Open Problems in Topology
J. van Mill and G.M. Reed (Editors)

CHAPTER 32

Problems in Topology arising from Analysis

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Contents

1. Topologically Equivalent Measures on the Cantor Space

Two measures μ and ν defined on the family of all Borel subsets of a topological space X are said to be *homeomorphic* or *topologically equivalent* provided there exists a homeomorphism $h: X \to X$ such that $\mu = \nu h^{-1}$. This means that for each Borel set E, $\mu(E) = \nu(h^{-1}(E))$. The measure μ is said to be a *continuous image* of ν if h is only required to be continuous. OXTOBY and ULAM [1941] characterized those probability measures, μ, on the finite dimensional cubes $[0,1]^n$, which are homeomorphic to Lebesgue measure—μ must give each point measure zero, each nonempty open set positive measure and the boundary of the cube must have μ measure zero. Later OXTOBY and PRASAD [1978] extended this theorem to the Hilbert cube. The situation regarding the Cantor set remains unsolved—even for product measures.

Let $X = \{0,1\}^{\mathbf{N}}$ and for each r, $0 \le r \le 1$, let $\mu(r)$ be the infinite product probability measure on X determined by r: $\mu(r) = \prod_{n=1}^{\infty} \mu_n$, where $\mu_n(0) = 1-r$ and $\mu_n(1) = r$, for all n. For each r, let $E(r) = \{s: \mu(r) \text{ is homeomorphic to } \mu(s)\}$.

First, let us note when one of these product measures is a continuous image of another.

1.1. THEOREM. *The measure $\mu(r)$ is a continuous image of $\mu(s)$ if and only if there is positive integer n and integers a_i, $0 \le i \le n$, such that*

$$0 \le a_i \le \binom{n}{i}, \tag{1}$$

and

$$r = \sum_{i=0}^{n} a_i s^i (1-s)^{n-i}. \tag{2}$$

PROOF. Suppose $f: \{0,1\}^{\mathbf{N}} \to \{0,1\}^{\mathbf{N}}$ is continuous and for each Borel set E,

$$\mu(r)(E) = \mu(s)(f^{-1}(E)). \tag{3}$$

Let $E = \langle 1 \rangle$. Then $f^{-1}(E)$ is a clopen subset of $\{0,1\}^{\mathbf{N}}$. Therefore, there is a positive integer n and a subset $\mathcal{E}$ of $\{0,1\}^n$ such that

$$f^{-1}(\langle 1 \rangle) = \bigcup \{\langle e \rangle : e \in \mathcal{E}\}. \tag{4}$$

For each i, $0 \le i \le n$, let a_i be the number of sequences $e = (q_1, \ldots, q_n)$ of $\mathcal{E}$ such that $\#(e) = \sum_{p=1}^{n} q_p = i$. Thus, $0 \le a_i \le \binom{n}{i}$ and if $\#(e) = i$, then $\mu(s) = (\langle e \rangle) = s^i (i-s)^{n-i}$. Thus

$$r = \mu(r)(\langle 1 \rangle) = \sum_{i=0}^{n} a_i s^i (1-s)^{n-i}. \tag{5}$$

Conversely, let us assume that (1) and (2) hold. Let $\mathcal{E}$ be a subset of $\{0,1\}^n$ such that $\mathcal{E}$ has exactly a_i members e with $\#(e) = i$. Notice that if $\sigma \in \{0,1\}^{\mathbb{N}}$, then σ has a unique representation as

$$\sigma = t_1 * t_2 * t_2 * t_3 \cdots, \tag{6}$$

where for each i, t_i is in $\{0,1\}^n$. Define $f: \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ by $f(\sigma)(i) = 1$, if and only if $t_i \in \mathcal{E}$. Clearly, f is a continuous map of $\{0,1\}^{\mathbb{N}}$ into $\{0,1\}^{\mathbb{N}}$ and for all k,

$$\begin{aligned} 1 - r &= \mu(r)(\{\sigma: \sigma(k) = 0\}) = \sum_{i=0}^{n} \left[\binom{n}{i} - a_i\right] s^i(1-s)^{n-i} \qquad (7) \\ &= \mu(s)(f^{-1}(\{\sigma: \sigma(k) = 0\}). \end{aligned}$$

From this it follows that $\mu(r)$ is the image of $\mu(s)$ under f. □

1.2. EXAMPLE. $\mu(1/2)$ is the image of $\mu(1/\sqrt{2})$.

Let us note that there are many maps which take $\mu(s)$ to $\mu(r)$. For if f is such map, then since $\mu(s) = \mu(s) \circ h$, where h is a homeomorphism induced by a permutation, $\mu(r) = \mu(s) \circ f \circ h$. Theorem 1.1 characterizes those shift invariant product measures $\mu(s)$ and $\mu(r)$ such that each is a continuous image of the other.

