\Phi)$ and $F(\Phi)$ and $O(\Phi)$ are closed under finite unions and intersections; i.e., $F(\Phi)$ and $O(\Phi)$ are sublattices of $P(X)$, the power set of X. Of course, $\varnothing$ and X are in $F(\Phi)$.

Remark 3.3. The family $F_\delta(\Phi)(O_\sigma(\Phi))$ contains $\varnothing$ and X and is a sublattice of $P(X)$ which is closed under countable intersections (unions), i.e., $F_\delta(O_\sigma)$ is a $\delta(\sigma)$-sublattice of $P(X)$.

Remark 3.4. If f is in Φ, then $f^{-1}(U) \in O_\sigma(\Phi)$ for every open set U and $f^{-1}(K) \in F_\delta(\Phi)$ for every closed set K.

Remark 3.5. Suppose $F = \bigcap_{n=1}^{\infty} F_n$, where $F_n = f_n^{-1}(O)$ and $f_n \in \Phi$, for each n. Then $F = f^{-1}(0)$, where $f = \sum_{n=1}^{\infty} 2^{-n}(|f_n| \wedge 1)$. Thus, if $F \in F_\delta(\Phi)$, then there is a nonnegative function f, which is the uniform limit of a sequence from Φ, such that $F = f^{-1}(0)$.

It follows from these remarks that if Φ is closed under uniform limits, then $Z(\Phi) = F(\Phi) = F_\delta(\Phi)$, and if $f \in \Phi$, then $f^{-1}(K) \in Z(\Phi)$, for each closed set K. However, even if Φ is closed under uniform limits, there may yet be functions f such that the inverse image of each closed number set is in the δ-lattice, $Z(\Phi)$, but the function f is not in Φ. An easy example of this may be seen by letting Φ be the family of bounded real-valued functions on the unit interval, and let f be any unbounded function on the interval.

In view of the previous remarks, we find that if Φ is a complete ordinary function system, then $F(\Phi) = Z(\Phi)$ is a δ-sublattice of $P(X)$ containing $\varnothing$ and X. Also, we have the following converse of this:

THEOREM 3.2. *Let $\mathscr{M}$ be a δ-sublattice of $P(X)$ containing $\varnothing$ and X, and let $\Phi(\mathscr{M})$ be the family of all real-valued functions f defined on X such that $f^{-1}(K) \in \mathscr{M}$, for every closed number set K. Then $\Phi(\mathscr{M})$ is a complete ordinary function system* [15].

Proof. Of course, the constant functions belong to $\Phi(\mathscr{M})$ and a function $f \in \Phi(\mathscr{M})$ if and only if $f^{-1}[a, \infty) \in \mathscr{M}$ and $f^{-1}(a, \infty) \in C\mathscr{M}$, for every number a, where $C\mathscr{M}$ denotes the family of complements of sets in $\mathscr{M}$. Using this fact it is not difficult to show that the family $\Phi(\mathscr{M})$ is an ordinary function system. In order to show that $\Phi(\mathscr{M})$ is complete, let us show that $\Phi(\mathscr{M})$ has the interposition property. If $\{f_n\}_{n=1}^{\infty}$ is a nondecreasing sequence from $\Phi(\mathscr{M})$ converging to f, then $f^{-1}(a, \infty) = \bigcup_{n=1}^{\infty} f_n^{-1}(a, \infty)$ and $f^{-1}(a, \infty) \in C\mathscr{M}$, for every a. Consequently, if $f \in US\Phi(\mathscr{M}) \cap LS\Phi(\mathscr{M})$, then $f \in \Phi(\mathscr{M})$. Finally, notice that if f is the uniform limit of a sequence from $\Phi(\mathscr{M})$, then f is in both $US\Phi(\mathscr{M})$ and $LS\Phi(\mathscr{M})$. Therefore, $\Phi(\mathscr{M})$ is a complete ordinary function system.

THEOREM 3.3. *Let Φ be a vector lattice containing constants, and let $\tilde{\Phi}$ be the family of all functions f such that $f^{-1}(K) \in F_\delta(\Phi)$, for every closed number set K. Then $\tilde{\Phi}$ is the smallest complete ordinary function system containing Φ* [15].

Proof. Since $F_\delta(\Phi)$ is a δ-lattice of $P(X)$ containing $\varnothing$ and X, it follows from Theorem 3.2 that $\tilde{\Phi}$ is a complete ordinary function system containing Φ. Let G be a complete ordinary function system containing Φ, and let φ be a function in $\tilde{\Phi}$ such that $\varphi(X) \leqslant [0, 1]$.

If $a < b$, then the sets $\varphi^{-1}(-\infty, a]$ and $\varphi^{-1}[b, \infty)$ are in $F_\delta(\Phi)$ and it follows from Remark 3.5 that there are nonnegative functions v_1 and v_2 which are the uniform limits of sequences from Φ such that $\varphi^{-1}(-\infty, a] = v_1^{-1}(0)$ and $\varphi^{-1}[b, \infty) = v_2^{-1}(0)$. Thus, $v_1/(v_1 + v_2)$ is a function in G such that

$$(*) \quad v^{-1}(0) = \varphi^{-1}(-\infty, a], v^{-1}(1) = \varphi^{-1}[b, \infty) \qquad \text{and} \qquad v(X) \subseteqq [0, 1].$$

For each positive integer n, let v_{nm} be a function in G such that (*) holds where $a = (m - 1)/n$ and $b = (m/n)$, $m = 1, 2, \ldots, n$, and let $v_n = 1/n \sum_{m=1}^{n} v_{nm}$. It follows that $| v_n - \varphi | \leqslant 1/n$, uniformly over X and φ is in G. From this it follows that if f is a bounded function in $\tilde{\Phi}$, then f is in G.

If f is an arbitrary function in $\tilde{\Phi}$, then $\varphi = f/(1 + | f |)$ is a bounded function in $\tilde{\Phi}$ and is in G. Thus, $f = \varphi/(1 - | \varphi |)$ is in G. This completes the argument that $\tilde{\Phi}$ is the complete ordinary function system generated by Φ.

Obviously we have:

COROLLARY 3.3a. *If Φ is a complete ordinary function system, then f is in Φ if and only if $f^{-1}(K) \in Z(\Phi) = F(\Phi)$, for every closed number set K.*

This corollary implies that the notion of a δ-sublattice of $P(X)$ containing $\varnothing$ and X is finer than the notion of a topology of closed sets on X, at least as far as complete ordinary function systems are concerned. This is meant in the sense that complete ordinary function systems Φ defined on X are distinguished by their families of zero sets, $Z(\Phi) = F(\Phi)$. This is in contrast to the fact that there are two complete ordinary function systems on the interval [0, 1] which generate the same weak topology on [0, 1]. For example, let Φ be the family $R^{[0,1]}$, and let Φ' be the family of all functions on [0, 1] which are continuous except for a countable set in the usual topology. This phenomenon is considered again in the last part of Section 5.

Theorem 3.3 describes the completion $\tilde{\Phi}$ in terms of inverse image sets. The next theorem gives an explicit construction of the completion from Φ. The next two lemmas will be needed.

LEMMA 3.4a. *If* $F \in F_\delta(\Phi)$, *then* $\xi_F \in US\Phi$.

Proof. Let $F = f^{-1}(0)$, where f is a nonnegative function in Φ_u, where Φ_u is the space of all uniform limits of sequences from Φ. Let $h = 1 - (f \wedge 1)$. Then h is a function from Φ_u such that $h^{-1}(1) = F$ and $h(X) \subseteq [0, 1]$. Since the bounded functions in Φ_u form an algebra [30], it follows that for each n, h^n is in Φ_u and $\xi_F = \lim h^n$. Therefore, $\xi_F \in US\Phi_u$. But it is easily verified that $US\Phi_u = US\Phi$.

LEMMA 3.4b. *In order that* $f \in US\Phi$, *it is necessary and sufficient that* (1) *there be a function* h *in* Φ *such that* $h \geqslant f$ *and* (2) $f^{-1}[a, \infty) \in F_\delta(\Phi)$, *for every number* a.

Proof. The necessity is clear from the argument given in Theorem 3.2. For the sufficiency, first suppose $0 \geqslant f$.

Let $\{r_p\}_{p=1}^{\infty}$ be a sequence of all the nonpositive rational numbers. Let $F_p = f^{-1}[r_p, \infty)$, and let $h_p = -r_p\xi_{F_p} + r_p$. From Lemma 3.4a, $h_p \in US\Phi$. Let $g_n = \bigwedge_{p=1}^{n} h_p$, $n = 1, 2, 3, \ldots$. Then $g_n \in US\Phi$ and $f = \lim_{n\to\infty} g_n$. Thus, $f \in US(US\Phi) = US\Phi$.

Now suppose $h \in \Phi$ and $h \geqslant f$. Then $0 \geqslant f - h$ and it follows that for each number a, $(f - h)^{-1}(-\infty, a) \in O_\sigma(\Phi)$. Therefore, $f - h \in US\Phi$ and hence $f = (f - h) + h \in US\Phi$.

