

Some complexity results in topology and analysis

by

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Abstract. If X is a compact metric space of dimension n , then $K(X)$, the n -dimensional kernel of X , is the union of all n -dimensional Cantor manifolds in X . Aleksandrov raised the problem of what the descriptive complexity of $K(X)$ could be. A straightforward analysis shows that if X is an n -dimensional complete separable metric space, then $K(X)$ is a Σ_2^1 or PCA set. We show (a) there is an n -dimensional continuum X in \mathbb{R}^{n+1} for which $K(X)$ is a complete Π_1^1 set. In particular, $K(X) \in \Pi_1^1 - \Sigma_1^1$; $K(X)$ is coanalytic but is not an analytic set and (b) there is an n -dimensional continuum X in \mathbb{R}^{n+2} for which $K(X)$ is a complete Σ_2^1 set. In particular, $K(X) \in \Sigma_2^1 - \Pi_2^1$; $K(X)$ is PCA, but not CPCA. It is also shown the Lebesgue measure as a function on the closed subsets of $[0, 1]$ is an explicit example of an upper semicontinuous function which is not countably continuous.

I. Complexity of $K(X)$. Let us recall that an n -dimensional Cantor manifold is a compact n -dimensional space which cannot be separated by a closed subset of dimension $\leq n - 2$. Since there are complete separable metric spaces of dimension n which contain no compact sets of dimension n , there are n -dimensional spaces which contain no Cantor manifolds of dimension n . However, every compact metric space of dimension n does contain an n -dimensional Cantor manifold. If X is a compact metric space of dimension n , then $K(X)$, the n -dimensional kernel of X , is the union of all n -dimensional Cantor manifolds in X . R. Pol [P], in answering a question of Aleksandrov [A], has given an example of a space X such that $X \setminus K(X)$ is still n -dimensional. As Pol pointed out to us, Aleksandrov also raised the question of what the descriptive complexity of $K(X)$ could be. For Pol's example, it turns out that $K(X)$ is a Borel set. We will determine the exact bounds on the complexity of $K(X)$ in this paper. We first note the following

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simple bound on the complexity of $K(X)$.

THEOREM 1. *Let X be an n -dimensional complete separable metric space. Then $K(X)$ is a Σ_2^1 or PCA set.*

PROOF. Let C be the set of all Cantor manifolds in the space, and $\mathcal{K}(X)$, of all compact subsets of X . Then $K(X) = \text{proj}_1\{(x, M) : x \in M \text{ and } M \in C\}$. Also, $\mathcal{K}(X) \setminus C = \text{proj}_1\{(M, L) \in \mathcal{K}(X) \times \mathcal{K}(X) : \dim L \leq n-2 \text{ and } L \text{ separates } M\}$. Clearly, this last set of pairs is an analytic set. Thus, C is a coanalytic or Π_1^1 set and $K(X)$ is a PCA set.

Our next theorem is to show that this simple estimate is fairly sharp. We assume all our spaces X are compact.

THEOREM 2. *Let $n \geq 2$.*

(a) *If $X \subset \mathbb{R}^n$ and $\dim X = n$, then $K(X)$ is an F_σ set.*

(b) *There is an n -dimensional compact set X in \mathbb{R}^{n+1} for which $K(X)$ is a complete Π_1^1 set. In particular, $K(X) \in \Pi_1^1 - \Sigma_1^1$; $K(X)$ is coanalytic but is not an analytic set.*

(c) *There is an n -dimensional compact set X in \mathbb{R}^{n+2} for which $K(X)$ is a complete Σ_2^1 set. In particular, $K(X) \in \Sigma_2^1 - \Pi_2^1$; $K(X)$ is PCA, but not CPCA.*

REMARKS. Parts (a) and (c) of this theorem are the best possible. We do not know whether part (b) can be improved to Σ_2^1 as well. Also, we could easily make the sets constructed for parts (b) and (c) continua.

