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Borel measurable selections of Paretian utility functions

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We show that there is a Borel measurable selection of Paretian utilities in 'markets with a continuum of traders'. *Theorem*: Let T and X be Polish spaces, R a Borel subset of $T \times X \times X$, and $B=\{(t,x)\colon (t,x,x)\in R\}$. Suppose that for each t, $R_t=\{(x,y)\colon (t,x,y)\in R\}$ is a preference order on $B_t=\{x\colon (t,x)\in B\}$. Then there is a Borel measurable function $f\colon B\to[0,1]$ such that for all $t\in T$, $f_t\colon B_t\to[0,1]$ is a Paretian utility or continuous representation of R_t. This improves earlier results showing that there are universally measurable f.

Key words: Preference orders; First separation principle; Borel measurable

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1. Introduction

Let Y be a topological space. A preference order, $\preceq$, on Y is a closed linear preorder on Y. This means $\preceq$ is a binary relation on Y which is a closed subset of $Y \times Y$ and which is transitive, reflexive and complete. In other words, $\preceq$ is a linear order on the equivalence classes of Y given by $x \sim y$ if and only if $x \preceq y$ and $y \preceq x$. We write $x \prec y$ provided $x \preceq y$ and $\neg(y \preceq x)$. We note that a linear preorder on Y is closed if and only if all sets of the form $\{y\colon y \preceq x\}$ or $\{y\colon x \preceq y\}$ are closed in Y. This concept was introduced by Debreu (1959) and is discussed in Hildenbrand (1974) and Hildenbrand and Kirman (1988).

Debreu (1964) showed that any preference order $\preceq$ on Y, where Y is second countable, can be realized by a continuous function $f\colon Y\to[0,1]$, i.e., for all $x,y\in Y$, $x\preceq y \leftrightarrow f(x)\leqq f(y)$. Thus, f is a continuous representation of the preference order $\preceq$. Such a function f is called a Paretian utility for $\preceq$. See also, Mauldin (1984) and Rader (1963) and the discussion in Hildenbrand (1974).

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A parameterized version of this problem or 'preference orders in markets with a continuum of traders' has been considered in Burgess (1985), Mauldin (1983) and Wesley (1976). Let T and X be Polish spaces and let R be a Borel measurable subset of $T\times X\times X$ such that for each $t\in T$, $R_t=\{[(x,y)\colon (t,x,y,)\in R\}$ is a preference order on the set $B_t=\{x\colon (t,x,x)\in R\}$. Here closed always means relatively closed in B_t. We refer to the set $B=\{(t,x)\colon (t,x,x)\in R\}$ as the field of R. Since the map $f\colon T\times X\to T\times X\times X$ given by $f(t,x)=(t,x,x)$ is continuous and $B=f^{-1}(R)$, B is a Borel subset of $T\times X$. For $t\in T$, we write $x\preceq_t y$ for $(t,x,y)\in R$. Thus, we are considering for each point or trade t, a preference order on the field B_t where both the field of the preference order and the preference order vary in a Borel measurable fashion with t. Wesley (1976), using forcing methods, showed that if $T=[0,1]$ and $X=R^n$, then there is a Borel measurable function $f\colon B\to R$ such that for Lebesgue measure almost all t, f_t is a utility for $\preceq_t$. Mauldin (1983) improved this result. Using standard methods of set theory, he showed in the general setting, the existence of a selection f of Paretian utilities such that f is measurable with respect to a known family of universally measurable sets, $\mathscr{S}(T\times X)$, the C-sets of Selivanovskii. Here $\mathscr{S}(T\times X)$ is the smallest family of subsets of $T\times X$ containing the open sets which is closed under complementation and Suslin's operation ($\mathscr{A}$). This result also yields Wesley's theorem as a corollary.

The main problem which remained unresolved until now is whether there exists a Borel measurable function $f\colon B\to[0,1]$ such that for all $t\in T$, the function $f_t\colon B_t\to[0,1]$ given by $f_t(x)=f(t,x)$ is continuous and represents $\preceq_t$. In other words, the question is whether there exist a Borel measurable selection of Paretian utilities. This would be the best possible result. In Cenzer and Mauldin (1983) it was shown that there is a Borel measurable f in a special case: each $\ _t$ is not only a linear preorder, but is a prewellorder. We will show here that indeed such a Borel measurable function f exists in the general case.

The starting point for the previous approaches involved first showing that there is a sequence, $\{f_n\}_{n=1}^{\infty}$, of universally measurable selections for B such that for each t, $\{f_n(t)\}_{n=1}^{\infty}$ is order dense in B_t. In particular, each function f_n is a 'uniformization' of B. This means the graph of each function f_n is a subset of B. Unless these functions can be chosen to be Borel maps [which is possible in certain cases as shown in Mauldin (1983)], the previous procedure cannot lead to a Borel measurable parameterization. We give an example at the end to show that it is not always possible for the maps f_n to be Borel measurable.

We state now the main theorem of this paper. Recall that a Polish space refers to a complete separable metric space. A function $f\colon X\to Y$ between Polish spaces is Borel measurable, or Borel for short, iff $f^{-1}(U)$ is Borel for all open $U\subset Y$. This is equivalent to saying [Kuratowski (1962), Theorem 1,

p. 384 and Theorem 2, p. 489), or Moschovakis (1980, 2E.4, p. 90] that graph $(f)=\{(x,y): f(x)=y\}$ is a Borel subset of $X \times Y$.

Theorem (boldface version). Let T and X be Polish spaces, R a Borel subset of $T \times X \times X$ such that for each t, $R_t=\{(x,y): (t,x,y) \in R\}$ is a preference order on $B_t=\{x: (t,x) \in B\}$, where $B=\{(t,x): (t,x,x) \in R\}$ is the field of R. Then there is a Borel measurable function $f: B \to [0,1]$ such that for all $t \in T$, $f_t: B_t \to [0,1]$ is a Paretian utility or continuous representation of R_t.

For the remainder of this section we fix some notation and present some standard results from descriptive set theory for the convenience of the reader. The reader wishing more background might consult Moschovakis (1980) or Kuratowski (1966). In section 2 we prove our main technical result, the existence of sectionwise open (or closed) Borel separating families. In section 3 we do the main 'Urysohn-like' construction. This does not quite produce a representing function, but what we call an 'almost representing' function. The arguments up to this point use only the 'easy uniformization theorem', discussed below, and notions and results from classical descriptive set theory (what modern descriptive set theorists refer to as the 'boldface' theory). Finally we must modify this function to obtain a true representing function, and thereby complete the proof of the main theorem. We present two proofs for this last part of the construction, one in section 4 and one in section 5. The proof in section 4 uses still only the easy uniformization theorem, but uses some notions from effective descriptive set theory (the 'lightface' theory). The proof in section 5 uses only the classical notions, but makes appeal to the full uniformization theorem. The lightface arguments may prove to be necessary elsewhere, so we believe it is worth presenting both arguments. The reader sufficiently familiar with descriptive set theory may now skip directly to section 2.

Throughout the rest of this paper T, X denote Polish spaces, $R \subset T \times X \times X$ is as stated in the theorem, and $B=\{(t,x): (t,x,x)\}$ is its field. For $t \in T$, $B_t=\{x: (t,x) \in B\}$ denotes the section of B at t. We rarely mention R but write instead $x \preceq_t y$ for $R(t,x,y)$ and $\prec_t$ for $R(t,x,y) \wedge \neg R(t,y,x)$. If $C \subset T \times X$ we will also write $C(t)$ for the section of C at t to ease notation. We employ frequently the 'logical notation' common to descriptive set theorists. For example, if $S \subset T \times X$ we write $S(t,x)$ interchangeably with $(t,x) \in S$. Also, $\neg S(t,x)$ means $(t,x) \notin S$. Throughout, ω denotes the set of natural numbers. Clopen means both closed and open.

Recall that a subset A of a Polish space is analytic [or in the terminology of Moschovakis (1980), $\utilde{\Sigma}_1^1$] if A is the image of a Borel set under a Borel measurable map [Kuratowski (1966, p. 478), Moschovakis (1980, p. 39)]. A set C is coanalytic (or $\utilde{\Pi}_1^1$) means its complement is an analytic set. Equivalently, $S \subset X$ is analytic if it be written in the form $S(x) \Leftrightarrow \exists y \in Y\, T(x,y)$

where $T \subset X \times Y$ is Borel (equivalently, closed), for some Polish space Y. Likewise, S is coanalytic if it can be written in the form $S(x) \Leftrightarrow \forall y \in Y T(x, y)$, for Borel T. The class of analytic sets is closed under countable unions, countable intersections, and existential quantification over Polish spaces. Likewise, coanalytic sets are closed under universal quantification. Suslin's theorem [Kuratowski (1966, p. 486) or Moschovakis (1980, p. 90)] says a set is Borel iff it is analytic and coanalytic.