THEOREM 3.4. *Let* Φ *be a linear lattice containing the constant functions, then* (1) *every bounded function in* $\tilde{\Phi}$ *belong to* $(US\Phi) \cap (LS\Phi)$ *and*

(2) *$f \in \tilde{\Phi}$ if and only if $f = f_1/f_2$, where f_1 and f_2 belong to* $(US\Phi) \cap (LS\Phi)$. Compare [15], [16], and [30].

Proof. If f is a bounded function in $\tilde{\Phi}$, then by Theorem 3.3 $f^{-1}[a, \infty) \in F_\delta(\Phi)$, for every number a, and by Lemma 3.4b, $f \in US\Phi$. Also, $f^{-1}(-\infty, a] \in F_\delta(\Phi)$, for every number a. Therefore, $-f \in US\Phi$ and $f \in (US\Phi) \cap (LS\Phi)$.

If f is an unbounded function in $\tilde{\Phi}$, then $f_1 = f/(1 + |f|)$ and $f_2 = 1 - |f_1|$ are bounded functions in Φ and $f_1/f_2 = f$. Finally, observe that $(US\Phi) \cap (LS\Phi) \subseteq \tilde{\Phi}$.

This completes the development given here of the complete ordinary function system generated by a linear lattice containing the constants and it connections with inverse image sets. We would like to remark at this point on the relationship between the families Φ_u and $(US\Phi) \cap (LS\Phi)$ and its dependence on the structure of Φ. As has been mentioned, if Φ is a vector lattice containing constants, then $\Phi_u \subseteq (US\Phi) \cap (LS\Phi)$. But it may be that this inclusion is proper even if Φ has the additional properties of being an algebra and closed under uniform limits. For example, if Φ is the family of all uniformly continuous functions on a dense subset D of the open interval $(0, 1)$, then Φ has the properties mentioned and yet $(US\Phi) \cap (LS\Phi)$ is the space of all bounded continuous functions on D. On the other hand, if Φ has the algebraic structure of an ordinary function system, then a bounded function f is in Φ_u if and only if f is in $(US\Phi) \cap (LS\Phi)$. An argument for this can be obtained from the argument of Theorem 3.3 and Lemma 3.4b.

The final theorems of this section allow us to establish some connections between the Baire classes generated by Φ and the Borel classes generated by $Z(\Phi)$. In Theorem 3.5, several necessary and sufficient conditions are given in order that a function be in Φ_1.

THEOREM 3.5. *Let Φ be a linear lattice containing the constant functions. The following statements are equivalent*:

(1) $f \in \Phi_1$

(2) $f \in LS(US\Phi) \cup US(LS\Phi)$;

(3) $f \in ((US\Phi) \cap (LS\Phi))_1$;

(4) $f \in (\tilde{\Phi})_1$;

(5) *$f^{-1}(F)$ is the intersection of countably many sets from $O_\sigma(\Phi)$, for each closed set F* [30];

(6) *f is the uniform limit of a sequence of functions each of which is the difference of two functions in $US\Phi$* [51].

Proof. It follows from Theorem 2.1 that the first two statements are equivalent. It follows from Corollary 2.2c that the first three statements are equivalent.

If $f \in (\tilde{\Phi})_1$, then f is the pointwise limit of a sequence of bounded functions in $\tilde{\Phi}$. But by Theorem 3.4, every bounded functions in $\tilde{\Phi}$ is in $US\Phi \cap LS\Phi$. It follows that the first four statements are equivalent.

If $f \in US(LS\Phi)$ and $\{f_n\}_{n=1}^{\infty}$ is a nonincreasing sequence from $LS\Phi$ converging to f, then the set $f^{-1}(-\infty, a)$ is $\bigcup_{n=1}^{\infty} f_n^{-1}(-\infty, a - (1/n)]$ and thus is the union of countably many sets in $F_\delta(\Phi)$.

If $f \in \Phi_1$, then it follows that for each set O, $f^{-1}(O)$ is the union of countably many sets in $F_\delta(\Phi)$. Thus, if $f \in \Phi_1$, then $f^{-1}(F)$ is the intersection of countably many sets from $O_\sigma(\Phi)$. This shows that the first statement implies the fifth.

Suppose that for each closed set F, $f^{-1}(F) \in (O_\sigma(\Phi))_\delta$. Let $\varphi = f/(1 + |f|) = (I/(1 + |I|)) \circ f$; $\varphi^{-1}(F) \in (O_\sigma(\Phi))_\delta$, for every closed set F.

Let $\{r_p\}_{p=1}^{\infty}$ be a countable dense subset of $(-1, 1)$, and let

$$\varphi^{-1}(-\infty, r_p) = \bigcup_{n=1}^{\infty} F_{pn},$$

where each $F_{pn} \in F_\delta(\Phi)$.

Let $g_{np} = (1 - r_p)\, \xi_{F'_{pn}} + r_p$, for each pair of integers p, n. From Lemma 3.4b, we have $g_{np} \in LS\Phi$ and $g_n = \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} g_{ij}$ is in $LS\Phi$, for each n. It follows that $\{g_n\}_{n=1}^{\infty}$ is a nonincreasing sequence from $LS\Phi$ converging to φ. Therefore, $\varphi \in US(LS\Phi)$. Similarly, $\varphi \in LS(US\Phi)$ and $\varphi \in \Phi_1$. Since Φ_1 is a complete ordinary function system, $f = \varphi/(1 - |\varphi|)$ is in Φ_1. The first five statements are equivalent.

Since Φ_1 is complete, the sixth statement implies the first.

Finally, suppose $\{f_p\}_{p=1}^{\infty}$ is a sequence of bounded functions from Φ converging to f. For each p, let $g_p = \bigvee_{i=1}^{n} f_{i+p-1}$ and $h_p = \bigwedge_{i=1}^{n} f_{i+p-1}$. For each p, $g_p \geqslant q_{p+1}$, $h_{p+1} \geqslant h_p$, $g_p \in LS\Phi$, and $h_p \in US\Phi$. The sequences $\{g_p\}_{p=1}^{\infty}$ and $\{h_p\}_{p=1}^{\infty}$ converge pointwise to f.

For each p, let $R_p = \{x: g_p(x) - h_p(x) \leqslant \epsilon\}$, and let $R_0 = \varnothing$. The sets $R_p \in F_\delta(\Phi)$ and $\xi_{R_p} \in US\Phi$. Let $f_0 = 0$ and for each p,

$$s_p = \sum_{h=1}^{p} (f_n - f_{n-1} \vee 0)$$

and

$$t_p = \sum_{n=1}^{p} (f_n - f_{n-1} \wedge 0).$$

For each p, s_p is a nonnegative bounded function in Φ. It follows that $s_p \cdot \xi_{R_q}$ is in $US\Phi$, for each pair p, q. Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of bounded functions in Φ decreasing to ξ_{R_q}. As noted before, $s_p \cdot k_n \in \Phi_u$ and $s_p \cdot \xi_{R_q} \in US\Phi_u = US\Phi$.

Let

$$h = \sum_{p=1}^{\infty} s_p \xi_{R_p} - \sum_{p=1}^{\infty} s_p \xi_{R_{p-1}},$$

and let

$$k = \sum_{p=1}^{\infty} t_p \xi_{R_p} - \sum_{p=1}^{\infty} t_p \xi_{R_{p-1}}.$$

Thus, $h + k$ is the difference of two functions in $US\Phi$. We have $X = \bigcup_{p=1}^{\infty} R_p$, and if $x \in R_p - R_{p-1}$, then $|f(x) - (h + k)(x)| = |f(x) - f_p(x)| \leqslant \epsilon$. This completes the proof of Theorem 3.5.

From Remark 3.5, we have:

COROLLARY 3.5. *If Φ is closed under uniform limits, then $f \in \Phi_1$ if and only if $f^{-1}(F) \in O_\delta(\Phi)$, for every closed set F.*

We are now in a position to give a development of the Baire classes generated by the family of functions Φ together with the classes encountered in a development of the Borel system or σ-algebra of sets generated by $F(\Phi)$. The developments will be in exact correspondence.

Let $\Phi_0 = \Phi$, and for each ordinal α, let Φ_α be the space of all pointwise limits from $\Phi_{\alpha-1}$, if α is not a limit ordinal, and let Φ_α be the complete ordinary function system generated by the ordinary function system $\bigcup_{\gamma<\alpha} \Phi_\gamma$, if α is a limit ordinal. Note that according to Theorem 3.5, if α is a limit ordinal, then $\Phi_{\alpha+1} = (\Phi_\alpha)_1 = (\bigcup_{\gamma<\alpha} \Phi_\gamma)_1$.

Note. The development described here differs slightly from the one given earlier. The description given here will be used throughout the remainder of the paper.

The families Φ_α can be connected with the σ-algebra or Borel system of sets generated by $Z(\Phi)$.

We now describe the σ-algebra generated by a family F of subsets of X.

