MULTIFRACTAL DECOMPOSITIONS OF DIGRAPH RECURSIVE FRACTALS

G. A. Edgar and R. Daniel Mauldin

The Ohio State University and University of North Texas

November 25, 1991

Abstract. We prove that the multifractal decomposition behaves as expected for a family of sets K known as digraph recursive fractals, using measures μ of Markov type. For each value of a parameter α between a minimum $\alpha_{\min}$ and maximum $\alpha_{\max}$, we define "multifractal components" $K^{(\alpha)}$ of K, and show that they are fractals in the sense of Taylor. The dimension $f(\alpha)$ of $K^{(\alpha)}$ is computed from the data of the problem. The typical concave "multifractal $f(\alpha)$" dimension spectrum curve results. Under appropriate disjointness conditions, the multifractal components $K^{(\alpha)}$ are given by:

$$K^{(\alpha)} = \left\{ x \in K : \lim_{\varepsilon \downarrow 0} \frac{\log \mu\big(B_\varepsilon(x)\big)}{\log \operatorname{diam}\, B_\varepsilon(x)} = \alpha \right\},$$

i.e., $K^{(\alpha)}$ consists of those points where μ has pointwise dimension α.

Contents

1991 *Mathematics Subject Classification.* 28A80.
Key words and phrases. Fractal, multifractal, Hausdorff dimension, packing dimension.
Research supported in part by NSF grants DMS 87-01120 and DMS 90-007035

Introduction

It has been argued in the physics literature (for example Halsey, et. al. [9]) that certain fractals carrying a natural measure may be analyzed in terms of the scaling properties of the measure. The fractal K should contain "singularities of strength α" for certain values of a parameter α; a fractal dimension $f(\alpha)$ describes how densely those singularities are distributed. Computations show a typical concave shape for this function $f(\alpha)$, sometimes known as a "dimension spectrum".

Here we attempt to provide a mathematical setting for this sort of "multifractal deomposition". We begin with a (possibly fractal) nonempty compact set K in Euclidean space $\mathbb{R}^n$, and a measure μ on K. For example, we might have a dynamical system where, in the limit, the trajectories approach a (strange) attractor K, and the ergodic time-averages along the process approach a corresponding measure μ. Or we might imagine an iterated function system, which approaches its attractor K as points are chosen according to the "random method" or the "chaos game"; the time-averages again converge to a natural measure on the attractor.

When a set K has fractal dimension d and supports a "natural" finite measure μ, we may expect "typically", for $x \in K$ and $\varepsilon > 0$, that the measure $\mu\big(B_\varepsilon(x)\big)$ of the ball of radius ε centered at x is roughly equal to $(2\varepsilon)^d$, the dth power of the diameter of the ball. This might mean that

$$0 < \limsup_{\varepsilon \downarrow 0} \frac{\mu\big(B_\varepsilon(x)\big)}{\big(\operatorname{diam} B_\varepsilon(x)\big)^d} < \infty$$

or, more generally, that

$$\lim_{\varepsilon \downarrow 0} \frac{\log \mu\big(B_\varepsilon(x)\big)}{\log \operatorname{diam} B_\varepsilon(x)} = d$$

for all $x \in K$. Multifractal decomposition will be interesting exactly when this does not happen—for many different values of the parameter α, the set

$$K^{(\alpha)} = \left\{ x \in K : \lim_{\varepsilon \downarrow 0} \frac{\log \mu\big(B_\varepsilon(x)\big)}{\log \operatorname{diam} \, B_\varepsilon(x)} = \alpha \right\}$$

is non-trivial. The sets $K^{(\alpha)}$ may be thought of as the multifractal components of K. (We use a slightly different definition below, however; it coincides with this when there is a disjointness property.) The Hausdorff dimension of $K^{(\alpha)}$ is called $f(\alpha)$; this function describes the dimension spectrum.

It is important to note that we are using the *Hausdorff dimension* of $K^{(\alpha)}$, and not the box dimension. Typically, all the sets $K^{(\alpha)}$ are dense in K, so they all have box dimension equal to the box dimension of K itself. But the Hausdorff dimension $f(\alpha)$ varies with α.

There are some special cases (the "digraph recursive fractals") when all of the Hausdorff dimension computations can be carried out explicitly. Those computations are carried out here. They do, indeed, show the characteristic convex multifractal dimension spectrum curve $f(\alpha)$. The packing dimension and the Hausdorff dimension of the multifractal component $K^{(\alpha)}$ agree, so it is a "fractal" in the sense of Taylor [16]. The work here extends the case computed by Cawley & Mauldin [4]. It was, in turn, suggested by the heuristics of Halsey, et. al. [9]. Some interesting digraph recursive fractals are found in [3].

1. The Setting

We will describe here: the fractals K_u to be investigated, the "string models" $E_u^{(\omega)}$ that will be used in the investigation, the measures $\hat{\mu}_u$ of Markov type used to define multifractal components $K_u^{(\alpha)}$ of the K_u, and the functions $\beta(q)$ and $f(\alpha)$ that describe the properties (such as the fractal dimension) of these components. [Note that $\beta(q)$ is often called $-\tau(q)$.]

1.1. The sets. First, a directed multigraph (V, E) should be fixed. The elements $v \in V$ are the **vertices** of the graph; the elements $e \in E$ are the **edges** of the graph. For $u, v \in V$, there is a subset E_{uv} of E, known as the edges from u to v. Each edge belongs to exactly one of these subsets. We will sometimes write $E_u = \bigcup_v E_{uv}$, the set of all edges leaving the vertex u.

We will often think of the set E as a set of "letters" that label the edges of the graph, so we will talk about "words" or "strings" made up of these letters. A **path** in the graph is a finite string $\gamma = e_1 e_2 \cdots e_k$ of edges, such that the terminal vertex of each edge e_i is the initial vertex of the next edge e_{i+1}. We write $E_{uv}^{(k)}$ for the set of all paths of length k that begin at u and end at v; and $E_u^{(k)}$ for the set of all paths of length k that begin at u; and $E_u^{(*)}$ for the set of all finite paths of any length that begin at u; and $E^{(*)}$ for the set of all finite paths.

A path that begins and ends at the same node is called a **cycle**. A cycle with no repeated nodes is a **simple cycle**. A cycle consisting of a single edge (from a node back to itself) is a **loop**.

We will assume that the graph (V, E) is **strongly connected**, that is, there is a path from any vertex to any other, along the edges of the graph (taken in the proper directions). We will also assume that there are at least two edges leaving each node. [We explain this assumption more fully below.]

Next, a **ratio** $r(e)$ should be specified for each edge $e \in E$. We will assume for simplicity that $0 < r(e) < 1$. (In the terminology of [5], it is "strictly contracting".) So if we write $r_{\min} = \min_e r(e)$ and $r_{\max} = \max_e r(e)$, then we have $0 < r_{\min} \leq r_{\max} < 1$. If $\gamma = e_1 e_2 \cdots e_k$ is a path, write $r(\gamma) = r(e_1) r(e_2) \cdots r(e_k)$.

Let J_u be nonempty compact subsets of Euclidean space $\mathbb{R}^n$ (one for each $u \in V$). The set J_u should be equal to the closure of its interior. We assume for simplicity that these seed sets have diameter 1. A **digraph recursive fractal**, or **Mauldin-Williams fractal**, based on seed sets J_u and ratios $r(e)$ is one of the sets

$$K_u = \bigcap_{k=0}^{\infty} \bigcup_{\gamma \in E_u^{(k)}} J(\gamma),$$

where the sets $J(\gamma)$ are chosen recursively:

(i) $J(\Lambda_u) = J_u$, where Λ_u is the empty path from u to u.

(ii) For $\gamma \in E^{(k)}$ with terminal vertex v, the set $J(\gamma)$ is geometrically similar to J_v with reduction ratio $r(\gamma)$.

(iii) For $\gamma \in E^{(k)}$, with terminal vertex v, the sets $J(\gamma e)$, $e \in E_v$, are nonoverlapping subsets of $J(\gamma)$.

(Note the word "nonoverlapping". This means that they intersect at most in their boundaries. Since the sets $J(\gamma)$ are similar to the original sets J_u, they, too, are equal to the closures of their interiors. Thus we are postulating the "open set condition", as in [12]: the interiors of the sets $J(\gamma)$ are disjoint (for γ of a given length), and their closures are the $J(\gamma)$.) Notice that there are many choices of how the sets $J(\gamma e)$ may be placed inside $J(\gamma)$. For the "graph self similar" fractals (as in [5] or [10]), start with similarities $\theta_e \colon \mathbb{R}^n \to \mathbb{R}^n$, one for each edge $e \in E$, and let

$$J(\gamma) = \theta_{e_1} \theta_{e_2} \cdots \theta_{e_k} [J_v]$$

where $\gamma = e_1 e_2 \cdots e_k \in E_{uv}^{(k)}$. The seed sets J_u must be chosen so that (iii) is satisfied. Another possibility (as in [8]) is a scheme that places the subsets $J(\gamma e)$ inside $J(\gamma)$ at random.

The terminology "Mauldin-Williams fractal" was introduced by the first author in [5]; the second author first studied them as "graph-directed constructions" in [10].

We will assume that our directed multigraph (V, E) has the property that each node has at least two edges leaving it. We claim that this restriction does not change the fractals K_u that can be constructed. If the entire graph is a simple cycle, then the sets K_u are singletons; so to ensure the system is nontrivial, we assume there is some node with two edges leaving.

Now suppose in (V, E, r) there is some node u_0 with only one edge e_0 leaving. Since the graph is strongly connected, that edge goes to some node other than u_0. Define a new directed multigraph as follows: $V' = V \setminus \{u_0\}$; the edges E' are of two kinds: the edges $e \in E$ that neither begin nor end at u_0, and the paths ee_0, where e is an edge ending at u_0. The ratios r' are given by $r'(e) = r(e)$ in the first case, and $r'(ee_0) = r(e)r(e_0)$ in the second case. The new system (V', E', r') constructs the same sets as (V, E, r), but has u_0 deleted. Continuing in this way, we never remove a node with two or more edges leaving, but (by the finiteness of V) the process eventually ends. When it does, we have a graph where every node has at least two edges leaving.

The Hausdorff dimension for digraph recursive fractals was computed in [10], see [5, Theorem 6.4.8]. This is done as follows: For each positive number s we define a square matrix $A(s)$, with rows and columns indexed by the set V: the entry in row u

and column v is

$$A_{uv}(s) = \sum_{e \in E_{uv}} r(e)^s.$$

The Hausdorff dimension d of all the sets K_u is the unique nonnegative number d such that the matrix $A(d)$ has spectral radius 1.

