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A Classification of Disintegrations of Measures

S. GRAF AND R. DANIEL MAULDIN

0. Introduction. Disintegration of measures is a very useful tool in ergodic theory (see, for example, von Neumann [7]), in the theory of conditional probabilities (see, for example, Parthasarathy [8]), and in descriptive set theory (see, for example, Mauldin [5] and Graf-Mauldin [2]).

The origins of disintegation are uncertain but the first rigorous definitions and results—to our knowledge—are due to von Neumann [7]. In the late forties Rokhlin [10] and Maharam [3] independently introduced canonical representations of disintegrations. Later Maharam [4] returned to the descriptive set theoretic aspects of these representations. The purpose of our paper is to offer a sharpening of her results. The improvements are that we can use an "ordinate set" in the place of Maharam's "almost ordinate set" to represent the non-atomic part of a disintegration and that we get rid of Maharam's "garbage set". As a main tool we use the classification for completely orthogonal transition kernels given by Mauldin-Preiss-v. Weizsäcker [6].

Our considerations still leave one major problem of Maharam [3] open which can be reformulated as follows: Is every conditional measure distribution consisting of σ-finite measures necessarily uniformly σ-finite (see section on "Preliminaries" for the definition of uniform σ-finiteness of a conditional measure distribution)?

1. Preliminaries. For a Hausdorff space X let $\mathscr{B}(X)$ denote the Borel σ-field on X. In the following X is always a Lusin space (i.e. a one-to-one continuous image of a Polish space) and Y is a non-empty Suslin space (i.e. a continuous image of a Polish space). For the basic facts about Suslin- and Lusin-spaces we refer the reader to [9]. Moreover, p is a $\mathscr{B}(X)-\mathscr{B}(Y)$-measurable (i.e., a Borel measurable) map from X *onto* Y. A family $(\mu_y)_{y\in Y}$

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is called a *conditional measure distribution* with respect to p if the following conditions hold:

(i) Each μ_y is a (not necessarily σ-finite) measure on $\mathscr{B}(X)$ with

$$\mu_y(X \backslash p^{-1}(y)) = 0.$$

(ii) For every $A \in \mathscr{B}(X)$ the map $y \to \mu_y(A)$ is Borel measurable on Y.

A conditional measure distribution $(\mu_y)_{y \in Y}$ is called finite (uniformly σ-finite) if $\mu_y(X) < \infty$ for all $y \in Y$ (if there is a sequence $(A_n)_{n \in \mathbf{N}}$ in $\mathscr{B}(X)$ with $\bigcup_{n \in \mathbf{N}} A_n = X$ and $\mu_y(A_n) < \infty$ for all $y \in Y$). It is called a *transition kernel* with respect to p if $\mu_y(X) = 1$ for all $y \in Y$.

If μ is a σ-finite measure on $\mathscr{B}(X)$ and ν is a σ-finite measure on $\mathscr{B}(Y)$ then a conditional measure distribution $(\mu_y)_{y \in Y}$ with respect to p is called a strict disintegration of μ with respect to (p, ν) if

$$\forall A \in \mathscr{B}(X) \colon \int \mu_y(A)\, d\nu(y) = \mu(A). \tag{1.1}$$

Since we will only deal with strict disintegrations we will use "disintegration" to mean "strict disintegration".

Maharam proved the following fundamental theorem.

THEOREM. *A σ-finite measure μ on $\mathscr{B}(X)$ has a uniformly σ-finite disintegration with respect to (p, ν) with ν σ-finite if and only if*

$$\forall B \in \mathscr{B}(Y) \colon \nu(B) = 0 \Rightarrow \mu(p^{-1}(B)) = 0 \tag{1.2}$$

i.e., the image measure $\mu \circ p^{-1}$ of μ with respect to p is absolutely continuous with respect to ν.

It is an *open question* whether, in the above situation, every disintegration $(\mu_y)_{y \in Y}$ consisting of σ-finite μ_y's is already uniformly σ-finite (see [3]). (The answer is "yes" for locally finite μ_y's!). We will return to this question at the end of the paper.

In what follows λ_B stands for the restriction of one-dimensional Lebesgue-measure to the Lebesgue-measurable subset B of $\mathbf{R}$. For a point y in Y the symbol ε_y denotes the Dirac measure concentrated at y.

2. The classification of atomless σ-finite conditional measure distributions.

2.1 LEMMA. *Let ν be a σ-finite atomless measure on the Borel field of a Lusin space Y with $\nu(Y) > 0$. Let $Y' = \{t \in \mathbf{R}_+ | t \leq \nu(Y)\}$. Then there exists a Borel isomorphism φ from Y onto Y' with $\nu \circ \varphi^{-1} = \lambda_{Y'}$.*

PROOF. There exists a sequence $(B_n)_n$ of pairwise disjoint Borel sets in Y with $\bigcup_n B_n = Y$ and $0 < \nu(B_n) < \infty$ for every n. As is well-known, for each n, there exist a Borel isomorphism φ_n of B_n onto $]0, \nu(B_n)]$ such that $\nu|_{B_n} \circ \varphi_n^{-1} = \lambda_{]0, \nu(B_n)]}$. Define $\tilde{\varphi} \colon Y \to]0, +\infty[$ by $\tilde{\varphi}(y) = \varphi_n(y) + \sum_{i=1}^{n-1} \nu(B_i)$ if $y \in B_n$.

Then φ is Borel measurable and one-to-one with $\tilde{\varphi}(Y) =]0, \nu(Y)] \cap \mathbf{R}$, hence a Borel isomorphism from Y onto $]0, \nu(Y)] \cap \mathbf{R}$. It is easy to check $\nu \circ$

$\tilde{\varphi}^{-1} = \lambda_{]0,\nu(Y)]\cap\mathbf{R}}$. Let $\tilde{\tilde{\varphi}}$ be a Lebesgue measure preserving Borel isomorphism of $]0,\nu(Y)]\cap\mathbf{R}$ onto $Y' = [0,\nu(Y)]\cap\mathbf{R}_+$. Then $\varphi = \tilde{\tilde{\varphi}} \circ \tilde{\varphi}$ has the desired properties.