Let F_0 be the family of all countable intersections of members of F together with X and $\varnothing$, and let O_0 be the family of all complements of sets in F_0. For each ordinal $\alpha > 0$, let F_α be the family of all countable intersections of sets from $\bigcup_{\gamma<\alpha} O_\gamma$ and let O_α be the family of all complements of sets in F_α. The family $O_{\omega_1} = F_{\omega_1}$ is the σ-algebra or Borel system of sets generated by the family F.

In particular, if $F = F(\Phi)$, then in consequence of the theorems already established, we have:

THEOREM 3.6. *If $\alpha > 0$, then f is in the family Φ_α if and only if $f^{-1}(K) \in F_\alpha$, for each closed set K. This condition also holds for $\alpha = 0$, if Φ is a complete ordinary function system.*

The functions in $\Phi_\alpha - \bigcup_{\gamma<\alpha} \Phi_\gamma$ are said to be of exactly class α. The sets in $A_\alpha = F_\alpha \cap O_\alpha$ are said to be ambiguous sets of class, α and the sets in $EA_\alpha = A_\alpha - \bigcup_{\gamma<\alpha} A_\gamma$ are said to be of exactly ambiguous class α.

Of course, $\Phi_{\omega_1} = \Phi_{\omega_1+1}$ and $F_{\omega_1} = O_{\omega_1} = A_{\omega_1}$. The Baire order of a family Φ is the first ordinal α such that $\Phi_\alpha = \Phi_{\alpha+1}$. Note that the Baire order of a family Φ is the first ordinal α such that $F_\alpha = O_\alpha$. One of the central problems in the Baire process is the determination of the Baire order of a given family.

As a final remark in this section, note that if Φ is a complete ordinary function system, then the bounded function in Φ forms a real Banach algebra under the uniform norm. A representation of the dual, Φ^*, of this space has been obtained by A. D. Alexandroff [1].

DEFINITION. Let $ba(X, \Sigma)$ be the space of all real-valued bounded additive set functions defined on Σ, the σ-algebra generated by $F(\Phi)$ under the variation norm. A function $\mu \in ba(X, \Sigma)$ is said to be regular if for each set A in the *algebra* generated by $F(\Phi)$ and for each $\epsilon > 0$, there is a set $F \in F(\Phi)$ and set $G \in O(\Phi)$ such that $F \subseteq A \subseteq G$ and the variation of μ over $G - F$, var$(\mu, G - F)$, is less than ϵ. Let $rba(X, \Sigma)$ be the subspace of a (X, Σ) consisting of all regular set functions.

THEOREM. *Suppose Φ is a complete ordinary function system and $T \in \Phi^*$. Then there is only one set function $\mu \in rba(X, \Sigma)$ such that* (t) *$T(f) = \int_X f\, d\mu$, for every bounded function f in Φ. Moreover, $| T | = \| \mu \|$ and Φ^* and $rba(S, \Sigma)$ are isometrically isomorphic by the mapping defined by* (*t*).

Note that this theorem includes the representation of the dual of $C(X, R)$ where X is a topological space and $C(X, R)$ is the space of all bounded real-valued continuous functions on X under the uniform norm. Also, this theorem may be applied to give the more well-known representation theorems [9].

4. The Baire Order of Spaces $C(X)$

Let X be a topological space, and let $C = C(X)$ be the family of all real-valued continuous functions on X. In this section the Baire order of $C(X)$ is determined for those X which are compact Hausdorff spaces or which are complete metric spaces which contain a dense-in-itself set.

By a Baire function on X is meant a function in C_{ω_1}. By a Baire set in X is meant a set in the σ-algebra generated by the family $Z(X)$ of all zero sets of functions in $C(X)$. By a Borel set on X is meant a set in the σ-algebra generated by the closed subsets of X. Of course, if X is perfectly normal, then the Borel sets and Baire sets coincide.

Let Φ be a linear lattice containing constants, and let $\Phi_b{}^*$ be the dual of the normed linear space consisting of all bounded functions in Φ under the uniform norm. The first theorem of this section, due to the author, gives sufficient conditions on $\Phi_b{}^*$ in order that the Baire order of Φ be no more than 1.

Theorem 4.1. *Suppose that for each $T \in \Phi_b{}^*$, there is a countable subset $\{x_n\}_{n=1}^{\infty}$ of X and a sequence $\{a_n\}_{n=1}^{\infty}$ from l_1 such that*

$$(*) \qquad T(f) = \sum_{n=1}^{\infty} a_n f(x_n)$$

for every bounded function f in Φ. Then the Baire order of Φ is no more than 1.

Proof. Let $f \in \Phi_2 = (\Phi_u)_2$, and let $\mathcal{A}$ be a closed and separable subalgebra of the Banach space of bounded functions in Φ_u such that $f \in \mathcal{A}_2$ and let $\mathcal{A}$ contain constants. Let $\bar{X}$ be the decomposition of X obtained by identifying the stationary sets of $\mathcal{A}$. Let $\bar{\mathcal{A}}$ be the Banach algebra defined on $\bar{X}$ obtained from $\mathcal{A}$ by the natural correspondence. Of course, f is constant on these stationary sets of $\mathcal{A}$ and the corresponding function $\bar{f}$ belongs to $\bar{\mathcal{A}}_2$.

The Banach algebra $\bar{\mathcal{A}}$ separates points of $\bar{X}$. Consequently, there is a compact Hausdorff space S and a 1–1 function τ from $\bar{X}$ onto a dense subset of S such that the transformation: $f \to f \circ \tau^{-1}$ for each $f \in C(S)$ is an isomorphism of $C(S)$ and $\bar{\mathcal{A}}$, as lattices and algebras, and is an isometry [9, p. 274].

Since $\bar{\mathcal{A}}$ is separable, $C(S)$ is separable and therefore S is a compact metric space. Since each functional on $\bar{\mathcal{A}}$ has the form (*), each functional on $C(S)$ has the same form. Therefore, S is countable.

Let $\bar{f}$ be any extension of $\bar{f}$ to all of S. But the Baire order of $C(S)$

for S countable, compact metric is no more than 1 and every real-valued function on S is in C_1 [30]. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence from $C(S)$ converging pointwise to $\bar{f}$. The sequence $\{f_n = g_n \circ \tau^{-1}\}_{n=1}^{\infty}$ converges pointwise to f and it follows that the Baire order of Φ is no more than 1.

COROLLARY 4.1 [Meyer, 33]. *If S is a dispersed compact Hausdorff space, then the Baire order of $C(S)$ is 0, if S is finite, and is 1, if S is not finite.*

This corollary follows directly from the preceding theorem and the fact that for dispersed compact T_2 spaces, $C^*(S) \cong l_1(S)$, which was first proved by W. Rudin. See [38].

The next theorems show that the Baire order of $C = C[0, 1]$ is ω_1. The arguments proceed by constructing "universal" functions and then applying a Cantor type argument on the diagonal.

THEOREM 4.2 [Lebesgue, 25]. *For each $\alpha < \omega_1$, there is a Baire function U_α on $[0, 1] \times [0, 1]$ such that if $f \in C_\alpha$ and $f[0, 1] \subseteq [0, 1]$, then there is a number x such that $U(x, y) = f(y)$, for every y in $[0, 1]$.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a countable dense subset of the positive part of the unit ball of $C[0, 1]$. Let

$$U_0(x, y) = \begin{cases} f_n(y), & \text{if } x = 1/n, \\ 0, & \text{if otnerwise.} \end{cases}$$

It is easy to show that U_0 is a function in Baire's class 1 on the unit square.

Let $h = (h_1, h_2, h_3, \ldots)$ be a continuous function from $[0, 1]$ onto $I^{\aleph_0}$, the Hilbert cube.

Let $U_1(x, y) = \overline{\lim}_{n\to\infty} U_0(h_n(x), y)$. It follows that for each n, $U_0(h_n(x), y)$ is a function in Baire's class 1 on the unit square and U_1 is a Baire function on the unit square. If f is a function in Baire's class 1 on the interval $[0, 1]$ and $f([0, 1]) \subseteq [0, 1]$, then there is a subsequence $\{f_{n_p}\}_{p=1}^{\infty}$ converging pointwise to f.

Let x be a number in $[0, 1]$ such that $h(x) = \{1/n_p\}_{p=1}^{\infty}$. Then $U_1(x, y) = f(y)$, for every y in $[0, 1]$.

Suppose α is a countable ordinal, $\alpha > 1$, and for each ordinal γ, $\gamma < \alpha$, a function U_γ having the prescribed properties has been designated.

If α is not a limit ordinal, let $U_\alpha(x, y) = \overline{\lim}_{n\to\infty} U_{\alpha-1}(h_n(x), y)$. Again, it follows that U_α is a Baire function on the unit square. If $f \in C_\alpha$ and $f([0, 1]) \subseteq [0, 1]$, then there is a sequence $\{x_p\}_{p=1}^{\infty}$ from $[0, 1]$ such that $f(y) = \lim_{p\to\infty} U_{\alpha-1}(x_p, y)$. Let x be a number in $[0, 1]$ such that $h(x) = \{x_p\}_{p=1}^{\infty}$. It follows $U_\alpha(x, y) = f(y)$, for each y in $[0, 1]$.