1.2. The models. We will use some "string models" in our investigation of these fractals. Write $E_u^{(\omega)}$ for the set of all *infinite* strings, using symbols from E, where the initial vertex of the first edge is u and the terminal vertex of each edge is the initial vertex of the next edge. These sets are naturally compact metric spaces. For each $\gamma \in E^{(*)}$, the **cylinder** $[\gamma]$ is the set of all infinite strings $\sigma \in E^{(\omega)}$ that begin with γ. Then the set $\left\{ [\gamma] : \gamma \in E_u^{(*)} \right\}$ is the base for the topology on $E_u^{(\omega)}$. For $\sigma \in E_u^{(\omega)}$ and a positive integer k, the **restriction** $\sigma{\restriction}k$ is the finite string made up of the first k letters of σ. The same notation $\gamma{\restriction}k$ is used for finite strings γ when k is less than the length of γ. As a special case of this: if γ has length k, then the **parent** of γ is $\gamma^- = \gamma{\restriction}(k-1)$, obtained by omitting the last letter of γ.

There is a **model map** $h_u \colon E_u^{(\omega)} \to \mathbb{R}^n$ for each u, defined so that $h_u(\sigma)$ is the unique element of the set

$$\bigcap_{k=1}^{\infty} J(\sigma{\restriction}k).$$

Then clearly $K_u = h_u\left[E_u^{(\omega)}\right]$. If $h_u(\sigma) = x$, then we say that the string σ is the **address** of the point x. In the case when the sets $J(\gamma e)$ that constitute $J(\gamma)$ are actually disjoint (not merely nonoverlapping), the model maps h_u are one-to-one. That means each point has a unique address.

1.3. Measures of Markov type. We begin with positive numbers $p(e)$, one for each edge $e \in E$. They are to be called **transition probabilities**. The probabilities of all edges leaving a given node u must sum to 1:

$$\sum_{v \in V} \sum_{e \in E_{uv}} p(e) = 1.$$

(Since each node has at least two edges leaving it, this implies that $p(e) < 1$ for all edges e.) Then we define products, which are to be thought of as probabilities of paths: if $\gamma \in E^{(*)}$, say $\gamma = e_1 e_2 \cdots e_k$, then

$$p(\gamma) = p(e_1) p(e_2) \cdots p(e_k).$$

These numbers satisfy an additivity condition: if $\gamma \in E^{(*)}_{uv}$, then

$$p(\gamma) = \sum_{e \in E_v} p(\gamma e).$$

Therefore, for each $u \in V$, there is a unique measure $\hat{\mu}_u$ on $E^{(\omega)}_u$ with

$$\hat{\mu}_u\big([\gamma]\big) = p(\gamma)$$

for all $\gamma \in E^{(*)}_u$. Measures of this kind will be called **measures of Markov type**. Discussion of them can be found under the heading "Markov chains" in many probability books. For example [2, Section 1.8], [14, Chapter 4].

We may think of this in more "probabilistic" language. Imagine a particle moving (at random) on our graph. At each tick of the clock, it traverses one of the edges (in the direction of the arrow) from one vertex to another. The number $p(e)$ gives the probability that the edge e will be chosen, among all the edges emanating from the vertex that is occupied at the present time. We have a sequence $(X_k)_{k=0}^{\infty}$ of random variables with values in V. Given that $X_k = u$, the conditional probability that $X_{k+1} = v$ is

$$\sum_{e \in E_{uv}} p(e).$$

(We keep a bit more information than is conventional. There may be several edges from u to v; we will not combine them into a single edge, so that we know not only where X_k moved to, but also which edge it traversed to get there. If necessary, this can be thought of as a larger Markov chain, where E is the set of states. In [10] there is at most one edge from one node u to another v. But here we allow several edges from u to v as in [5].) So this means: If $X_0 = u$, then the conditional probability that the process traverses edges $e_1, e_2, \cdots, e_k$ in the first k steps is 0 unless the string $\gamma = e_1 e_2 \cdots e_k$ belongs to $E_u^{(k)}$, and in that case the conditional probability is

$$p(\gamma) = \prod_{i=1}^{k} p(e_i).$$

The measure $\hat{\mu}_u$ on $E_u^{(\omega)}$ corresponds to a measure μ_u on $K_u \subseteq \mathbb{R}^n$: For $F \subseteq \mathbb{R}^n$, define $\mu_u(F) = \hat{\mu}_u\big(h_u^{-1}[F]\big)$. Another way to think of this measure involves the construction of K_u using the sets $J(\gamma)$. We begin by assigning mass 1 to the set J_u. Then that mass is distributed among the subsets $J(e)$, $e \in E_u$, so that $J(e)$ has mass $p(e)$. Once the mass for a set $J(\gamma)$ has been assigned, then it is distributed among the subsets $J(\gamma e)$, according to the values of $p(e)$.

1.4. Hausdorff and packing dimensions. The two fractal dimensions that we will be concerned with here are the **Hausdorff dimension** and the **packing dimension**. We briefly review their definitions.

Let $F \subseteq \mathbb{R}^n$ be a set. Fix positive real numbers s and ε. Define

$$\mathcal{H}^s_\varepsilon(F) = \inf \sum_i (\text{diam } A_i)^s,$$

where the infimum is over all countable families $\{A_i\}_{i=1}^\infty$ of sets with $\bigcup_i A_i \supseteq F$ and diam $A_i < \varepsilon$ for all i. Define the **s-dimensional Hausdorff outer measure** of F by:

$$\mathcal{H}^s(F) = \lim_{\varepsilon \downarrow 0} \mathcal{H}^s_\varepsilon(F) = \sup_{\varepsilon > 0} \mathcal{H}^s_\varepsilon(F).$$

There is a unique critical value d, with $0 \leq d \leq n$, such that

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } s < d \\ \\ 0 & \text{if } s > d. \end{cases}$$

This critical value d is called the **Hausdorff dimension** of the set F; we will write $d = \dim F$. For more complete discussions, see [1, Section 5.4], [5, Section 6.1], [6, Section 2.1]. We will need to know that $\mathcal{H}^s$ is a countably-additive measure on the Borel sets of $\mathbb{R}^n$.

Let $F \subseteq \mathbb{R}^n$ be a set. Fix positive real numbers s and ε. Define

$$\widetilde{\mathcal{P}}^s_\varepsilon(F) = \sup \sum_{i=1}^\infty (2\varepsilon_i)^s,$$

where the supremum is over all countable disjoint families $\left\{B_{\varepsilon_i}(x_i)\right\}_{i=1}^\infty$ of balls with $\varepsilon_i < \varepsilon$ and $x_i \in F$. Define the **s-dimensional packing pre-measure** of F by:

$$\widetilde{\mathcal{P}}^s(F) = \lim_{\varepsilon \downarrow 0} \widetilde{\mathcal{P}}^s_\varepsilon(F) = \inf_{\varepsilon > 0} \widetilde{\mathcal{P}}^s_\varepsilon(F).$$

Then define the **s-dimensional packing outer measure** of F by:

$$\mathcal{P}^s(F) = \inf \sum_{i=1}^\infty \widetilde{\mathcal{P}}^s(F_i),$$

where the infimum is over all countable families $\{F_i\}_{i=1}^{\infty}$ of sets with $\bigcup_i F_i \supseteq F$. There is a unique critical value d, with $0 \leq d \leq n$, such that

$$\mathcal{P}^s(F) = \begin{cases} \infty & \text{if } s < d \\ \\ 0 & \text{if } s > d. \end{cases}$$

This critical value d is called the **packing dimension** of the set F; we will write $d = \operatorname{Dim} F$. For more complete discussions, see [5, Section 6.5], [6, Section 3.4]. We will need to know that $\mathcal{P}^s$ is a countably-additive measure on the Borel sets of $\mathbb{R}^n$.

For any set $F \subseteq \mathbb{R}^n$, we have $\dim F \leq \operatorname{Dim} F$. (For example, [5, Proposition 6.5.7].) Taylor [16] has proposed that the term "fractal" be used for a set $F \subseteq \mathbb{R}^n$ with $\dim F = \operatorname{Dim} F$. We will prove that the "multifractal components" $K_u^{(\alpha)}$ of our digraph recursive fractals satisfy this criterion.

1.5. Multifractal decomposition. Now consider the digraph recursive fractals K_u defined above, and the measures $\hat{\mu}_u$ on the model spaces $E_u^{(\omega)}$. The "balls" in $E_u^{(\omega)}$ are the cylinders $[\gamma]$; the measure of a cylinder is $\hat{\mu}_u([\gamma]) = p(\gamma)$; the diameter of a cylinder may be considered to be $r(\gamma)$. (The diameter of the image $h_u\big[[\gamma]\big]$ is $\leq r(\gamma)$ and $\geq c\,r(\gamma)$ for some constant c.) Given a real number α, we will be interested in the sets

$$\begin{aligned} \widehat{K}_u^{(\alpha)} &= \left\{ \sigma \in E_u^{(\omega)} : \lim_{k\to\infty} \frac{\log p(\sigma{\restriction}k)}{\log r(\sigma{\restriction}k)} = \alpha \right\} \\ K_u^{(\alpha)} &= h_u\left[\widehat{K}_u^{(\alpha)}\right]. \end{aligned}$$

They will be called the **multifractal components** of K_u (with respect to $\hat{\mu}_u$.). Note that $\widehat{K}_u^{(\alpha)}$ is a Borel set (actually, an $F_{\sigma\delta}$-set). $K_u^{(\alpha)}$ is at least an analytic set.

At least in a special case, there is another (more natural) description of the multifractal components. Suppose the K_u exhibit "graph self-similarity": there is a similarity $\theta_e\colon \mathbb{R}^n \to \mathbb{R}^n$ for each edge $e \in E$, such that for $\gamma = e_1 e_2 \cdots e_k \in E_{uv}^{(k)}$,

$$J(\gamma) = \theta_{e_1}\theta_{e_2}\cdots\theta_{e_k}[J_v].$$

Suppose that, for each $u \in V$, the sets $J(e)$ for $e \in E_u$ are disjoint. Then the multifractal components $K_u^{(\alpha)}$ defined above satisfy

$$K_u^{(\alpha)} = \left\{ x \in K_u : \lim_{\varepsilon\downarrow 0} \frac{\log \mu\big(B_\varepsilon(x)\big)}{\log \operatorname{diam}\, B_\varepsilon(x)} = \alpha \right\}.$$

Indeed, two disjoint compact sets are separated by positive distance. Let

$$c = \min\left\{ \operatorname{dist}\big(J(e), J(e')\big) : u \in V, e, e' \in E_u, e \neq e' \right\}.$$

Then, for $\sigma, \tau \in E_u^{(\omega)}$ we have

$$c\, r(\gamma) \leq \big|h_u(\sigma) - h_u(\tau)\big| \leq r(\gamma),$$

where γ is the longest common prefix of σ and τ. [This can be proved by induction on the length of γ.] Now if $x = h_u(\sigma) \in K_u$ and $\varepsilon > 0$, then we have

$$K_u \cap B_\varepsilon(x) \subseteq J(\sigma{\restriction}k_1)$$

where k_1 is the largest integer with $\varepsilon < cr(\sigma{\restriction}k_1)$, and

$$B_\varepsilon(x) \supseteq K_u \cap J(\sigma{\restriction}k_0),$$

where k_0 is the least integer with $\varepsilon > r(\sigma{\restriction}k_0)$. Writing $r_{\min} = \min_e r(e)$, we may deduce that

$$\frac{\log p(\sigma{\restriction}k_0)}{\log\big((2/r_{\min})r(\sigma{\restriction}k_0)\big)} \leq \frac{\log \mu_u\big(B_\varepsilon(x)\big)}{\log \operatorname{diam}\, B_\varepsilon(x)} \leq \frac{\log p(\sigma{\restriction}k_1)}{\log\big(2cr_{\min}r(\sigma{\restriction}k_1)\big)}.$$

These inequalities show that

$$\lim_{k\to\infty} \frac{\log p(\sigma{\restriction}k)}{\log r(\sigma{\restriction}k)} = \lim_{\varepsilon\to 0} \frac{\log \mu_u\big(B_\varepsilon(x)\big)}{\log \operatorname{diam} B_\varepsilon(x)}$$

whenever one of these limits exsits.

