THE JOURNAL OF SYMBOLIC LOGIC
Volume 56, Number 2, June 1991

NONUNIFORMIZATION RESULTS FOR THE PROJECTIVE HIERARCHY

STEVE JACKSON[1] AND R. DANIEL MAULDIN[2]

Abstract. Let X and Y be uncountable Polish spaces. We show in ZF that there is a coanalytic subset P of $X \times Y$ with countable sections which cannot be expressed as the union of countably many partial coanalytic, or even PCA $= \Sigma^1_2$, graphs. If $X = Y = \omega^\omega$, P may be taken to be Π^1_1. Assuming stronger set theoretic axioms, we identify the least pointclass such that any such coanalytic P can be expressed as the union of countably many graphs in this pointclass. This last result is extended (under suitable hypotheses) to all levels of the projective hierarchy.

Introduction. Let X and Y be uncountable Polish spaces. It is a well-known result of Novikov and Kondo that any $\boldsymbol{\Pi}^1_1$ (i.e., coanalytic) subset P of $X \times Y$ can be uniformized by a $\boldsymbol{\Pi}^1_1$ relation $P' \subset P$. That is, $\forall x \in X\, [\exists y \in Y\, P(x, y) \leftrightarrow \exists$ a unique y $P'(x, y)]$. Although uniformization fails for $\boldsymbol{\Sigma}^1_1$ (analytic) sets, Lusin (see [Lu, p. 247]) and Novikov [No] did obtain the result that every analytic (Borel) set $P \subset X \times Y$ with countable sections can be written as a countable union of analytic (Borel) graphs, i.e., $P = \bigcup_n G_n$ where $G_n \in \boldsymbol{\Sigma}^1_1\, (\boldsymbol{\Delta}^1_1)$ is a graph (throughout this paper, "graph" will denote a partial graph, i.e., $\forall x\, \exists$ at most one y $G_n(x, y)$). Similarly, assuming $\boldsymbol{\Delta}^1_{2n}$ determinacy, each $P \in \boldsymbol{\Sigma}^1_{2n+1}$ can be written as a countable union of $\boldsymbol{\Sigma}^1_{2n+1}$ graphs. On the other hand, if P is $\boldsymbol{\Pi}^1_1$, then P can be expressed as the union of ω_1 Borel sets B_α, $\alpha < \omega_1$. If, in addition, each section of P is countable, then each set B_α has countable sections and can therefore be expressed as the union of countably many $\boldsymbol{\Delta}^1_1$ graphs. Thus, each $\boldsymbol{\Pi}^1_1$ set P with countable sections can be expressed as the union of ω_1 Borel graphs. A natural question then, raised by Mauldin, is the following.

Question. Can every $\boldsymbol{\Pi}^1_1$ set $P \subset X \times Y$ with countable sections be written as the countable union of $\boldsymbol{\Pi}^1_1$ graphs, $P = \bigcup_n G_n$?

We show, by working in ZF, that the answer is no in a strong way—our Theorem 1. We are grateful to W. H. Woodin for pointing out to us that our original result (which ruled out coverings by $\boldsymbol{\Sigma}^1_2$ graphs) could be extended to include graphs

Received May 7, 1990; revised August 15, 1990.
[1]Research supported by National Science Foundation Grant DMS-90-07808.
[2]Research supported by National Science Foundation Grant DMS-88-03361.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 04A15, 03E60; Secondary 28A05.
Key words and phrases. Coanalytic, countable uniformizations, game quantifier.

0022-4812/91/5602-0027/$01.70

in the σ-algebra generated by the Σ^1_2 sets. In fact, assuming Π^1_1-determinacy, we identify in Theorem 3 the least pointclass such that any coanalytic set P with countable sections can be expressed as the union of countably many graphs each in this pointclass. Moreover, in Theorem 3 we extend this result (under suitable hypotheses) to all levels of the projective hierarchy. On the other hand, in Theorem 2 we show that the answer to the question is yes, provided each section of P is finite. Finally, we make an observation concerning the pointclass where we pick up a covering assuming $V = L$, and raise some questions.