If α is a limit ordinal, let $\{\gamma_n\}_{n=1}^{\infty}$ be an increasing sequence of ordinals converging to α, and let $U_\alpha(x, y) = \overline{\lim}_{n\to\infty} U_{\gamma_n}(h_n(x), y)$. If $f \in C_\alpha$ and $f([0, 1]) \subseteq [0, 1]$, then there is a sequence $\{f_n\}_{n=1}^{\infty}$ such that for each n, $f_n \in C_{\gamma_n}$, $f_n([0, 1]) \subseteqq [0, 1]$, and $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f. Let x be a number such that $U_{\gamma_n}(h_n(x), y) = f_n(y)$ for each n and for each y in $[0, 1]$. Thus, U_α are the prescribed properties for the ordinal α. This completes the argument for Theorem 4.2.

The functions U_α are said to be universal functions with respect to the positive part of the unit ball of C_α. There is no universal function with respect to the positive part of the unit ball of $C[0, 1]$ which is a continuous function on the unit square. In the argument for the next theorem, it will become apparent that there is no universal function with respect to the positive part of the unit ball of C_{ω_1} which is a Baire function on the unit square. Also, see [18].

THEOREM 4.3 [Lebesgue]. *The Baire order of* $C[0, 1]$ *is* ω_1.

Proof. Suppose the Baire order of $C[0, 1]$ is α, $\alpha < \omega_1$. Let U_α be a universal function as described in Theorem 4.2 which is a Baire function on the unit square. Let $g(x) = 1 - U_\alpha(x, x)$, x in $[0, 1]$. Then g is a Baire function on the unit interval. Let f be the pointwise limit of the sequence $\{g^n\}_{n=1}^{\infty}$. Then $f \in C_\alpha$ and $f([0, 1]) \subseteq [0, 1]$. Therefore, there is some x such that $U_\alpha(x, y) = f(y)$ for every y in $[0, 1]$. In particular, $f(x) = U_\alpha(x, x)$. But, if $U_\alpha(x, x) = 0$, then $f(x) = 1$, and if $U_\alpha(x, x) > 0$, then $f(x) = 0$. This is a contradiction and the Baire order of $C[0, 1]$ is ω_1. The following theorem is also due essentially to Lebesgue [21]. This particular formulation of the theorem has had some recent applications [4]

Recall from Section 3 that if G is a family of subsets of X, then the Borel system by G may be constructed as follows: Let G_0 be the family of all countable unions of sets in G together with $\varnothing$ and X, and let F_0 be the family of all complements of sets in G_0. For each $\alpha > 0$, let F_α be the family of all countable intersections of sets from $\bigcup_{\gamma<\alpha} G_\gamma$, and let G_α be the family of all complements of sets in F_α. Then $F_{\omega_1} = G_{\omega_1}$ is the Borel system or σ-algebra generated by the family G. The Borel order of a family G is the first ordinal α such that $F_\alpha = G_\alpha$. Some special universal functions are now described in

THEOREM 4.4. *Let G be a countable family of subsets of* $\mathcal{N}$, *the space of all irrational numbers between* 0 *and* 1. *If the Borel system generated by G includes all the Borel subsets of* $\mathcal{N}$, *then the Borel order of G is* ω_1. Compare [21, p. 368].

Again, the proof is a diagonal type argument applied to universal functions for the generated Borel classes.

Proof. For each $Z \in \mathcal{N}$, let $(Z_1, Z_2, Z_3, \ldots)$ be the sequence of integers appearing in the continued fraction expansion of Z. This defines a reversible transformation from $\mathcal{N}$ onto the set of all sequences of positive integers. Let

$$\begin{aligned} Z^1 &= (Z_1, Z_3, Z_5, \ldots) \\ Z^2 &= (Z_2, Z_6, Z_{10}, \ldots) \\ &\vdots \\ Z^n &= (Z_{2^{n-1}}, Z_{3\cdot 2^{n-1}}, Z_{5\cdot 2^{n-1}}, \ldots) \\ &\vdots \end{aligned}$$

for each n. This defines a homeomorphism between $\mathcal{N}$ and $\mathcal{N}^{\aleph_0}$. Also, note that if f is a continuous function from $\mathcal{N}$ into $\mathcal{N}$, then the functions f_n from $\mathcal{N}$ into the space of positive integers are continuous, where $f(Z) = (f_1(Z), f_2(Z), f_3(Z), \ldots)$.

Let $O_1, O_2, O_3, \ldots$ be a sequence of all the members of G together with $\varnothing$ and X.

Let $U_0(Z) = \bigcup_{n=1}^{\infty} O_{Z_n}$, for each $Z \in \mathcal{N}$. Clearly, U_0 is a universal function for the class G_0; U_0 is a function from $\mathcal{N}$ onto G_0.

Let f be a continuous function from $\mathcal{N}$ into $\mathcal{N}$. We have

$$A_f = \{Z\colon Z \in U_0(f(Z))\} = \left(Z\colon Z \in \bigcup_{n=1}^{\infty} O_{f_n(Z)}\right);$$

$$A_f = \bigcup_{n=1}^{\infty} \{Z\colon Z \in O_{f_n(Z)}\}.$$

For each n, we have

$$\{Z\colon Z \in O_{f_n(Z)}\} = \bigcup_{i=1}^{\infty} \{J_{ni} \cap O_i\},$$

where $J_{ni} = \{Z\colon f_n(Z) = i\}$. Since each f_n is continuous, J_{ni} is open and therefore the set A_f belongs to G_{ω_1}.

Suppose $0 < \alpha < \omega_1$ and for each ordinal γ, $0 \leqslant \gamma < \alpha$, a universal function U_γ from $\mathcal{N}'$ onto the class G_α has been given such that if f is a continuous function from $\mathcal{N}$ into $\mathcal{N}$, then the set

$$A_f = \{Z\colon Z \in U_\gamma f(Z)\} \in G_{\omega_1}.$$

If α is a positive integer n, let

$$V_\alpha(Z) = \bigcap_{k=0}^{\infty} \left(\bigcap_{p=1}^{n} U_{p-1}(Z^{kn+p}) \right).$$

If $\alpha \geqslant \omega_0$, let $\{\gamma_p\}_{p=1}^{\infty}$ be a sequence of all ordinals less than α, and let

$$V_\alpha(Z) = \bigcap_{p=1}^{\infty} U_{\gamma_p}(Z),$$

for each $Z \in \mathscr{N}$.

Clearly, V_α is a universal function for the class F_α. Let f be a continuous function from $\mathscr{N}$ into $\mathscr{N}$. It will be shown that the set

$$B_f = \{Z\colon Z \in V_\alpha(f(Z))\} \in G_{\omega_1}.$$

The argument for $\alpha < \omega_0$ is similar.

We have

$$B_f = \{Z\colon Z \in V_\alpha(f(Z))\} = \left(Z\colon Z \in \bigcap_{p=1}^{\infty} U_{\gamma_p}((f(Z))^p)\right);$$

$$B_f = \bigcap_{p=1}^{\infty} \{Z\colon Z \in U_{\gamma_p}(f(Z))^p\}.$$

But for each p, the function $Z \to (f(Z))e$ is a continuous function from $\mathscr{N}$ into $\mathscr{N}$. So, $B_f \in G_{\omega_1}$.

Let $U_\alpha(Z) = X - V_\alpha(Z)$, for $Z \in \mathscr{N}$. It follows that U_α is a universal function for the class G_α such that if f is a continuous function from $\mathscr{N}$ into $\mathscr{N}$, then $A_f = \{Z\colon Z \in \bigcup_\alpha (f(Z))\} \in G_{\omega_1}$.

Finally, suppose the Borel order of G is α, $\alpha < \omega_1$. Let $I_\alpha = \{Z\colon Z \in U_\alpha(Z)\}$.

Since $G_\alpha = F_\alpha = G_{\omega_1}$, $X - I_\alpha \in G_\alpha$, and there is some Z such that $U_\alpha(Z) = X - I_\alpha$. But this is a contradiction and the Borel order of G is ω_1. This completes the argument for Theorem 4.3.

Since the Baire order of $C[0, 1]$ is no less than the Baire order of $C[\mathscr{N}]$, we have

Corollary 4.4. *The Baire order of* $C[0, 1]$ *is* ω_1.

In order to show that the Baire order of $C(X)$ is ω_1, where X is a compact Hausdorff space which is not dispersed, we will employ the fact that perfect mappings between subsets of complete and separable metric spaces preserve the classes $EZ_\alpha(ECZ_\alpha)$, $\alpha \geqslant \omega_0$. Recall that if X is a topological space, then $Z_0 = Z_0(X)$ is the family of all zero sets of real-valued continuous functions defined on X and $EZ_\alpha(ECZ_\alpha)$ is the family of all those sets which are *exactly* of class $Z_\alpha(CZ_\alpha)$ in the iterative process described previously.