The Hausdorff dimensions of the multifractal components may be computed as follows. (Details are given below, in Section 3.) Let $A(q,\beta)$ be a square matrix with rows and columns indexed by V. The entry in row u, column v, is

$$A_{uv}(q,\beta) = \sum_{e\in E_{uv}} p(e)^q r(e)^\beta.$$

For given q, there is a unique β so that $A(q,\beta)$ has spectral radius 1. This defines β as an analytic function of q. Define $\alpha = -d\beta/dq$ and $f = q\alpha + \beta$. We note that in much of the literature, what we call β is known as $-\tau$.

1.6. Theorem. *Let (V,E) be a strongly connected directed multigraph. Let $r(e)$, $0 < r(e) < 1$, be a system of ratios for the graph, and let $p(e)$, $0 < p(e) < 1$, be a system of transition probabilities for the graph, defining measures $\hat{\mu}_u$ of Markov type on the string models $E_u^{(\omega)}$. Let q, β, α, f be four numbers related as above. Then for each $u \in V$, the multifractal component $K_u^{(\alpha)}$ is a fractal with dimension f:*

$$\dim K_u^{(\alpha)} = \operatorname{Dim} K_u^{(\alpha)} = f.$$

This theorem is proved below, in Section 4.

2. An Example

Let us consider a particular example, before we proceed to the general case. The "two-part dust" (Figure 1) is from [5, Section 6.4]. It is related to the graph shown in Figure 2. The set of vertices is $\{\mathsf{U},\mathsf{V}\}$. The set of edges is $\{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\}$. The ratios are given by: $r(\mathsf{a})=1/2$, $r(\mathsf{b})=1/4$, $r(\mathsf{c})=1/2$, and $r(\mathsf{d})=3/4$. The two digraph recursive fractals K_{U} and K_{V} are fractals with dimension $d=1$ (both Hausdorff and packing dimension).

Figure 1. The two-part dust. *Figure 2.* The directed graph.

Now consider the natural measures $\hat{\mu}_{\mathsf{U}},\hat{\mu}_{\mathsf{V}}$ of Markov type, obtained by assigning $p(e)=1/2$ for all edges e; for our Markov chain, at each step, we use each of the two possible edges leaving the present node with equal probability. Then the matrix A becomes

$$A(q,\beta)=\begin{bmatrix} \left(\frac{1}{2}\right)^q\left(\frac{1}{2}\right)^\beta & \left(\frac{1}{2}\right)^q\left(\frac{1}{4}\right)^\beta \\ \left(\frac{1}{2}\right)^q\left(\frac{1}{2}\right)^\beta & \left(\frac{1}{2}\right)^q\left(\frac{3}{4}\right)^\beta \end{bmatrix}.$$

We want the spectral radius to be equal to 1. So q and β must satisfy the equation $\det(A-I)=0$, or

$$2^{-2q-3\beta}3^\beta-2^{-q-\beta}-2^{-q-2\beta}3^\beta+1-2^{-2q-3\beta}=0. \tag{1}$$

This may be considered a quadratic equation for 2^{-q}. It may be solved for q. We have chosen the appropriate one of the two roots:

$$q=\frac{\log\left(2^{-\beta-1}+3^\beta 2^{-2\beta-1}+2^{-2\beta-1}\sqrt{2^{2\beta}-3^\beta 2^{\beta+1}+3^{2\beta}+2^{\beta+2}}\right)}{\log 2}.$$

Figure 3. Graph of $\beta(q)$. *Figure 4.* Graph of $f(\alpha)$.

Then $\alpha = -d\beta/dq$, so (differentiating (1) implicitly) $\alpha =$

$$\frac{(2^{-\beta-q} + 3^{\beta}2^{-2\beta-q} + 2^{-3\beta-2q+1}(-3^{\beta}+1))\log 2}{(2^{-\beta-q} + 3^{\beta}2^{-2\beta-q+1} + 3\cdot 2^{-3\beta-2q}(-3^{\beta}+1))\log 2 + 3^{\beta}(-2^{-2\beta-q} + 2^{-3\beta-2q})\log 3}.$$

As usual, $f = q\alpha + \beta$.

Here are some special values:

$q = 0$, $\beta = 1$, $\alpha = 6\log 2/(10\log 2 - 3\log 3) \approx 1.1439$ and $f = 1$.

$q = 1$, $\beta = 0$, $\alpha = f = 4\log 2/(6\log 2 - \log 3) \approx 0.90599$.

$q \to \infty$, $\beta \to -\infty$, $\alpha \to \alpha_{\min} = 2/3$ and $f \to 0$.

$q \to -\infty$, $\beta \to \infty$, $\alpha \to \alpha_{\max} = \log 2/(2\log 2 - \log 3) \approx 2.4094$ and $f \to 0$.

The graph of $\beta(q)$ shows that it is decreasing and convex, with oblique asymptotes at both ends. The graph of $f(\alpha)$ shows the typical concave "dimension spectrum" shape, with maximum value $1 = \dim K_{\mathbf{U}} = \dim K_{\mathbf{V}}$ at $\alpha = 1.1439$ (and $q = 0$).

3. Auxiliary Functions

Now we consider more carefully how the auxiliary functions β and α behave. Let $A(q,\beta)$ be as above. The entry in row u, column v, is

$$A_{uv}(q,\beta) = \sum_{e \in E_{uv}} p(e)^q r(e)^\beta.$$

Let $\Phi(q,\beta)$ be the spectral radius of $A(q,\beta)$. The arguments dealing with $\Phi(q,\beta)$ rely on the theory of nonnegative matrices, known as Perron-Frobenius theory: see for example [7], [11], [14]. Here are the basic properties of the function Φ:

3.1. Proposition.

(i) $\Phi\colon \mathbb{R}\times\mathbb{R} \to (0,\infty)$ *is continuous (in fact, analytic).*

(ii) Φ *is strictly decreasing in each variable separately; that is: if* $q_1 < q_2$, *then* $\Phi(q_1,\beta) > \Phi(q_2,\beta)$; *and if* $\beta_1 < \beta_2$, *then* $\Phi(q,\beta_1) > \Phi(q,\beta_2)$.

(iii) *For fixed* q *we have* $\lim_{\beta\to\infty}\Phi(q,\beta) = 0$ *and* $\lim_{\beta\to-\infty}\Phi(q,\beta) = \infty$. *For fixed* β *we have* $\lim_{q\to\infty}\Phi(q,\beta) = 0$ *and* $\lim_{q\to-\infty}\Phi(q,\beta) = \infty$.

(iv) Φ *is log-convex; that is: if* $q_1, q_2, \beta_1, \beta_2 \in \mathbb{R}$, $a_1, a_2 \geq 0$, $a_1 + a_2 = 1$, *then*

$$\Phi\big(a_1 q_1 + a_2 q_2, a_1\beta_1 + a_2\beta_2\big) \leq \Phi\big(q_1,\beta_1\big)^{a_1}\Phi\big(q_2,\beta_2\big)^{a_2}.$$

Proof. (i) Each entry $A_{uv}(q,\beta)$ is continuous (analytic). The largest zero of a polynomial is an analytic function of the coefficients of the polynomial in the region where that zero is a simple zero. Since the graph is strongly connected, the matrix $A(q,\beta)$ is irreducible, so the spectral radius $\Phi(q,\beta)$ is a simple zero of the characteristic polynomial.

(ii) Fix β, and let $q_1 < q_2$. There is a positive Perron-Frobenius eigenvector $(\rho_v)_{v\in V}$ for the matrix $A(q_1,\beta)$ with

$$\sum_v \sum_{e\in E_{uv}} p(e)^{q_1} r(e)^\beta \rho_v = \Phi(q_1,\beta)\,\rho_u$$

for all $u \in V$. Now $p(e) < 1$, so $p(e)^{q_1} > p(e)^{q_2}$. Therefore

$$\sum_v \sum_{e\in E_{uv}} p(e)^{q_2} r(e)^\beta \rho_v < \sum_v \sum_{e\in E_{uv}} p(e)^{q_1} r(e)^\beta \rho_v = \Phi(q_1,\beta)\,\rho_u.$$

This is a strict inequality since there is a nonzero entry in row u. Therefore (as in the Perron-Frobenius theorem), we conclude $\Phi(q_2,\beta) < \Phi(q_1,\beta)$.

The proof that $\Phi(q,\beta)$ is strictly decreasing in β is the same.

(iii) Fix q. When $\beta \to \infty$, all of the entries of $A(q,\beta)$ approach 0, so $\Phi(q,\beta) \to 0$. Similarly, when we let $\beta \to -\infty$, the nonzero entries of $A(q,\beta)$ approach ∞; there is at least one nonzero entry in each row of $A(q,\beta)$, so also $\Phi(q,\beta) \to \infty$.

The proof of the other case is the same.

(iv) Fix values $q_1, q_2, \beta_1, \beta_2 \in \mathbb{R}$ and $a_1, a_2 \geq 0$ with $a_1 + a_2 = 1$. There are positive eigenvectors (ρ_{1v}) and (ρ_{2v}) with

$$\sum_v \sum_{e\in E_{uv}} p(e)^{q_i} r(e)^{\beta_i} \rho_{iv} = \Phi(q_i,\beta_i)\,\rho_{iu}$$

for all $u \in V$, $i = 1, 2$. Write $q = a_1 q_1 + a_2 q_2$, $\beta = a_1\beta_1 + a_2\beta_2$. Let $\rho_u = \rho_{1u}^{a_1}\rho_{2u}^{a_2}$ for $u \in V$. Then (using Hölder's inequality):

$$\begin{aligned}
\sum_v \sum_{e\in E_{uv}} p(e)^q r(e)^\beta \rho_v &= \sum\sum \left(p(e)^{q_1} r(e)^{\beta_1} \rho_{1v}\right)^{a_1} \left(p(e)^{q_2} r(e)^{\beta_2} \rho_{2v}\right)^{a_2} \\
&\leq \left(\sum\sum p(e)^{q_1} r(e)^{\beta_1} \rho_{1v}\right)^{a_1} \left(\sum\sum p(e)^{q_2} r(e)^{\beta_2} \rho_{2v}\right)^{a_2} \\
&= \left(\Phi(q_1,\beta_1)\,\rho_{1u}\right)^{a_1} \left(\Phi(q_2,\beta_2)\,\rho_{2u}\right)^{a_2} \\
&= \Phi(q_1,\beta_1)^{a_1}\,\Phi(q_2,\beta_2)^{a_2}\,\rho_u.
\end{aligned}$$

Now the vector (ρ_u) is positive, so by Perron-Frobenius theory we conclude $\Phi(q,\beta) \leq \Phi(q_1,\beta_1)^{a_1}\Phi(q_2,\beta_2)^{a_2}$. $\square$

Now for fixed q, the function $\Phi(q,\beta)$ is a continuous function of β. Its values range from 0 (when $\beta \to \infty$) to ∞ (when $\beta \to -\infty$). Therefore by the intermediate value theorem there is a real number β such that

$$\Phi(q,\beta) = 1.$$

The solution β is unique, since Φ is a strictly decreasing function of β. This defines β implicitly as a function of q.