1.3. THEOREM. *Each of $\mu(r)$ and $\mu(s)$ is the continuous image of the other if and only if there are positive integers n and m, integers a_i, $0 \leq i \leq n$, integers b_j, $0 \leq j \leq m$ such that*

$$0 \leq a_i \leq \binom{n}{i}, 0 \leq b_j \leq \binom{m}{j}, \tag{8}$$

$$r = \sum_{i=0}^{n} a_i s^i (1-s)^{n-i}, \tag{9}$$

and

$$s = \sum_{j=0}^{m} b_j r^j (1-r)^{n-j}. \tag{10}$$

□

? 1065. **Problem 1.4.** *Is it true that $\mu(r)$ and $\mu(s)$ are homeomorphic if and only if equations* (8), (9) *and* (10) *hold?*

Let us note that for integers a_i and b_j satisfying the given constraints, there is always a solution of equations (9) and (10). This may be seen by applying

Brouwer's fixed point theorem to the map given by:

$$(r,s) \to \left(\sum_{i=0}^{n} a_i s^i (1-s)^{n-i}, \sum_{j=0}^{m} r^j (1-r)^{n-j}\right).$$

A number of references can be drawn from Theorem 1.1. For each r, let $F(r) = \{s$: each of $\mu(r)$ and $\mu(s)$ is a continuous image of the other$\}$. NAVARRO-BERMUDEZ [1979, 1984] showed:

1.5. THEOREM. *For each r, $F(r)$ is countable and $F(r) \supseteq E(r)$. If r is rational or transcendental, then $E(r) = F(r)$ and consists of exactly its obvious members: $E(r) = \{r, 1-r\}$.* □

Huang extended this theorem by proving the same result in case r is an algebraic integer of degree two. The situation is more complicated for the other algebraic numbers. For example, HUANG [1986] proved:

1.6. THEOREM. *For each $n > 2$, there is an algebraic integer $r \in (0,1)$ of degree n and a number $s \in (0,1)$ such that r and s satisfy relations of the form (9) and (10) and $s \neq r$ and $s \neq 1-r$.*

Let us examine Huang's algebraic integer of degree three. It is the unique real solution of

$$r^3 + r^2 - 1 = 0 \qquad (A)$$

(It is perhaps worth noting that $1/r$ is the smallest Pisot-Vijayaraghavan number.) Now, set

$$s = r^2. \qquad (B)$$

Clearly, $s \neq r$ and $s \neq 1-r$. OXTOBY and NAVARRO-BERMUDEZ [1988] showed that for this r and s, the measures $\mu(r)$, $\mu(1-r)$, $\mu(s)$, and $\mu(1-s)$ are topologically equivalent.

Problem 1.7. *Let r be the root of eq.* (A) *between 0 and 1. Does $E(r)$ or $F(r)$ consists of exactly the four numbers r, $1-r$, r^2 and $1-r^2$?* 1066. ?

Problem 1.8. *For each r, is it true that there are only finitely many numbers s such that $\mu(s)$ and $\mu(r)$ are homeomorphic?* 1067. ?

2. Two-Point Sets

MAZURKIEWICZ [1914] showed that there is a "two-point" subset M of $\mathbb{R}^2$, i.e., M meets each line in exactly 2 points. Direct generalizations of this result were given by ERDÖS and BAGEMIHL [1957]. The axiom of choice plays a central role in the construction of M. The problem naturally arises as to how effective such a construction can be.

? 1068. **Problem 2.1.** *Is there a Borel set M in $\mathbb{R}^2$ which meets each straight line in exactly two points? Can M be a G_δ set?*

LARMAN [1968] has shown that M cannot be an F_σ set. But, even whether M can be a G_δ set is unknown. It is known that if M is an analytic set then M is a Borel set. This follows for example from the fact that every analytic subset A of $\mathbb{R}^2$ such that each vertical fiber A_x has cardinality ≤ 2 lies in a Borel set B such that each vertical fiber has cardinality ≤ 2. Miller has shown that $V = L$ implies that M can be taken to be a coanalytic set (MILLER [1989]).