Let R be the space of real numbers provided with the usual topology. The following lemmas and Theorem 4.5 are due to A. D. Taimanov [48].

LEMMA 4.5a. *Suppose φ is a lower semicontinuous function defined on a closed number set K and Γ is the graph of φ. If $A \in CZ_\alpha(R^2)$, then the projection of the set $A \cap \Gamma$ onto the x-axis is of class $CZ_{\alpha+1}(R)$ if α is finite, and of class $CZ_\alpha(R)$ if $\alpha \geqslant \omega_0$. If $A \in Z_\alpha(R^2)$, then the projection of the set $A \cap \Gamma$ onto the x-axis is of class $Z_{\alpha+1}(R)$ if α is finite and $\alpha > 0$, and of class $Z_\alpha(R)$ if $\alpha \geqslant \omega_0$.*

Proof. Let $B = \Pi_x(A \cap \Gamma)$. The proof proceeds by transitive induction.

Case 1. First, let A be open. For each $x \in B$, let $\Delta_x = [r_1, r_2] \times (r_3, r_4]$ (where the r_i $(i = 1, 2, 3, 4)$ are rational numbers) contain $(x, \varphi(x))$, and let $\Delta_x \subseteq A$. Also, let V_x be the projection of $\Delta_x \cap \Gamma$ onto the x axis.

$$V_x = \{t\colon r_1 \leqslant t \leqslant r_2 \text{ and } r_3 < \varphi(t) \leqslant r_4\}.$$

From the lower semicontinuity of φ it follows that each set V_x is the intersection of an open and closed set. Thus, each set V_x is an F_σ set and the set B, being the union of the countable family V_x, is an F_σ set. If $A \in CZ_0(R^2)$, then $B \in CZ_1(R)$.

Now, let $A \in Z_1(R^2)$. Thus $A = \bigcap_{n=1}^\infty A_n$, where each A_n is open. Then $B = \Pi_x(A \cap \Gamma) = \bigcap_{n=1}^\infty \Pi_x(A_n \cap \Gamma)$ and B is an $F_{\sigma\delta}$ set. If $A \in Z_1$, then $B \in Z_2$.

Case 2. First suppose A is compact. Let $A = \bigcap_{n=1}^\infty U_n$, where $\bar{U}_1$ is compact and for each n, $\bar{U}_{n+1} \subseteq U_n$ and U_n is open. Let $\{f_n\}_{n=1}^\infty$ be a nondecreasing sequence of continuous functions converging to φ.

For each n, let $O_n = \{x\colon (x, f_s(x)) \in U_n$, for some $s \geqslant n\}$. For each n, O_n is an open set and $B = \bigcap_{n=1}^\infty O_n$. If A is compact, then B is a G_δ set.

Now, let $A \in CZ_1$. Thus $A = \bigcup_{n=1}^\infty A_n$, where, for each n, A_n is compact. Then $B = \Pi_x(A \cap \Gamma) = \bigcup_{n=1}^\infty \Pi_x(A_n \cap \Gamma)$ and B is a $G_{\delta\sigma}$ set. If $A \in CZ_1$, then $B \in CZ_2$.

Case 3. Suppose n is a positive integer, the lemma holds for each $\alpha \leqslant n$, and $A \in Z_{n+1}$.

Let $A = \bigcap_{p=1}^{\infty} A_p$, with $A_p \in CZ_n$. Then $B = \Pi_x(A \cap \Gamma) = \bigcap_{p=1}^{\infty} (\Pi_x(A_p \cap \Gamma))$. By assumption, $\Pi_x(A_p \cap \Gamma) \in CZ_{n+1}$. Thus, if $A \in Z_{n+1}$, then $B \in Z_{n+2}$.

The remaining cases are proven in a similar fashion.

LEMMA 4.5b. *Suppose φ is a lower semicontinuous function defined on a closed subset K of a Hilbert cube H and Γ is the graph of φ. Then the conclusions of Lemma* 4.5a *still hold with H substituted for R^2.*

A proof of this lemma follows the proof of the preceding lemma exactly.

LEMMA 4.5c. *Suppose f is a perfect map from a subset A of $I = [0, 1]$ onto a subset B of a Hilbert cube H. If $A \in CZ_\alpha$, then $B \in CZ_{\alpha+1}$, if $\alpha > 1$ and α is finite, and $B \in CZ_\alpha$, if $\alpha \geqslant \omega_0$. If $A \in Z_\alpha$, then $B \in Z_{\alpha+1}$, if $\alpha > 0$ and α is finite, and $B \in Z_\alpha$, if $\alpha \geqslant \omega_0$.*

Proof. Let L denote the graph of the mapping f in $H \times I$: $L = \{(f(y), y): y \in A\}$.

Since the sets CZ_α, $\alpha > 1$, and the sets Z_α, $\alpha > 0$, are intrinsic invariants [21, p. 432] and the set L is homeomorphic to A, we need only consider the projection ψ of L onto H. The closure $\bar{L}$ of L is a compact set to which ψ can be extended. Let ψ^* denote the mapping of the set $\bar{L}$ onto the set $\bar{B}$.

The set $\bar{L}$ decomposes into the union of sets $\psi^{*-1}(x)$, where $x \in \bar{B}$. Suppose there is some $x \in \bar{B}$ such that $\psi^{*-1}(x) \cap L \neq \varnothing$ and $\psi^{*-1}(x) \cap (\bar{L} - L) \neq \varnothing$. Let $(x, y) \in \psi^{*-1}(x) \cap (\bar{L} - L)$, and let $(f(y_n), y_n)$ be a sequence from L converging to the point (x, y). It follows that $x \in B$ and since f is perfect, x is a limit point of the infinite set $\{f(y_n)\}_{n=1}^{\infty}$. Thus, the set $M = \{y_n\}_{n=1}^{\infty}$ is closed in A, but the set $f(M)$ is not closed in B. This is a contradiction.

So, for each point $x \in \bar{B}$, let $(x, \varphi(x))$ be the lowest point of $\bar{L}$ on the set $\{x\} \times I$. The function φ is a lower semicontinuous function defined on the closed subset $\bar{B}$ of a Hilbert cube H. Let Γ be the graph of φ in $H \times I$. The projection of the set $L \cap \Gamma$ onto H is B and this lemma follows from the preceding lemma.

THEOREM 4.5. *Suppose A is a subset of a complete separable metric space S, B is a subset of a complete separable metric space S_1, and f is a* *perfect map from A onto B. If $A \in CZ_\alpha$, then $B \in CZ_{\alpha+1}$, if $1 < \alpha < \omega_0$, and $B \in CZ_\alpha$, if $\alpha \geqslant \omega_0$. If $A \in Z_\alpha$, then $B \in Z_{\alpha+1}$, if $0 < \alpha < \omega_0$, and $B \in Z_\alpha$, if $\alpha \geqslant \omega_0$.*

Proof. We may assume that A lies in a Hilbert cube H and B lies in a Hilbert cube H_1. Let θ be a continuous function from the interval [0, 1] onto H. The composition $f \circ \theta$ is a perfect mapping of $\theta^{-1}(A)$ onto B and the theorem follows from the preceding lemma.

Let $EZ_\alpha = Z_\alpha - \bigcup_{\gamma<\alpha} Z\gamma$ and let $ECZ_\alpha = CZ_\alpha - \bigcup_{\gamma<\alpha} CZ_\gamma$, for each $\alpha > 0$. The next theorem is due to I. A. Vainstein [52].

THEOREM 4.6. *Suppose A is a subset of a complete separable metric space S, B is a subset of a complete separable metric space S_1, and f is a perfect map of A onto B. If $A \in EZ_\alpha(S)$ $(ECZ_\alpha(S))$ with $\alpha > 2$, then the class of the set B is not lowered.*

Proof. Let A lie in a Hilbert cube H, and let B lie in a Hilbert cube H_1. Let Γ be the graph of the mapping f in $H \times H_1$, and let ψ be the natural projection of $\bar{\Gamma}$ onto the closure of B in H_1, $\bar{B}$.

Suppose $A \in EZ_\alpha(S)$. Since Γ is homeomorphic to A and the classes $Z_\gamma(CZ_\gamma)$ $\gamma > 1$ are intrinsic invariants, $\Gamma \in EZ_\alpha(\bar{\Gamma})$.

Suppose $B \in Z_\gamma(S_1)$ $(CZ_\gamma(S_1))$, $1 < \gamma < \alpha$. Then $B \in Z_\gamma(\bar{B})(CZ_\gamma(\bar{B}))$, $1 < \gamma < \alpha$, and since ψ^{-1} preserves unions, intersections, and complements, we have $\psi^{-1}(B) \in Z_\gamma(\bar{\Gamma})(CZ_\gamma(\bar{\Gamma}))$.