For the convenience of the reader we provide a quick review of the lightface notions. The reader wishing more background could consult chapter three of Moschovakis (1980). The reader intent on avoiding any lightface arguments can skip section 4, reading section 5 instead. In sections 2–4 we also use the notation of the lightface theory, but the reader may read these sections replacing Σ^1_1 by 'analytic', Π^1_1 by 'coanalytic', and Δ^1_1 by 'Borel'.

As classical (boldface) descriptive set theory is developed in the context of Polish spaces, the effective (lightface) theory is developed in the context of recursively presented Polish spaces. Roughly, this means a Polish space with a countable dense set $D = \{r_i \colon i \in \omega\}$ such that the metric on D is effectively computable. All of the familiar Polish spaces (e.g. the reals $\mathbb{R}$, the Baire space, $C[0, 1]$) are recursively presented. Below, X, Y, T denote recursively presented Polish spaces. The classical notions of analytic, coanalytic, and Borel have lightface analogs Σ^1_1, Π^1_1, and Δ^1_1, respectively. Briefly, a set $S \subset X$ is Σ^1_1 (or more generally $\Sigma^1_1(t)$, i.e., 'Σ^1_1 in the parameter $t \in T$') iff it can written in the form $S(x) \Leftrightarrow \exists y \in Y T(x, y)$ for some space Y, where $T \subset X \times Y$ is Π^0_1 [resp. $\Pi^0_1(t)$]. Π^0_1 is the effective refinement of the notion of closed, which is sometimes denoted $\underset{\sim}{\Pi}^0_1$. For $x \in T$, the set $A \subset X$ is $\Pi^0_1(t)$ if there is a $B \subset T \times X$ in Π^0_1 such that $A = B_t$. S is Π^1_1 iff its complement is Σ^1_1, and Δ^1_1 iff it is both Σ^1_1 and Π^1_1. A basic fact is that $\underset{\sim}{\Sigma}^1_1 = \bigcup_{t \in T} \Sigma^1_1(t)$, and similarly for $\underset{\sim}{\Pi}^1_1$ and $\underset{\sim}{\Delta}^1_1$. The collection of Σ^1_1 sets is closed under existential and universal quantification over the integers, and existential quantification over recursively presented Polish spaces. Similarly, Π^1_1 is closed under universal quantification over Polish spaces. A function $f \colon X \to Y$ is Δ^1_1 iff graph $(f) \subset X \times Y$ is Δ^1_1. Essentially all of the theorems of 'classical' descriptive set theory about the boldface classes have 'effective' analogs, replacing $\underset{\sim}{\Pi}^1_1$ by Π^1_1, etc. Because the lightface classes refine the boldface classes, one obtains a sharper result by proving the lightface version of a theorem. In fact, the lightface version implies the boldface version for all Polish spaces (not just recursively presented spaces) since any Polish space X has a presentation $\{r_i \colon i \in \omega\}$ which is recursive in some parameter $z \in \mathbb{R}$.

We in fact actually prove the following sharper form of the main theorem:

Theorem (*lightface version*). *Let T, X be recursively presented Polish spaces, and R a Δ^1_1 subset of $T \times X \times X$ such that for each t, R_t is a preference order on B_t where $B = \{t, x)\colon (t, x, x) \in R\}$ is the field of R. Then there is a Δ^1_1 function*

f: $B \to [0,1]$ such that for all $t \in T$, f_t: $B_t \to [0,1]$ is a continuous representation of R_t.

As mentioned before, the arguments of this paper may be read by either the boldface or lightface reader. The latter will obtain the above lightface version of the theorem and the former, the boldface version. The lightface reader should read sections 1 through 4 for the complete proof. The boldface reader should read sections 2 and 3 replacing 'Σ^1_1' by 'analytic', 'Π^1_1' by 'coanalytic', 'Δ^1_1' by 'Borel', and 'recursive' by 'continuous', and then read section 5. We caution the lightface reader that when dealing with a sequence of sets A_n, if we say A_n is Π^1_1 (or Σ^1_1, etc.) we always mean that the sequence of sets is uniformly Π^1_1, that is, the set $A(x,n) \Leftrightarrow x \in A_n$ is Π^1_1. Also, 'Polish space' means 'recursively presented Polish space' to the lightface reader.

Finally, let us recall in detail some facts which will be used several times in the proof. They are consequences of the so-called 'easy uniformization theorem', but we give direct proofs below using only the countable reduction property for Π^1_1 sets. For the sake of completeness recall first the Novikov–Kondo uniformization theorem [Moschovakis (1980, p. 235)]: if $S \subset X \times Y$ is $\underset{\sim}{\Pi}^1_1$, then there is a function f with $\underset{\sim}{\Pi}^1_1$ graph which uniformizes S, that is, $\mathrm{dom}(f) = \{x \in X\colon \exists y \in Y S(x,y)\}$ and $\forall x \in \mathrm{dom}(f) S(x, f(x))$. The lightface version for Π^1_1 sets is the Novikov–Kondo–Addison theorem. If Y is countable (e.g. $Y = \omega$) the theorem is easier, and sometimes referred to as the 'easy uniformization theorem' [Moschovakis (1980, p. 202)]. We need only this easier version to establish the facts below which we need.

Recall the reduction property for Π^1_1 [Kuratowski (1966, p. 508), Moschovakis (1980, p. 204)] says that if A, B are Π^1_1 subsets of a Polish space, then there are Π^1_1 sets $C \subset A$, $D \subset B$ such that $C \cap D = \varnothing$, $C \cup D = A \cup B$. There is also a countable reduction theorem for infinitely many Π^1_1 sets A_n [see again Kuratowski (1966, p. 508)]. To see this, let $A_n \subset X$ be Π^1_1 sets. Define $R \subset X \times \omega$ by $R(x,n) \Leftrightarrow (x \in A_n)$. Thus, R is Π^1_1. Let $R' \subset R$ uniformize R by the easy uniformization theorem. Let $B_n = \{x\colon (x,n) \in R'\}$. Then $B_n \subset A_n$, the B_n are pairwise disjoint, and $\bigcup_n B_n = \bigcup_n A_n$ since R' was a uniformization of R.

We prove now two facts which will be used repeatedly in the paper [c.f. the Δ selection priciple [Moschovakis (1980, p. 203)]. We give direct proofs using only the countable reduction property for Π^1_1 sets.

Fact 1. Let $S \subset X \times \omega$ be Π^1_1 and with $\mathrm{dom}(S) = \{x \in X\colon \exists n \in \omega S(x,n)\}$ a Δ^1_1 subset of X. Then there are Δ^1_1 functions f_m: $X \to \omega$ such that $\mathrm{dom}(f_m) \subset \mathrm{dom}(S)$, $\forall m \in \omega\, \forall x \in \mathrm{dom}(f_m) S(x, f_m(x))$, and $S = \bigcup_m \mathrm{graph}(f_m)$.

Proof. Let $A_m = \{x \in X\colon S(x,m)\}$, so $A_m \in \Pi^1_1$. By the countable reduction theorem for Π^1_1 let $B_m \subset A_m$ be disjoint Π^1_1 sets with $\bigcup_m B_m = \bigcup_m A_m = \mathrm{dom}(S)$. Since $\mathrm{dom}(S)$ is Δ^1_1, the B_m are actually Δ^1_1. Let $f_m = B_m \times \{m\}$. □

Note that the domains of the f_m above are Δ_1^1 sets (it is actually true that the domain of an arbitrary Δ_1^1 function $f\colon X\to Y$ is Δ_1^1). Another useful fact which follows from the above is the 'Δ selection principle':

Fact 2. If $S\subset X\times\omega$ *is* Π_1^1 *with* $\operatorname{dom}(S)\in\Delta_1^1$, *then there is a* Δ_1^1 *function* f: $\operatorname{dom}(S)\to\omega$ *such that* $\forall x\in\operatorname{dom}(S)$, $S(x,f(x))$.