PROOF. For the proof of (a), let us recall that every compact n -dimensional Cantor manifold in X lies in a maximal Cantor manifold in X , a *dimensional component* of X [E, p. 99]. Since every n -dimensional subset of \mathbb{R}^n contains an open set and no two dimensional components of X can have interior points in common, X has only countably many dimensional components. Thus, $K(X)$ is an F_σ set.

To prove parts (b) and (c), we assume for simplicity that $n = 2$. The general case differs little from the arguments below.

To prove (b), let $B = \{(0, 0, z) : z \in C\}$, where C is the middle third Cantor set. Let $P \subset B$ be Π_1^1 . We will produce a compactum $X \subset \mathbb{R}^3$ such that $K(X) \cap B = P$. This easily gives that $K(X)$ is not necessarily a Borel set and that $K(X)$ may be Π_1^1 complete. We will define X by defining its “sections”, $X_z = \{(x, y) : (x, y, z) \in X\}$, for each $z \in C$.

Suppose now that $z \in C$. We first construct a set D_z , “a framework”, which will always be a subset of X_z . Figure 1a illustrates this framework. We begin with the triangle which is the union of the line segments joining $(0, 0, z)$, $(1, 1, z)$ and $(-1, 1, z)$. We then adjoin the line segments L_k from $(1 - 1/2^k, 1, z)$ to $(0, 0, z)$, for $k = 0, 1, 2, \dots$. We next add a series of

2-dimensional “tubes” as follows. We first add tubes $T(1), T(2), T(3), \dots$ where $T(i)$ is the intersection of the above triangular region with the rectangular region $\{(x, y, z) : |x| \leq 1 \text{ and } |y - 1/2^i| < 1/4 \cdot 2^i\}$. For convenience, we also set $T(0)$ to be the top edge of the triangular region. The top edge is the only tube which is not 2-dimensional. We then add tubes $T(i, j)$ as follows: first $T(i, 0) = T(i)$ and if $j \neq 0$, then:

(a) $T(i, j)$ is the intersection of a rectangular region with the triangular region formed by the line segment L_0 and the top and right edges of the original triangular region.

(b) Each $T(i, j)$ lies strictly between $T(i)$ and $T(i+1)$.

(c) $T(i, j)$ lies above $T(i, j+1)$.

(d) $T(i, j) \cap T(i', j') = \emptyset$, for $(i, j) \neq (i', j')$.

(e) For each i , the only limit points of $\bigcup_j T(i, j)$ not in $\bigcup_j T(i, j)$ lie on $T(i+1)$.

In general, we construct tubes $T(i_1, \dots, i_n)$ for all $(i_1, \dots, i_n) \in \omega^{<\omega}$ such that $T(i_1, \dots, i_{n-1}, 0) = T(i_1, \dots, i_{n-1})$ and for $i_n \neq 0$ satisfying the analogous requirements:

(a) $T(i_1, \dots, i_n)$ is the intersection of a rectangular region with the triangular region formed by the line segment L_{n-2} and the top and right edges of the original triangular region.

(b) Each $T(i_1, \dots, i_n)$ lies strictly between $T(i_1, \dots, i_{n-1})$ and $T(i_1, \dots, i_{n-1} + 1)$.

(c) $T(i_1, \dots, i_n)$ lies above $T(i_1, \dots, i_n + 1)$.

(d) $T(i_1, \dots, i_n) \cap T(i'_1, \dots, i'_n) = \emptyset$, for $(i_1, \dots, i_n) \neq (i'_1, \dots, i'_n)$.

(e) For each (i_1, \dots, i_{n-1}) , the only limit points of $\bigcup_j T(i_1, \dots, i_{n-1}, j)$ not in $\bigcup_j T(i_1, \dots, i_{n-1}, j)$ lie on $T(i_1, \dots, i_{n-1} + 1)$.

The framework D_z is completed by adding all the sets $T(i_1, \dots, i_n)$ to those already adjoined.