§1. **Results in ZF.** We first recall that any two uncountable Polish spaces are Δ^1_1 isomorphic. As the notions of Δ^1_1, Σ^1_1, Π^1_1, "countable", etc., are closed under Δ^1_1 isomorphisms, we assume without loss of generality for the remainder of this paper that $X = Y = \omega^\omega$. Following the common abuse of language among logicians, we refer to ω^ω as the "reals". We let $\mathscr{B}(\Sigma^1_2)$ denote the σ-algebra generated by the Σ^1_2 sets, that is, the smallest pointclass containing the Σ^1_2 sets and closed under countable unions, intersections, and complements.

THEOREM 1 (ZF). *There is a Π^1_1 set $G \subset \omega^\omega \times \omega^\omega$ with each section $G_x = \{y \in \omega^\omega\colon G(x,y)\}$ countably infinite, and such that G is not the union of countably many $\mathscr{B}(\Sigma^1_2)$ (partial) graphs.*

PROOF. Let us fix two recursive bijections. One is a coding $y \to (y_n)^\infty_{n=0}$ of ω^ω onto $(\omega^\omega)^\omega$, and the other $z \to (z^0, z^1)$ a coding of ω^ω onto $\omega^\omega \times \omega^\omega$. We denote the inverse of the second bijection by $(u, v) \to \langle u, v\rangle$. These recursive bijections are used rather than general Borel isomorphisms or homeomorphisms in order to simplify the proofs.

For ease of notation, we consider the case where each of the graphs lies in the pointclass $\Sigma^1_2 \wedge \Pi^1_2$. Let $U \subset \omega^\omega \times (\omega^\omega \times \omega^\omega)$ be Σ^1_2 and universal for Σ^1_2 subsets of $\omega^\omega \times \omega^\omega$. We proceed to define a Π^1_1 set $G \subset \omega^\omega \times \omega^\omega$ witnessing Theorem 1. Throughout, ZF_N will denote a sufficiently large fragment of ZF so that the relevant ZF theorems we require in our argument are provable in ZF_N (in particular, Π^1_1 and Σ^1_1 statements are absolute for transitive models of ZF_N). Let also φ be a Σ^1_2 formula which defines U. For ease of notation, we let $\psi(y_m, y, x)$ abbreviate $\varphi(y^0_m, y, x) \wedge \neg\varphi(y^1_m, y, x)$. Here y^0_m means $(y_m)^0$, and similarly for y^1_m. Also, we call an ordinal β *good* (with respect to y) if $L_\beta(y) \models \mathrm{ZF}_N + (V = L)$.

We first define G' by: $G'(y, w) \leftrightarrow \exists\beta < \omega_1[\beta \text{ is good } \& \forall\gamma < \beta\,(\neg(\gamma \text{ is a good ordinal which is a limit of good ordinals}) \vee \exists\gamma' > \gamma\ \exists m \in \omega\ \exists x \in L_\gamma(y)\ (L_\gamma(y) \models \neg\psi(y_m, y, x) \wedge L_{\gamma'}(y) \models \psi(y_m, y, x)) \wedge w \in L_\beta(y)]$. Clearly G' is Σ^1_2. Let $\Omega'(y, w)$ be the above Σ^1_2 formula defining G'. We claim also that G'_y is countable for each y. To see this, fix $y \in \omega^\omega$, and let β^* be the least ordinal $< \omega_1$ such that β^* is a good ordinal which is a limit of good ordinals, and $L_{\beta^*}(y)$ is a Σ_2 elementary substructure of $L(y)$ (the set of ordinals having these properties contains a c.u.b. set). Notice that, for any $x \in L_{\beta^*}(y)$ and $\beta > \beta^*$,

$$L_{\beta^*}(y) \models \psi(y_m, y, x) \leftrightarrow L_\beta(y) \models \psi(y_m, y, x) \leftrightarrow L(y) \models \psi(y_m, y, x).$$

We use here only that ψ is a Boolean combination of Σ^1_2 and Π^1_2 formulas. It follows that $G'_y \subset L_{\beta^*}(y)$, and hence is countable.

Now let $G \subset \omega^\omega \times \omega^\omega$ be such that, for all y,

$$G'(y, w) \leftrightarrow \exists z\, G(y, \langle w, z\rangle) \leftrightarrow \exists! z\, G(y, \langle w, z\rangle).$$

Let Ω be a Π^1_1 formula defining G. We may assume, in fact, that it is a theorem of ZF_N that

$$\forall y \forall w(\Omega'(y, w) \leftrightarrow \exists z\, \Omega(y, \langle w, z\rangle) \leftrightarrow \exists! z\, \Omega(y, \langle w, z\rangle)).$$

Note that all sections of G are countable.