I have proven the following.

2.2. THEOREM. *A two point set M must always be totally disconnected, i.e., every connected subset of M consists of a single point.* □

Larman's Theorem follows from this since each σ-compact subset of $\mathbb{R}^2$ which meets each vertical line in two points contains the graph of a continuous function defined on some interval.

? 1069. **Problem 2.3.** *Must a two-point set M always be zero-dimensional?*

Note that if E is a subset of the plane which meets each line in 2^ω points then there is a two-point set M lying in E. Since there is such a subset E of the plane which is both zero-dimensional and of planar Lebesgue measure 0, M can be both zero-dimensional and of Lebesgue measure 0. On the other hand, one can construct M such that M meets each closed subset of $\mathbb{R}^2$ which has positive Lebesgue measure. Thus, M can also be taken to be non-Lebesgue measurable. It should be noted that the property of being a partial two-point set cannot necessarily be extended. For example, the unit circle meets each line in no more than two points but of course we cannot even add a single point to this set and retain this property.

? 1070. **Problem 2.4.** *Can a zero-dimensional partial two-point set always be extended to a two-point set?*

(van Mill and I note that this is true assuming **CH** holds).

3. Pisot-Vijayaraghavan Numbers

Let S be the set of all Pisot-Vijayaraghavan numbers. Thus, $x \in S$ if and only if x is an algebraic number, $x > 1$ and all its conjugates have moduli less than 1. SALEM [1983] proved that the countable set S is also a closed subset of $\mathbb{R}$. SIEGEL [1944] showed that the smallest element of S is the root of $x^3 - x^2 - 1$. PISOT and DUFRENOY [1953] showed that the smallest number in the Cantor-Bendixson derived set of S is the root of $x^2 - x - 1$.

Problem 3.1. *What is the order type of the set S of all Pisot-Vijayaraghavan numbers?* 1071. ?

Problem 3.2. *What is the Cantor-Bendixson derived set order of S?* 1072. ?

4. Finite Shift Maximal Sequences Arising in Dynamical Systems

A particular countable linear order type arises in one-dimensional dynamics. A simple case occurs in the iteration of the critical point in a scaled family of unimodal maps of the unit interval one-dimensional dynamics. For example, consider the quadratic map $q(x) = 4x(1-x)$ on the unit interval, $[0,1]$. For each λ, $0 \leq \lambda \leq 1$, consider the itinerary, $I_{\lambda q}(1/2)$, of the critical point of the scaled map, λq. Thus

$$I_{\lambda q}(1/2)(i) = \begin{cases} R, & \text{if } (\lambda q)^i(1/2) > 1/2, \\ C, & \text{if } (\lambda q)^i(1/2) = 1/2, \\ L, & \text{if } (\lambda q)^i(1/2) < 1/2. \end{cases}$$

We make the convention that the sequence stops at the first C if there is a C in the sequence. Thus, a finite itinerary arises from a value of λ such that $1/2$ is periodic under λq. The set of all possible itineraries has been abstractly characterized as follows. First, consider the parity-lexicographic order on the space S of all finite and infinite sequences of R, L and C such that if the sequence has a C there is only one C and it is the last term of the sequence. Thus, if $A = (A_1, A_2, \ldots)$ and $B = (B_1, B_2, \ldots)$ are elements of S, then $A \leq B$ provided $A_i < B_i$, where i is the first place where A and B disagree and we use the order $L < C < R$ if there are an even number of R's preceding A_i in A and we use the reverse order if there are an odd number. An element A of S is said to be *shift maximal* provided A is not less than any of its shifts, $\sigma^i(A) = (A_{i+1}, A_{i+2}, \ldots)$ in the parity-lexicographic order.

4.1. THEOREM. *An element $A = (A_1, A_2, A_3, \ldots)$ is the itinerary of $1/2$ under the quadratic map, q, for some value of λ if and only if A is shift maximal.* □

This theorem is true not only for the quadratic map but for a general wide class of maps of $[0,1]$ onto $[0,1]$ (See COLLET and ECKMAN [1980] and BEYER, MAULDIN and STEIN [1986].)

Problem 4.2. *What is the order type of the countable set of finite shift-maximal sequences in the parity-lexicographic order?* 1073. ?