2.2 THEOREM. *Let $(\mu_y)_{y\in Y}$ be an atomless transition kernel with respect to p. Then Y is a Lusin space and there exists a Borel isomorphism ψ from X onto $Y\times[0,1]$ such that $\psi(p^{-1}(y)) = \{y\}\times[0,1]$ and $\mu_y\circ\psi = \varepsilon_y\otimes\lambda_{[0,1]}$ for every $y\in Y$.*

PROOF. Since $Y\neq\varnothing$ and each μ_y is atomless it follows that X is uncountable. Hence there exists a Borel isomorphism τ from X onto $[0,1]$. Since Y is Suslin there exists an analytic subset Y' of $[0,1]$ and a Borel isomorphism j from Y onto Y'. Define $\varphi\colon X\to[0,1]\times[0,1]$ by $\varphi(x) = (j(p(x)),\tau(x))$. Then φ is Borel measurable and one-to-one. Hence $X' := \varphi(X)$ is a Borel subset of $[0,1]\times[0,1]$. Let $\pi\colon[0,1]\times[0,1]\to[0,1]$ be the projection onto the first coordinate. Using the proof of Theorem 2.2 in **[5]** one can show that there is a Borel subset M of X' with $\pi_1(M) = \pi_1(X')$ and $M_y = \{x\in[0,1] | (y,x)\in M\}$ compact. By a result of Novikov (see, for instance, Dellacherie **[1]**, Sect. 2, Th. 19) it follows that $\pi_1(M)$ is a Borel subset of $[0,1]$. Since $\pi_1(M) = \pi_1(X') = j(Y)$ it follows that Y is a Lusin space.

If Y is countable then one may apply Lemma 2.1 to each fiber $p^{-1}(y)$ and each μ_y separately and thus obtain the conclusion of the theorem.

If Y is uncountable then there exists a Borel isomorphism φ from Y onto $[0,1]$. Let $B = \{(p(x),x) | x\in X\}$. By our assumptions Theorem 2.3 of **[6]** is applicable. Hence there exists a Borel isomorphism $\tilde{\psi}$ from B onto $[0,1]\times[0,1]$ with $\tilde{\psi}(\{y\}\times B_y) = \{\varphi(y)\}\times[0,1]$ and $(\varepsilon_y\otimes\mu_y)\circ\tilde{\psi}^{-1} = \varepsilon_{\varphi(y)}\otimes\lambda_{[0,1]}$ for all $y\in Y$. Let $\tilde{\tilde{\psi}}\colon X\to B$ be defined by $\tilde{\tilde{\psi}}(x) = (p(x),x)$. Then $\tilde{\tilde{\psi}}$ is a Borel isomorphism from X onto B. Let $\psi' = \tilde{\psi}\circ\tilde{\tilde{\psi}}$ and let ψ_1' and ψ_2' be the components of ψ'. Define $\psi\colon X\to Y\times[0,1]$ by $\psi(x) = (\varphi^{-1}\circ\psi_1'(x),\psi_2'(x))$. Then ψ is a Borel isomorphism with the required properties.

2.3 THEOREM. *Let $(\mu_y)_{y\in Y}$ be a finite atomless conditional measure distribution with respect to p with $\mu_y\neq 0$ for every $y\in Y$. For $y\in Y$ let*

$$I_y := [0,\mu_y(X)].$$

Let $X' = \bigcup_{y\in Y}\{y\}\times I_y$. Then Y is a Lusin space and there exists a Borel isomorphism ψ from X onto X' such that

$$\psi(p^{-1}(y)) = \{y\}\times I_y \textit{ and } \mu_y\circ\psi^{-1} = \varepsilon_y\otimes\lambda_{I_y}$$

for every $y\in Y$.

PROOF. Let $\mu_y' = (1/\mu_y(X))\mu_y$. Then $(\mu_y')_{y\in Y}$ is a transition kernel with respect to p. By Theorem 2.2 the space Y is Lusin and there exists a Borel isomorphism ψ' from X onto $Y\times[0,1]$ such that $\psi'(p^{-1}(y)) = \{y\}\times[0,1]$ and $\mu_y'\circ\psi'^{-1} = \varepsilon_y\otimes\lambda_{[0,1]}$ for every $y\in Y$. Let ψ_1',ψ_2' be the components of ψ'. Then $\psi_1' = p$. Define $\psi\colon X\to Y\times\mathbf{R}$ by $\psi(x) = (p(x),\mu_{p(x)}(X)\cdot\psi_2(x))$.

Then ψ is Borel measurable and one-to-one with $\psi(X) = X'$. Hence ψ is a Borel isomorphism from X onto X'. Moreover,

$$\psi(p^{-1}(y)) = \{y\} \times I_y.$$

Obviously

$$\mu_y \circ \psi^{-1} = \varepsilon_y \otimes \lambda'_y$$

where λ'_y is the image of $\lambda_{[0,1]}$ with respect to $t \to \mu_y(X) \cdot t$ multiplied by $\mu_y(X)$. Thus λ'_y is the Lebesgue measure on $[0, \mu_y(X)[$, i.e.,

$$\mu_y \circ \psi^{-1} = \varepsilon_y \otimes \lambda_{I_y}.$$

2.4 *Remark.* In the terminology of Maharam [3] the set X' is called an ordinate set (or the ordinate set of the Borel-measurable function $y \to \mu_y(X)$).