Suppose, towards a contradiction, that G could be written as the countable union of graphs G_m each in the pointclass $\boldsymbol{\Sigma}^1_2 \wedge \boldsymbol{\Pi}^1_2$. Fix y_m for each m such that

$$\forall y \forall x(G_m(y, x) \leftrightarrow \psi(y_m, y, x)).$$

Let y be the real coding the y_m. Now let β^* be the least ordinal such that (β^* is good and a limit of good ordinals) $\wedge\ \forall m\ \forall x \in L_{\beta^*}(y)(L_{\beta^*}(y) \models \neg\psi(y_m, y, x) \rightarrow \forall \gamma > \beta^*\ L_\gamma(y) \models \neg\psi(y_m, y, x))$. We easily have that, for any formula θ which is a Boolean combination of Σ^1_2 and Π^1_2 formulas, and $x \in L_{\beta^*}(y)$,

$$L_{\beta^*}(y) \models \theta(x) \leftrightarrow L(y) \models \theta(x).$$

From the definition of G'_y it follows that $\omega^\omega \cap L_{\beta^*}(y) \subset G'_y$. Moreover, if $w \in L_{\beta^*}(y)$ then $L_{\beta^*}(y) \models \Omega'(y, w)$. To see this, suppose $w \in L_{\beta^*}(y)$. Then, for some good ordinal $\beta < \beta^*$, $w \in L_\beta(y)$. By minimality of β^* it follows that $\forall \gamma < \beta[\neg(\gamma$ is good and a limit of good ordinals) $\vee\ \exists m\ \exists x \in L_\gamma(y)\ \exists \gamma' > \gamma(L_\gamma(y) \models \neg\psi(y_m, y, x) \wedge L_{\gamma'}(y) \models \psi(y_m, y, x))]$. From the definition of β^* it then follows that we may replace "$\exists \gamma' > \gamma$" in the above by "$\exists \gamma' > \gamma, \gamma' < \beta^*$". Hence β witnesses that $\Omega'(y, w)$ is satisfied in $L_{\beta^*}(y)$.

However, using the last clause in the definition of β^*, which guarantees $L_{\beta^*}(y)$ is a Σ_2-elementary substructure of $L(y)$, it follows that β^* is good, $L_{\beta^*}(y) \models$ "$\{x: \exists m\, \psi(y_m, y, x)\}$ is countable". So, let $w, z \in L_{\beta^*}(y)$ be such that $L_{\beta^*}(y) \models \Omega(y, \langle w, z\rangle)$ and $L_{\beta^*}(y) \models \forall m\ \neg\psi(y_m, y, \langle w, z\rangle)$. From the definition of β^* it follows that $L(y) \models \forall m\ \neg\psi(y_m, y, \langle w, z\rangle)$, and, by absoluteness, $V \models \forall m\ \neg\psi(y_m, y, \langle w, z\rangle)$. Hence, $\forall m\ \neg G_m(y, \langle w, z\rangle)$, which is a contradiction as $V \models \Omega(y, \langle w, z\rangle)$ by absoluteness, and so $\langle w, z\rangle \in G_y$. This completes the proof of Theorem 1.

For the sake of completeness, we note the following.

THEOREM 2. *Let $P \subset \omega^\omega \times \omega^\omega$ be $\boldsymbol{\Pi}^1_1$ with each section finite. Then $P = \bigcup_n G_n$, where each G_n is a $\boldsymbol{\Pi}^1_1$ graph.*

PROOF. Let φ be a $\boldsymbol{\Pi}^1_1$ norm on P and let $<_{\mathrm{lex}}$ be the lexicographic linear order on ω^ω. For each x, define a well-order on P_x by

$$y <_x z \rightarrow \varphi(x, y) < \varphi(x, z) \vee \varphi(x, y) = \varphi(x, z) \text{ and } y <_{\mathrm{lex}} z.$$