Let $P \subset B$ be Π_1^1 and let T be a tree on $\{0, 1\} \times \omega$ which codes P . In other words, $T \subset \bigcup_{n=0}^{\infty} \{0, 1\}^n \times \omega^n$ with the property that $(r, s) \in T \rightarrow (r|k, s|k) \in T$, for all $k < \ln(r)$ and T is such that for all $z \in \{0, 1\}^\omega (\simeq C)$ we have $(0, 0, z) \in P \leftrightarrow T_z$ is well-founded, where $T_z = \{s \in \omega^{<\omega} : (z|n(s), s) \in T\}$.

For a fixed $z \in \{0, 1\}^\omega$, we obtain X_z by adjoining to D_z certain regions as follows:

Let $f : \omega \rightarrow \omega$ be an onto map such that $f^{-1}(n)$ is infinite for all n . For each (i_1, \dots, i_n) such that $(f(i_1), \dots, f(i_n)) \notin T_z$ we add to D_z the open region $R(i_1, \dots, i_n)$ which is bounded by the line segment L_{n-2} and the line segment from $(0, 0, z)$ to $(1, 1, z)$ (where L_{-1} is defined to be the left hand boundary of the triangular region) and by the tubes $T(i_1, \dots, i_n)$ and

$T(i_1, \dots, i_n + 1)$. Of course, the boundary of this region is already in X_z . This second stage is illustrated in Fig. 1b.

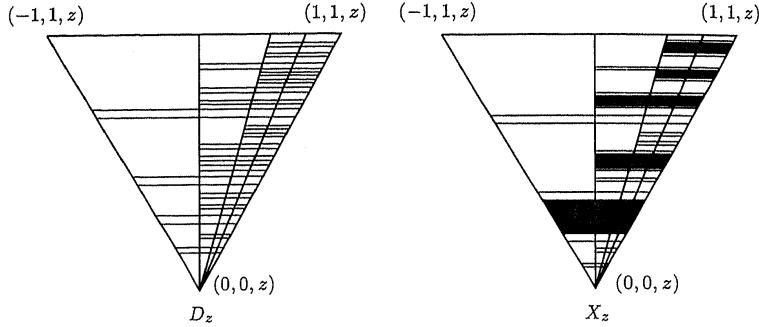


Fig. 1a. The framework

Fig. 1b. Adjoining regions

This completes the definition of X_z and hence of X . Since $X \subset \mathbb{R}^2 \times C$, $\dim X \leq 2$ and since the sections of X are 2-dimensional, $\dim X = 2$.

Clearly, each section X_z of X is closed. In order to see that X itself is closed, let $\{(x_n, y_n, z_n)\}_{n=1}^\infty$ be a sequence of points of X converging to a point $p = (x, y, z)$. If $(x, y, z) \in D_z$, then $p \in X$. So, suppose p lies in some open region $R^2(i_1, \dots, i_m)$. For n sufficiently large, $(x_n, y_n, z_n) \in R^{2n}(i_1, \dots, i_m)$. This implies that for n sufficiently large, $(f(i_1), \dots, f(i_m)) \notin T_{z_n}$. Since $z_n \rightarrow z$ in $\{0, 1\}^\omega$, we have $(f(i_1), \dots, f(i_m)) \notin T_z$. This implies $(x, y, z) \in X_z$ and X is closed.

We now show $(0, 0, z) \in P \leftrightarrow (0, 0, z) \in K(X)$. Note that if M is a Cantor manifold lying in X then M must be a subset of some section X_z , since any two sections of X can be separated from each other by a zero-dimensional set—namely the empty set.