Proof. Let $f(x)=m\Leftrightarrow x\in B_m$ where the B_m are as above. □

2. The separating families

We say the family $\{B_n\colon n\in\omega\}$ of subsets of B is a separating family for the preference order if whenever (t,x), $(t,y)\in B$ and $x\prec_t y$ then for some $n\in\omega$ we have $(t,x)\notin B_n$ and $(t,y)\in B_n$. We construct in this section two families $\{B_n\colon n\in\omega\}$ (resp. $\{\tilde{B}_n\colon n\in\omega\}$) of Δ_1^1 subsets of B, where each B_n (resp. $\tilde{B}_n$) is upward saturated (see below) and sectionwise open (resp. closed). Recall that B_n sectionwise open means that each section $B_n(t)$ is open in the relative topology on $B_t=\{x\colon (t,x)\in B\}$.

Let us make some more terminology and note some basic facts. A set $C\subset B$ is 'upward saturated' means if $(t,x)\in C$ and $x\quad{}_t y$, then $(t,y)\in C$. If $E\subset B$, then the smallest upward saturated set containing C is, $\operatorname{sat}^{\mathrm{u}}(C)=\{(t,y)\colon \exists x[(t,x)\in C$ and $x\quad{}_t y]$. A set C is saturated means if $(t,x)\in C$ and $x\sim_t y$, then $(t,y)\in C$. An upward saturated set is, of course, saturated. There is a similar notion, $\operatorname{sat}^{\mathrm{d}}(C)$, for the downward saturation of C. Note that the saturation, in any of the above senses, of a Δ_1^1 set is Σ_1^1. We will also consider the sectionwise versions of these notions. For example, if $E\subset X$ and $t\in T$, then $\operatorname{sat}_t(E)=\{z\colon \exists y\in E(z\sim_t y)\}$. Let L be the 'strict' preference relation; $L=\{(t,x,y)\colon x\prec_t y\}$ and let $E=\{(t,x,y)\colon x\sim_t y\}$. Both E and L are Δ_1^1 subsets of $T\times X\times X$. To see this, set $\hat{R}=\{(t,x,y)\colon (t,y,x)\in R\}$. Clearly, $\hat{R}$ is a Δ_1^1 set, since $\hat{R}=f^{-1}(R)$, where f is the permutation map $T\times X\times X$ into itself which interchanges the second and third coordinates (Δ_1^1,Σ_1^1, and Π_1^1 sets are all closed under inverse images by recursive functions, just as Borel, analytic, and coanalytic sets are closed under inverse images by continuous functions). Now, $E=R\cap\hat{R}$ and $L=R\backslash E$. Note that $\hat{L}=\{(t,x,y)\colon y\prec_t x\}$ is also a Δ_1^1 set.

There is one technical observation which will play a role in our considerations. Fix t and note that the closure of a downward saturated set may not be downward saturated (although this is the case if each $\prec_t$ is a linear ordering). So, for $A\subset B_t$, $\operatorname{cl}_t(\operatorname{sat}_t^{\mathrm{d}}(A))$ is not necessarily downward saturated. Fig. 1 indicates how this can occur. It can happen that there is some $z\in\operatorname{cl}_t(\operatorname{sat}_t^{\mathrm{d}}(A))$ and some $w\sim_t z$ such that $w\notin\operatorname{cl}_t(\operatorname{sat}_t^{\mathrm{d}}(A))$. However, in this case, note that $\operatorname{sat}_t^{\mathrm{d}}(\operatorname{cl}_t(\operatorname{sat}_t^{\mathrm{d}}(E))=\{u\colon u\quad{}_t z\}$. Thus, $\operatorname{sat}_t^{\mathrm{d}}(\operatorname{cl}_t(\operatorname{sat}_t^{\mathrm{d}}(E))$ is always both downward saturated and closed with respect to t.

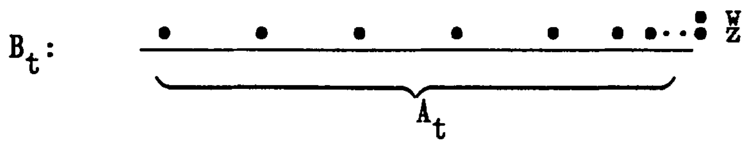

Fig. 1. Shown is a section B_t of a preference order on some $B \subset T \times \mathbb{R}^2$ where points are preferred according to the values of their x-coordinates. Here, A_t converges to, but does not contain z. We have $\mathrm{cl}_t(\mathrm{sat}_t^{\mathrm{d}}(A_t)) = A_t \cup \{z\}$.

The main ingredient in the proof of the theorem is the following lemma.

Lemma 1. There is a sequence of Δ_1^1 subsets $\{B_n\}_{n=0}^{\infty}$ of $B \subset T \times X$ satisfying:

(i) *for each n and each t, $B_n(t)$ is upward saturated (and, therefore, saturated) and $B_n(t)$ is open relative to B_t, and*

(ii) *if $x \prec_t y$, then, for some n, $(t,x) \notin B_n$ and $(t,y) \in B_n$.*

More succinctly, Lemma 1 asserts the existence of a Δ_1^1, upward saturated, sectionwise open separating family.

Proof. Let $U_1, U_2, U_3, \ldots$ be a recursive presentation for the topology of X [see Moschovakis (1980, p. 128)]. (In the boldface case, simply take any enumeration of a base for X.) Define $G \subset T \times \omega \times \omega$ by

$$G(t,i,j) \Leftrightarrow \forall z_1 \forall z_2 [\{z_1 \in \mathrm{sat}_t^{\mathrm{d}}(\mathrm{cl}_t(\mathrm{sat}_t^{\mathrm{d}}(U_i \cap B_t))) \wedge z_2 \in U_j \cap B_t\} \rightarrow z_1 \prec z_2].$$

We claim that G is a Π_1^1 subset of $T \times \omega \times \omega$. This is immediate from the closure properties for Π_1^1 once we observe that the relation $H \subset T \times \omega \times X$ defined by $H(t,i,z_1) \Leftrightarrow z_1 \in \mathrm{sat}_t^{\mathrm{d}}(\mathrm{cl}_t(\mathrm{sat}_t^{\mathrm{d}}(U_i \cap B_t)))$is Σ_1^1. To see this, note that $H(t,i,z_1) \Leftrightarrow \exists w \exists \{y_n : n \in \omega\}[w \sim_t z_1 \wedge (\forall n \in \omega,\ y_n \in \mathrm{sat}_t^{\mathrm{d}}(U_i \cap B_t)) \wedge (y_n$ converges in X to $w)]$. Here $\{y_n : n \in \omega\}$ denotes an element of the Polish space $X^\omega =$ the set of sequences from X [note: X^ω is a (recursively presented) Polish space if X is – c.f. Moschovakis (1980, 3B.3, p. 133)]. From the closure properties of Σ_1^1 sets and the fact that the downward saturation of a Σ_1^1 set is Σ_1^1 we see that H is Σ_1^1.

Define now $A \subset T \times X \times X \times \omega \times \omega$ by $A(t,x,y,i,j) \Leftrightarrow [(x \prec_t y) \wedge x \in U_i \wedge y \in U_j \wedge G(t,i,j)]$. Thus, A is also Π_1^1. We claim that for all (t,x,y) such that $x \prec_t y$ that $\exists i \exists j\ A(t,x,y,i,j)$. To see this, fix t,x,y with $x \prec_t y$. First suppose that for some z that $x \prec_t z \prec_t y$. Then let i,j be such that $x \in U_i \cap B_t \subset \{w : m \prec_t z\}$, and $y \in U_j \cap B_t \subset \{w : w \succ_t z\}$. Then $A(t,x,y,i,j)$ since $\mathrm{sat}_t^{\mathrm{d}}(\mathrm{cl}_t(\mathrm{sat}_t^{\mathrm{d}}(U_i \cap B_t) \subset \{w : w \quad_t z\}$. The case where there is no z strictly between x and y is similar, since in this case $\{w : w \quad_t x\}$ is (relatively) clopen.

Applying Fact 2 of section 1 to A (viewing the domain of A as $L = \{(t,x,y) : x \prec_t y\})$ now yields a Δ_1^1 function $h : L \rightarrow \omega \times \omega$ such that for all

$(t,x,y)\in L$, $A(t,x,y,h_0(t,x,y),h_1(t,x,y))$, where $h(t,x,y)=(h_0(t,x,y),h_1(t,x,y))$. Let $S\subset T\times\omega\times\omega$ be defined by $S(t,i,j)\Leftrightarrow\exists x\exists y[x\prec_t y \wedge h(t,x,y)=(i,j)]$. Thus S is Σ^1_1. Clearly $S\subset G$. Hence by the separation property of Σ^1_1 sets [Moschovakis (1980, p. 204)] there is a Δ^1_1 set U such that $S\subset U\subset G$. Apply now Fact 1 of section 1 to U. This yields a sequence of Δ^1_1 functions f_n with $\mathrm{dom}(f_n)\subset\mathrm{dom}(U)\subset T$ such that $U=\bigcup_n\mathrm{graph}(f_n)$.