Here are a few useful properties of this function:

3.2. Proposition. *Let $\beta = \beta(q)$ be defined by $\Phi(q,\beta) = 1$. Then*

(i) *β is an analytic function of the real variable q.*

(ii) *β is strictly decreasing; that is: if $q_1 < q_2$, then $\beta(q_1) > \beta(q_2)$.*

(iii) *$\lim_{q\to-\infty} \beta(q) = \infty$ and $\lim_{q\to\infty} \beta(q) = -\infty$.*

(iv) *β is a convex function; that is: if $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$, then*

$$\beta\big(a_1q_1 + a_2q_2\big) \leq a_1\beta\big(q_1\big) + a_2\beta\big(q_2\big).$$

Proof. (i) Φ is an analytic function of its two variables. Neither of its partial derivatives vanishes. Therefore by the implicit function theorem, β is an analytic function of q.

(ii) Let $q_1 < q_2$. We must show that $\beta(q_1) > \beta(q_2)$. Suppose not: $\beta(q_1) \leq \beta(q_2)$. Then $1 = \Phi\big(q_1, \beta(q_1)\big) > \Phi\big(q_2, \beta(q_1)\big) \geq \Phi\big(q_2, \beta(q_2)\big) = 1$, a contradiction.

(iii) follows from Proposition 3.1(ii).

(iv) Let q_1, q_2, a_1, a_2 be given with $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$. Write $\beta_1 = \beta(q_1)$ and $\beta_2 = \beta(q_2)$. Then

$$\begin{aligned}
\Phi(a_1q_1 + a_2q_2, a_1\beta_1 + a_2\beta_2) &\leq \Phi(q_1, \beta_1)^{a_1}\Phi(q_2, \beta_2)^{a_2} \\
&= 1^{a_1}1^{a_2} = 1 \\
&= \Phi\big(a_1q_1 + a_2q_2, \beta(a_1q_1 + a_2q_2)\big).
\end{aligned}$$

Therefore $a_1\beta_1 + a_2\beta_2 \geq \beta(a_1q_1 + a_2q_2)$, as claimed. $\square$

Note that the probabilities $p(e)$ were postulated to satisfy

$$\sum_{v \in V} \sum_{e \in E_{uv}} p(e) = 1$$

for all $u \in V$. So by the Perron-Frobenius theorem, the spectral radius of $A(1, 0)$ is 1. So $\beta(1) = 0$. As noted above, the Haudorff (and packing) dimensions of the sets K_u are the number d with $\Phi(0, d) = 1$. So $\beta(0) = d$.

Now let us consider the derivative $\beta'(q)$. We know that q and β satisfy the equation $\Phi(q, \beta) = 1$. The matrix is irreducible, so there is a one-dimensional eigenspace for eigenvalue 1 in the matrix $A(q, \beta)$. We normalize to obtain a unique vector $(\rho_v)_{v \in V}$ satisfying

$$\sum_{v \in V} \sum_{e \in E_{uv}} p(e)^q r(e)^\beta \rho_v = \rho_u \quad \text{for all } u \in V;$$

$$\sum_{v \in V} \rho_v = 1.$$

Cramer's rule shows that the entries ρ_v are analytic functions of q. The graph is strongly connected, so the matrix is irreducible; so by the Perron-Frobenius theorem,

$\rho_v > 0$ for all v. Similarly, there exists a left eigenvector λ_u for the matrix. This time we will normalize slightly differently:

$$\sum_{u\in V}\sum_{e\in E_{uv}} \lambda_u p(e)^q r(e)^\beta = \lambda_v \quad \text{for all } v \in V;$$
$$\sum_{u\in V} \lambda_u \rho_u = 1.$$

Again, the entries λ_u are positive analytic functions of q.

To simplify the notation, we use a prime $'$ for derivative with respect to q. First note

$$\sum_{u\in V} \lambda_u \rho_u = 1$$
$$\left(\sum_{u\in V} \lambda_u \rho_u\right)' = 0$$
$$\sum_{u\in V} \lambda'_u \rho_u + \sum_{u\in V} \lambda_u \rho'_u = 0.$$

Consider the expression

$$S = \sum_u \sum_v \sum_{e\in E_{uv}} \lambda_u p(e)^q r(e)^\beta \rho_v.$$

Then, of course $S = \sum_v \lambda_v \rho_v = 1$. Differentiating, we obtain

$$\begin{aligned}
0 &= \sum_u \sum_v \sum_{e\in E_{uv}} \lambda'_u p(e)^q r(e)^\beta \rho_v + \sum_u \sum_v \sum_{e\in E_{uv}} \lambda_u p(e)^q \big(\log p(e)\big) r(e)^\beta \rho_v \\
&\quad + \sum_u \sum_v \sum_{e\in E_{uv}} \lambda_u p(e)^q r(e)^\beta \big(\log r(e)\big) \beta' \rho_v + \sum_u \sum_v \sum_{e\in E_{uv}} \lambda_u p(e)^q r(e)^\beta \rho'_v \\
&= \sum_u \lambda'_u \rho_u + \sum_u \sum_v \sum_{e\in E_{uv}} \big(\lambda_u p(e)^q r(e)^\beta \rho_v\big)\big(\log p(e) + \beta' \log r(e)\big) + \sum_v \lambda_v \rho'_v \\
&= \sum_u \sum_v \sum_{e\in E_{uv}} \big(\lambda_u p(e)^q r(e)^\beta \rho_v\big)\big(\log p(e) + \beta' \log r(e)\big).
\end{aligned}$$

Therefore

$$\beta'(q) = -\frac{\sum_u \sum_v \sum_{e\in E_{uv}} \big(\lambda_u p(e)^q r(e)^\beta \rho_v\big) \log p(e)}{\sum_u \sum_v \sum_{e\in E_{uv}} \big(\lambda_u p(e)^q r(e)^\beta \rho_v\big) \log r(e)}.$$

So we conclude that $\beta'(q) < 0$, agreeing with Proposition 3.2(ii).

We will write $\alpha = -d\beta/dq$. Thus $\alpha > 0$, and

$$\alpha = \frac{\sum_u \sum_v \sum_{e\in E_{uv}} \big(\lambda_u p(e)^q r(e)^\beta \rho_v\big) \log p(e)}{\sum_u \sum_v \sum_{e\in E_{uv}} \big(\lambda_u p(e)^q r(e)^\beta \rho_v\big) \log r(e)}. \tag{2}$$

Computations below will use some other numerical parameters. If $\gamma = e_1 e_2 \cdots e_k \in E^{(k)}$ is a path, let

$$\eta(\gamma) = \frac{\log p(\gamma)}{\log r(\gamma)} = \frac{\log\big(p(e_1)p(e_2)\cdots p(e_k)\big)}{\log\big(r(e_1)r(e_2)\cdots r(e_k)\big)}.$$

Write

$$\eta_{\min} = \min\{\, \eta(\zeta) : \zeta \text{ is a simple cycle}\,\}$$

$$\eta_{\max} = \max\{\, \eta(\zeta) : \zeta \text{ is a simple cycle}\,\}.$$

We will see below that α ranges from $\eta_{\min}$ to $\eta_{\max}$.

The last auxiliary function is $f = q\alpha + \beta$. Its behavior depends on the behavior of β. Now β is a convex function of q, so $d^2\beta/dq^2 \geq 0$, and therefore $d\alpha/dq \leq 0$. Actually, there are two rather different possibilities: β is a linear function or β is a strictly convex function.

3.3. Proposition. *Let $(x_v)_{v\in V}$ be the "Perron numbers": $x_v > 0$ and*

$$\sum_{v\in V} \sum_{e\in E_{uv}} r(e)^d x_v^d = x_u^d, \qquad \textit{for all } u \in V.$$

(A) Suppose $p(e) = (x_u^{-1} r(e) x_v)^d$ for all $u, v \in V$ and $e \in E_{uv}$. Then:

(i) *β is a linear function: $\beta(q) = d - dq$.*

(ii) *$\alpha = d$ is constant.*

(iii) *$f = d$ is constant.*

(iv) *$K_u^{(d)} = K_u$ and $K_u^{(\alpha)} = \varnothing$ for all $\alpha \neq d$.*

(B) Suppose $p(e) \neq (x_u^{-1} r(e) x_v)^d$ for at least one edge e. Then:

(i) *β is a strictly convex function of q.*

(ii) *α is a strictly decreasing function of q, so we may consider q as a function of α defined on an interval $(\alpha_{\min}, \alpha_{\max})$.*

(iii) *f is a strictly concave function of α.*

(iv) *$K_u^{(\alpha)} \neq \varnothing$ if and only if $\eta_{\min} \leq \alpha \leq \eta_{\max}$.*

(v) *The function $\beta(q) + \eta_{\min} q$ is nonincreasing and has a limit ≥ 0 as $q \to \infty$; the function $\beta(q) + \eta_{\max} q$ is nondecreasing and has a limit ≥ 0 as $q \to -\infty$. And α is a decreasing function of q, with $\alpha \to \eta_{\min}$ as $q \to \infty$ and $\alpha \to \eta_{\max}$ as $q \to -\infty$. (So $\alpha_{\min} = \eta_{\min}$ and $\alpha_{\max} = \eta_{\max}$.)*

Proof. The Perron numbers $x_v > 0$ exist by the Perron-Frobenius theorem. So if we write $x_{\min} = \min_v x_v$ and $x_{\max} = \max_v x_v$, then we have $0 < x_{\min} \leq x_{\max} < \infty$.

(A)(i) Let q be given. We claim that $\beta(q) = d - dq$. If we write $\rho_v = x_v^{d-dq}$, then we have

$$\begin{aligned}\sum_v \sum_{e \in E_{uv}} p(e)^q r(e)^{d-dq} \rho_v &= \sum \sum x_u^{-dq} r(e)^{dq} x_v^{dq} r(e)^{d-dq} x_v^{d-dq} \\ &= x_u^{-dq} \sum \sum r(e)^d x_v^d = x_u^{d-dq} = \rho_u.\end{aligned}$$

Therefore $\Phi(q, d - dq) = 1$, so $\beta(q) = d - dq$.