(By convention, if $P(x, y)$ and $\neg P(y, z)$, then $\varphi(x, y) < \varphi(x, z)$.) For each $n \in \omega$, define the partial graph G_n by $G_n(x, y) \leftrightarrow y$ is the nth element in the order $<_x$. We show by induction that the G_n's are $\boldsymbol{\Pi}^1_1$ sets. We have

$$G_0(x, y) \leftrightarrow P(x, y) \wedge \forall z(\varphi(x, y) \leq \varphi(x, z)) \wedge \forall z(\varphi(x, y) = \varphi(x, z) \rightarrow y <_{\mathrm{lex}} z).$$

Thus, G_0 is $\boldsymbol{\Pi}^1_1$. For $n > 0$, we have

$$G_n(x, y) \leftrightarrow P(x, y) \wedge \exists z_0, \ldots, z_{n-1} \in \Delta^1_1(x, y)[G_0(x, z_0) \wedge \cdots \wedge G_{n-1}(x, z_{n-1}) \wedge \forall i(z_i <_x y) \wedge \forall w(w <_x y \rightarrow \exists i(w = z_i))].$$

G_n is $\boldsymbol{\Pi}^1_1$ since the last conjunct can be written in a $\boldsymbol{\Pi}^1_1$ manner (using Kleene's theorem on restricted quantification [Mo, Theorem 4D.3]).

§2. Further results. Assuming stronger set theoretic axioms, we now locate the smallest pointclass $\boldsymbol{\Gamma}$ such that every $\boldsymbol{\Pi}^1_1$ set with countable sections can be written as a union of countably many graphs, each in $\boldsymbol{\Gamma}$. In fact, we do this for all levels $\boldsymbol{\Pi}^1_{2n+1}$. In order to do this, let $\Game$ denote the game quantifier (see [Mo]), and $\Game^p$ its p-fold iterate. To be specific, if $A \subset (\omega^\omega)^n$, then $\Game A \subset (\omega^\omega)^{n-1}$ is defined by $\Game A(x_1, \ldots, x_{n-1}) \leftrightarrow$ I wins the integer game $G_{x_1,\ldots,x_{n-1}}$ where I plays $y_0, y_2, \ldots$ and II plays $y_1, y_3, \ldots$ and I wins $G_{x_1,\ldots,x_{n-1}} \leftrightarrow (x_1, \ldots, x_{n-1}, y) \in A$. Also, $\Game^{p+1}A = \Game\Game^p A$, $A \subset (\omega^\omega)^n$, $n \geq p+2$. An easy computation shows that, assuming the relevant games are determined, $\Game^p\omega \cdot k\text{-}\boldsymbol{\Pi}^1_1 \subset \Delta^1_{p+2}$.

THEOREM 3. *Assume $\bigcup_k \Game^{2n}\omega \cdot k\text{-}\boldsymbol{\Pi}^1_1$ determinacy. There is a Π^1_{2n+1} set $P \subset \omega^\omega \times \omega^\omega$ with all sections countable and such that for each $k \in \omega$, P cannot be written as a countable union of graphs $P = \bigcup_m G_m$ with each $G_m \in \Game^{2n+1}\omega \cdot k\text{-}\boldsymbol{\Pi}^1_1$. However, every $P \in \Sigma^1_{2n+2}$ with countable sections can be written as $P = \bigcup_m G_m$ with each G_m in $\bigcup_k \Game^{2n+1}\omega \cdot k\text{-}\boldsymbol{\Pi}^1_1$.*

Before beginning the proof, let us note that for $n = 0$, we need $\boldsymbol{\Pi}^1_1$-determinacy. It is conjectured (see [KMS]) that, for $n \geq 0$, $\bigcup_k \Game^{2n}\omega \cdot k\text{-}\boldsymbol{\Pi}^1_1$ determinacy is equivalent to $\boldsymbol{\Pi}^1_{2n+1}$ determinacy. For $n = 0$, this equivalence is a result of Martin and Harrington. The proof of Theorem 4 uses heavily the following theorem of Martin (see [Ma, Theorem 3.5]), as well as Martin's technique for handling the iterated game quantifier.