Temporarily fix an $n\in\omega$, and we define B_n. The construction uses an infinite iteration of the operations of descriptive set theoretic separation, upward saturation, and adding sets sectionwise open. Define first two subsets D^1_n, D^2_n of $T\times X$ by

$$(t,x)\in D^1_n\leftrightarrow\exists(i,j)[f_n(t)=(i,j)\wedge x\in\mathrm{sat}^{\mathrm{d}}_t(\mathrm{cl}_t(\mathrm{sat}^{\mathrm{d}}_t(U_i\cap B_t)],$$

and similarly.

$$(t,x)\in D^2_n(t,x)\leftrightarrow\exists(i,j)[f_n(t)=(i,j)\wedge x\in\mathrm{sat}^{\mathrm{u}}_t(U_j\cap B_t)].$$

Notice that D^1_n and D^2_n are disjoint Σ^1_1, D^1_n is downward saturated, D^2_n is upward saturated, and D^1_n is sectionwise (relatively) closed. By the separation theorem for Σ^1_1, there is a Δ^1_1 set F_0 such that $F_0\supset D^2_n$ and $F_0\cap D^1_n=\varnothing$.

Next we claim that there is a Σ^1_1 set $G_0\subset T\times X$ with $F_0\subset G_0$, $G_0\cap D^1_n=\varnothing$, and G_0 sectionwise open. To see this consider the relation $K\subset T\times X\times\omega$ defined by $K(t,x,i)\Leftrightarrow F_0(t,x)\wedge(x\in U_i)\wedge\forall z(z\in(U_i\cap B_t)\rightarrow z\notin D^1_n)$. So K is Π^1_1. Also, for every $(t,x)\in F_0$ there is an $i\in\omega$ such that $K(t,x,i)$, since D^1_n is sectionwise closed. Applying Fact 2 of section 1 to K produces a Δ^1_1 function τ: $F_0\rightarrow\omega$ such that for all $(t,x)\in F_0$, $K(t,x,\tau(t,x))$. Then let $G_0(t,x)\Leftrightarrow\exists y\exists m\in\omega[(F_0(t,y)\wedge\tau(t,y)=m\wedge x\in U_m\cap B_t]$. Finally, let $H_0=\mathrm{sat}^{\mathrm{u}}(G_0)$. So, H_0 is also Σ^1_1, is upward saturated, and disjoint from D^1_n. We are now in a position to repeat the above arguments starting with D^1_n and H_0 instead of D^1_n and D^2_n. Continuing, we produce sets $F_0\subset G_0\subset H_0\subset F_1\subset G_1\subset H_1\subset\cdots\subset F_k\subset G_k\subset H_k\subset\ldots$. Let $B_n=\bigcup_k F_k$. Then B_n is disjoint from D^1_n since all the F_k are, is upward saturated since all the H_k are, is sectionwise open since all the G_k are, and is Δ^1_1 since all the F_k are Δ^1_1 and the relation $F(t,x,k)\Leftrightarrow(t,x)\in F_k$ can be seen to be Δ^1_1 (note: in the boldface case we use only that a countable union of Borel sets is Borel). □

It is easy to see that a slight variation of the above construction produces a separating family $\{\tilde{B}_n: n\in\omega\}$ of Δ^1_1 subsets of B which are upward saturated and sectionwise *closed* (e.g. do the above argument making the B_n downward saturated, instead of upward saturated, and then take complements relative to B).

3. The main construction

Fix for the rest of the argument a separating family $\{B_n: n\in\omega\}$ of Borel, upward saturated, sectionwise open subsets of B, and a separating family $\{\tilde{B}_n\colon n\in\omega\}$ of Borel, upward saturated, sectionwise closed subsets of B. Recall that $B_n(t)=\{x\colon (t,x)\in B\}$ denotes the section of B_n at t, and similarly for other sets. Recall also that when referring to a section of a set, such as $B_n(t)$, open, closed always refer to the relative topology on the section.

We produce in this section a function $F\colon B\to[0,1]$ which is an 'almost representing' function. To be precise:

Definition. $F\colon B\to[0,1]$ is an almost representing function for the preference order R if F is a Δ_1^1 function which is sectionwise continuous, invariant with respect to $\sim_t$, non-decreasing on each section with respect to $\prec_t$, and for all t,x,y if $x\prec_t y$ and there are at least five distinct $\sim_t$ classes strictly between x and y, then $F(t,x)<F(t,y)$.

It is because of this last clause that we only have an almost representation. In section 4 we modify F to obtain a true representing function.

The idea of the construction in this section is quite similar to that of Urysohn's lemma in topology. We construct a family of sets C_r, and use this family to define F.

We first introduce two relations, which we denote $\subset\subset$ and $\subset_{\mathrm{cl}}$, between the sections $B_n(t)$, $\tilde{B}_n(t)$. Define $F\subset T\times\omega\times\omega\times\omega$ by $F(t,n,m,i)\Leftrightarrow (i=0 \wedge \forall x[B_n(t,x)\to\tilde{B}_m(t,x)]) \vee (i=1 \wedge \forall x[\tilde{B}_m(t,x)\to B_n(t,x)]$. So F is Π_1^1 and $\forall t\forall n\forall m\exists i\ F(t,n,m,i)$. By Fact 2, let $f\colon T\times\omega\times\omega\to\{0,1\}$ be Δ_1^1 and uniformize F on the last coordinate. For $t\in T$, n, $m\in\omega$ define $B_n(t)\subset\subset\tilde{B}_m(t)$ iff $f(t,n,m)=0$. Note that $\tilde{B}_m(t)\subset B_n(t)$ implies $B_n(t)\subset\subset\tilde{B}_m(t)$, and if $B_n(t)\subset\tilde{B}_m(t)$ but not $B_n(t)\subset\subset\tilde{B}_m(t)$ then $B_n(t)\subset\tilde{B}_m(t)$. Also, for any n, $m\in\omega$, $\{t\in T\colon B_n(t)\subset\subset\tilde{B}_m(t)\}$ is Δ_1^1, since f is Δ_1^1.

Similarly define $G(t,n,m,i)\subset T\times\omega\times\omega\times\omega$ using $\tilde{B}_n(t)$, $B_m(t)$ instead of $B_n(t)$, $\tilde{B}_m(t)$, and get a corresponding Δ_1^1 function $g\colon T\times\omega\times\omega\to\{0,1\}$. Define $\tilde{B}_n(t)\subset\subset B_m(t)$ analogously.

For $t\in T$, n, $m\in\omega$ define $B_n(t)\subset_{\mathrm{cl}}B_m(t)$ ['$B_n(t)$ closure is strongly contained in $B_m(t)$'] to hold provided $\exists k\in\omega\ [B_n(t)\subset\subset\tilde{B}_k(t)\subset\subset B_m(t)]$. Note that $\subset_{\mathrm{cl}}$ need not be transitive. However, for n, $m\in\omega$, $\{t\in\mathrm{T}: B_n(t)\subset_{\mathrm{cl}}B_m(t)\}$ is Δ_1^1.

Without loss of generality assume that $B_0=\tilde{B}_0=B$ and $B_1=\tilde{B}_1=\emptyset$. Let $\mathscr{D}$ be the set of dyadic rational numbers in $[0,1]$, i.e., rationals r of the form $r=i/2^n$ for some n and $0\leq i\leq 2^n$. Let $\mathscr{D}_n\subset\mathscr{D}$ be those rationals of the form $i/2^n$ for $0\leq i\leq 2^n$.

Our next goal is to define the family $\{C_r\colon r\in\mathscr{D}\}$ of Borel, upward saturated, sectionwise open subsets of B. The C_r will be nested in that if $r<s$ are in $\mathscr{D}$ then $C_s\subset C_r$, and in fact we will have that on each section $\mathrm{cl}_{B_t}(C_s)\subset C_r$.