(ii) Differentiate the result of (i): $\alpha = -d\beta/dq = d$.

(iii) $f = q\alpha + \beta = d$.

(iv) Let $\gamma = e_1 e_2 \cdots e_k$ be a path in $E_{uv}^{(k)}$. Since the terminal vertex of each edge e_i is the initial vertex of the following edge e_{i+1}, most of the x_v's cancel in the product

$$p(\gamma) = \left(x_u^{-1} r(e_1) r(e_2) \cdots r(e_k) x_v\right)^d = \left(x_u^{-1} r(\gamma) x_v\right)^d.$$

so that

$$\frac{\log p(\gamma)}{\log r(\gamma)} = d\left(1 + \frac{\log(x_v/x_u)}{\log r(\gamma)}\right).$$

Note that $\log(x_v/x_u)$ is bounded above by $\log(x_{\max}/x_{\min})$ and bounded below by $\log(x_{\min}/x_{\max})$. Now let $\sigma \in E_u^{(\omega)}$. If we write $\gamma = \sigma{\restriction}k$, we have

$$\frac{\log p(\sigma{\restriction}k)}{\log r(\sigma{\restriction}k)} = d\left(1 + \frac{C_k}{\log r(\sigma{\restriction}k)}\right),$$

where C_k remains bounded as $k \to \infty$, while $\log r(\sigma{\restriction}k) \to -\infty$ as $k \to \infty$. This shows that $\log p(\sigma{\restriction}k)/\log r(\sigma{\restriction}k) \to d$. Therefore $E_u^{(\omega)} = \widehat{K}_u^{(d)}$, so $K_u = K_u^{(d)}$.

(B) On the other hand, suppose that $p(e) \neq (x_u^{-1}r(e)x_v)^d$ for some edge e. We claim that β is a strictly convex function of q. Since β is real-analytic, if β is linear on some interval, then it is linear everywhere. That is, if equality holds in Proposition 3.2 (iv) for some $q_1 \neq q_2$, then it holds for all. So suppose equality holds for $q_1 = 0$, $q_2 = 1$. Now $\beta(0) = d$ and $\beta(1) = 0$. The right eigenvectors ρ_{iv} are:

$$\rho_{1v} = x_v^d,$$

$$\rho_{2v} = 1.$$

Equality in Proposition 3.2 (iv) means equality in Hölder's inequality in the proof of Proposition 3.1 (iv); thus there are constants a_u with

$$p(e)^0 r(e)^d x_v^d = a_u p(e)^1 r(e)^0 1 \qquad \text{for } e \in E_{uv}.$$

Summing over $v \in V$ and $e \in E_{uv}$, we get $x_u^d = a_u$. Therefore

$$p(e) = \left(x_u^{-1} r(e) x_v\right)^d$$

for all $u, v \in V$ and $e \in E_{uv}$. So we are in case (A).

(i) Thus, in case (B), β is strictly convex. (ii) It follows that α is strictly decreasing. And (iii)

$$f' = \alpha + q\alpha' - \alpha = q\alpha'.$$

So $df/d\alpha = f'/\alpha' = q$. In the graph of f as a function of α (as in Figure 4) the parameter q is the slope of the tangent line to the curve. At the endpoints, where $q \to \infty$ and $q \to -\infty$, the graph has vertical tangent lines. Also,

$$\frac{d^2 f}{d\alpha^2} = \frac{dq}{d\alpha} = \frac{1}{d\alpha/dq} = \frac{-1}{d^2\beta/dq^2} < 0.$$

So $f(\alpha)$ is a strictly concave function.

(iv) Suppose $\alpha < \eta_{\min}$. We must show that $K_u^{(\alpha)} = \varnothing$, or equivalently that

$$\liminf_{k\to\infty} \frac{\log p(\sigma\restriction k)}{\log r(\sigma\restriction k)} \geq \eta_{\min}$$

for all $\sigma \in E_u^{(\omega)}$. Now for all simple cycles ζ we have $\log p(\zeta)/\log r(\zeta) \geq \eta_{\min}$, so $\log p(\zeta) \leq \eta_{\min} \log r(\zeta)$, and thus

$$p(\zeta) \leq r(\zeta)^{\eta_{\min}}.$$

Now any cycle may be partitioned into finitely many simple cycles. Indeed, if a cycle ζ is not simple, some node is repeated, so it contains a shorter cycle; the shortest cycle contained in ζ is a simple cycle. When this simple cycle is removed from ζ, what remains is a cycle shorter than ζ. Thus, if $\zeta = e_1 e_2 \cdots e_k$, then the indices $1, \cdots, k$ may be partitioned as a disjoint union

$$\{1, 2, \cdots, k\} = \bigcup_{j=1}^{m} I_j$$

so that $\{ e_i : i \in I_j \}$ is a simple cycle ζ_j for each j.

Now if ζ is any cycle, partition it into simple cycles $\zeta_1, \zeta_2, \cdots, \zeta_m$. Then

$$p(\zeta) = p(\zeta_1) \cdots p(\zeta_m) \leq r(\zeta_1)^{\eta_{\min}} \cdots r(\zeta_m)^{\eta_{\min}} = r(\zeta)^{\eta_{\min}}.$$

Therefore $\eta(\zeta) \geq \eta_{\min}$.

There are only finitely many nodes in the graph; say there are N nodes. The same argument as above shows that any finite path γ may be partitioned into cycles plus at most N edges. Thus, if $C = \max\{ 1, p(e)/r(e)^{\eta_{\min}} : e \in E \}$, then

$$p(\gamma) \leq r(\gamma)^{\eta_{\min}} C^N.$$

Now for given σ, we have $r(\sigma{\restriction}k) \to 0$ and $p(\sigma{\restriction}k) \to 0$, so the term $N \log C$ disappears in the limit, and

$$\liminf_{k\to\infty} \frac{\log p(\sigma{\restriction}k)}{\log r(\sigma{\restriction}k)} \geq \eta_{\min}$$

as required.

A similar argument shows that $K_u^{(\alpha)} = \varnothing$ if $\alpha > \eta_{\max}$.

Now there is a cycle ζ_0 with $\eta(\zeta_0) = \eta_{\min}$ and a cycle ζ_1 with $\eta(\zeta_1) = \eta_{\max}$. The infinite path $\sigma = \zeta_0\zeta_0\cdots$ obtained by repeating ζ_0 achieves

$$\lim_{k\to\infty} \frac{\log p(\sigma{\restriction}k)}{\log r(\sigma{\restriction}k)} = \eta_{\min}.$$

The path obtained by repeating ζ_1 achieves $\eta_{\max}$. And any value of α between $\eta_{\min}$ and $\eta_{\max}$ is achieved by an infinite path that intersperses the two cycles ζ_0 and ζ_1 in the proper proportions. So $K_u^{(\alpha)} \neq \varnothing$ for such α.

(v) Now we analyze the asymptotic properties of $\beta(q)$: when $q \to \infty$, and $\beta \to -\infty$, we claim that α decreases to $\eta_{\min}$ and $\beta + \eta_{\min} q$ is nonincreasing. (The other asymptote $q \to -\infty$ may be done similarly.)

Now if $\gamma \in E^{(k)}$ is a path of length k, it may be partitioned into cycles $\zeta_1, \zeta_2, \cdots, \zeta_m$ plus at most N additional edges. Then, as before

$$\begin{aligned} &\log p(\gamma) \le \eta_{\min} \log r(\gamma) + N \log C, \\ &\frac{\log p(\gamma)}{\log r(\gamma)} \ge \eta_{\min} + \frac{N \log C}{\log r(\gamma)} \ge \eta_{\min} + \frac{N \log C}{k \log r_{\max}}. \end{aligned}$$

Now if we write for each k

$$\eta_k = \min \left\{ \eta(\gamma) : \gamma \in E^{(k)} \right\},$$

we may conclude

$$\eta_k \ge \eta_{\min} + \frac{N \log C}{k \log r_{\max}}.$$

But $k \log r_{\max} \to -\infty$ as $k \to \infty$, so we have

$$\liminf_{k \to \infty} \eta_k \ge \eta_{\min}.$$

On the other hand, for given large k, we may construct a path γ of length k by repeating the simple cycle ζ_0 (which achieves $\eta_{\min}$) many times, followed by the first few edges of ζ_0. Now if $c = \min \{ 1, p(e)/r(e)^{\eta_{\min}} : e \in E \}$, we have

$$\eta(\gamma) \le \eta_{\min} + \frac{N \log c}{\log r(\gamma)};$$

thus

$$\eta_k \le \eta_{\min} + \frac{N \log c}{\log r(\gamma)}.$$

This shows

$$\lim_{k \to \infty} \eta_k = \eta_{\min}.$$

For a given positive integer k, the power $A\big(q,\beta(q)\big)^k$ of the matrix also has spectral radius 1 with the same left and right eigenvectors. So for each node u,

$$\rho_u = \sum_v \sum_{\gamma \in E_{uv}^{(k)}} p(\gamma)^q r(\gamma)^\beta \rho_v. \tag{3}$$

Thus (as in the case $k = 1$) we have

$$1 = \sum_u \lambda_u \rho_u = \sum_u \sum_v \sum_{\gamma \in E_{uv}^{(k)}} \lambda_u p(\gamma)^q r(\gamma)^\beta \rho_v,$$

and may differentiate to conclude

$$\beta'(q) = -\frac{\sum_u \sum_v \sum_{\gamma \in E_{uv}^{(k)}} \big(\lambda_u p(\gamma)^q r(\gamma)^\beta \rho_v\big) \log p(\gamma)}{\sum_u \sum_v \sum_{\gamma \in E_{uv}^{(k)}} \big(\lambda_u p(\gamma)^q r(\gamma)^\beta \rho_v\big) \log r(\gamma)}.$$

But for all γ with length k, $\eta(\gamma) \geq \eta_k$, so $p(\gamma) = r(\gamma)^{\eta(\gamma)} \leq r(\gamma)^{\eta_k}$, and thus $\log p(\gamma) \leq \eta_k \log r(\gamma)$. Therefore we have

$$\beta'(q) \leq -\frac{\eta_k \sum_u \sum_v \sum_{\gamma \in E_{uv}^{(k)}} \big(\lambda_u p(\gamma)^q r(\gamma)^\beta \rho_v\big) \log r(\gamma)}{\sum_u \sum_v \sum_{\gamma \in E_{uv}^{(k)}} \big(\lambda_u p(\gamma)^q r(\gamma)^\beta \rho_v\big) \log r(\gamma)} = -\eta_k.$$

Take the limit as $k \to \infty$ to obtain $\beta'(q) \leq -\eta_{\mathrm{min}}$. This implies that $\alpha \geq \eta_{\mathrm{min}}$; and also $\beta + \eta_{\mathrm{min}} q$ is a nonincreasing function of q.