We will show that $\psi^{-1}(B) = \Gamma$ and through this contradiction prove the theorem.

Suppose $(x, y) \in \psi^{-1}(B) \cap \bar{\Gamma} - \Gamma$. Since $f^{-1}(y)$ is compact in A, (x, y) is not a limit point of $f^{-1}(y) \times \{y\}$. It follows that there is an infinite subset $M = \{x_n\}_{n=1}^\infty$ of $A - f^{-1}(y)$ such that $\{x_n, f(x_n)\}_{n=1}^\infty$ converges to (x, y). Of course, $x \in A$ and the set M is closed in A. However, the set $f(M) = \{y_n\}_{n=1}^\infty$ is not closed in B and this contradicts the fact that the mapping f is closed.

By combining Theorem 4.6 and Theorem 4.5, we have:

COROLLARY 4.6. *Suppose f is a perfect map of a subset A of a complete separable metric space S onto a subset B of a complete separable metric space S_1. Then f preserves the classes $Z_\alpha(CZ_\alpha)$ if $\alpha \geqslant \omega_0$ and does not change the class by more than 1 if $2 < \alpha < \omega_0$.*

The next theorem is due to J. E. Jayne [17] and M. M. Choban [6].

THEOREM 4.7. *Suppose X is a compact Hausdorff space and contains a perfect set. Then the Baire order of X is ω_1.*

Proof. Let f be a continuous function from X onto the interval $[0, 1]$ [38]. Let α be an infinite countable ordinal, and let M be a Borel subset of $[0, 1]$ belonging to the family $EZ_\alpha[0, 1]$. It follows that $A = f^{-1}(M)$ belongs to $EZ_\gamma(X)$, for some $\gamma \leqslant \alpha$. It will be shown that $\gamma = \alpha$.

Suppose $\gamma < \alpha$. Let $K = \{g_n^{-1}(0)\}_{n=1}^{\infty}$ be a countable collection of zero sets such that A belongs to the family K_γ, where for each n, $g_n(X) \subseteq [0, 1]$.

Let $g(x) = (g_1(x), g_2(x), g_3(x),...)$ for each x in X. The space $Y = g(X)$ is a compact metric space, $g^{-1}g(g_n^{-1}(0))) = g_n^{-1}(0)$ for each h; and $g^{-1}(g(A)) = A$. (Note that if h is a map from a set X to a set Y, then the family W of all sets B such that $h^{-1}(h(B)) = B$ is a σ-algebra).

Let $\psi(x) = (f(x), g(x))$, for each x in X. The space $H = \psi(X) \subseteq [0, 1] \times Y$ is a compact metric space. For each n, $\psi^{-1}((\psi g^n_{-1}(0))) = g_n^{-1}(0)$, $\psi^{-1}(\psi(A)) = A$ and $\psi(A) \in EZ_\gamma(H)$. Let Π be the restriction to H of the natural projection of $[0, 1] \times Y$ onto $[0, 1]$. Then $\Pi(\psi(A)) = M$ and $\Pi^{-1}(\Pi(\psi(A))) = \psi(A)$. This implies that Π is a perfect map. This contradicts the fact that perfect maps in complete and separable metric spaces preserve the classes EZ_α, $\alpha \geqslant \omega_0$ [Theorems 4.5 and 4.6].

COROLLARY 4.7. *Suppose X is a complete metric space which contains a dense-in-itself set. Then the Baire order of $C(X)$ is ω_1.*

This follows from the facts that such a space contains a compact perfect set P and that every Baire function on P has an extension to a Baire function on X. Compare with [10].

Notice that if X and Y are compact, T_2 spaces, then the Baire order of $C(X \times Y)$ is the maximum of the order of $C(X)$ and the order of $C(Y)$. This naturally leads to the following:

Question. Is it true that the Baire order of $C(X \times Y)$ is the maximum of the order of $C(X)$ and $C(Y)$, where X and Y are completely regular spaces?

In closing this section, the author would like to point out that many topological spaces may be characterized by relationships among the lower (upper) semicontinuous functions and the families $C = C(X)$, USC, and LSC. Some of the characterizations follows.

THEOREM 4.8 (Tong [49]). *In order that X be normal, it is necessary* *and sufficient if f is upper semicontinuous and g is lower semicontinuous and $f \leqslant g$, then there is a continuous function h such that $f \leqslant h \leqslant g$.*

THEOREM 4.9 (Tong [49]). *In order that X be perfectly normal, it is necessary and sufficient that if f is upper semicontinuous, then f belongs to USC.*

THEOREM 4.10 (Dowker [8]). *In order that X be normal and countably paracompact, it is necessary and sufficient that if f is upper semicontinuous, g is lower semicontinuous, and $f < g$, then there is a continuous function h such that $f < h < g$.*

The author notes:

THEOREM 4.11. *In order that X be completely regular, it is necessary and sufficient that if f is a bounded upper semicontinuous function, then for each x in X,*

$$f(x) = \text{g.l.b.}\{g(x) : g \geqslant f \text{ and } g \in C(X)\}.$$

Also, we have:

THEOREM 4.12 (Ross and Stromberg [43]). *A compact Baire set F in a topological space X is the zero-set of some continuous function.*

From Theorem 4.12, can be easily derived:

THEOREM 4.13. *A compact, T_2 space is perfectly normal if and only if every Baire set is a Borel set.*

Theorem 4.13 leads to the following:

Question. Is it true that the order of the Borel subsets of a completely regular T_2 space X is the order of the Baire subsets of X? In particular, what if X is compact?

There seems to be very little information on this problem. For some interesting examples see [53].

Finally, we have:

THEOREM 4.14 (Frolik [12]). *A space X is Lindelof if and only if the family of Baire sets is the smallest countably additive and countably multiplicative collection $\mathscr{M}$ of sets such that X locally belongs to $\mathscr{M}$.*

A space X locally belongs to a collection $\mathcal{M}$ of subsets of X if each point of X has arbitrarily small neighborhoods in $\mathcal{M}$.

5. Special Spaces and Baire Embeddings of Function Systems

In this section the Baire order of various families is obtained and some examples are given. The paper concludes with an embedding of an arbitrary complete ordinary function system Φ, defined on a set X in the space $C = C(\delta X)$, where δX is a "topological extension" of X which extends to give an embedding of Φ_α in C_α. From this a sufficient condition in order that the Baire order of Φ be ω_1 is derived [Theorem 5.10].

Definition. If X is a topological space, let $W(X)$ be the family of all real-valued functions f defined on X for which there is a first-category subset P of X such that the partial function $f \mid X - P$ is continuous (such a function f is said to have the Baire property in the wide sense [21]).

Lemma 5.1 [Baire, 2]. *If f is a function of Baire's class 1, then the set of points of discontinuity of f is of the first category.*

Proof. The set D of points of discontinuity of f satisfies the formula: $D = \bigcup_R (\overline{f^{-1}(\overline{F_n})} - f^{-1}(F_n))$, where the sets $\{F_n\}_{n=1}^{\infty}$ form a countable base for the closed number sets.

Since f is of Baire's class 1, $f^{-1}(F_n)$ is a G_δ set. Thus, $\overline{f^{-1}(\overline{F_n})} - f^{-1}(F_n)$ is an F_σ set and is also a boundary set. Therefore, for each n, $\overline{f^{-1}(\overline{F_n})} - f^{-1}(F_n)$ is of the first category, and the set D is of the first category.

Theorem 5.1. *The family, $W(X)$, of all functions having the Baire property in the wide sense is closed under pointwise limits.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence from $W(X)$ converging pointwise to a function f. For each n, let P_n be a first-category set such that $f_n \mid X - P_n$ is continuous. Let $P_0 = \bigcup_{n=1}^{\infty} P_n$, P_0 is of the first category, and for each n, $f_n \mid X - P_0$ is continuous. Let P_d be the set of points of discontinuity of $f \mid X - P_0$. The set P_d is of the first category in $X - P_0$ and hence in X. Therefore, $P = P_0 \cup P_d$ is of the first category and $f \mid X - P$ is continuous.

DEFINITION. If X is a topological space, let $R(X)$ be the family of all real-valued functions f defined on X such that for every closed set F, the partial function $f \mid F$ has the Baire property in the wide sense (such a function is said to have the Baire property in the restricted sense [21]).

COROLLARY 5.1. *The family, $R(X)$, of functions having the Baire property in the restricted sense is closed under pointwise limits.*

Remarks. It is always true that $C(X) \subseteq R(X) \subseteq W(X)$, and since the last two families are closed under pointwise limits, $C_{\omega_1} \subseteq R(X) \subseteq W(X)$. Every Baire function has the Baire property in the restricted sense and hence in the wide sense. For a wide variety of spaces, the inclusions given above are proper [21]. It should be remarked that the family, $A(X)$, of A-functions on X is also closed under pointwise limits and $C_{\omega_1} \subseteq A(X) \subseteq R(X)$ [21, pp. 95 and 512].