We define the C_r in stages. At step n we define C_r for each $r\in\mathscr{D}_n$. At the same time we define at step n a Δ_1^1 function $\theta_n\colon T\times\mathscr{D}_n\to\omega$. We will also have that θ_{n+1} extends θ_n. The C_r will be related to the B_n by $C_r(t)=B_{\theta_n(t,r)}(t)$, for $r\in\mathscr{D}_n$.

First, set $C_0=B_0=B$, $C_1=B_1=\varnothing$. Also, set $\theta_0(t,0)=0$ and $\theta_0(t,1)=1$.

Suppose now that $C_{k/2^n}$ and θ_n have been defined, where $0\leqq k\leqq 2^n$. We define $C_{l/2^{n+1}}$ for $0\leqq l\leqq 2^{n+1}$ and extend θ_n to θ_{n+1}. If l is even, we set $C_{l/2^{n+1}}=C_{(l/2)/2^n}$, and set $\theta_{n+1}(t,l/2^{n+1})=\theta_n(t,(l/2)/2^n)$. For $l=2m+1$, we define $C_{(2m+1)/2^{n+1}}\subset B$ as follows:

$$(t,x)\in C_{(2m+1)/2^{n+1}}\Leftrightarrow\exists p[p\notin\{\theta_n(t,r):$$

$$r\in\mathscr{D}_n\}\wedge(C_{(2m+2)/2^{n+1}}(t)\subset_{\mathrm{cl}}B_p(t)\subset_{\mathrm{cl}}C_{2m/2^{n+1}}(t))$$

$$\wedge\ \text{for the least such } p \text{ we have}$$

$$(t,x)\in B_p]\vee[\neg\exists p\{\cdots\}\wedge(t,x)\in C_{2m/2^{n+1}}]$$

$$\Leftrightarrow\exists p[p\notin\{\theta_n(t,r):$$

$$r\in\mathscr{D}_n\}\wedge(B_{\theta_n(t,(2m+2)/2^{n+1})}(t)\subset_{\mathrm{cl}}B_p(t)\subset_{\mathrm{cl}}B_{\theta_n(t,\,2m/2^{n+1})}(t))$$

$$\wedge\,(t,x)\in B_p \text{ for the least such } p]$$

$$\vee\,[\neg\exists p\{\cdots\}\wedge(t,x)\in B_{\theta_n(t,\,2m/2^{n+1})}(t)].$$

Thus $C_{(2m+1)/2^{n+1}}$ is Δ^1_1. In fact, inspection of the above argument shows that the C_r are uniformly Δ^1_1, that is, the relation $C(r,t)\Leftrightarrow C_r(t)$ is also Δ^1_1. Also, $C_{(2m+1)/2^{n+1}}$ is upward saturated and sectionwise open since each B_n is.

We define $\theta_{n+1}(t,(2m+1)/2^{n+1})=\text{least}\quad p\notin\{\theta_n(t,r):r\in\mathscr{D}_n\}$ such that $C_{(2m+2)/2^{n+1}}\subset_{\mathrm{cl}}B_p(t)\subset_{\mathrm{cl}}C_{2m/2^{n+1}}$ if such a p exists, and otherwise set $\theta_{n+1}(t,(2m+1)/2^{n+1})=\theta_n(t,2m/2^{n+1})$. Clearly θ_{n+1} is a Δ^1_1 function extending θ_n.

This defines C_r for all $r\in\mathscr{D}$. We let $\theta=\bigcup_n\theta_n$: $T\times\mathscr{D}\to\omega$. Thus, $C_r(t)=B_{\theta(t,r)}(t)$ holds for all $t\in T$, $r\in\mathscr{D}$. In summary, the C_r are Δ^1_1, upward saturated, sectionwise open, and if $r<s$ are in $\mathscr{D}$ then $C_s\subset C_r$ (the last property following easily from the construction).

We prove some additional facts about the C_r.

Lemma 2. Let $r<s\in\mathscr{D}$, $t\in\mathscr{D}$, and suppose $C_r(t)-C_s(t)$ has infinitely many $\sim_t$ classes. Then for some $p\in\mathscr{D}$ with $r<p<s$ we have $C_s(t)\subsetneqq C_p(t)\subsetneqq C_r(t)$.

Note that it follows that $C_s(t)\subset_{\mathrm{cl}}C_p(t)\subset_{\mathrm{cl}}C_r(t)$ since from the definition of the C_r we have that whenever $a<b$ are in $\mathscr{D}$ then either $C_b(t)\subset_{\mathrm{cl}}C_a(t)$ or else $C_b(t)=C_a(t)$.

Proof. Take n so that $r=i/2^n$, $s=j/2^n$ where $0\leqq i<j\leqq 2^n$. Without loss of generality we may assume that $j=i+1$. Let d,u,v,w be such that $d\prec_t u\prec_t v \prec_t w$ and $d,u,v,w\in C_r(t)-C_s(t)$. Since the B_k, $\tilde{B}_k$ were separating families we have that for some $l,p,q\in\omega$ that $w\in\tilde{B}_p(t)$, $v\notin\tilde{B}_p(t)$, $v\in B_q(t)$, $u\notin B_q(t)$, $u\in\tilde{B}_1(t)$, $d\notin\tilde{B}_1(t)$. It follows that $C_s(t)\subset_{\text{cl}}C_{(2i+1)/2^{n+1}}\subset_{\text{cl}}C_r(t)$. If $C_{(2i+1)/2^{n+1}}\neq C_s(t)$, $C_r(t)$ we are done. Otherwise, set $s_1=s$ and $r_1=(2i+1)/2^{n+1}$ if $C_{(2i+1)/2^{n+1}}(t)=C_r(t)$, and set $s_1=(2i+1)/2^{n+1}$ and $r_1=r$ if $C_{(2i+1)/2^{n+1}}(t)=C_s(t)$. Repeat the argument using s_1, r_1 instead of s,r to get s_2,r_2, etc. Note that for all k we have

$$\theta\left(t,\frac{s_k+r_k}{2}\right)\notin\{\theta(t,r_0),\theta(t,s_0),\ldots,\theta(t,r_k),\theta(t,s_k)\},$$

since in the definition of $C_{(s_k+r_k)/2}(t)$ we are in the case (in definition of C_r) where $\exists p\ \{\cdots\}$. Hence from the minimality clause in the definition of C_r it follows that for some k we must have $C_{(s_k+r_k)/2}(t)=B_q(t)$. Hence, $C_s(t)=C_{s_k}(t)\subsetneqq C_{(s_k+r_k)/2}(t)=B_q(t)\subsetneqq C_{r_k}(t)=C_r(t)$. $\square$

Lemma 3. For all $r<s\in\mathscr{D}$ and $t\in T$ we have $\text{cl}_{B_t}(C_s(t))\subset C_r(t)$.

Proof. Let $t\in T$, $n\in\omega$, $0\leqq i<2^n$, and consider $r=i/2^n$, $s=(i+1)/2^n$. Let $u=(r+s)/2=(2i+1)/2^{n+1}$. It is enough to show that $\text{cl}_{B_t}(C_u(t))\subset C_r(t)$ and $\text{cl}_{B_t}(C_s(t))\subset C_u(t)$. If in the definition of $C_u(t)$ the first case applies (i.e. $\exists p\{\cdots\}$) then $C_s(t)\subset_{\text{cl}}C_u(t)\subset_{\text{cl}}C_r(t)$ and we are done since $C_s(t)\subset_{\text{cl}}C_u(t)$ implies $\text{cl}_{B_t}(C_s(t))\subset C_u(t)$, etc. Otherwise, from the definition of $C_u(t)$ we have $C_u(t)=C_r(t)$. By Lemma 2, $C_r(t)-C_s(t)$ has only finitely many $\sim_t$ classes. If $C_r(t)=C_s(t)$, then by induction we have $\text{cl}_{B_t}(C_s(t))\subset C_r(t)$, so $C_r(t)=C_s(t)$ is (relatively) clopen and we are done. If $C_r(t)\neq C_s(t)$, then $C_r(t)$ must have a $\sim_t$ least class, and again $C_r(t)$ is clopen and we are done. $\square$

We define now the function $F: B\to[0,1]$ which will be the almost representing function.

Define $F: B\to[0,1]$ by $F(t,x)=\inf\{r\in\mathscr{D}: (t,x)\notin C_r\}$. Clearly F is well defined and maps B into $[0,1]$. F is also non-decreasing on each section since each of the C_r is upward saturated, and F is also invariant with respect to $\sim_t$ since each of the C_r is.

We prove a few lemmas concerning the function F.

Lemma 4. F is a Δ_1^1 function.