2.5 COROLLARY. *Let $(\mu_y)_{y\in Y}$ be a finite atomless conditional measure distribution with $\mu_y \neq 0$ for every $y \in Y$. Let $X'' = \bigcup_{y\in Y}\{y\} \times]0, \mu_y(X)]$. Then there exists a Borel isomorphism ψ from X onto X'' such that*

$$\psi(p^{-1}(y)) = \{y\} \times]0, \mu_y(X)]$$

and

$$\mu_y \circ \psi = \varepsilon_y \otimes \lambda_{]0,\mu_y(X)]}$$

for every $y \in Y$.

PROOF. This corollary follows immediately from Theorem 2.3 if one observes that there is a Lebesgue measure preserving Borel isomorphism of $[0, 1]$ onto $]0, 1]$.

2.6 LEMMA. *Let $(\mu_y)_{y\in Y}$ be a uniformly σ-finite conditional measure distribution with respect to p with $\mu_y \neq 0$ for every $y \in Y$. Then there exists a (finite or infinite) sequence $(A_n)_n$ of pairwise disjoint Borel sets in X such that*

(i) $\bigcup_n A_n = X$.
(ii) $p(A_n) \in \mathscr{B}(Y)$.
(iii) $\forall y \in p(A_n)$: $0 < \mu_y(A_n) < \infty$.

PROOF. Let $(B_n)_{n\in \mathbf{N}}$ be a sequence in $\mathscr{B}(X)$ with $\mu_y(B_n) < \infty$ for all n and all $y \in Y$ and $\bigcup_n B_n = X$. Without loss of generality we may assume that the B_n's are pairwise disjoint. Let $Y_n := \{y \in Y : 0 < \mu_y(B_n)\}$. Then Y_n is a Borel set in Y and, by our assumption on $(\mu_y)_y$, $\bigcup_n Y_n = Y$. Set $Y_0 = \varnothing$ and define $B'_n := B_n \cap p^{-1}(Y_n)$ and

$$N := X \backslash \bigcup_n B'_n.$$

Define

$$A_n = \left[(N \cap p^{-1}(Y_n)) \backslash \bigcup_{i=1}^{n-1} N \cap p^{-1}(Y_i)\right] \cup B'_n.$$

Let $I = \{n \in \mathbf{N} | Y_n \neq \emptyset\}$. Then $(A_n)_{n \in I}$ is a finite or infinite sequence of pairwise disjoint Borel subsets of X. It is easy to check $X = \bigcup_{n \in I} A_n$ and $p(A_n) = Y_n \in \mathscr{B}(Y)$. For $y \in Y_n$ we have:

$$\begin{aligned}\mu_y(A_n) \geq \mu_y(B_n') &= \mu_y(B_n' \cap p^{-1}(y)) = \mu_y(B_n \cap p^{-1}(y)) \\ &= \mu_y(B_n) > 0\end{aligned}$$

and $\mu_y(A_n) \leq \mu_y(N) + \mu_y(B_n)$.

Since $\mu_y(N) \leq \sum_m \mu_y(N \cap B_m) = \sum_m \mu_y(B_m \backslash B_m') = 0$ we obtain $\mu_y(A_n) < \infty$.

2.7 THEOREM. *Let $(\mu_y)_{y \in Y}$ be a uniformly σ-finite atomless measure distribution with respect to p with $\mu_y \neq 0$ for every $y \in Y$. For $y \in Y$ let*

$$I_y = \begin{cases} [0, +\infty[, & \mu_y(X) = \infty \\ [0, \mu_y(X)], & \mu_y(X) < \infty. \end{cases}$$

Let $X' := \bigcup_{y \in Y} \{y\} \times I_y$. Then Y is a Lusin space and there exists a Borel isomorphism ψ from X onto X' such that $\psi(p^{-1}(y)) = \{y\} \times I_y$ and $\mu_y \circ \psi^{-1} = \varepsilon_y \otimes \lambda_{I_y}$ for every $y \in Y$.

PROOF. Let $(A_n)_n$ be chosen as in the conclusion of Lemma 2.6. By Theorem 2.3 $p(A_n)$ is a Lusin space. Hence $Y = \bigcup_n p(A_n)$ is a Lusin space. Let $A_n' = \bigcup_{y \in p(A_n)} \{y\} \times]0, \mu_y(A_n)]$. By Cor. 2.5 there exists a Borel isomorphism ψ_n from A_n onto A_n' such that

$$\psi_n(p^{-1}(y) \cap A_n) = \{y\} \times]0, \mu_y(A_n)]$$

and

$$\mu_{y|A_n} \circ \psi_n^{-1} = \varepsilon_y \otimes \lambda_{]0, \mu_y(A_n)]}$$

for every $y \in p(A_n)$.

Let $\psi_n^{(1)}, \psi_n^{(2)}$ be the components of ψ_n. Then $\psi_n^{(1)} = p|_{A_n}$. Define $\tilde{\psi}\colon X \to Y \times \mathbf{R}$ by

$$\tilde{\psi}(x) = (p(x), \psi_n^{(2)}(x) + \sum_{i=1}^{n-1} \mu_y(A_i))$$

if $x \in A_n$. Then $\tilde{\psi}$ is Borel measurable and one-to-one. It is easy to check

$$\tilde{\psi}(p^{-1}(y)) = \{y\} \times (]0, \mu_y(X)] \cap \mathbf{R})$$

and, therefore, $\tilde{\psi}(X) = X'' := \bigcup_{y \in Y} \{y\} \times (]0, \mu_y(X)] \cap \mathbf{R})$. Hence $\tilde{\psi}$ is a Borel isomorphism from X onto X''. We have $\mu_y \circ \tilde{\psi}^{-1} = \varepsilon_y \otimes \lambda_{]0, \mu_y(X)] \cap \mathbf{R}}$. To prove this it is enough to show that

$$\mu_{y|A_n} \circ (\tilde{\psi}_{|A_n})^{-1} = \varepsilon_y \otimes \lambda \left] \sum_{i=1}^{n-1} \mu_y(A_i), \sum_{i=1}^{n} \mu_y(A_i) \right]$$

for every $y \in Y$. But this is obviously true.