First, assume $(0, 0, z) \notin P$. The tree T_z is ill-founded meaning there is a sequence $(j_1, j_2, \dots) \in \{0, 1\}^\omega$ such that $(z|_m, (j_1, \dots, j_m)) \in T$ for all m . Thus, for any (i_1, \dots, i_m) such that $f(i_1) = j_1, \dots, f(i_m) = j_m$, the portion of the open region $R(i_1, \dots, i_m)$ lying between L_{m-2} and L_{m-1} is disjoint from X_z . However, by removing countably many points from X_z (along each line L_k and the left and right edges of the triangular region), we may then clearly disconnect X_z by a countable set. Since the sets $f^{-1}(j_q)$ are infinite, we may, in fact, disconnect X_z by a countable set C so that $X_z \setminus C = U \cup V$ where $V \subset B_\varepsilon(0, 0, z)$, for ε arbitrarily small. Thus, there is no Cantor manifold M lying in X and containing $(0, 0, z)$. So, $(0, 0, z) \notin P$ implies $(0, 0, z) \notin K(X)$.

Assume now that $(0, 0, z) \in P$, or equivalently, the tree T_z is well-founded. Let $M_z = T \cup R$, where T is the union of all 2-dimensional

tubes in the framework and R is the union of all adjoined regions. Clearly, $(0, 0, z) \in M_z$. We claim that M_z is a Cantor 2-manifold, which completes the proof.

By way of contradiction, assume there is a 0-dimensional set $I \subset M_z$ such that $M_z \setminus I = U \cup V$, where U and V are nonempty, disjoint and relatively open sets in $M_z \setminus I$.

LEMMA 3. *There exist a pair of "consecutive" tubes, i.e., tubes of the form $T(i_1, \dots, i_{m-1}, i_m)$ and $T(i_1, \dots, i_{m-1}, i_m + 1)$, such that $T(i_1, \dots, i_{m-1}, i_m) \setminus I \subset U$ and $T(i_1, \dots, i_{m-1}, i_m + 1) \setminus I \subset V$ or vice versa.*

Proof. It is easy to see that each of U and V must contain a tube (mod I), and the tubes are partitioned according to whether they are a subset of U or of V (mod I). Suppose that the lemma fails and let $T(i_1, \dots, i_m)$ be a tube in U (more precisely, $T(i_1, \dots, i_m) \setminus I \subset U$). Since the lemma fails, we have $T(i_1, \dots, i_{m-1}, j) \setminus I \subset U$, for all $j \geq 1$. This then implies that $T(i_1, \dots, i_{m-1} + 1) \setminus I \subset U$. Likewise, we then get $T(i_1, \dots, i_{m-2} + 1) \setminus I \subset U, \dots, T(i_1 + 1) \setminus I \subset U$. Similarly, starting with a tube contained in V , we get a tube $T(j_1 + 1)$ such that $T(j_1 + 1) \setminus I \subset V$. This easily gives the lemma.

Assume now, without loss of generality, that $T(i_1, \dots, i_m) \setminus I \subset U$ and $T(i_1, \dots, i_m + 1) \setminus I \subset V$. Since $T(i_1, \dots, i_m + 1) \setminus I \subset V$, it follows that $T(i_1, \dots, i_m, i_{m+1}) \setminus I \subset V$, for some $i_{m+1} \geq 1$ (in fact for all sufficiently large i_{m+1}). From this we easily get consecutive tubes $T(i_1, \dots, i_m, i_{m+1}) \setminus I \subset U$ and $T(i_1, \dots, i_{m+1} + 1) \setminus I \subset V$ (note that we may have $i_{m+1} = 0$). Continuing, we define a sequence $(i_1, i_2, \dots) \in \omega^\omega$. Since T_z is well-founded, some $(f(i_1), \dots, f(i_k)) \notin T_z$ with $k \geq m$, and thus $R(i_1, \dots, i_k) \subset M_z$. This is impossible, since $T(i_1, \dots, i_k) \setminus I \subset U$, $T(i_1, \dots, i_k + 1) \setminus I \subset V$, and $T(i_1, \dots, i_k) \cup T(i_1, \dots, i_k + 1) \cup R(i_1, \dots, i_k) \setminus I$ is connected. This establishes part (b) of the theorem except for the fact that $K(X)$ is Π_1^1 . A straightforward computation shows that P is the complicated part of $K(X)$; the part of $K(X)$ not on the z -axis is an F_σ set.