Proof. We have $F(t,x)=y\Leftrightarrow\forall N\in\omega\exists r\in\mathscr{D}[r<y+1/N \wedge (t,x)\notin C_r] \wedge$

$\forall r \in \mathscr{D}[(t,x) \notin C_r \to y \leqq r]$. Clearly the graph of F is Δ_1^1, hence F is a Δ_1^1 function. □

Lemma 5. For all $t \in T$, $F_t = F(t, \cdot)$ is continuous (from B_t to $[0,1]$).

Proof. Fix $t \in T$ and $x \in B_t$. First we show that F_t is right continuous at x. If there is a $\prec_t$ least class greater than x the result is trivial, so assume otherwise.

Case 1. For some $r \in \mathscr{D}$, $C_r(t) = \{y \in B_t\colon x \prec_t y\}$. If there are two distinct r_1, $r_2 \in \mathscr{D}$ with $C_{r_1}(t) = C_{r_2}(t) = \{y \in B_t\colon x \prec_t y\}$, then by Lemma 3, $C_r(t)$ is (relatively) clopen, and the result is trivial. So, assume there is a unique r such that $C_r(t) = \{y \in B_t\colon x \prec_t y\}$. In this case clearly $F(t,x) = r$. Since there is no least class greater than x (more precisely $[x]_{\sim_t}$) it follows that for all $r' \in \mathscr{D}$ with $r' > r$ that $\exists y \succ_t x (y \notin C_{r'}(t))$. Hence for any $r' > r$ $\exists y \succ_t x$ with $F_t(y) \leqq r'$. So F_t is right continuous at x in this case.
Case 2. For all $r \in \mathscr{D}$, $C_r(t) \neq \{y \in B_t\colon x \prec_t y\}$. Given $\varepsilon > 0$ let $r \in \mathscr{D}$ be such that $(t,x) \notin C_r$ and $|F(t,x) - r| < \varepsilon$. By the assumption of this case there is a $y \succ_t x$ such that $(t,y) \notin C_r$. Thus $F_t(y) \leqq r$, so $|F_t(y) - F_t(x)| < \varepsilon$.

We now show that F_t is left continuous at x. We assume w.l.o.g. that there is no $\sim_t$ largest class less than x.

Let r_0, $s_0 \in \mathscr{D}$ be such that $x \in C_{r_0}$, $x \notin C_{s_0}(t)$ and $|F_t(x) - s_0| < \varepsilon$, where $\varepsilon > 0$ is given. Without loss of generality we may assume that $r_0 = i/2^n$, $s_0 = (i+1)/2^n$ for some i, n. Let $u_0 = (r_0 + s_0)/2 = (2i+1)/2^{n+1}$. Set $r_1 = r_0$, $s_1 = u_0$ if $x \notin C_{u_0}(t)$, and set $r_1 = u_0$, $s_1 = s_0$ if $x \in C_{u_0}(t)$. Thus, $x \notin C_{s_1}(t)$, $x \in C_{r_1}(t)$, $x \in C_{r_1}(t)$, and $s_1 - r_1 = \frac{1}{2}(s_0 - r_0)$. Continuing, we get $r_k < s_k \in \mathscr{D}$ with $x \notin C_{s_k}(t)$, $x \in C_{r_k}(t)$, and $|s_k - r_k| < \varepsilon$. If now $y \preceq_t x$ and $y \in C_{r_k}(t)$ then $F_t(y) \geqq r_k$, so $|F_t(y) - F_t(x)| < \varepsilon$. □

Thus, F is sectionwise continuous. Our next lemma completes the proof that F is an almost representation.

Lemma 6. For all $t \in T$, x, $y \in B_t$ with $x \prec_t y$ if $\exists z_1, z_2, z_3, z_4, z_5$ such that $x \prec_t z_1 \prec_t \dots \prec_t z_5 \prec_t y$, then $F_t(x) < F_t(y)$.

Proof. Fix $t, x, y, z_1, \dots, z_5$ as above. We first claim that for some $s \in \mathscr{D}$ that $y \in C_s(t)$ and $x \notin C_s(t)$. To see this, first take r_0, $s_0 \in \mathscr{D}$ with $r_0 = 0/1$, $s_0 = 1/1$ so that $y \notin C_{s_0}(t)$ and $x \in C_{r_0}(t)$. Assume $r_k < s_k \in \mathscr{D}_k$ have been defined with $y \notin C_{s_k}(t)$ and $x \in C_{r_k}(t)$. Let $u_k = (r_k + s_k)/2$. Note that there are integers p, q, and e such that $y \in \tilde{B}_e$, $z_5 \notin \tilde{B}_e$, $z_5 \in B_q$, $z_4 \notin B_q$, $z_4 \in \tilde{B}_p$, and $z_3 \notin \tilde{B}_p$. It follows that $\exists p[p \notin \{\theta(t,r)\colon r \in \mathscr{D}_k\} \wedge C_{s_k}(t) \subset_{\mathrm{cl}} B_p(t) \subset_{\mathrm{cl}} C_{r_k}(t)]$ [recall here that if $B_a(t) \subsetneqq \tilde{B}_b(t)$ then $B_a(t) \subset\subset \tilde{B}_b(t)$]. Therefore, $\theta(t, u_k) \notin \{\theta(t,r)\colon r \in \mathscr{D}_k\}$. Let $r_{k+1} = r_k$, $s_{k+1} = u_k$ if $y \notin C_{u_k}(t)$, and $r_{k+1} = u_k$, $s_{k+1} = s_k$ if $x \in C_{u_k}(t)$ [if $y \in C_{u_k}(t)$

and $x \notin C_{u_k}(t)$ we are done]. For large enough k it follows using the minimality clause in the definition of C_r that $y \in C_{u_k}(t)$ and $x \notin C_{u_k}(t)$, since for each k $C_{s_k}(t) \subset_{\text{cl}} B_q(t) \subset_{\text{cl}} C_{r_k}(t)$, and hence $B_q(t)$ must be $C_{u_k}(t)$ for some large enough k.

Fix now $s \in \mathscr{D}$ such that $y \in C_s(t)$ and $x \notin C_s(t)$. We claim that for some $r \in \mathscr{D}$ with $r \neq s$ that $y \in C_r(t)$, $x \notin C_r(t)$ (note: we may have $r<s$ or $r>s$ here). To see this, repeat the above argument using the fact that either $z_1, z_2, z_3 \notin C_s(t)$ or $z_3, z_4, z_5 \in C_s(t)$ and that only three z points are sufficient for the above argument. By renaming r, s if necessary we may assume that $r<s$. Thus we have $r<s$ and $y \in C_s(t)$, $C_r(t)$, and $x \notin C_s(t)$, $C_r(t)$. Hence, $F(t,y) \geqq s$ and $F(t,x) \leqq r$, so $F(t,x) < F(t,y)$. □

In summary, we have produced a Δ^1_1 $F\colon B \to [0,1]$ which is sectionwise continuous, invariant, non-decreasing, and iff $x \prec_t y$ with at least five distinct $\sim_t$ classes strictly between x and y then $F(t,x) < F(t,y)$. In particular, note that if each section B_t is order dense in itself with respect to $\prec_t$ then F is already a representation.

4. Getting the representing function

We now proceed to modify our 'almost representing' function F to obtain a representing function $\tilde{F}$. We need nothing from sections 2 and 3 other than the existence of the almost representing function F.

Fix for the rest of this section an almost representing function $F\colon B \to [0,1]$. The idea of this section is to add to F countably many new functions f_n which will separate the 'gaps' which F fails to separate. The set of gaps (i.e., (t,x,y) with $x \prec_t y$ and such that there do not exist five distinct $\sim_t$ classes strictly between x and y), however, is not in general Δ^1_1. Instead, we exploit the fact that we already have the almost representing function F, which allows us to do a separation argument.

Define $U \subset T \times X \times X$ by $U(t,x,y) \Leftrightarrow x \prec_t y \wedge (F(t,x) = F(t,y))$. Clearly U is Δ^1_1 and saturated. Note that if $U(t,x,y)$ then there are at most four $\sim_t$ classes strictly between x and y.