Next we claim that $\beta(q) + \eta_{\mathrm{min}} q$ converges to a finite nonnegative limit as $q \to \infty$. If $\zeta \in E_{uu}^{(k)}$ is a simple cycle, then from (3) we have

$$\rho_u > p(\zeta)^q r(\zeta)^{\beta(q)} \rho_u.$$

The inequality is strict since there is more to the graph than this single simple cycle. Therefore $r(\zeta)^{q\eta(\zeta)+\beta(q)} < 1$, so $q\eta(\zeta) + \beta(q) > 0$. This holds for all simple cycles, so

$$q\eta_{\mathrm{min}} + \beta(q) > 0.$$

Now since $q\eta_{\min}+\beta(q)$ converges to a finite limit, its graph has a horizontal asymptote. But its derivative $\eta_{\min}-\alpha(q)$ is negative and increasing, so that derivative must converge to 0. Therefore $\alpha(q)\to\eta_{\min}$ as $q\to\infty$. That is, $\alpha_{\min}=\eta_{\min}$.

The asymptotic properties as $q\to-\infty$ are proved in the same way. □

4. Proof of the Dimension Theorem

We now come to the proof of Theorem 1.6. Fix a real number q. There are corresponding values β, α, f as above. For each $u\in V$ we will prove that the set $K_u^{(\alpha)}$ is a fractal with dimension f:

$$\dim K_u^{(\alpha)} = \operatorname{Dim} K_u^{(\alpha)} = f.$$

The proof is divided into the "upper bound" $\operatorname{Dim} K_u^{(\alpha)} \le f$ and the "lower bound" $\dim K_u^{(\alpha)} \ge f$. Since the inequality $\dim F \le \operatorname{Dim} F$ is true for any set $F\subseteq\mathbb{R}^n$, these two bounds suffice to prove the result.

Before we proceed with the proof, we will consider the auxiliary measures.

The matrix $A(q,\beta)$ has spectral radius 1. So (as above) there exist positive right and left eigenvectors: ρ_v,λ_v with

$$\sum_{v\in V}\sum_{e\in E_{uv}} p(e)^q r(e)^\beta \rho_v = \rho_u \quad \text{for all } u\in V;$$
$$\sum_{u\in V}\sum_{e\in E_{uv}} \lambda_u p(e)^q r(e)^\beta = \lambda_v \quad \text{for all } v\in V.$$

By the Perron-Frobenius theorem, $\lambda_v,\rho_v>0$. So if we write $\rho_{\min}=\min_v\rho_v$ and $\rho_{\max}=\max_v\rho_v$, then $0<\rho_{\min}\le\rho_{\max}$. If we let

$$P(e)=\rho_u^{-1}p(e)^q r(e)^\beta\rho_v$$

for all $e \in E_{uv}$, then we have

$$\sum_{v \in V} \sum_{e \in E_{uv}} P(e) = 1$$

for all $u \in V$. These can be used as transition probabilities for some measure of Markov type, called $\hat{\mu}_u^{(q)}$. Equivalently, for $\gamma \in E_{uv}^{(k)}$, the cylinder $[\gamma]$ is given measure

$$\hat{\mu}_u^{(q)}\big([\gamma]\big) = \rho_u^{-1} p(\gamma)^q r(\gamma)^\beta \rho_v.$$

There is a corresponding measure $\mu_u^{(q)}$ on K_u defined by $\mu_u^{(q)}(F) = \hat{\mu}_u^{(q)}\big(h_u^{-1}[F]\big)$.

4.1. Lemma. *Let $u \in V$. The measure $\hat{\mu}_u^{(q)}$ is concentrated on the set $\widehat{K}_u^{(\alpha)}$; that is,*

$$\hat{\mu}_u^{(q)}\left(\widehat{K}_u^{(\alpha)}\right) = 1.$$

Proof. Consider the Markov chain (X_k) with transition probabilities $P(e)$, as above. In probabilistic language, we are required to prove that (conditioned on $X_0 = u$) the "trajectory" of the Markov chain (that is, the string made up of the successive edges traversed by the process) almost surely belongs to $\widehat{K}_u^{(\alpha)}$.

Now the numbers $\pi_u = \lambda_u \rho_u$ constitute a "stationary distribution" for the Markov chain. That is, for every $v \in V$,

$$\sum_{u \in V} \sum_{e \in E_{uv}} \pi_u P(e) = \pi_v.$$

We next apply the ergodic theorem for Markov chains: [7, Theorem 11′, p. 95] or [14, Theorems 4.1, 4.2]. The graph (V, E) is strongly connected, so the Markov chain is ergodic; thus π_u is the unique stationary distribution, and it occurs as the long-run frequencies of the process. This means that, in the long run, each state v is visited a fraction π_v of the time. Consequently, for each edge $e \in E_v$, the fraction $\pi_v P(e)$ of all edge traversals occur on the edge e. Precisely, if $g\colon E \to \mathbb{R}$ is any function, then for almost all $\sigma \in E_u^{(\omega)}$,

$$\frac{1}{k}\sum_{i=1}^{k} g(\sigma_i) \to \sum_{u\in V}\sum_{v\in V}\sum_{e\in E_{uv}} \pi_u P(e) g(e) \qquad \text{as } k \to \infty.$$

We have written σ_i for the ith letter of the string σ.

Now let us apply this with $g(e) = \log p(e)$. Then $\sum_{i=1}^{k} \log p(\sigma_i) = \log \prod_{i=1}^{k} p(\sigma_i) = \log p(\sigma\restriction k)$. We conclude: for almost all σ,

$$\begin{aligned}\frac{1}{k}\log p(\sigma\restriction k) &\to \sum_{u\in V}\sum_{v\in V}\sum_{e\in E_{uv}} \pi_u P(e)\log p(e)\\ &= \sum_{u\in V}\sum_{v\in V}\sum_{e\in E_{uv}} \lambda_u p(e)^q r(e)^\beta \rho_v \log p(e).\end{aligned}$$

Similarly,

$$\frac{1}{k}\log r(\sigma\restriction k) \to \sum_{u\in V}\sum_{v\in V}\sum_{e\in E_{uv}} \lambda_u p(e)^q r(e)^\beta \rho_v \log r(e).$$

Therefore, for almost all $\sigma \in E_u^{(\omega)}$, we have the ratio limit

$$\frac{\log p(\sigma\restriction k)}{\log r(\sigma\restriction k)} \to \frac{\sum_{u\in V}\sum_{v\in V}\sum_{e\in E_{uv}} \lambda_u p(e)^q r(e)^\beta \rho_v \log p(e)}{\sum_{u\in V}\sum_{v\in V}\sum_{e\in E_{uv}} \lambda_u p(e)^q r(e)^\beta \rho_v \log r(e)} = \alpha$$

by (2). That is, $\sigma \in \widehat{K}_u^{(\alpha)}$. $\square$

4.2. Proposition (upper bound). *Let $u \in V$. The packing dimension inequality* $\operatorname{Dim} K_u^{(\alpha)} \le f$ *holds.*

Proof. (i) First, in the case $q = 0$, we have $f = \beta = d$; this is the usual computation of the packing dimension of a digraph recursive fractal, for example [5, Theorem 6.5.10].

(ii) Next, consider the case $q > 0$. Fix $\delta > 0$. Define

$$\begin{aligned} \widehat{S}_u^{(k)} &= \left\{ \sigma \in E_u^{(\omega)} : \frac{\log p(\sigma\restriction k)}{\log r(\sigma\restriction k)} \leq \alpha + \frac{\delta}{q} \right\}, \\ \widehat{T}_u^{(N)} &= \bigcap_{k=N}^{\infty} \widehat{S}_u^{(k)}, \\ T_u^{(N)} &= h_u\left[\widehat{T}_u^{(N)}\right]. \end{aligned}$$

Then we have

$$\begin{aligned} \widehat{K}_u^{(\alpha)} &\subseteq \bigcup_{N=1}^{\infty} \widehat{T}_u^{(N)} \\ K_u^{(\alpha)} = h_u\left[\widehat{K}_u^{(\alpha)}\right] &\subseteq \bigcup_{N=1}^{\infty} T_u^{(N)}. \end{aligned}$$

We will show $\operatorname{Dim} T_u^{(N)} \leq f + \delta$. This is true for all N, so $\operatorname{Dim} K_u^{(\alpha)} \leq f + \delta$. And this is true for all $\delta > 0$, so $\operatorname{Dim} K_u^{(\alpha)} \leq f$, as claimed.

So fix N and consider the $(f+\delta)$-dimensional packing measure of $T_u^{(N)}$. Let $\varepsilon > 0$ be so small that $\varepsilon < r(\gamma)$ for all $\gamma \in E_u^{(N)}$. Let

$$B_{\varepsilon_i}(x_i), \qquad i = 1, 2, \cdots$$

be a countable disjoint collection of balls with $x_i \in T_u^{(N)}$ and $\varepsilon_i < \varepsilon$. There exist $\sigma_i \in \widehat{T}_u^{(N)}$ for $i = 1, 2, \cdots$ so that $h_u(\sigma_i) = x_i$. Let $k_i \in \mathbb{N}$ be such that $r(\sigma\restriction k_i) < \varepsilon_i \leq r(\sigma\restriction(k_i - 1))$. Then $k_i \geq N$ by the choice of ε. Now $\sigma_i \in \widehat{S}_u^{(k_i)}$, so:

$$\begin{aligned} \frac{\log p(\sigma_i\restriction k_i)}{\log r(\sigma_i\restriction k_i)} &\leq \alpha + \frac{\delta}{q} \\ \log p(\sigma_i\restriction k_i) &\geq \left(\alpha + \frac{\delta}{q}\right) \log r(\sigma_i\restriction k_i) \\ p(\sigma_i\restriction k_i) &\geq r(\sigma_i\restriction k_i)^{\alpha+\delta/q} \\ p(\sigma_i\restriction k_i)^q &\geq r(\sigma_i\restriction k_i)^{\alpha q+\delta} \\ p(\sigma_i\restriction k_i)^q r(\sigma_i\restriction k_i)^{\beta} &\geq r(\sigma_i\restriction k_i)^{f+\delta}. \end{aligned}$$

By the choice of the k_i, the cylinders $\left[\sigma_i{\restriction}k_i\right]$ are disjoint. Now

$$\operatorname{diam} B_{\varepsilon_i}(x_i) = 2\varepsilon_i \leq 2r(\sigma_i{\restriction}(k_i-1)) \leq (2/r_{\min})r(\sigma_i{\restriction}k_i).$$

Thus

$$\begin{aligned}
\left(\frac{r_{\min}}{2}\right)^{f+\delta} \sum_i (2\varepsilon_i)^{f+\delta} &\leq \sum_i r(\sigma_i{\restriction}k_i)^{f+\delta} \leq \sum_i p(\sigma_i{\restriction}k_i)^q r(\sigma_i{\restriction}k_i)^\beta \\
&\leq \left(\frac{\rho_{\max}}{\rho_{\min}}\right) \sum_i \hat{\mu}_u^{(q)}\big([\sigma_i{\restriction}k_i]\big) \\
&= \left(\frac{\rho_{\max}}{\rho_{\min}}\right) \hat{\mu}_u^{(q)}\left(\bigcup_i [\sigma_i{\restriction}k_i]\right) \leq \left(\frac{\rho_{\max}}{\rho_{\min}}\right).
\end{aligned}$$

This shows

$$\widetilde{\mathcal{P}}_\varepsilon^{f+\delta}\big(T_u^{(N)}\big) \leq \left(\frac{\rho_{\max}}{\rho_{\min}}\right)\left(\frac{2}{r_{\min}}\right)^{f+\delta}.$$

Let $\varepsilon \to 0$ to obtain $\widetilde{\mathcal{P}}^{f+\delta}\big(T_u^{(N)}\big) \leq (\rho_{\max}/\rho_{\min})(2/r_{\min})^{f+\delta} < \infty$ and $\mathcal{P}^{f+\delta}\big(T_u^{(N)}\big) < \infty$. So $\operatorname{Dim} T_u^{(N)} \leq f+\delta$.