Proof of (c). We work now in \mathbb{R}^4 . We will use the construction given in the proof of part (b). Let C be the Cantor middle third set (we again will think of C as $\{0, 1\}^\omega$). Let $P \subset C$ be Σ_2^1 . We define a compact subset X of \mathbb{R}^4 such that for $z \in C$, $(0, 0, z, z) \in K(X) \leftrightarrow z \in P$. This will certainly show $K(X)$ may be Σ_2^1 complete.

Since P is Σ_2^1 , we have $P(z) \leftrightarrow H(z, w)$ where $H \subset \{0, 1\}^\omega \times \{0, 1\}^\omega$ is Π_1^1 . The idea is to associate with each z in C a collection of triangular regions, one for each $w \in \{0, 1\}^\omega$, with common point $(0, 0, z, z)$. Each such triangular region $T_{z,w}$ is filled in exactly as in part (b), using now the pair (z, w) instead of z . Thus, we might envision such a set as in Fig. 2.

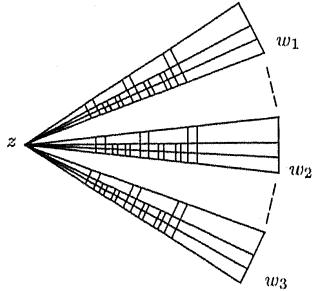


Fig. 2

We run into a problem trying to carry out this construction in \mathbb{R}^3 , however, as we cannot guarantee that the sets $T_{z,w}$ corresponding to different z 's will be disjoint. We found it necessary to add one more dimension to assure this last disjointness property holds.

To be specific, for each $z, w \in \{0, 1\}^\omega$, we define a "triangle" in \mathbb{R}^4 , $T_{z,w}$, to be the union of the closed line segments joining the points $(0, 0, z, z)$, $(1, 1, w, z)$ and $(-1, 1, w, z)$. For each such $(z, w) \in \{0, 1\}^\omega \times \{0, 1\}^\omega$, we define $E_{z,w}$ to be the "filled-in" version of $T_{z,w}$ exactly as X_z was the filling of D_z in case (b). We then set $X = \bigcup_{z,w} E_{z,w}$.

An argument analogous to that given in part (b) can be given to show that X is closed. Also, note that X is still 2-dimensional, since it is a subset of $\mathbb{R}^2 \times C \times C$.

Also, if $z \in P$, then for some $w \in \{0, 1\}^\omega$, $(z, w) \in H$. But then, arguing exactly as in case (b), $M_{z,w}$ is a Cantor 2-manifold containing $(0, 0, z, z)$. Thus, $(0, 0, z, z) \in K(X)$.

Finally, suppose $z \notin P$. So, for all $w \in \{0, 1\}^\omega$, $(z, w) \notin H$. We show that $(0, 0, z, z) \notin K(X)$. Suppose to the contrary that M were a Cantor 2-manifold with $(0, 0, z, z) \in M \subset X$. If for some $w \in \{0, 1\}^\omega$, we have $M \subset E_{z,w}$, then we would have a contradiction exactly as in (b). Thus, for some $(z', w') \neq (z, w)$, $M \cap E_{z,w} \neq \emptyset$ and $M \cap E_{z',w'} \neq \emptyset$. If $z \neq z'$, we have a contradiction as we may define a separation of M by $U = M \cap \{(a, b, c, d) : d > s\}$, $V = M \cap \{(a, b, c, d) : d < s\}$, where s is between z and z' and $s \notin C$. So, we may assume $M \subset \bigcup_w E_{z,w}$ and $M \cap E_{z,w} \not\subseteq \{(0, 0, z, z)\}$ and $M \cap E_{z,w'} \not\subseteq \{(0, 0, z, z)\}$ for some $w \neq w'$. However, we may then clearly separate $M \setminus \{(0, 0, z, z)\}$, a contradiction. This completes the proof of Theorem 2.

As we remarked, the construction for part (c) of the theorem seems to require working with \mathbb{R}^4 .