Now let φ_1 be a Lebesgue measure preserving Borel isomorphism of $]0,1]$ onto $[0,1]$ and let φ_2 be a Lebesgue measure preserving Borel isomorphism of $]0,+\infty[$ onto $[0,+\infty[$. Define $\tilde{\tilde{\psi}}\colon X'' \to X'$ by

$$\tilde{\tilde{\psi}}(y,t) = \begin{cases} (y, \varphi_2(t)), & \mu_y(X) = \infty \\ (y, \mu_y(X)\varphi_1(\frac{1}{\mu_y(X)}t)), & \mu_y(X) < \infty. \end{cases}$$

Then $\psi = \tilde{\tilde{\psi}} \circ \tilde{\psi}$ has the required properties.

2.8 *Remark.* In the situation of Theorem 2.7 the space Y is either at most countable or Borel isomorphic to $[0,1]$.

3. The classification of arbitrary uniformly σ-finite conditional measure distributions.

3.1 Lemma. *Let $(\mu_y)_{y\in Y}$ be a conditional measure distribution with respect to p. Let X_a denote the set of atoms of $(\mu_y)_{y\in Y}$, i.e.,*

$$X_a = \{x \in X \mid \exists y \in Y\colon \mu_y(\{x\}) > 0\}.$$

Then X_a is a Borel set.

Proof. Without loss of generality we may assume that X is a Polish space. Since μ_y is concentrated on $p^{-1}(y)$ we obviously have

$$X_a = \{x \in X \mid \mu_{p(x)}(\{x\}) > 0\}$$

Let $\mathscr{L}$ be a countable base for the topology of X. Then

$$X_a = \bigcup_{n\in\mathbf{N}} X_a^{(n)}$$

with

$$\begin{aligned} X_a^{(n)} &:= \left\{x \in X \mid \forall U \in \mathscr{L}\colon x \in U \Rightarrow \mu_{p(x)}(U) \geq \frac{1}{n}\right\} \\ &= \bigcap_{U\in\mathscr{L}} (X\setminus U) \cup \left\{x \in X \mid \mu_{p(x)}(U) \geq \frac{1}{n}\right\}. \end{aligned}$$

Thus $X_a^{(n)}$ and hence X_a is a Borel set.

3.2 Theorem. *Let $(\mu_y)_{y\in Y}$ be a uniformly σ-finite conditional measure distribution with respect to p. For every $y \in Y$ let $n_y \in \mathbf{N}\cup\{\infty\}$ be the number of atoms of μ_y and*

$$Z_y = \begin{cases} \varnothing, & n_y = 0 \\ \{-1,\dots,-n_y\}, & 0 < n_y < \infty \\ \{-1,-2,\dots\}, & n_y = \infty. \end{cases}$$

Let $X_a = \{x \in X \mid \exists y \in Y\colon \mu_y(\{x\}) > 0\}$ and μ_y^{na} the atomless part of μ_y, i.e. $\mu_y^{na} = \mu_y|_{X\setminus X_a}$. Let

$$I_y = \begin{cases} [0, \mu_y^{na}(X)], & \mu_y(X\setminus X_a) < \infty \\ [0,+\infty[, & \mu_y(X\setminus X_a) = \infty. \end{cases}$$

Let $Y_0 = \{y \in Y | \mu_y(X \backslash X_a) = 0\}$.

Then $Y \backslash Y_0$ *is a Lusin space and there exists a Borel subset* X' *of* $Y \times \mathbf{R}$ *and a Borel isomorphism* ψ *from* X *onto* X' *such that the following conditions hold:*

(i) $\forall y \in Y$: $\psi(p^{-1}(y)) = \{y\} \times \{t \in \mathbf{R} | (y, t) \in X'\}$.

(ii) $\forall y \in Y$: $\psi(p^{-1}(y) \cap X_a) = \{y\} \times Z_y$.

(iii) $\forall y \in Y \backslash Y_0$: $\psi(p^{-1}(y) \backslash X_a) = \{y\} \times I_y$ *and* $\mu_y^{na} \circ \psi^{-1} = \varepsilon_y \otimes \lambda_{I_y}$.

(iv) $\forall y \in Y_0$: $\psi(p^{-1}(y) \backslash X_a) \subset \{y\} \times [0, +\infty[$ *and*

$$\varepsilon_y \otimes \lambda_{[0,+\infty[}(\psi(p^{-1}(y)) \backslash X_a) = 0.$$

PROOF. By Lemma 3.1 the set X_a is in $\mathscr{B}(X)$. This implies $Y_0 \in \mathscr{B}(Y)$ and therefore, $(X \backslash X_a) \cap p^{-1}(Y \backslash Y_0) \in \mathscr{B}(X)$. Since X is a Lusin space the same is true for $X_{na} := (X \backslash X_a) \cap p^{-1}(Y \backslash Y_0)$. Since $p(X_{na}) = Y \backslash Y_0$ it follows from Theorem 2.7 applied with X_{na} in the place of X, $Y \backslash Y_0$ in the place of Y, $p_{|X_{na}}$ in the place of p, and μ_y^{na} in the place of μ_y that $Y \backslash Y_0$ is a Lusin space and that there exists a Borel isomorphism ψ_1 from X_{na} onto $X_1 = \bigcup_{y \in Y \backslash Y_0} \{y\} \times I_y$ with

$$\psi(p^{-1}(y) \cap X_{na}) = \{y\} \times I_y$$

and

$$\mu_y^{na} \circ \psi_1^{-1} = \varepsilon_y \otimes \lambda_{I_y} \quad \text{for every } y \in Y.$$

By a theorem of Lusin (see, for instance, [1]) there are Borel sets $D_n \subset Y$ and Borel measurable maps $f_n \colon D_n \to X_a$ with $f_n(y) \in p^{-1}(y)$ for every $y \in Y, \bigcup_{n \in \mathrm{N}} f_n(D_n) = X_a$ and $f_n(D_n) \cap f_m(D_m) = \varnothing$ if $n \neq m$. Define $\psi_2 \colon X_a \to Y \times \{-n | n \in \mathrm{N}\}$ by $\psi_2(x) = (p(x), -n)$ if $x \in f_n(D_n)$.