In the case of metric spaces, the family $W(X)$ has a particularly clear realization as the following theorem of Kuratowski [22] shows:

THEOREM 5.2. *If X is a metric space, then a function f is in the family $W(X)$ if and only if the pointwise limit of a sequence of functions each continuous except for a first-category set.*

Proof. Let $f \in W(X)$, and let $\{F_n\}_{n=1}^{\infty}$ be an increasing sequence of closed nowhere-dense subsets of X such that the partial function $f \mid X - D$ is continuous, where $D = \bigcup_{n=1}^{\infty} F_n$. Let $F_0 = \varnothing$.

For each $x \in D$, let $k(x)$ be the integer such that $x \in F_{k(x)} - F_{k(x)-1}$. If $\{x_p\}_{p=1}^{\infty}$ is a sequence from D converging to a point x of $X - D$, then $\{k(x_p)\}_{p=1}^{\infty} \to \infty$.

For each $x \in D$, let $\varphi(x)$ be a point of $X - D$ such that $\rho(x, \varphi(x)) < \rho(x, X - D) + 1/k(x)$. For each $x \in X - D$, let $\varphi(x) = x$. The mapping φ from X onto $X - D$ is continuous at each point of $X - D$.

Let g be the composition of f with φ, $f \circ \varphi$. The function g is continuous at each point of $X - D$.

$$g(x) = \begin{cases} f(x), & x \in X - D \\ f(\varphi(x)), & x \in D. \end{cases}$$

The function g is continuous at each point of $X - D$.

For each n, let

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in F_n, \\ g(x), & \text{if } x \in F_n'. \end{cases}$$

The functions f_n are continuous at each point of $X - D$ and the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f.

DEFINITION. If X is a topological space, let $M(X)$ be the family of all functions f such that the set of points of discontinuity of f is a first-category set.

A restatement of the preceding theorem gives:

THEOREM 5.2. *If X is a metric space, $W(X) = M_1(x)$ and the Baire order of $M(X)$ is at most 1.*

The following example is due to Sierpinski [47];

EXAMPLE 5.1. If $\aleph_1 = 2^{\aleph_0}$, then there is a number set X such that the Baire order of $C(X)$ is 2.

Proof. Let X be a Lusin set; X is a set of cardinality c such that every set of the first category in X is countable.

If $f \in M(X)$, then f is continuous except for a countable set and it follows that $f \in C_1(X)$. Thus, $W(X) = M_1(X) = C_2(X)$ and the Baire order of X is no more than 2.

Let D be a countable dense subset of X. The characteristic function ξ_D of D has the Baire property in the wide sense. Therefore, $\xi_D \in C_2(X)$. However, since ξ_D is not continuous at any point, $\xi_D \notin C_1(X)$. Thus, the Baire order of $C(X)$ is 2.

This example is due to Szpilrajn [47].

EXAMPLE 5.2. If $\aleph_1 = 2^{\aleph_0}$, then there is an uncountable number set X such that the Baire order of $C(X)$ is 1.

Proof. Suppose $\aleph_1 = 2^{\aleph_0}$. Let X be an uncountable number set whose intersection with each set of Lebesgue measure 0 is countable.

Suppose M is a Borel set with respect to X. Then $M = B \cap X$, where B is a Borel subset of the space of all real numbers. Since B is Lebesgue measurable, B can be expressed as the union of an F_σ set K and a set N of Lebesgue measure 0. Thus,

$$M = (K \cap X) \cup (N \cap X).$$

But, $N \cap X$ being a countable set is an F_σ set in X. Therefore every Borel subset of X is a F_σ set. It follows that the order of $C(X)$ is 1.

There has been some work done on the Baire system generated by the family of functions on a space X which are continuous except for a negligible set [24, 31, 32, 44, 50]. Recently, the author has shown that the Baire order of all function which are continuous except for a set of Lebesgue measure zero is ω_1.

The author does not know the answer to the following question: For each countable ordinal α, is there a metric space X or a completely regular space X such that the Baire order of $C(X)$ is α?

Let Φ be a complete ordinary function system on a set X, and let X be given the weak topology generated by Φ. Let $U\Phi$ be the family of all functions f for which there is a subfamily A of Φ such that for each x in X,

$$f(x) = \text{g.l.b.}\{g(x): g \in A\}.$$

Let $L\Phi$ be the family of all f such that $-f \in U\Phi$. Of course, $(U\Phi) \cap (L\Phi) \subseteq C(X)$, and it can be shown that each bounded function in $C(X)$ is in $(U\Phi) \cap (L\Phi)$. The author notes the following:

THEOREM 5.3. *A function f on X is in $C(X)$ if and only if $f = f_1/f_2$, where f_1 and f_2 are in $(U\Phi) \cap (L\Phi)$.*

If $\Phi = (U\Phi) \cap (L\Phi)$, then the Baire order problem for Φ is the same as the Baire order problem for $C(X)$. However, this is not usually the case. The following lemmas will be used to construct a topological extension, δX, of X by adding ideal points to X and obtaining a space of continuous functions on δX which has the same Baire order as Φ. It will be assumed that the family Φ separates points of X. This is no restriction for our considerations, since if Φ does not separate points of X, we may consider the stationary sets of Φ as our points and obtain a function system which has the same structure as Φ.

The construction of the space δX follows the construction of the space eX given in Gilman and ^{x}erison [13] and the lemmas are stated here for completeness.

DEFINITION. The statement that $\mathscr{F}$ is a $Z(\Phi)$-filter means $\mathscr{F} \subseteq Z(\Phi)$ such that (1) if $Z_1, Z_2 \in \mathscr{F}$, then $Z_1 \cap Z_2 \in \mathscr{F}$, (2) if $Z_1 \in \mathscr{F}$, $Z \in Z_1$, and $Z \in Z(\Phi)$, then $Z \in \mathscr{F}$, and (3) $\infty \notin \mathscr{F}$. $\mathscr{U}$ is a $Z(\Phi)$-ultrafilter mean $\mathscr{U}$ is a maximal $Z(\Phi)$-filter.

LEMMA 5.4a. *If A_1 and A_2 are disjoint zero sets, then there is a zero* *set* Z_1 *and a zero set* Z_2, Z_1 *disjoint from* A_1, Z_2 *disjoint from* A_2 *such that* $Z_1 \cup Z_2 = X$.

LEMMA 5.4b. *Let* $\mathscr{F}$ *be a* $Z(\Phi)$*-filter. In order that* $\mathscr{F}$ *be maximal, it is necessary and sufficient that if a zero set* Z *meets every member of* $\mathscr{F}$, *then* $Z \in \mathscr{F}$. *If* $x \in X$, *then* $\mathscr{U}_x = \{Z: x \in Z\}$ *is an ultrafilter.*

LEMMA 5.4c. *Let* $\mathscr{U}$ *be a* $Z(\Phi)$*-ultrafilter. Then* $Z_1 \cup Z_2 \in \mathscr{U}$ *if and only if either* Z_1 *or* Z_2 *belongs to* $\mathscr{U}$ *and* $Z_1 \cap Z_2 \in \mathscr{U}$ *if and only if* Z_1 *and* Z_2 *belongs to* $\mathscr{U}$.

LEMMA 5.4d. *If* $\mathscr{U}_1$ *and* $\mathscr{U}_2$ *are distinct ultrafilters, then there are disjoint zero sets* A_1 *and* A_2 *such that* $A_1 \in \mathscr{U}_1$ *and* $A_2 \in \mathscr{U}_2$.

Let βX be the set of all $Z(\Phi)$-ultrafilters. For each $Z \in Z(\Phi)$, let $\bar{Z} = \{\mathscr{U}: \mathscr{U} \in \beta X \text{ and } Z \in \mathscr{U}\}$.

LEMMA 5.4e. *If* $Z_1, Z_2 \in Z(\Phi)$, *then* $\bar{Z}_1 \cup \bar{Z}_2 = \overline{Z_1 \cup Z_2}$ *and* $\bar{Z}_1 \cap \bar{Z}_2 = \overline{Z_1 \cap Z_2}$.

It follows from Lemma 5.4e that the family $\{\bar{Z}: Z \in Z(\Phi)\}$ is a closed base for a topology on βX. Let βX be given this topology. Let φ be the mapping of X into βX defined by $\varphi(x) = \mathscr{U}_x$, $x \in X$.

THEOREM 5.4. *If* $Z \in Z(\Phi)$, *then* $\text{cl}_{\beta X}\, \varphi(Z) = \bar{Z}$. *In particular,* $\varphi(X)$ *is dense in* βX *and* βX *is a compact Hausdorff space.*

Let f be a function from X into a compact Hausdorff space Y such that for each closed set F of Y, $f^{-1}(F) \in Z(\Phi)$. For each $\mathscr{U} \in \beta(X)$, let $f^{\#}(\mathscr{U}) = \{F = \bar{F} \subset Y: f^{-1}(F) \in \mathscr{U}\}$.