QUESTION 1. If $X \subset \mathbb{R}^3$, is $K(X)$ always Π_1^1 ?

QUESTION 2. Simply from the general theory of projective sets we know there is some Σ_2^1 norm (or Σ_2^1 operator) on X which yields $K(X)$. It would be very interesting to give some natural topological definition of this norm or operator. Perhaps such a norm or operator would be Π_1^1 for continua in \mathbb{R}^3 . This would give a positive solution to our Question 1.

QUESTION 3. Is there an n -dimensional compactum X such that $K(X)$ is Σ_2^1 complete and $X \setminus K(X)$ is also n -dimensional? We conjecture the answer to this question is yes. If so, can X be taken to be homogeneous?

QUESTION 4. Is there a continuum M such that the homeomorphism group X of M is n -dimensional and $X \setminus K(X)$ is Σ_2^1 complete? (We note that the existence of totally disconnected homeomorphism groups of dimension n is not completely settled [B, M].)

II. A result about Lebesgue measure. A real-valued function f defined on a complete separable metric space X is said to be *countably continuous* provided X can be partitioned into countably many disjoint Borel sets B_1, B_2, \dots such that for each i , the partial function $f|B_i$ is continuous on B_i . Clearly, every continuous function is countably continuous and there are countably continuous Borel measurable functions of arbitrarily high Borel class.

Note. In this definition of countably continuous we required the sets B_1, B_2, \dots to be Borel sets. However, this is no restriction *within the class of Borel measurable functions*, in view of the fact that a Borel measurable function f is countably continuous in our sense if and only if there is a partition of X into sets C_1, C_2, \dots such that $f|C_i$ is continuous on C_i . To prove this equivalence one uses the fact that if f is continuous on $E \subset X$, then there is a Borel set $\hat{E} \supset E$ and a continuous extension \hat{f} of f to \hat{E} .

Recall that f is *lower semicontinuous* if f is the pointwise supremum of a sequence of continuous functions. Adyan and Novikov showed there exists a lower semicontinuous function which is not countably continuous [A-N]. We reprove this result here. We then give an explicit example by showing that Lebesgue measure regarded as a function on the space of compact subsets of $[0, 1]$, which is upper semicontinuous, is not countably continuous.

We first provide a quick proof of the result of Adyan and Novikov which illustrates our proof for Lebesgue measure.

THEOREM 4. *Let X be an uncountable complete separable metric space. Then there is a lower semicontinuous $f : X \rightarrow \mathbb{R}$ which is not countably continuous.*

Proof. Fix a Σ_1^0 set $U \subset X \times \omega$ which is universal in the sense that for each $x \in X$, $U_x = \{n : (x, n) \in U\}$ is a complete $\Sigma_1^0(x)$ set. For each $x \in X$, let $f(x) \in \mathbb{R}$ be defined by $f(x) = \sum_{n=0}^{\infty} 3^{-n} a_n$, where $a_n = 1$ if $n \in U_x$ and $a_n = 0$ if $n \notin U_x$.

First, note that f is lower semicontinuous. To see this, write $U(n, x) \leftrightarrow \exists m R(x, m, n)$, where $R \subset X \times \omega \times \omega$ is recursive. Then, for each $k \in \omega$, define $f_k : X \rightarrow \mathbb{R}$ by $f_k(x) = \sum_{n=0}^k 3^{-n} a_{n,k}$, where $a_{n,k} = 1$ if $\exists n \leq k R(x, m, n)$ and 0 otherwise. Clearly, $f = \sup_k f_k$, and each f_k is continuous (in fact recursive).