Then ψ_2 is Borel measurable and one-to-one with

$$\psi_2(X_a) = X_2 := \bigcup_{y \in Y} \{y\} \times Z_y,$$

hence a Borel isomorphism from X_a onto X_2. It follows from the definition of ψ_2 that $\psi_2(p^{-1}(y) \cap X_a) = \{y\} \times Z_y$ for every $y \in Y$.

Consider $X_{na}^0 := p^{-1}(Y_0) \cap (X \backslash X_a)$. Then X_{na}^0 is a Borel set in X. Hence there exists a Borel isomorphism $j \colon X_{na}^0 \to C$ from X_{na}^0 onto a Borel subset of the classical Cantor set $C \subset [0, 1]$. Define $\psi_3 \colon X_{na}^0 \to Y \times \mathbf{R}$ by $\psi_3(x) = (p(x), j(x))$. Then ψ_3 is a Borel isomorphism of X_{na}^0 onto a Borel subset B of $Y \times \mathbf{R}$. For $y \in Y_0$ we get

$$\psi_3(p^{-1}(y) \backslash X_a) \subset \{y\} \times [0, +\infty[$$

and $\varepsilon_y \otimes \lambda_{[0,+\infty[}(\psi_3(p^{-1}(y) \backslash X_a) = 0$, since $\psi_3(p^{-1}(y) \backslash X_a)) \subset \{y\} \times C$. Define $\psi \colon X \to Y \times \mathbf{R}$ by $\psi_{|(X \backslash X_a) \cap p^{-1}(Y \backslash Y_0)} = \psi_1$, $\psi_{|(X \backslash X_a) \cap p^{-1}(Y_0)} = \psi_3$, $\psi_{|X_a} = \psi_2$, and $X' := \psi(X)$. Then ψ and X' have the required properties.

4. The classification of uniformly σ-finite disintegrations.

4.1 *Definition.* Let X' be a Lusin space, Y' a Suslin space, $p' \colon X' \to Y'$ Borel measurable and onto. Let μ' and ν' be σ-finite measures on $\mathscr{B}(X')$

and $\mathscr{B}(Y')$ respectively. Let $(\mu'_y)_{y\in Y'}$ be a disintegration of μ' w.r.t. (p',ν'). Let μ and ν be σ-finite measures on $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ respectively and let $(\mu_y)_{y\in Y}$ be a disintegration of μ w.r.t. (p,ν). Then $(X,Y,p,(\mu_y),\nu)$ and a $(X',Y',p',(\mu'_y),\nu')$ are called *isometric* if there exist Borel isomorphisms φ from Y onto Y' and ψ from X onto X' such that

(i) $\varphi\circ p = p'\circ\psi$.

(ii) $\nu\circ\varphi^{-1} = \nu'$.

(iii) $\mu_y\circ\psi^{-1} = \mu'_y$.

4.2 *Remark.* In the situation of Def. 4.1 it follows that $\mu\circ\psi^{-1} = \mu'$.

4.3 *Definition.* A *model disintegration* consists of a Borel set $X'\subset\mathbf{R}^2$, an analytic set $Y'\subset\mathbf{R}$, the canonical projection p' from X' onto the first coordinate, a conditional measure distribution $(\mu'_y)_{y\in Y'}$ with respect to p', and a measure ν' on $\mathscr{B}(Y')$ such that the following conditions hold:

(i) $p'(X') = Y'$.

(ii) $Y'\cap]-\infty,3]$ equals $]-\infty,3]$ or $[-a,-3]$ for some $a>3$ or is empty.

(iii) $Y'\cap]-3,-2]$ is an analytic Lebesgue nullset.

(iv) $Y'\cap]-2,0[$ is contained in $\{-1\pm\frac{1}{n+1}|n\in\mathbf{N}\}$.

(v) $Y'\cap[0,+\infty[$ equals $[0,+\infty[$ or $[0,b]$ for some $b>0$ or is a Borel set of Lebesgue measure 0.

(vi) The atomless part of ν' is one-dimensional Lebesgue measure restricted to Y'.

(vii) The set of atoms of ν' equals $Y'\cap]-2,0[$.

(viii) For $y\in Y'$ with $y>-1$ the set $X'_y\cap[0,+\infty[$ is a non-trivial closed interval containing 0 (possibly a half-line).

(ix) For $y\in Y'$ with $y<-1$ the set $X'_y\cap[0,+\infty[$ is a Borel set of Lebesgue measure 0.

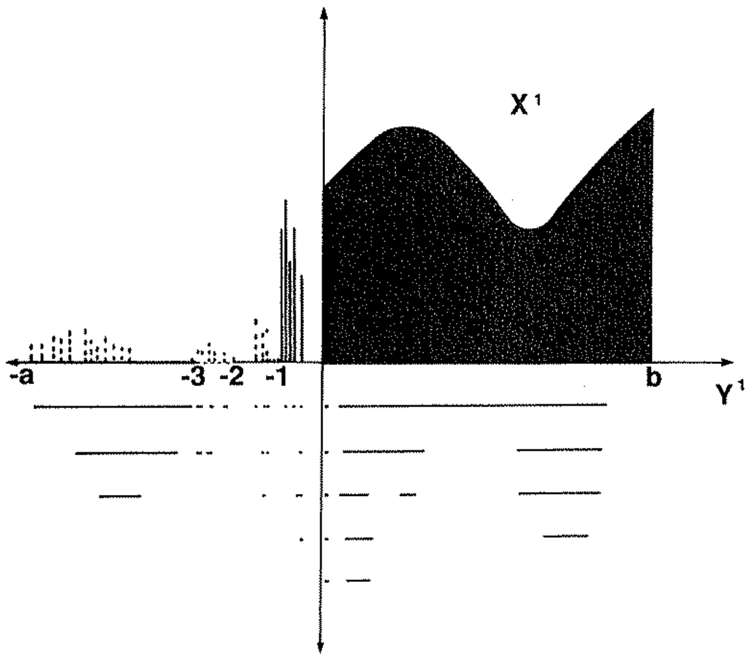
(x) For $y\in Y'$ the set $X_y\cap]-\infty,0[$ is contained in $\{-n|n\in\mathbf{N}\}$.