5. Borel Selectors and Matchings

Consider the hyperspace of all compact subsets of the unit interval, $\mathcal{K}(I)$. There are exactly 2 continuous selectors. If $f\colon \mathcal{K}(I) \to I$ is continuous and for each compact set K, $f(K) \in K$, then either $f(K) = \max(K)$ for all K or else $f(K) = \min(K)$ for all K. In MAULDIN [1980], I showed that there are ω_1 Borel measurable selectors $f_\alpha : \mathcal{K}(I) \to I$ such that if K is an uncountable compact set, then the values $f_\alpha(K)$ are distinct.

? 1074. **Problem 5.1.** *Can one prove in* **ZFC** *that there are continuum many Borel measurable selectors on $\mathcal{K}(I)$ such that for each uncountable compact set K, the selected points of K are all distinct?*

There does exist such a family of Borel selectors if instead of the uncountable compact sets, one considers the family of compact perfect sets (MAULDIN [1979]).

? 1075. **Problem 5.2.** *Let B be a Borel subset of $[0,1] \times [0,1]$ such that each horizontal and each vertical fiber of B is co-meager. Can B be filled up by a collection of pairwise disjoint graphs of Borel isomorphisms of $[0,1]$ onto $[0,1]$?*

DEBS and SAINT-RAYMOND [1989] have shown that B does contain a Borel matching—the graph of some Borel isomorphism. This result is false if co-meager is replaced by Lebesgue measure one. An example of such a set is given in GRAF and MAULDIN [1985] and in more detail in MAULDIN and SCHLEE [1989]. More problems on this theme are given in MAULDIN [1989].

6. Dynamical Systems on $S^1 \times \mathbb{R}$—Invariant Continua

Fix $a > 0$ and $B > 0$ and define a map $T: S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ by

$$T(e^{i2\pi x}, y) = (e^{i2\pi ax}, B(y - A(x)).$$

In order for the map to be well-defined and continuous, we assume $A: \mathbb{R} \to \mathbb{R}$ is continuous, has period 1 and that a is a positive integer. For convenience, we assume $\|A\| = 1$. Note that T maps the fiber $\{e^{i2\pi x}\} \times \mathbb{R}$ one-to-one and onto $\{e^{i2\pi ax}\} \times \mathbb{R}$. Also, T restricted to the fiber is an orientation preserving similarity map with similarity ratio

$$B: \|T(e^{2\pi ix}, y) - T(e^{2\pi ix}, z)\| = B|y - z|.$$

This map or close relatives have been studied by KAPLAN, MALLET-PARET and YORKE [1984], MOSER [1969] and FREDRICKSON ET AL [1983]. In order to examine the dynamics of T, note that

$$T^n(e^{i2\pi x}, y) = (e^{2\pi i a^n x}, B^n y - \sum_{p=0}^{n-1} B^{n-p} A(a^p x)).$$

If $a = 1$, then the dynamics are quite simple. If $B = 1$, then $T^n(e^{2\pi ix}, y) = (e^{2\pi ix}, y - nA(x))$ and the asymptotic behaviour is clear. If $B \neq 1$, then the graph $\mathcal{G}$, of

$$f(x) = (\frac{B}{B-1})A(x)$$

lifted to the cylinder is invariant. If $0 < B < 1$, this graph is a universal attractor. In fact, for each x and y, $T^n(e^{2\pi ix}, y) \to (e^{2\pi ix}, \mathcal{G}(x))$. If $B > 1$, this graph is a repeller. The points of the cylinder above the graph iterate to $+\infty$ and those below iterate to $-\infty$.

From this point on, we assume $a \geq 2$. Now the map T is a-to-1:

$$T^{-1}(e^{2\pi ix}, y) = \left\{(e^{2\pi i((x+k)/a)}, B^{-1}y + A((x+k)/a)) \colon k = 0, \ldots, a-1\right\}.$$

If $B > 1$, then the graph of the continuous, period 1 function f which satisfies the functional equation:

$$f(ax) = B(f(x) - A(x))$$

is invariant. Or, setting $b = 1/B$,

$$f(x) = A(x) + bf(x).$$

The unique solution of this equation is the Weierstrass function:

$$f(x) = \sum_{p=0}^{\infty} b^p A(a^p x).$$

The graph of f on the cylinder is a nowhere differentiable invariant 1-torus. It is also a universal repeller. The points of the cylinder above the graph iterate to $+\infty$ and those below iterate to $-\infty$. The capacity dimension of this graph is $2 + \log b / \log a$, in some cases (KAPLAN, MALLET-PARET and YORKE [1984]). The Hausdorff dimension of this set is a long standing unsolved problem. It is widely believed that the capacity dimension is the Hausdorff dimension. The best estimates in the general case are given in MAULDIN and WILLIAMS [1986].