Suppose that f were countably continuous, and let $\alpha_1, \alpha_2 \dots \in X$ code Borel sets B_1, B_2, \dots which have disjoint union X and such that $f|_{B_i}$ is continuous for each i . Since $f|_{B_i}$ is continuous, there is a "real" $\beta_i \in X$ providing a code for $f|_{B_i}$; i.e., the relation $\{(s, t) : s, t \in \omega, s, t \text{ code basic neighborhoods } N_s, N_t \text{ of } x \text{ such that } f(B_i \cap N_s) \subset N_t\}$ is $\Delta_1^0(\beta_i)$. Let $\gamma = \langle \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \rangle$. Then $\gamma \in B_i$ for some i , and thus, $f(\gamma) \in \Delta_1^0(\gamma, \beta_i) = \Delta_1^0(\gamma)$. This contradicts the fact that $f(\gamma)$ is $\Sigma_1^0(\gamma)$ complete.

We turn now to the corresponding question for Lebesgue measure. We now let Y denote the compact metric space $\{0, 1\}^\omega$ with Lebesgue or Haar measure, μ . This is for convenience only as the proof below will work with minor modification for $Y = [0, 1]$ and the usual Lebesgue measure.

THEOREM 5. *Let X denote the space of closed subsets of Y with the Hausdorff metric. Then $\mu : X \rightarrow \mathbb{R}$ is upper semicontinuous but is not countably continuous.*

Proof. It is well known that μ is upper semicontinuous. Let us assume that μ is countably continuous. Following the outline of the above argument, it is enough to show that for any $\gamma' \in Y = \{0, 1\}^\omega$, there is a closed $F \subset Y$ such that if $\gamma \in Y$ is the real coding F (i.e., γ codes the tree $T \subset \{0, 1\}^{<\omega}$ representing F), then $\gamma =_T \gamma'$ and $\mu(F) \notin \Delta_1^0(\gamma)$, where $=_T$ denotes Turing equivalence. Since the argument below works uniformly in any degree, we may now take $\gamma' =_T 0$. That is, we produce an $F \subset Y$ with recursive code but such that $\mu(F)$ is not recursive.

To do this, let $A \subset \omega$ be a Σ_1^0 complete set, so $A(n) \leftrightarrow \exists m R(m, n)$ where $R \subset \omega \times \omega$ is recursive. Define a tree $T \subset \{0, 1\}^{<\omega}$ by $T(s) \leftrightarrow s \in \{0, 1\}^{<\omega}$ and $s = t \wedge 0 \wedge u$ where t is a sequence of 1's of even length $2j$, $u \in \{0, 1\}^{<\omega}$, and $\forall k \leq \text{length}(u) [\exists m \leq k R(j, m) \rightarrow u(k) = 1]$. Clearly T can be coded by a recursive real γ . Let $F \subset Y$ be the closed set corresponding to T . We claim that $\mu(F)$ is not recursive. For suppose $\mu(F)$ were recursive. Then we could show A is recursive as follows: Given $n \in \omega$, consider all possible sums of the form $s = \sum_{p=0}^{n-1} \varepsilon_p (1/4)^{p+1}$ where $\varepsilon_p = 0$ or 1. For each of these, determine if $|\mu(F) - s| < 1/4^{n+2}$. If any of the s satisfy this inequality, then

$n \in A$, otherwise $n \notin A$. This follows since $\mu(F) = \sum_{p=0}^{\infty} \varepsilon_p (1/4)^{p+1}$ where $\varepsilon_p = 1$ if $p \in A$ and is 0 otherwise. Since $\mu(F)$ was assumed recursive, this is a recursive procedure, violating the assumption that A is Σ_1^0 complete. This completes the proof of the theorem.

QUESTION 5. We conjecture that almost all derivatives are not countably continuous. Consider the Banach space of all bounded derivatives on $[0, 1]$. Is it true that the set of functions in this space which are countably continuous is meager?

In fact, we conjecture the same is true for functions in Baire class 1:

QUESTION 6. Consider the Banach space of all bounded Baire class 1 functions on $[0, 1]$. Is it true that the set of functions in this space which are countably continuous is meager?

Added in proof. Results of related interest concerning countable continuity have been obtained by J. Cichoń, M. Morayne, J. Pawlikowski and S. Solecki, *Decomposing Baire functions*, J. Symbolic Logic 56 (1991), 1273–1283.

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