(xi) For each $y\in Y'$ the atomless part of μ'_y equals one-dimensional Lebesgue measure restricted to $p'^{-1}(y)$.

(xii) The set of atoms of μ_y equals $p'^{-1}(y)\cap(\{y\}\times]-\infty,0[)$.

4.4 *Remark.* Define $\mu' = \int\mu'_y\,d\nu'(y)$. Then the set of atoms of μ' equals $X'\cap]-2,0[x]-\infty,0[$, the atomless part of $\mu'_{|X'\cap(]-\infty,0]\times\mathbf{R}\cup]0,+\infty[\times]-\infty,0[)}$ equals one-dim. Lebesgue measure restricted to $X'\cap(]-\infty,0]\times\mathbf{R}\cup]0,+\infty[\times]-\infty,0[)$ and $\mu'_{|X'\cap]-0,+\infty[\times[0,+\infty[}$ equals two-dim. Lebesgue measure restricted to $X'\cap(]-0,+\infty[\times[0,+\infty[)$.

4.5 THEOREM. *Let μ be a σ-finite measure on $\mathscr{B}(X)$ and let ν be a σ-finite measure on $\mathscr{B}(Y)$. Let $(\mu_y)_{y\in Y}$ be a uniformly σ-finite disintegration of μ with respect to (p,ν). Let $X_a = \{x\in X|\exists y\in Y\colon \mu_y(\{x\})>0\}$, $Y_0 = \{y\in Y\colon \mu_y(X\backslash X_a)=0\}$, $Y_a = \{y\in Y|\nu(\{y\})>0\}$. Let μ_y^{na} be the atomless part of μ_y and ν^{na} the atomless part of ν. Let I_y and Z_y be defined as in Theorem 3.2. Let $\pi_1\colon\mathbf{R}^2\to\mathbf{R}$ be the projection onto the first component. Then there exist Borel isomorphisms φ from Y onto an analytic subset Y' of $\mathbf{R}$ and ψ from X onto a Borel subset X' of $\mathbf{R}^2$ with the following properties:*



EXAMPLE OF A MODEL DISINTEGRATION

(i) $\pi_1 \circ \psi = \varphi \circ p$.

(ii) $\forall y \in Y \colon \psi(p^{-1}(y)) = \pi_1^{-1}(\varphi(y)) \cap X'$.

(iii) $\varphi(Y \backslash (Y_0 \cup Y_a))$ *is a Borel-measurable Lebesgue nullset in* $[0,1]$ *if* $\nu(Y \backslash (Y_0 \cup Y_a)) = 0$ *and*

$$\varphi(Y \backslash (Y_0 \cup Y_a)) = \{t \in \mathbf{R} | 0 \leq t \leq \nu(Y \backslash (Y_0 \cup Y_a))\} \textit{ if } \nu(Y \backslash (Y_a \cup Y_0)) > 0.$$

Moreover, $\nu^{na} \circ (\varphi_{|Y-(Y_0 \cup Y_a)})^{-1} = \lambda_{\varphi(Y \backslash (Y_0 \cup Y_a))}$.

(iv) $\varphi(Y_0 \backslash Y_a)$ *consists of an analytic Lebesgue nullset in* $]-3,-2]$ *and, if* $\nu(Y_0 \backslash Y_a) > 0$, *of* $[-3 - \nu(Y_0 \backslash Y_a), -3] \cap \mathbf{R}$. *Moreover,* $\nu \circ (\varphi_{|Y_0 \backslash Y_a})^{-1} = \nu^{na} \circ (\varphi_{|Y_0 \backslash Y_a})^{-1} = \lambda_{\varphi(Y_0 \backslash Y_a)}$.

(v) $\varphi(Y_0 \cap Y_a) = \{-1 - 1/(n+1) | n \in \mathbf{N}$ *and* $1 \leq n \leq \operatorname{card}(Y_0 \cap Y_a)\}$.

(vi) $\varphi(Y_a \backslash Y_0) = \{-1 + 1/(n+1) | n \in \mathbf{N}$ *and* $1 \leq n \leq \operatorname{card}(Y_a \backslash Y_0)\}$.

(vii) $\forall y \in Y \colon \psi(p^{-1}(y) \cap X_a) = \{\varphi(y)\} \times Z_y$.

(viii) $\forall y \in Y \backslash Y_0 \colon \psi(p^{-1}(y) \backslash X_a) = \{\varphi(y)\} \times I_y$ *and* $\mu_y^{na} \circ \psi^{-1} = \varepsilon_{\varphi(y)} \otimes \lambda_{I_y}$.