THEOREM (Martin). *Assume $\bigcup_k \Game^{2n-2}\omega \cdot k\text{-}\boldsymbol{\Pi}^1_1$ determinacy. Let C_{2n} = the largest countable Σ^1_{2n} set (which exists from Π^1_{2n-1}-determinacy, a weaker assumption). Then the reals in C_{2n} are precisely those $x \in \omega^\omega$ which are in $\bigcup_k \Game^{2n-1}\omega \cdot k\text{-}\boldsymbol{\Pi}^1_1$ (as subsets of ω).*

PROOF OF THEOREM 3. For the first part of the theorem, let $P \subset \omega^\omega \times \omega^\omega$ be Π^1_{2n+1} and such that, for all $x \in \omega^\omega$, $P_x = C_{2n+1}(x)$ = largest countable $\Pi^1_{2n+1}(x)$ set (which exists from $\boldsymbol{\Pi}^1_{2n+1}$ determinacy [Mo, 6E.9, p. 344]). Suppose, for some fixed $k \in \omega$, that $P = \bigcup_m G_m$ with each G_m a graph in $\Game^{2n+1}\omega \cdot k\text{-}\boldsymbol{\Pi}^1_1$. Fix $x \in \omega^\omega$ such that each G_m is in $\Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1(x)$. Then, since $P_x = \bigcup_m G_m(x)$, P_x would consist entirely of $\Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1(x)$ singletons, i.e., if $y \in P_x$, then $\{y\}$ is in $\Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1(x)$. Relativizing now to x everywhere, we would have that C_{2n+1} consists only of $\Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1$ singletons. We will show now that this is not the case. It is worthwhile to isolate this fact, which is of independent interest.

THEOREM 4. *Assume $\bigcup_k \Game^{2n}\omega \cdot k\text{-}\boldsymbol{\Pi}^1_1$ determinacy. Let C_{2n+1} be the largest countable Π^1_{2n+1} set. Then, for each fixed $k \in \omega$, C_{2n+1} does not consist entirely of $\Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1$ singletons.*

Note that from the theorem of Martin above, C_{2n+1} does consist entirely of $\bigcup_p \Game^{2n+1}\omega \cdot p\text{-}\Pi^1_1$ singletons (as a $\Game^{2n+1}\omega \cdot p\text{-}\Pi^1_1$ real is a $\Game^{2n+1}\omega \cdot (p+1)\text{-}\Pi^1_1$ singleton). Also, each $x \in C_{2n+2}$ is recursive in some Π^1_{2n+2} singleton ($\Pi^1_{2n+2} \subset \Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1$, for all k). See Theorem 11.4 of [KMS].

REMARK. For $n = 0$, we could prove Theorem 4 directly by considering the model L. For $n > 0$, however, the straightforward attempt to generalize this by considering $L(C_{2n+2})$ does not seem to work.

PROOF OF THEOREM 4. Towards a contradiction, fix $k \in \omega$ such that C_{2n+1} consists entirely of $\Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1$ singletons. For ease of notation, let $n = 1$. Write

C_4 as $C_4(x) \leftrightarrow \exists y\, P(x,y) \leftrightarrow \exists! y\, P(x,y)$ with $P \in \Pi^1_3$. Since $P \subset C_3$, P consists, by assumption, of $\Game^3\omega \cdot k\text{-}\Pi^1_1$ singletons. Now, there is a formula φ such that for all $\Game^3\omega \cdot k\text{-}\Pi^1_1$ singletons x,

$$\exists n \in \omega\ \forall s(s = x \leftrightarrow \Game y \Game z L[\langle s, y, z\rangle] \models \varphi(n, \langle s, y, z\rangle, \omega_1, \dots, \omega_k)).$$

Here, $\omega_1, \dots, \omega_k$ are the $\omega_1, \dots, \omega_k$ of V. We refer the reader to [Ma]. Also, φ is absolute for any transitive model containing $\langle s, y, z\rangle$, $\omega_1, \dots, \omega_k$ (φ asserts that player I has a winning strategy for a certain closed ordinal game mentioning $\omega_1, \dots, \omega_k$ (see [Ma, Lemma 2.1])). Let $\mathscr{D}$ denote the set of Turing degrees, and μ the Martin measure on $\mathscr{D}$ (more precisely, filter on $\mathscr{D}$). So, if $A \subset \mathscr{D}$, then $\mu(A) = 1 \leftrightarrow \exists d \in \mathscr{D}\ \forall d' \geq_{\mathrm{T}} d\ d' \in A$. Here $\leq_{\mathrm{T}}$ denotes the Turing reducibility partial order. We have enough determinacy (in fact, $\boldsymbol{\Pi}^1_{2n+1}$-determinacy will suffice) so that the sets of degrees we will consider will be measured by μ. Let $\forall^*$ denote "for almost all with respect to the filter μ".