We need to introduce a more or less standard coding for the $\Delta^1_1(t)$ subsets of X, uniformly in t. Although this is quite a standard construction, this is the place in the argument where we seem to need to invoke the lightface classes. We recall the construction. Let $A \subset \omega \times T \times X$ be universal Π^1_1 for subsets of $T \times X$ [see Moschovakis (1980, section 3F)], i.e., every Π^1_1 subset of $T \times X$ is of the form $\{(t,x)\colon A(n,t,x)\}$ for some $n \in \omega$. It follows that for each $t \in T$ that $A_t = \{(n,x)\colon A(n,t,x)\}$ is universal for $\Pi^1_1(t)$ subsets of X. Let $n \to (n_0, n_1)$ be a recursive bijection between ω and $\omega \times \omega$, and let $(a,b) \to \langle a,b \rangle$ be its (recursive) inverse. Define A_0, $A_1 \subset \omega \times T \times X$ by $A_0(n,t,x) \Leftrightarrow A(n_0,t,x)$, and $A_1(n,t,x) \Leftrightarrow A(n_1,t,x)$. So, A_0, A_1 are Π^1_1. Let C, D be Π^1_1 and reduce A_0,

A_1, that is, $C \cap D = \varnothing$, $C \cup D = A_0 \cup A_1$, $A_0 - A_1 \subset C$, and $A_1 - A_0 \subset D$. We define a set CO which is our set of codes for $\Delta^1_1(t)$ subsets of X. Define $CO \subset \omega \times T$ by $CO(n,t) \Leftrightarrow \forall x[C(n,t,x) \vee D(n,t,x)]$ $(\Leftrightarrow \forall x[A_0(n,t,x) \vee A_1(n,t,x)])$. Thus, CO is Π^1_1. If $(n,t) \in CO$, we say that n is the code of a $\Delta^1_1(t)$ subset of X. Specifically, if $(n,t) \in CO$, let $S_{n,t} = \{x \in X\colon C(n,t,x)\} = \{x \in X\colon \neg D(n,t,x)\}$. Thus if $(n,t) \in CO$, $S_{n,t} \subset X$ is $\Delta^1_1(t)$.

Returning to the proof, define $V \subset T \times X \times X \times \omega \times \omega$ by:

$$V(t,x,y,i,j) \Leftrightarrow U(t,x,y) \wedge CO(i,t) \wedge CO(j,t) \wedge x \in S_{i,t}$$

$$\wedge\ y \in S_{j,t} \wedge (S_{i,t} \cap S_{i,ty} = \varnothing) \wedge (S_{i,t} \subset B_t \wedge S_{j,t} \subset B_t)$$

$$\wedge\ (S_{j,t} \wedge \text{ is downward saturated with respect to } \preceq_t)$$

$$\wedge\ (S_{j,t} \text{ is upward saturated with respect to } \preceq_t)$$

$$\wedge\ (S_{i,t} \cup S_{j,t} = B_t) \wedge (S_{i,t},\ S_{j,t} \text{ are relatively open in } B_t).$$

Writing this out in full we have

$$V(t,x,y,i,j) \Leftrightarrow U(t,x,y) \wedge CO(i,t) \wedge CO(j,t) \wedge C(i,t,\ x)$$

$$\wedge\ C(j,t,y) \wedge \forall z(D(i,t,z) \vee D(j,t,z)) \wedge \forall z(\neg D(i,t,z) \rightarrow B(t,\ z))$$

$$\wedge\ \forall z(\neg D(j,t,z) \rightarrow B(t,z))$$

$$\wedge\ \forall z \forall w[(\neg D(i,t,z) \wedge w \preceq_t z) \rightarrow C(i,t,\ w)] \wedge \forall z \forall w[(\neg D(j,t,z)$$

$$\wedge\ w \succeq_t z) \rightarrow C(j,t,w)] \wedge \forall z[B(t,z) \rightarrow (C(i,t,\ z) \vee C(j,t,w))]$$

$$\wedge\ \forall z[\neg D(i,t,z) \rightarrow \exists k \in \omega(z \in U_k \wedge \forall w\{w \in\ U_k \cap B_t \rightarrow C(i,t,w)\})]$$

$$\wedge\ \forall z[\neg D(j,\ t,z) \rightarrow \exists k \in \omega(z \in U_k \wedge \forall w\{w \in U_k \cap B_t \rightarrow C(j,t,z)\})].$$

Written out it is evident that V is Π^1_1.

Note that for all t, x, y such that $U(t,\ x,\ y)$ there are $i, j \in \omega$ such that $V(t,\ x,\ y,\ i,\ j)$. To see this, suppose $U(t,\ x,\ y)$. Let $z_1 \prec_t \ldots \prec_t z_k$, $k \leqq 4$, be representatives from the $\sim_t$ classes strictly between x and y (we allow $k=0$, i.e., there are no $\sim_t$ classes strictly between x and y). Let U, V be disjoint basic open sets with $x \in U$, $z_1 \in V$ (or $y \in V$ if $k=0$), and such that $U \subset \{w\colon w \quad {}_t x\}$ and $V \subset \{w\colon w \succeq_t z_1\}$. Let $E = \mathrm{sat}^d_t(U \cap B_t)$, and $F = \mathrm{sat}^u_t(V \cap B_t)$. Then E, F are disjoint $\Sigma^1_1(t)$ sets with $E \cup F = B_t$. Also, E and F are both relatively open in B_t. Since $B_t \in \Delta^1_1(t)$, we actually have $E, F \in \Delta^1_1(t)$.

Consequently there are $i, j \in \omega$ such that $CO(i, t)$, $CO(j, t)$, $S_{i,t}=E$, and $S_{j,t}=F$. Then $V(t,x,y,i,j)$ holds.

By Fact 2 of section 1, there is a Δ_1^1 function g: $U \to \omega \times \omega$ uniforming V on the last two coordinates. That is, for all $(t,x,y) \in U$, $V(t,x,y,g_0(t,x,y), g_1(t,x,y))$ where g_0, g_1 are defined by $g(t,x,y)=(g_0(t,x,y), g_1(t,x,y))$ for $(t,x,y) \in U$. Let g': $U \to \omega$ be defined by $g'(t,x,y)=\langle g_0(t,x,y), g_1(t,x,y)\rangle$. Thus, for all $(t,x,y) \in U$, $V(t,x,y,g'(t,x,y)_0, g'(t,x,y)_1)$ holds. Define $H \subset T \times \omega$ by $H(t,n) \Leftrightarrow \exists x \, \exists y \; (U(t,x,y) \wedge g'(t,x,y)=n)$. Thus H is Σ_1^1. Define also $I \subset T \times \omega$ by

$$
\begin{aligned}
I(t,n) \Leftrightarrow\, & CO(n_0,t) \wedge CO(n_1,t) \wedge (S_{n_0,t} \cap S_{n_1,t}=\varnothing) \\
& \wedge (S_{n_0,t} \subset B_t \wedge S_{n_1,t} \subset B_t) \\
& \wedge (S_{n_0,t} \text{ is downward closed with respect to } \preceq_t) \\
& \wedge (S_{n_1,t} \text{ is upward closed with respect to } \preceq_t) \\
& \wedge (S_{n_0,t} \cup S_{n_1,t}=B_t) \wedge
\end{aligned}
$$

$(S_{n_0,t}, S_{n_1,t})$ are relatively open in B_t). I is Π_1^1 by a computation similar to that for V. Clearly $H \subset I$. So, let $J \subset T \times \omega$ be Δ_1^1 and separate H, $(T \times \omega)-I$, that is, $H \subset J \subset I$.

Since J is Δ_1^1, we may easily express J as the countable union of graphs of partial Δ_1^1 functions j_n [proof: define j_n by $j_n(t)=p \Leftrightarrow p=n \wedge (t,n) \in J$]. That is, $J=\bigcup_n \operatorname{graph}(j_n)$. Note that for each n, the domain of j_n is a Δ_1^1 subset of T. Also, the sequence of j_n is Δ_1^1, that is, the relation of three arguments $j_n(t)=p$ is Δ_1^1.

For each $n \in \omega$, define now a function f_n: $B \to [0,1]$ by

$$
f_n(t,x)=\begin{cases} 0 & \text{if } t \notin \operatorname{dom}(j_n) \\ 0 & \text{if } t \in \operatorname{dom}(j_n) \text{ and } x \in S_{j_n(t)_0,t}. \\ 1 & \text{if } t \in \operatorname{dom}(j_n) \text{ and } x \in S_{j_n(t)_1,t} \end{cases}
$$

Since for $t \in \operatorname{dom}(j_n)$ we have $x \in S_{j_n(t)_0,t}$ iff $C(j_n(t)_0,t,x)$ iff $\neg D(j_n(t)_0,t,x)$ [since $CO(j_n(t)_0,t)$], and similarly for $S_{j_n(t)_1,t}$, it follows that the graph of f_n is Δ_1^1. Hence f_n is a Δ_1^1 function. Moreover, as a relation of four arguments, $f_n(t,x)=i$ is Δ_1^1.