(ix) $\forall y \in Y_0 \colon \psi(p^{-1}(y) \backslash X_a) \subset \{\varphi(y)\} \times [0, +\infty[$ *and*

$$\varepsilon_{\varphi(y)} \otimes \lambda_{[0,+\infty[}(\psi(p^{-1}(y) \backslash X_a)) = 0.$$

(x) $\mu^{na}_{|p^{-1}(Y \backslash Y_0)} \circ \psi^{-1} = \lambda^2_{|X'}$ *where* λ^2 *denotes two-dimensional Lebesgue measure,* $\mu^{na}_{|p^{-1}(Y_a)} \circ \psi^{-1}$ *is one-dimensional Lebesgue measure on*

$$X' \cap ([-1,0] \times \mathbf{R}_+),$$

ψ *maps the set of atoms of* μ *onto* $X' \cap ([-2,0] \times]-\infty, 0[)$.

(xi) $\nu^{na} \circ \varphi^{-1} = \lambda_{|Y'}$. φ *maps the set of atoms of* ν *onto* $Y' \cap]-2,0]$.

PROOF. That X_a, Y_a and Y_0 are Borel sets in X and Y resp. either follows from Lemma 3.1 or is easy to verify. Since Y_a is at most countable and since

$Y\backslash Y_0$ is a Lusin space by Theorem 3.2, it follows that $Y\backslash(Y_0 \cup Y_a)$ is a Lusin space. If $\nu(Y\backslash(Y_0 \cup Y_a)) = 0$ let φ_1 be a Borel isomorphism of $Y\backslash(Y_0 \cup Y_a)$ onto a Borel subset of $[0, 1]$ with Lebesgue measure zero. If

$$\nu(Y\backslash(Y_0 \cup Y_a)) > 0,$$

then Lemma 2.1 implies that there is a Borel isomorphism φ_1 from $Y\backslash(Y_0 \cup Y_a)$ onto $\{t \in \mathbf{R} | 0 \leq t \leq \nu(Y\backslash(Y_0 \cup Y_a))\}$ with

$$\nu_{|Y\backslash(Y_0\cup Y_a)} \circ \varphi_1^{-1} = \lambda_{\{t\in\mathbf{R}|0\leq t\leq \nu(Y\backslash(Y_0\cup Y_a))\}}.$$

Since ν is σ-finite on a Suslin space there exists a sequence (K_n) of compact subsets of $Y_0\backslash Y_a$ with $\nu((Y_0\backslash Y_a)\backslash \bigcup K_n) = 0$. $S = \bigcup K_n$ is a Lusin space. Without loss of generality we take $S = \emptyset$ if $\nu(Y_0\backslash Y_a) = 0$. If $\nu(Y_0\backslash Y_a) > 0$ then, by Lemma 2.1, there exists a Borel-isomorphism $\varphi_2^{(1)}$ from S onto $[-3 - \nu(Y_0\backslash Y_a), -3] \cap \mathbf{R}$ with $\nu_{|Y_0\backslash Y_a} \circ (\varphi_2^{(1)})^{-1} = \lambda_{[-3-\nu(Y\backslash Y_a),-3]\cap\mathbf{R}}$. Moreover there exists a Borel-isomorphism $\varphi_2^{(2)}$ from $(Y_0\backslash Y_a)\backslash S$ onto an analytic subset of $]-3, -2]$ of Lebesgue measure 0. Let φ_3 be a bijection from $Y_0 \cap Y_a$ to $\{-1 - 1/n + 1 | n \in \mathbf{N}, 1 \leq n \leq \operatorname{card}(Y_0 \cap Y_a)\}$. Let φ_4 be a bijection from $Y_a\backslash Y_0$ to $\{-1 + 1/n + 1 | n \in \mathbf{N}, 1 \leq n \leq \operatorname{card}(Y_a\backslash Y_0)\}$. Define $\varphi: Y \to \mathbf{R}$ by

$$\varphi|Y\backslash(Y_0 \cup Y_a) = \varphi_1, \varphi_{|S} = \varphi_2^{(1)} \text{ if } S \neq \emptyset, \varphi_{|(Y_0\backslash Y_a)\backslash S} = \varphi_2^{(2)},$$
$$\varphi_{|Y_0\cap Y_a} = \varphi_3, \varphi_{|Y_a\backslash Y_0} = \varphi_4.$$

Then φ is Borel measurable and one-to-one. Let $\tilde{\psi}: X \to Y \times \mathbf{R}$ be the Borel isomorphism in the conclusion of Theorem 3.2. Let $\tilde{\psi}_1, \tilde{\psi}_2$ be the components of $\tilde{\psi}$. Then $\tilde{\psi}_1 = p$. Define $\psi: X \to \mathbf{R}^2$ by $\psi(x) = (\varphi(p(x)), \tilde{\psi}_2(x))$. Set $X' = \psi(X)$. Then φ, ψ and X' obviously satisfy conditions (i) through (xii).

Theorem 4.5 can be rephrased as follows:

4.6 THEOREM. *Let μ be a σ-finite on $\mathscr{B}(X)$. Every uniformly σ-finite disintegration of μ w.r.t. p and a σ-finite measure ν is isometric to a model disintegration.*

Appendix. On the uniform σ-finiteness of disintegrations consisting of σ-finite measures.

Here we deal with the problem whether a conditional measure distribution consisting of σ-finite measures is necessarily uniformly σ-finite. The first counterexample shows that, if we drop the condition that the σ-finite measures of the conditional measure distribution live on different fibers of a measurable map, then the conditional measure distribution is, in general, not uniformly σ-finite.

5.1 THEOREM. *Let $X = Y = \mathbf{R}$. For $y \in \mathbf{R}$ let $\mu_y: \mathscr{P}(\mathbf{R}) \to [0, +\infty]$ be defined by*

$$\mu_y(A) = \operatorname{card}[(A - y) \cap \mathbf{Q}],$$

where $\mathbf{Q}$ denotes the set of rational numbers.

Then each μ_y *is a* σ*-finite measure but there does not exist a sequence* $(A_n)_{n\in\mathbf{N}}$ *of Lebesgue measurable subsets of* $\mathbf{R}$ *with the following properties*

(i) $\bigcup_{n\in\mathbf{N}} A_n = \mathbf{R}$.