Now, for degrees $d_0 \leq_{\mathrm{T}} d_1 \in \mathscr{D}$ consider the model $M_{d_0 d_1} = \mathrm{HOD}^{L[d_1]}_{d_0}$ = the sets hereditarily ordinal definable in $L[d_1]$ using the degree d_0 as a parameter. Thus, $\forall d_0 d_1\ M_{d_0 d_1} \models \mathrm{ZFC}$. Let F be defined on $\mathscr{D} \times \mathscr{D}$ by $F(d_0, d_1) = M_{d_0 d_1}$. Also, let $\mathscr{M} = [F]_{\mu \times \mu}$ = the set represented by F in the ultrapower by the measure $\mu \times \mu$ on $\mathscr{D} \times \mathscr{D}$. Assuming full determinacy, it follows that $\mathscr{M}$ is well-defined and a model of ZFC. $\boldsymbol{\Pi}^1_3$-determinacy, however, is enough to get the following: for every formula $\varphi(x, \omega_1, \dots, \omega_k)$ with parameters a real x and cardinals $\omega_1, \dots, \omega_k$ of V, we have either

$$\forall^* d_0 \forall^* d_1\ L[d_1] \models \varphi(x, \omega_1, \dots, \omega_k) \quad \text{or} \quad \forall^* d_0 \forall^* d_1\ L[d_1] \models \neg\varphi(x, \omega_1, \dots, \omega_k).$$

CLAIM. $\mathscr{M} \cap \omega^\omega = C_4$. (More precisely, $x \in C_4 \leftrightarrow \forall^* d_0 \forall^* d_1\ x \in M_{d_0 d_1}$.)

First, we have that $\mathscr{M} \cap \omega^\omega \subset C_4$. To see this, note that $x \in \mathscr{M} \to \forall^* d_0 \forall^* d_1$ $\exists \alpha_0, \alpha_1, \dots, \alpha_k < \omega_1$ (x is definable in $L_{\alpha_0}[d_1]$ from parameters $d_0, \alpha_1, \dots, \alpha_k$). By the argument of Martin (see [Ma, Lemma 2.3]), this shows that $\mathscr{M} \cap \omega^\omega$ is contained in a countable Σ^1_4 set. (The above definition shows that the set is Σ^1_4, and it is countable as it can be well-ordered in a Σ^1_4 way, assuming $\boldsymbol{\Pi}^1_3$ determinacy.) Thus, $\mathscr{M} \cap \omega^\omega \subset C_4$.

For the other direction, suppose $x \in C_4$. Then, from the theorem of Martin cited above, $\exists n \in \omega$ such that

$$\forall p, q \in \omega\ (x(p) = q \leftrightarrow \Game y \Game z\ L[\langle y, z\rangle] \models \varphi(n, \langle y, z\rangle, \omega_1, \dots, \omega_k)).$$

However, we then easily have

$$\forall^* d_0 \forall^* d_1\ \forall p, q \in \omega\ (x(p) = q \leftrightarrow L[d_1] \models \exists \sigma_0 \leq_{\mathrm{T}} d_0\ \forall y_1 \leq_{\mathrm{T}} d_0\ \exists \sigma_1 \forall z_1$$
$$\varphi(n, \langle p, q, \sigma_0 \bigstar y_1, \sigma_1 \bigstar z_1\rangle, \omega_1, \dots, \omega_k)),$$

where $\sigma_0 \bigstar y_1$ is the result of the following strategy σ_0 for I against the play of y_1 by II, and similarly for $\sigma_1 \bigstar z_1$. This shows that $\forall^* d_0 \forall^* d_1\ x \in M_{d_0 d_1}$, hence $x \in \mathscr{M}$. This completes the proof of the claim.