Therefore, as noted above, we have $\operatorname{Dim} K_u^{(\alpha)} \leq f$.

(iii) Finally, consider the case $q < 0$. Fix $\delta > 0$. (So $\delta/q < 0$.) Define

$$\begin{aligned}
\widehat{S}_u^{(k)} &= \left\{ \sigma \in E_u^{(\omega)} : \frac{\log p(\sigma{\restriction}k)}{\log r(\sigma{\restriction}k)} \geq \alpha + \frac{\delta}{q} \right\}, \\
\widehat{T}_u^{(N)} &= \bigcap_{k=N}^{\infty} \widehat{S}_u^{(k)}, \\
T_u^{(N)} &= h_u\left[\widehat{T}_u^{(N)}\right].
\end{aligned}$$

As in the previous case, $K_u^{(\alpha)} \subseteq \bigcup_{N=1}^\infty T_u^{(N)}$. The rest of the proof proceeds as before. With the reversed inequality in the definition of $\widehat{S}_u^{(k)}$ and $q < 0$, we again obtain the estimate

$$p(\sigma_i{\restriction}k_i)^q r(\sigma_i{\restriction}k_i)^\beta \geq r(\sigma_i{\restriction}k_i)^{f+\delta}. \quad \square$$

4.3. Proposition (lower bound). *Let $u \in V$. The Hausdorff dimension inequality* $\dim K_u^{(\alpha)} \geq f$ *holds.*

Proof. (i) First, in the case $q = 0$, we have $f = \beta = d$. So the Hausdorff dimension inequality is essentially the computation of Mauldin and Williams. (Recall that we assume the open set condition.) See [5, Theorem 6.4.8] or [10, Theorem 3]. The measure $\hat{\mu}_u^{(0)}$ is the measure used in these references, and the set $\widehat{K}_u^{(\alpha)}$ supports this measure, so the computation actually shows $\dim K_u^{(\alpha)} \geq d$; indeed, $\mathcal{H}^d\big(K_u^{(\alpha)}\big)$ is positive and finite.

(ii) Next, consider the case $q > 0$. We must show that $\dim K_u^{(\alpha)} \geq f$. More generally, let F be any (Borel) set in $\mathbb{R}^n$ with $\mu_u^{(q)}(F) = a > 0$. We will show that $\dim F \geq f$. [That is, in terminology discussed below, we show that $\dim \mu_u^{(q)} \geq f$.] Let $\delta > 0$ be given. We will investigate the $(f - \delta)$-dimensional Hausdorff measure of the set F. Write $\widehat{F} = h_u^{-1}[F]$. Then let

$$\begin{aligned}
\widehat{S}_u^{(k)} &= \left\{ \sigma \in \widehat{F} : \frac{\log p(\sigma \restriction k)}{\log r(\sigma \restriction k)} \geq \alpha - \frac{\delta}{q} \right\}, \\
\widehat{T}_u^{(N)} &= \bigcap_{k=N}^{\infty} \widehat{S}_u^{(k)}, \\
T_u^{(N)} &= h_u \left[\widehat{T}_u^{(N)} \right].
\end{aligned}$$

Now $\widehat{F}$ is contained in the increasing union $\bigcup_{N=1}^{\infty} \widehat{T}_u^{(N)}$ and $\hat{\mu}_u^{(q)}\big(\widehat{F}\big) = a > 0$, so by the countable additivity of the measure,

$$\lim_{N \to \infty} \hat{\mu}_u^{(q)}\big(\widehat{T}_u^{(N)}\big) = a.$$

Choose N so large that $\hat{\mu}_u^{(q)}\big(\widehat{T}_u^{(N)}\big) > a/2$, then choose $\varepsilon > 0$ so small that $\varepsilon < r(\gamma)$ for all $\gamma \in E_u^{(N)}$.

Now suppose that $\{A_i\}$ is a countable cover of F by sets with diam $A_i < \varepsilon$. For each i, let

$$H_i = \left\{ \gamma \in E_u^{(*)} : r(\gamma) < \text{diam } A_i \leq r(\gamma^-), h_u\big[[\gamma]\big] \cap A_i \cap F \neq \varnothing \right\}.$$

There is a geometrical lemma ([10, Lemma V] or [5, pp. 172–3]) that shows there is a finite constant C such that each set H_i has at most C elements. If we write $H = \bigcup_{i=1}^{\infty} H_i$, then

$$\sum_{\gamma \in H} r(\gamma)^{f-\delta} \leq C \sum_i (\text{diam } A_i)^{f-\delta}.$$

Now $\{\, [\gamma] : \gamma \in H \,\}$ covers $\widehat{F} \supseteq \widehat{T}_u^{(N)}$. We need to construct a cover of $\widehat{T}_u^{(N)}$ more efficient than H. First, there is no need for the sets that do not meet $\widehat{T}_u^{(N)}$; let $H' = \left\{ \gamma \in H : [\gamma] \cap \widehat{T}_u^{(N)} \neq \varnothing \right\}$. Also, we need to cover the set only once: if two cylinders $[\gamma]$ are not disjoint then one of them is contained in the other, so we may discard the smaller one. So there is a set $H'' \subseteq H'$ such that $\{\, [\gamma] : \gamma \in H'' \,\}$ is a disjoint cover of $\widehat{T}_u^{(N)}$.

Now for each $\gamma \in H''$ there exists $\sigma \in \widehat{T}_u^{(N)}$ and $k \geq N$ so that $\sigma{\restriction}k = \gamma$. Then $\sigma \in \widehat{S}_u^{(k)}$, so

$$\begin{aligned}
\frac{\log p(\gamma)}{\log r(\gamma)} &\geq \alpha - \frac{\delta}{q} \\
\log p(\gamma) &\leq \left(\alpha - \frac{\delta}{q}\right) \log r(\gamma) \\
p(\gamma) &\leq r(\gamma)^{\alpha - \delta/q} \\
p(\gamma)^q &\leq r(\gamma)^{\alpha q - \delta} \\
p(\gamma)^q r(\gamma)^\beta &\leq r(\gamma)^{f-\delta}.
\end{aligned}$$

Thus, if $\gamma \in H''$ and $\gamma \in E_{uv}^{(k)}$, then

$$\hat{\mu}_u^{(q)}\big([\gamma]\big) = \rho_u^{-1} p(\gamma)^q r(\gamma)^\beta \rho_v \leq \frac{\rho_{\max}}{\rho_{\min}}\, r(\gamma)^{f-\delta}.$$

Now

$$\begin{aligned}
\frac{a}{2} &\leq \hat{\mu}_u^{(q)}\left(\widehat{T}_u^{(N)}\right) \leq \hat{\mu}_u^{(q)}\left(\bigcup_{\gamma\in H''}[\gamma]\right) = \sum_{\gamma\in H''} \hat{\mu}_u^{(q)}\big([\gamma]\big) \\
&\leq \left(\frac{\rho_{\max}}{\rho_{\min}}\right) \sum_{\gamma\in H''} r(\gamma)^{f-\delta} \leq \left(\frac{\rho_{\max}}{\rho_{\min}}\right) \sum_{\gamma\in H} r(\gamma)^{f-\delta} \\
&\leq \left(\frac{C\rho_{\max}}{\rho_{\min}}\right) \sum_i (\text{diam } A_i)^{f-\delta}.
\end{aligned}$$

Thus we have $\mathcal{H}^{f-\delta}\big(\widehat{F}\big) \geq \rho_{\min} a/2C\rho_{\max} > 0$. So dim $F \geq f - \delta$. This is true for all $\delta > 0$, so we have dim $F \geq f$, as required.

(iii) Finally, consider the case $q < 0$. Let $\delta > 0$ be given, so $\delta/q < 0$. Define now

$$\begin{aligned}
\widehat{S}_u^{(k)} &= \left\{ \sigma \in \widehat{F} : \frac{\log p(\sigma{\restriction}k)}{\log r(\sigma{\restriction}k)} \leq \alpha - \frac{\delta}{q} \right\}, \\
\widehat{T}_u^{(N)} &= \bigcap_{k=N}^{\infty} \widehat{S}_u^{(k)}, \\
T_u^{(N)} &= h_u\left[\widehat{T}_u^{(N)}\right].
\end{aligned}$$

Then proceed as in the previous case. The reversed inequality and $q < 0$ mean that the estimate

$$p(\gamma)^q r(\gamma)^\beta \leq r(\gamma)^{f-\delta}$$

remains correct. $\square$

5. Other Remarks

5.1. Hausdorff dimension of a measure. Some of the proofs given here actually deal with the Hausdorff dimension of a *measure* in order to estimate the Hausdorff dimension of a set. The proof of Proposition 4.3 shows not only that $\dim K_u^{(\alpha)} \geq f$, but that $\dim F \geq f$ for any set F with $\mu_u^{(q)}(F) > 0$. This may be interpreted as saying that the "Hausdorff dimension" of the measure $\mu_u^{(q)}$ is at least f.

There is more than one possible definition for the "Hausdorff dimension of a measure". We will show here that two of them coincide.

Let S be a metric space, and let μ be a finite measure defined on the Borel subsets of S. The **Hausdorff dimension** of the measure μ is the minimum of the dimensions of the sets that support μ:

$$\dim_1 \mu = \inf \{ \dim F : F \subseteq S,\ \mu(S \setminus F) = 0 \} .$$

There is another natural definition. Fix positive real numbers s and ε. Define

$$\mathcal{H}^s_\varepsilon(\mu) = \inf \sum_i (\operatorname{diam} A_i)^s,$$

where the infimum is over all countable families $\{A_i\}_{i=1}^\infty$ of sets with $\operatorname{diam} A_i < \varepsilon$ for all i that **almost cover** S in the sense that

$$\mu\left(S \setminus \bigcup_i A_i\right) = 0.$$

Define

$$\mathcal{H}^s(\mu) = \lim_{\varepsilon \downarrow 0} \mathcal{H}^s_\varepsilon(\mu) = \sup_{\varepsilon > 0} \mathcal{H}^s_\varepsilon(\mu).$$

There is a unique critical value s_0 such that

$$\mathcal{H}^s(\mu) = \begin{cases} \infty & \text{if } s < s_0 \\ \\ 0 & \text{if } s > s_0. \end{cases}$$

This critical value s_0 is called the **Hausdorff dimension** of the measure μ; we will write $s_0 = \dim_2 \mu$.