LEMMA 5.5. $f^{\#}(\mathscr{U})$ *is a filter of closed subsets of* Y *such that if* $F_1 \cup F_2 \in f^{\#}(\mathscr{U})$, *then* $F_1 \in f^{\#}(\mathscr{U})$ *or* $F_1 \in f^{\#}(\mathscr{U})$.

THEOREM 5.5. *For each* $\mathscr{U} \in \beta X$, *there is exactly one point* p *of* Y *such that* p *meets every set in* $f^{\#}(\mathscr{U})$.

For each $\mathscr{U} \in \beta X$, let $\hat{f}(\mathscr{U})$ be the point p of Y described in Theorem 5.5.

THEOREM 5.6. *For each* $x \in X$, $\hat{f}(\mathscr{U}_x) = f(x)$ *and* $\hat{f}$ *is a continuous function from* βX *to* Y. *In particular, if* f *is a bounded function in* Φ, *then* f

has a continuous extension to βX and the bounded functions in Φ are mapped onto $C(\beta X)$. Compare [14].

Of course, if X is a completely regular T_2 space and $\Phi = C(X)$, then βX is the Stone–Cech compactification of X.

Let δX be the subspace of βX consisting of all real $Z(\Phi)$-ultrafilters with the relative topology. An ultrafilter $\mathscr{U}$ is real means that if $Z_n \in \mathscr{U}$, for each n, then $\bigcap_{n=1}^{\infty} Z_n \in \mathscr{U}$. We have that $\varphi(X)$ is a dense subset of δX.

Suppose $f \in \Phi$. Consider f as a mapping from X into R^*, the one-point compactification of R. Suppose $\mathscr{U} \in \delta X$ and the limit of the filter $f^{\#}(\mathscr{U})$ is ∞. Let $\{F_n\}_{n=1}^{\infty}$ be a monotonic sequence from $f^{\#}(\mathscr{U})$ whose intersection is ∞. Let $Z = \bigcap_{n=1}^{\infty} f^{-1}(F_n)$. Since $\mathscr{U} \in \delta X$, $Z \in \mathscr{U}$ and $Z \neq \varnothing$. But, if $x \in Z$, then $f(x) = \infty$. This contradiction proves:

THEOREM 5.7. *Each function in Φ has a unique extension to a continuous function in $C(\delta X)$. Thus, every complete ordinary function system may be realized as a space of continuous functions on a completely regular T_2 space.*

Again, if X is a completely regular T_2 space and $\Phi = C(X)$, then δX is υX, the Hewitt real compactification of X.

Remarks. Some necessary and sufficient conditions for Φ to be mapped onto $C(\delta X)$ are given in [17]. However, the author does not know of any clear characterization of the systems Φ which are mapped onto $C(\delta X)$. For a study of this problem see]16].

The following theorem was essentially proven by Jayne [18].

THEOREM 5.8. *Each function in Φ_α has a unique extension to a function in $\hat{\Phi}_\alpha$ and the mapping $f \to \hat{f}$ takes $\hat{\Phi}_\alpha$ onto $\hat{\Phi}_\alpha$.*

Proof. Theorem 5.7 proves that this theorem holds for $\alpha = 0$. Note that if $\bar{Z} = \hat{f}^{-1}(0)$, for some $\hat{f} \in \hat{\Phi}$, then the set $\bar{Z}$ intersects X. A set S is said to be a W-analytic set with respect to a subfamily W of $P(X)$ if

$$S = \bigcup_{z \in \mathscr{N}} \left(\bigcap_{n=1}^{\infty} F_{(z_1, z_2, \ldots, z_n)} \right),$$

where $F_{(z_1,\ldots,z_n)} \in W$, for each finite sequence $(Z_1, \ldots, Z_n)$ of positive integers, and $\mathscr{N}$ is the space of all irrational members between 0 and 1. The family of all W-analytic sets is closed under countable unions and intersections [15].

Since each set in $CZ_0(\hat{\Phi})$ is the union of countably many sets in

$Z_0(\hat{\Phi})$, it follows that each set in $C_{\omega_1}(\hat{\Phi})$ is a $Z(\hat{\Phi})$-analytic set. Since each $Z(\hat{\Phi})$-analytic set intersects X, each set in $Z_{\omega_1}(\hat{\Phi})$ intersects X.

The proof of the theorem now proceeds by transfinite induction.

Suppose $f \in \Phi_1$ and $\{f_n\}_{n=1}^{\infty}$ is a sequence from Φ converging to f. Let $\{\hat{f}_n\}_{n=1}^{\infty}$ be the sequence of their extensions to δX, and let z be a point of δX. For each n, let $\hat{Z}_n = \{x: \hat{f}_n(x) = \hat{f}_n(z)\}$. Let $\hat{Z} = \bigcap_{n=1}^{\infty} \hat{Z}_n$. Of course, $z \in \hat{Z}$ and the set $\hat{Z}$ intersects X. Let $x \in \hat{Z} \cap X$. It follows that the sequence $\{\hat{f}_n(z)\}_{n=1}^{\infty}$ converges to $f(x)$ and the function f has an extension to a function f in the family $\hat{\Phi}_1$.

Suppose $\hat{f}$ and h are two extensions of f to $\hat{\Phi}_1$. The set $D = \{x: \hat{f}(x) \neq h(x)\}$ is a $Z_{\omega_1}(\hat{\Phi})$ set which does not intersect X. This contradiction proves the theorem for the ordinal 1. Also, if the theorem holds for a countable ordinal α, this argument may be employed to prove the theorem for the ordinal $\alpha + 1$.

Suppose α is a limit ordinal and the theorem holds for every $\gamma < \alpha$. Let $f \in \Phi$. Then $f = f_1 f_2$, where f_1 and f_2 belong to both $US(\bigcup_{\gamma<1} \Phi_\gamma)$ and $LS(\bigcup_{\gamma<\alpha} \Phi_\gamma)$. Let $\{g_n\}_{n=1}^{\infty}$ be a nondecreasing sequence from $\bigcup_{\gamma<\alpha} \Phi_\gamma$ converging to f_1, and let $\{h_n\}_{n=1}^{\infty}$ be a nonincreasing sequence from $\bigcup_{\gamma<\alpha} \Phi_\gamma$ converging to f_1. It follows that the sequence $\{\hat{g}_n\}_{n=1}^{\infty}$ is nondecreasing on δX and converges to a function $\hat{g}$ in $LS(\bigcup_{\gamma<\alpha} \hat{\Phi}_\gamma)$. Similarly, the sequence $\{\hat{h}_n\}_{n=1}^{\infty}$ is nonincreasing on δX and converges to a function $\hat{h}$ in $US(\bigcup_{\gamma<\alpha} \Phi_\gamma)$. Also, $\hat{g} = \hat{h}$ and f_1 has a unique extension to a function $\hat{f}_1$ which is in both $LS(\bigcup_{\gamma<\alpha} \Phi_\gamma)$. The theorem follows.

COROLLARY 5.8. *If X is a completely regular T_2 space, then each function in $C_\alpha(\upsilon X)$ has a unique extension to a function in $C_\alpha(\upsilon X)$.*

Remark. The fact that the real compactification of a completely regular space preserves to Baire classes was proven by Paul Meyer in 1961 (unpublished).

Theorem 5.8 leads to the following:

Question. For each ordinal α, what are necessary and sufficient conditions on Φ in order that Φ_α be mapped onto $C_\alpha(\delta X)$? For some initial studies into this problem, see [16] and [12].

THEOREM 5.9. *If K is a compact subset of δX, then every bounded Baire function defined on the space K has an extension to a bounded function in $\hat{\Phi}_{\omega_1}$.*

Proof. Let $\mathcal{Q}$ be the family of all restrictions of bounded functions in Φ_0 to K. It follows from the Stone–Weierstrauss Theorem that $\mathcal{Q}$ is $C(K)$.

Let G be the family of all bounded Baire functions defined on K which have extensions to bounded functions in $\hat{\Phi}_{\omega_1}$. Let g be a bounded function in G_1, and let $\{g_n\}_{n=1}^{\infty}$ be a sequence from G converging to g such that for each n, $\| g_n \| \leqslant \| g \|$. For each n, let f_n be an extension of g_n to $\hat{\Phi}_{\omega_1}$ such that $\| f_n \| \leqslant \| g_n \|$.

Let $h = \overline{\lim}_{n=\infty} f_n$ and let $k = \underline{\lim}_{n\to\infty} f_n$. By Theorem 2.2, there is a function l in $\hat{\Phi}_{\omega_1}$ such that $k \leqslant l \leqslant h$. The function l is a bounded extension of g to a function in Φ_{ω_1}. It follows that G is the family of all bounded Baire functions defined on K.

From the theorems of the preceding section and Theorem 5.9, we have:

THEOREM 5.10. *If the space δX contains a compact perfect set, then the Baire order of the complete ordinary function system Φ is ω_1.*

COROLLARY 5.10a. *Let X be a completely regular T_2 space. If the Hewitt real compactification of X contains a compact perfect set, then the Baire order of $C(x)$ is ω_1.*

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