Note also that f_n: $B \to [0,1]$ is sectionwise continuous. This follows since for all $t \in T$ the function $f_{n,t}=f_n(t,\cdot)$: $B_t \to [0,1]$ is either the constant 0 function or else the characteristic function of the (relatively) clopen set $S_{j_n(t)_1,t}=B_t-S_{j_n(t)_0,t}$. Also, f_n is clearly non-decreasing on each section and invariant under $\sim_t$.

Let $\tilde{F}$: $B \to [0,1]$ be defined by $\tilde{F}(t,x)=F(t,x)+\sum_{n \in \omega}(1/2^n) f_n(t,x)$. Clearly $\tilde{F}$ is sectionwise continuous since F and the f_n are. $\tilde{F}$ is also easily Δ_1^1 using the fact that F is Δ^1_1 and the relation $f_n(t,x)=i$ is Δ^1_1. Clearly $\tilde{F}$ is sectionwise non-decreasing with respect to $_t$ and invariant with respect to $\sim_t$.

Finally, we show that $\tilde{F}$ represents the preference order. Suppose (t,x), (t,y) are in B and $x \prec_t y$. If $(t,x,y) \notin U$ then $F(t,x) < F(t,y)$ and we are done. So, suppose $U(t,x,y)$. Then for some $n \in \omega$ we have $j_n(t) = g'(t,x,y)$ since range $g' \mid U_t = H_t \subset J_t$ and $J = \bigcup_n \text{graph}(j_n)$. But for this n we have $x \in S_{j_n(t)_0, t}$, $y \in S_{j_n(t)_1, t}$. Hence $f_n(t,x)=0$ and $f_n(t,y)=1$ and so $\tilde{F}(t,x) < \tilde{F}(t,y)$ and we are done.

This completes the proof of the theorem.

5. An alternate boldface or 'classical' argument

Let F: $B \to [0,1]$ be a Borel almost representing function for the preference order. We give in this section an alternate construction of the representing function $\tilde{F}$ from F, using only classical (boldface) arguments.

As before, define $U \subset T \times X \times X$ by $U(t,x,y) \Leftrightarrow x \prec_t y \wedge (F(t,x) = F(t,y))$. So U is Borel, saturated, and if $U(t,x,y)$ then there are at most four $\sim_t$ classes strictly between x and y. Note that for each $t \in T$ there are only countably many pairs of equivalence classes $([x]_t, [y]_t)$ such that $U(t,x,y)$, since each such pair is determined by a pair of basic open sets in B_t. Let $\{B_n: n \in \omega\}$ be a Borel separating family for the preference order, for example, the family constructed in section 2. Let h_n: $B \to [0,1]$ be the characteristic function of B_n, and let $H = \sum_n (1/2^n) h_n$. H is a Borel function (being a sum of Borel functions), invariant with respect to $\sim_t$, and if $x \prec_t y$ then $H(t,x) < H(t,y)$. Define $W \subset T \times \mathbb{R}$ by $W(t,r) \Leftrightarrow \exists x \, \exists y [U(t,x,y) \wedge H(t,x) = r_0 \wedge H(t,y) = r_1]$. Here $r \to (r_0, r_1)$ denotes a Borel bijection between $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$. Clearly W is analytic. Also, each section of W is countable since H is invariant. By a classical result of Lusin, every analytic set with countable sections in a product of Polish spaces can be written as a countable union of analytic graphs of functions [see Lusin (1930), Maitra (1980) or Mauldin (1978)]. Let g_n be functions with $\text{dom}(g_n) \subset T$, graph$(g_n)$ is analytic, and $W = \bigcup_n \text{graph}(g_n)$.

For $n \in \omega$, set $E_n = \text{graph}(g_n) \subset T \times \mathbb{R}$. So, E_n is analytic. Define $P_n \subset T \times \mathbb{R}$ by $P_n(t,r) \Leftrightarrow [\{z: H(t,z) \geqq r_1\}$ is relatively clopen in $B_t]$. Written out: $P_n(t,r) \Leftrightarrow \forall z [H(t,z) \geqq r_1 \to \{\exists i \in \omega \, z \in U_i \wedge \forall w \ (w \in U_i \cap B_t) \to H(t,w) \geqq r_1\}]$. Since H is Borel, P_n is easily coanalytic. Note that $E_n \subset P_n$ since if $E_n(t,r)$ then for some x, y such that $U(t,x,y)$ we have $\{z: H(t,z) \geqq r_1\} = \{z: z \succeq_t y\}$. Define $Q_n \subset T \times \mathbb{R}$ by $Q_n(t,r) \Leftrightarrow P_n(t,r) \wedge \forall s \in \mathbb{R}[E_n(t,s) \to (s=r)]$. Q_n is also coanalytic. Using the fact that E_n is the graph of a function, it follows easily that $E_n \subset Q_n$, and if $E_n(t,r)$ then r is the unique real such that $Q_n(t,r)$ holds as well. Let $F_n \subset Q_n$ be coanalytic and uniformize Q_n by the Novikov–Kondo uniformization theorem. Thus $E_n \subset F_n$. By the first separation theorem, let K_n be Borel with $E_n \subset K_n \subset F_n$. Hence, K_n is the graph of a Borel function which extends g_n. Let graph$(h_n) = K_n$.

Summarizing, we have produced Borel functions h_n such that $W \subset \bigcup_n \text{graph}(h_n)$, and $\forall t \in \text{dom}(h_n)[\{z\colon H(t,z) \geqq h_n(t)_1\}$ is relatively clopen in B_t]. Define now $f_n\colon B \to [0,1]$ by

$$f_n(t,x) = \begin{cases} 0 & \text{if} \quad t \notin \text{dom}(h_n) \\ 0 & \text{if} \quad t \in \text{dom}(h_n) \text{ and } H(t,x) < h_n(t)_1. \\ 1 & \text{if} \quad t \in \text{dom}(h_n) \text{ and } H(t,x) \geqq h_n(t)_1 \end{cases}$$

Since H and h_n are Borel, and the domain of a Borel function is Borel, it follows easily that f_n is Borel. Also, f_n is invariant since H is, and f_n is sectionwise continuous from the properties of h_n.

As before, let $\tilde{F}\colon B \to [0,1]$ be defined by $\tilde{F}(t,x) = F(t,x) + \sum_{n \in \omega} (1/2^n) f_n(t,x)$. To see $\tilde{F}$ represents the preference order, suppose (t,x), (t,y) are in B and $x \prec_t y$. If $(t,x,y) \notin U$, then $F(t,x) < F(t,y)$ and we are done. So, suppose $U(t,x,y)$. Then for some n we have $E_n(t,r)$ where $r_0 = H(t,x)$ and $r_1 = H(t,y)$. For this n we have $h_n(t) = r$. Since $\{z\colon H(t,z) \geqq h_n(t)_1\} = \{z\colon H(t,z) \geqq H(t,y)\} = \{z\colon z \succeq_t y\}$ in this case, it follows that $f_n(t,x) = 0$ and $f_n(t,y) = 1$. Hence, $\tilde{F}(t,x) < \tilde{F}(t,y)$ and we are done.

6. Concluding remarks

We point out again that the theorem applies even when the field $B \subset T \times X$ of the preference order R has no Borel uniformization.

Example. There is a Borel preference order $R \subset [0,1] \times [0,1] \times [0,1]$ with domain $[0,1]$, field $B \subset [0,1] \times [0,1]$, with each section R_t a linear ordering, yet B has no Borel uniformization.

Simply let B be a Borel subset of $[0,1] \times [0,1]$ which has no Borel uniformization and with projection onto the first coordinate being all of $[0,1]$. [One such example is given in [Mauldin (1979, Example 3.1).] Let $R(t,x,y) \leftrightarrow ((t,x),(t,y) \in B \wedge x \leqq y)$, where $\leqq$ is the usual order on $[0,1]$ (Note that in this case we may get a representing function f directly, namely $f(t,x) = x$).

Let us close this paper with a question. Our representing function does not seem to be invariant under the equivalence of the preference orders.

Question. Can the Borel measurable representing function F be chosen such that if $R_t = R_{t'}$, then $F(t,\cdot) = F(t',\cdot)$?

We believe this question should have a positive answer.

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