Proposition. *Let S be a metric space, and let μ be a finite measure defined on the Borel subsets of S. Then* $\dim_1 \mu = \dim_2 \mu$.

Proof. Let $F \subseteq S$ with $\mu(S \setminus F) = 0$. To estimate the s-dimensional Hausdorff measure of F, cover F:

$$F \subseteq \bigcup_i A_i, \qquad \text{diam } A_i < \varepsilon.$$

Then certainly $\{A_i\}$ almost covers S. So $\mathcal{H}^s_\varepsilon(\mu) \leq \sum \big(\text{diam } A_i\big)^s$. Therefore $\mathcal{H}^s_\varepsilon(\mu) \leq \mathcal{H}^s_\varepsilon(F)$. Let $\varepsilon \to 0$ to obtain $\mathcal{H}^s(\mu) \leq \mathcal{H}^s(F)$. If $s > \dim F$, then $\mathcal{H}^s(F) = 0$, so $\mathcal{H}^s(\mu) = 0$. Thus $\dim_2 \mu \leq \dim F$. This shows that $\dim_2 \mu \leq \dim_1 \mu$.

For the reverse inequality, let $s > \dim_2 \mu$, so that $\mathcal{H}^s(\mu) = 0$. Given $n \in \mathbb{N}$, find an almost cover $\{A_{ni}\}_{i=1}^\infty$ of S with diam $A_{ni} < 2^{-n}$ and $\sum_i (\text{diam } A_{ni})^s < 2^{-n}$. Then the set

$$F = \bigcap_n \bigcup_i A_{ni}$$

satisfies $\mu(S \setminus F) = 0$. But for each n, the family $\{A_{ni}\}_{i=1}^\infty$ covers F. So

$$\mathcal{H}^s_{2^{-n}}(F) \leq 2^{-n},$$

so that $\mathcal{H}^s(F) = 0$. Therefore $\dim F \leq s$, so $\dim_1 \mu \leq s$. This is true for every $s > \dim_2 \mu$, so we have $\dim_1 \mu \leq \dim_2 \mu$. $\square$

Use of measures such as $\mu_u^{(q)}$ in our proof of Theorem 1.6 is suggested in [4] by Cawley and Mauldin; they consider the case corresponding to a graph with one node. An independent calculation of the dimension of such a measure was done in [15] by Strichartz; he considers the case corresponding to a graph with one node, using a fixed family of similarity transformations, and the case $q = 1$, so that $\beta = 0$, and $f = \alpha$ is a ratio like (2).

5.2. Cross-cuts. Suppose C_u is a **cross-cut** of $E_u^{(*)}$. That is, when $E_u^{(*)}$ is given the structure of a tree, C_u is a maximal antichain in $E_u^{(*)}$. Or, the cylinders $[\gamma]$ for $\gamma \in C_u$ are disjoint and their union is dense in $E_u^{(\omega)}$. If C_u is infinite, we assume also that

$$1 = \sum_{\gamma \in C_u} \hat{\mu}_u^{(q)}\big([\gamma]\big).$$

So if we write $C_{uv} = C_u \cap E_{uv}^{(*)}$, then

$$1 = \sum_{v \in V} \sum_{\gamma \in C_{uv}} \rho_u^{-1} p(\gamma)^q r(\gamma)^\beta \rho_v \tag{4}$$

or

$$\rho_u = \sum_{v \in V} \sum_{\gamma \in C_{uv}} p(\gamma)^q r(\gamma)^\beta \rho_v.$$

Of course the sets $E_u^{(k)}$ used for (3) are cross-cuts. We may deduce formulas from the general (4) in the same way as from (3); for example

$$\begin{aligned} 1 &= \sum_u \sum_v \sum_{\gamma \in C_{uv}} \lambda_u p(\gamma)^q r(\gamma)^\beta \rho_v \\ \alpha &= \frac{\sum_u \sum_v \sum_{\gamma \in C_{uv}} \big(\lambda_u p(\gamma)^q r(\gamma)^\beta \rho_v\big) \log p(\gamma)}{\sum_u \sum_v \sum_{\gamma \in C_{uv}} \big(\lambda_u p(\gamma)^q r(\gamma)^\beta \rho_v\big) \log r(\gamma)}. \end{aligned}$$

Our graph (V, E) is strongly connected, so almost every $\sigma \in E^{(\omega)}$ returns eventually to its starting node. Thus

$$Z_u = \Big\{ \gamma \in E_{uu}^{(*)} : \gamma{\restriction}k \notin E_{uu}^{(*)} \text{ for } 1 \le k < |\gamma| \Big\}$$

is a cross-cut consisting only of cycles. Then the eigenvectors drop out of (4): for all u

$$1 = \sum_{\zeta \in Z_u} p(\zeta)^q r(\zeta)^\beta.$$

So we have

$$\alpha = \frac{\sum_{\zeta \in Z_u} p(\zeta)^q r(\zeta)^\beta \log p(\zeta)}{\sum_{\zeta \in Z_u} p(\zeta)^q r(\zeta)^\beta \log r(\zeta)}.$$

Another example of a cross-cut: fix $\varepsilon > 0$, and let

$$C_u = \left\{ \gamma \in E_u^{(*)} : r(\gamma) \leq \varepsilon < r(\gamma^-) \right\}.$$

This is useful for study of decomposition of K_u into sets of the same size.

5.3. Questions. There are several questions suggested by this work that we do not answer here. For example:

(a) When $q \to \infty$, is there some sort of "limiting construction", with dimension $f(\alpha_{\min})$? In [4], where the case of a graph with one node is considered, the limiting construction is composed of those edges e with equality $\eta(e) = \eta_{\min}$. Perhaps in our case one should delete everything except the simple cycles ζ with $\eta(\zeta) = \eta_{\min}$. In particular, if there is a unique such cycle ζ_0, then the limiting construction consists only of that cycle, so its attractors are single points when u lies on ζ_0. How are these points related to the components $K_u^{(\alpha_{\min})}$?

(b) What happens when the graph is not strongly connected (so the matrix is reducible)? Reading [10] would suggest that we should analyze all of the the strongly connected components of the graph, and then take the maximum of these dimensions.

(c) Is there a completeness theorem? Does

$$\bigcup_{\alpha_{\min} \leq \alpha \leq \alpha_{\max}} K_u^{(\alpha)}$$

have measure 1?

(d) Under what (disjointness) conditions, on the sets $J(\gamma)$ in the construction of the fractals K_u, can we replace the sets $K_u^{(\alpha)} = h_u\big[\widehat{K}_u^{(\alpha)}\big]$ used in this paper with the more naturally defined sets

$$\left\{ x \in K_u : \lim_{\varepsilon \downarrow 0} \frac{\log \mu\big(B_\varepsilon(x)\big)}{\log \operatorname{diam}\, B_\varepsilon(x)} = \alpha \right\} \quad ?$$

We have seen above that this can be done when we use a fixed family θ_e of similarities and have a strong disjointness property. (It was shown to be true in [4] in the case corresponding to a graph with a single node.) Is the open set condition enough?

(e) In Proposition 3.3, we see in case (B) that: β is a strictly convex function of q, α is a strictly decreasing function of q, f is a strictly concave function of α. Thus, we have strict inequalities:

$$\frac{d^2\beta}{dq^2} > 0, \qquad \frac{d\alpha}{dq} < 0, \qquad \frac{d^2 f}{d\alpha^2} < 0$$

except possibly at isolated points where equality occurs. Do these strict inequalities in fact hold everywhere?

(f) It would be interesting to investigate the relations between the computations in this paper and the "thermodynamic formalism" for dimension spectra. For example, D. A. Rand [13] investigates the dimension spectrum for "cookie-cutter" Cantor set fractals.

Consider these two classes of fractals: the cookie-cutter fractals and the digraph recursive fractals. Neither class contains the other: The digraph recursive fractals utilize only affine transformations, while the cookie-cutter fractals allow non-affine transformations. The cookie-cutter fractals are constructed in the line $\mathbb{R}$, while digraph recursive fractals are in Euclidean space of any dimension. The graph directing the construction of a cookie-cutter fractal is the graph with one node and two loops; while digraph recursive fractals may be defined by more general graphs.

For sets that are both digraph recursive fractals and cookie-cutter fractals, Rand's results agree with ours: Compare our Theorem 1.6 with Rand's Theorem 1. This agreement should extend much farther. Can the spectral radius $\Phi(q, \beta)$ be considered (the logarithm of) a "pressure" for the general digraph recursive fractal?

References

1. M. F. Barnsley, *Fractals Everywhere*, Academic Press, 1988.
2. P. Billingsley, *Probability and Measure*, Wiley-Interscience, 1979.
3. K. M. Brucks, *Hausdorff dimension and measure of basin boundaries*, Advances in Math. **78** (1989), 168–191.
4. R. Cawley and R. D. Mauldin, *Multifractal decompositions of Moran fractals*, Advances in Math. (to appear).
5. G. A. Edgar, *Measure, Topology, and Fractal Geometry*, Undergraduate Texts in Mathematics, Springer-Verlag New York, 1990.
6. K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, 1990.
7. F. R. Gantmacher, *The Theory of Matrices*, Volume 2, Chelsea, 1959, Chapter XIII: Matrices with non-negative elements.
8. S. Graf, R. D. Mauldin, and S. C. Williams, *The exact Hausdorff dimension in random recursive constructions*, Memoirs of the A. M. S. 381, American Mathematical Society, 1988.
9. T. Halsey, M. Jensen, L. Kadanoff, I. Procaccia, B. Shraiman, *Fractal measures and their singularities: The characterization of strange sets*, Phys. Rev. A **33** (1986), 1141–1151.
10. R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (1988), 811–829.
11. H. Minc, *Nonnegative Matrices*, Wiley, 1988.
12. P. A. P. Moran, *Additive functions of intervals and Hausdorff measure*, Proc. Cambridge Phil. Soc. **42** (1946), 15–23.
13. D. Rand, *The singularity spectrum $f(\alpha)$ for cookie-cutters*, Ergodic Theory and Dynamical Systems **9** (1989), 527–541.
14. E. Seneta, *Non-Negative Matrices*, Wiley, 1973.
15. R. Strichartz, *Self-similar measures and their Fourier transforms, I*, Indiana Univ. Math. J. **39** (1990), 797–817.
16. S. J. Taylor, *The measure theory of random fractals*, Math. Proc. Camb. Phil. Soc. **100** (1986), 383–406.

Department of Mathematics, The Ohio State University, Columbus, OH 43210-1174
E-mail address: edgar@mps.ohio-state.edu

Department of Mathematics, University of North Texas, Denton, TX 76203-3886
E-mail address: MAULDIN@UNTVAX