(ii) *For each* $y \in \mathbf{R}$ *and each* $n \in \mathbf{N}$, $\mu_y(A_n) < \infty$.

PROOF. First we will show that, for every Lebesgue mesurable $A \subset \mathbf{R}$, the map $y \to \mu_y(A)$ is Lebesgue measurable. To this end let I be a finite subset of $\mathbf{Q}$. Now, for an arbitrary subset A of $\mathbf{R}$:

$$\begin{aligned}&\{y \in \mathbf{R} | (A - y) \cap \mathbf{Q} = I\}\\ &\quad = \{y \in \mathbf{R} | y + I \subset A \text{ and } y + (\mathbf{Q}\backslash I) \subset \mathbf{R}\backslash A\}\\ &\quad = \bigcap_{q\in I}[A - q] \cap \bigcap_{r\in\mathbf{Q}\backslash I} [(\mathbf{R}\backslash A) - r].\end{aligned}$$

This implies, that for every $\alpha \in \mathbf{R}$, and every Lebesgue measurable $A \subset \mathbf{R}$, the set

$$\{y \in \mathbf{R} | \mu_y(A) < \alpha\} = \bigcup_{\substack{I\subset\mathbf{Q}\\ \text{card}\, I<\alpha}} \{y \in \mathbf{R} | (A - y) \cap \mathbf{Q} = I\} \in \mathscr{B}(\mathbf{R})$$

is Lebesgue measurable.

Therefore the map $y \to \mu_y(A)$ is Lebesgue measurable.

Now assume that there exists a sequence (A_n) of Lebesgue measurable sets satisfying conditions (i) and (ii). We will show that this assumption leads to a contradiction. From (ii) it follows that, for every $y \in \mathbf{R}$, the set $(A_n - y) \cap \mathbf{Q}$ has only finitely many elements. Let $(I_k)_{k\in\mathbf{N}}$ be an enumeration of the finite and nonempty subsets of $\mathbf{Q}$ and set

$$B_{nk} = \{y \in \mathbf{R} | (A_n - y) \cap \mathbf{Q} = I_k\}.$$

Our previous calculations show that each B_{nk} is Lebesgue measurable and these sets cover $\mathbf{R}$. Hence there are $k_0, n_0 \in \mathbf{N}$ with $\lambda(B_{n_0k_0}) > 0$, where λ denotes one-dimensional Lebesgue measures. Fix $r \in I_{k_0}$. For every $y \in B_{n_0k_0}$ we have $r \in A_{n_0} - y$, hence $r + B_{n_0k_0} \subset A_{n_0}$. Since $\lambda(r + B_{n_0k_0}) > 0$ we know by Steinhaus' theorem, that there exists an $\varepsilon > 0$ with

$$\begin{aligned}(-\varepsilon, \varepsilon) &\subset r + B_{n_0k_0} - (r + B_{n_0k_0})\\ &\subset A_{n_0} - B_{n_0k_0} - r.\end{aligned}$$

Thus

$$\begin{aligned}(-\varepsilon, \varepsilon) \cap \mathbf{Q} \subset (A_{n_0} - B_{n_0k_0} - r) \cap \mathbf{Q} &= [(A_{n_0} - B_{n_0k_0}) \cap \mathbf{Q}] - r\\ &= I_{k_0} - r.\end{aligned}$$

But the right hand side of this inclusion is a finite set while the left hand side is infinite, a contradiction.

The following theorem shows that a conditional measure distribution $(\mu_y)_{y\in Y}$ with respect to a measurable map $p\colon X \to Y$ consisting of σ-finite measures μ_y is uniformly σ-finite in a very weak sense.

5.2 THEOREM. *Let X, Y and p be as in the preliminaries. Let $(\mu_y)_{y\in Y}$ be a conditional measure distribution with respect to p such that each μ_y is σ-finite. Then there exists a sequence $(A_n)_{n\in\mathbf{N}}$ of subsets of X with the following properties:*

(i) $X = \bigcup_{n\in\mathbf{N}} A_n$.

(ii) *For each $n \in \mathbf{N}$ and each $y \in Y$ the set A_n is μ_y-measurable with $\mu_y(A_n) < \infty$.*

PROOF. Due to the σ-finiteness of μ_y, for each $y \in Y$, there exists a sequence $(A_n^y)_{n\in\mathbf{N}}$ of Borel subsets of X with $\bigcup_{n\in\mathbf{N}} A_n^y = p^{-1}(y)$ and $\mu_y(A_n^y) < \infty$. Define $A_n = \bigcup_{y\in Y} A_n^y$. Since μ_y is concentrated on $p^{-1}(y)$ we have $\mu_y^*(A_n \backslash p^{-1}(y)) = 0$ and, therefore, $A_n \backslash p^{-1}(y)$ is μ_y-measurable. (Here μ_y^* is the outer measure corresponding to μ_y.) On the other hand $A_n \cap p^{-1}(y) = A_n^y$ is a Borel set in X. Thus A_n is μ_y-measurable and $\mu_y(A_n) = \mu_y(A_n^y) < \infty$.

5.3 *Open problems.* In the preceding theorem, can the sets $(A_n)_{n\in\mathbf{N}}$ be chosen to be Borel subsets of X, i.e., is $(\mu_y)_{y\in Y}$ uniformly σ-finite? A slightly weaker problem is whether the sets (A_n) can be chosen to be universally measurable or to be in the σ-field generated by the Suslin subsets of X.

The first of the above questions was asked by Maharam [**4**].

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S. GRAF
FAKULTÄT FÜR MATHEMATIK UND INFORMATIK
UNIVERSITÄT PASSAU
POSTFACH 2540
D-8390 PASSAU
FEDERAL REPUBLIC OF GERMANY

R. D. MAULDIN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH TEXAS
P.O. BOX 5116
DENTON, TX 76203