We now define a particular real $\bar{x} \in \mathscr{M}$. For each $d_0, d_1 \in \mathscr{D}$, let $A_{d_0 d_1} = \{\langle n, x\rangle$: $n \in \omega$, $x \in \mathrm{HOD}^{L[d_1]}_{d_0} \cap \omega^\omega$, and $\exists w \in \mathrm{HOD}^{L[d_1]}_{d_0} \cap \omega^\omega$ such that if $x' = \langle x, w\rangle$, then x' is the unique real in $\mathrm{HOD}^{L[d_1]}_{d_0}$ such that $L[d_1] \models \exists \sigma_0 \leq_{\mathrm{T}} d_0\ \forall y_1 \leq_{\mathrm{T}} d_0\ \exists \sigma_1\ \forall z_1$ $\varphi(n, \langle x', \sigma_0 \bigstar y_1, \sigma_1 \bigstar z_1\rangle, \omega_1, \dots, \omega_k)\}$. Clearly, $A_{d_0 d_1} \subset M_{d_0 d_1}$. Let $\bar{x}_{d_0 d_1}$ = least real in $M_{d_0 d_1}$ and not in $A_{d_0 d_1}$, which exists since $A_{d_0 d_1}$ is countable in $M_{d_0 d_1}$. Let

$\bar{x} = [\bar{x}_{d_0 d_1}]_{\mu \times \mu}$. So $\bar{x} \in C_4$. Hence, for some $\bar{w} \in \omega^\omega$, $\langle \bar{x}, \bar{w} \rangle \in C_3$, and, so, for some fixed $\bar{n} \in \omega$, $\bar{x}' = \langle \bar{x}, \bar{w} \rangle$ is the unique real such that

$$\Game y \Game z\, L[\langle \bar{x}', y, z \rangle] \models \varphi(\bar{n}, \langle \bar{x}', y, z \rangle, \omega_1, \ldots, \omega_k).$$

We claim now that $\forall^* d_0 \forall^* d_1\ \bar{x} \in A_{d_0 d_1}$, a contradiction.

First, let

$$\psi(\bar{x}', d_0, d_1) \leftrightarrow L[d_1] \models \exists \sigma_0 \leq_T d_0\ \forall y_1 \leq_T d_0\ \exists \sigma_1\ \forall z_1\ \varphi(\bar{n}, \langle \bar{x}', \sigma_0 \bigstar y_1, \sigma_1 \bigstar z_1 \rangle, \omega_1, \ldots, \omega_k).$$

We easily have that

$$[\forall^* d_0 \forall^* d_1\ L[d_1] \models \exists \sigma_0 \leq_T d_0\ \forall y_1 \leq_T d_0\ \exists \sigma_1\ \forall z_1\ \varphi(\bar{n}, \langle \bar{x}', \sigma_0 \bigstar y_1, \sigma_1 \bigstar z_1 \rangle, \omega_1, \ldots, \omega_k)] \leftrightarrow \forall^* d_0 \forall^* d_1\ \psi(\bar{x}', d_0, d_1).$$

We must show that, for almost all d_0 and d_1, $\bar{x}'$ is the unique real s in $\mathrm{HOD}_{d_0}^{L[d_1]}$ satisfying $\psi(s, d_0, d_1)$. If this fails, then for almost all d_0, $d_1 \in \mathscr{D}$ let $u'_{d_0 d_1} \neq \bar{x}'_{d_0 d_1}$ be in $\mathrm{HOD}_{d_0}^{L[d_1]}$ and such that $\psi(u'_{d_0 d_1}, d_0, d_1)$ holds. Let $u' = [u'_{d_0 d_1}]_{\mu \times \mu}$. Hence, $u' \neq x'$. So,

$$\Game' y \Game' z\, L[\langle u', y, z \rangle] \models \neg \varphi(\bar{n}, \langle u', y, z \rangle, \omega_1, \ldots, \omega_k).$$

(Here, $\Game'$ denotes the game quantifier for player II.) Hence,

$$\forall^* d_0 \forall^* d_1\ L[d_1] \models \exists \tau_0 \leq_T d_0\ \forall y_0 \leq_T d_0\ \exists \tau_1\ \forall z_0\ \neg \varphi(\bar{n}, \langle u', \tau_0 \bigstar y_0, \tau_1 \bigstar z_0 \rangle, \omega_1, \ldots, \omega_k),$$

where $\tau_0 \bigstar y_0$ is the result of following strategy τ_0 for II against y_0, and similarly for $\tau_1 \bigstar z_0$. Since we also have that

$$\forall^* d_0 \forall^* d_1\ L[d_1] \models \exists \sigma_0 \leq_T d_0\ \forall y_1 \leq_T d_0\ \exists \sigma_1\ \forall z_1\ \varphi(\bar{n}, \langle u', \sigma_0 \bigstar y_1, \sigma_1 \bigstar z_1 \rangle, \omega_1, \ldots, \omega_k),$$

this is a contradiction.