Problem 6.1. *Find the Hausdorff dimension, γ, of this graph. Moreover, find the exact Hausdorff dimension function—if there is one. In other words, find a slowly varying function $L(t)$ such that $0 < \mathcal{H}^h(f) < \infty$, where $h(t) = t^\gamma L(t)$.* 1076. ?

If $0 < B < 1$, then T has an attracting continuum M. This is seen by noticing that if $|y| \leq \frac{B}{1-B}$, then

$$|B(y - A(x))| \leq B(|y| + |A(x)|) \leq B(\frac{B}{1-B} + 1) = \frac{B}{1-B}.$$

Thus the "can",

$$K = S^1 \times \left[\frac{-B}{1-B}, \frac{B}{1-B}\right],$$

is mapped into itself, $T(K) \subseteq K$. Set

$$M = \bigcap_{n=0}^{\infty} T^n(K).$$

Then M is an invariant continuum which separates $S^1 \times \mathbb{R}$ and M attracts the orbit of all points. Pat Carter and I have shown that T acts chaotically on the continuum M. The case $0 < B < 1$ is very different from the case $1 < B$, in fact I conjecture:

? 1077. **Problem 6.2.** *Is it true that M is a Sierpiński curve? In particular, is this true if A is the tent map on $[0,1]$?*

Let us remark that in general M is not a graph in this case. Let us assume M is the graph of a function from S^1 into $\mathbb{R}$. Since the graph is compact, there is a continuous period one map $f\colon \mathbb{R} \to \mathbb{R}$ such that M is the graph of the lift of f to the cylinder. Since

$$T(e^{i2\pi x}, f(x)) = (e^{i2\pi ax}, B(f(x) - A(x))),$$

the function f must satisfy the functional equation

$$f(ax) = B(f(x) - A(x)),$$

for all x. Or,

$$f(x) = A(x) + \frac{1}{B} f(ax).$$

However, Pat Carter and I have shown that for some functions, the unique solution of this equation which is continuous at zero does not have period one. This class includes the case when A is nonnegative. In particular, if A is the tent map, M is not a graph.

? 1078. **Problem 6.3.** *Let A be a non-constant, continuous, period one map of $\mathbb{R}$ into $\mathbb{R}$ with $\|A\| = 1$, a is an integer, $a \geq 2$ and $0 < B < 1$. Is it true that the unique continuous solution of*

$$f(x) = A(x) + \frac{1}{B} f(ax)$$

does not have period one, or more generally, is not periodic?

7. Borel Cross-Sections

Let X be an indecomposable continuum and consider the decomposition of X into its composants and let R be the corresponding equivalence relation: R is a Borel subset of $X \times X$ and each equivalence class is a meager, dense, F_σ subset of X. I have raised the following question over the past fifteen years, but it probably has been known much earlier.

Problem 7.1. *Is there a Borel subset B of X which meets each equivalence class in exactly one point?* 1079. ?

While this question remains unsolved, there is one case for which the answer is no. The continuum X is said to be *strictly transitive in the sense of category* provided that for each subset E of X which has the Baire property and which can be expressed as the union of some composants either E or $X \setminus E$ is meager (KURATOWSKI [1968]).

7.2. THEOREM. *Let X be an indecomposable continuum which is strictly transitive in the sense of category. There is no Borel cross-section for the composants of X.*

PROOF. Assume that there is a Borel cross-section B. For each subset E of X, let $sat(E)$ be the union of all composants which meet E. Notice that if E is a Borel set, then $sat(E)$ of E is analytic, since $sat(E) = \text{proj}_2(R \cap (E \times X))$ and, therefore, $sat(E)$ has the Baire property. Define a probability measure, μ, on the Borel subsets of B as follows: $\mu(E) = 1$, if $sat(E)$ is co-meager, and $\mu(E) = 0$, otherwise. Then μ gives each singleton measure 0, and each Borel subset of B has measure 0 or 1. This is impossible. □

There are a number of indecomposable continua which are strictly transitive: Knaster continua (KURATOWSKI [1968]) and those admitting a Polish group action for which the orbit decomposition consists of the composants (ROGERS [1986]).

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