This completes the proof of Theorem 4.

We turn to the proof of the second part of Theorem 3. Suppose $P \in \Sigma^1_{2n+2}$ with each section countable. By relativization, we assume without loss of generality that P is Σ^1_{2n+2}. Thus, for each $x \in \omega^\omega$,

$$P_x \subset C_{2n+2}(x) = \bigcup_k \Game^{2n+1} \omega \cdot k\text{-}\Pi^1_1.$$

For each fixed $p, k \in \omega$, we then define

$$\begin{aligned} G_{p,k}(x, y) &\leftrightarrow P(x, y) \wedge \text{“}y \text{ is the } p\text{th } \Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1(x) \text{ real”} \\ &\leftrightarrow P(x, y) \wedge \forall i, j \in \omega [y(i) = j \leftrightarrow M_k(p, i, j, x)], \end{aligned}$$

where $M_k \subset \omega \times \omega \times \omega \times \omega^\omega$ is in $\Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1$ and is universal for the $\Game^{2n+1}\omega \cdot k\text{-}\Pi^1_1$ subsets of $\omega \times \omega \times \omega^\omega$. Clearly each $G_{p,k}$ is a graph and is in $\Game^{2n+1}\omega \cdot (k+1)\text{-}\Pi^1_1$. Also,

$$\bigcup_{p,k} (G_{p,k})_x = P_x \cap \left(\bigcup_m \Game^{2n+1}\omega \cdot m\text{-}\Pi^1_1(x) \right) = P_x,$$

and we are done.

REMARK. For $n = 0$, Theorem 3 (which has a stronger hypothesis) gives a stronger result than Theorem 1, as $\mathscr{B}(\Sigma_2^1) \subsetneq \mathfrak{D}\omega \cdot k\text{-}\boldsymbol{\Pi}_1^1$.

We finish with an observation and some questions. First, a straightforward observation.

THEOREM 5. *If $V = L$, then any $\boldsymbol{\Pi}_1^1$ set with countable sections can be written as a countable union of Σ_3^1 graphs.*

PROOF. Let $P \subset \omega^\omega \times \omega^\omega \in \boldsymbol{\Pi}_1^1$ with each section countable, and define

$$P'(s, w) \leftrightarrow [\forall n\, P(s, w_n) \wedge \forall y (P(s, y) \rightarrow \exists m\, y = w_m)].$$

Then $P' \in \boldsymbol{\Pi}_2^1$ and hence can be uniformized by a Σ_3^1 set $P'' \subset \omega^\omega \times \omega^\omega$. Define $G_m \subset \omega^\omega \times \omega^\omega$ by

$$G_m(s, y) \leftrightarrow \exists w [P''(s, w) \wedge y = w_m].$$

Then each $G_m \in \Sigma_3^1$, and $P = \bigcup_m G_m$.

In view of the argument just given and a theorem of Levy [Le] which asserts the consistency of the existence of a $\boldsymbol{\Pi}_2^1$ set with no projective uniformization, it is natural ask if it is consistent that there be a $\boldsymbol{\Pi}_1^1$ set $P \subset \omega^\omega \times \omega^\omega$ with each section countable which cannot be written as a countable union of projective graphs.

In fact, Woodin [W] has informed us of the following theorem:

THEOREM (Woodin). *The following two statements are equiconsistent:*

(1) ZFC + $\exists$ *an inaccessible cardinal.*

(2) ZFC + $\exists$ Π_1^1 $P \subset \omega^\omega \times \omega^\omega$ *with countable sections such that P cannot be written as a countable union of projective (or even definable) graphs.*

One might also ask how far we can extend Theorem 1 in ZF.

Question. How much further can Theorem 1 be extended in ZF?

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH TEXAS
DENTON, TEXAS 76203