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MATHEMATICS OF DIMENSION MEASUREMENT FOR GRAPHS OF FUNCTIONS

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A stable ("real boxes") box-counting procedure introduced previously to measure the capacity dimension of the graph of a non-differentiable function or random process data-series, is examined for a test case example due to Besicovitch and Ursell, where the limit necessary for the definition doesn't exist.

There are a number of questions about dimensions of graphs of functions and their numerical measurement. These questions bear, in particular, upon issues of fractal modeling of data presumed to be represented by random processes [1-3]. We have discussed one of these questions earlier [3], addressing issues associated with the (at best) self-affinity of the graph of a function, f(t) against t [4,5]. Here we address a second. The box-counting notion of dimension, for the mathematical case, that a limit $\epsilon \to 0$ is required, is called the capacity dimension, $\dim_{cap}$; it is never less than the Hausdorff dimension, $\dim_H$, which was introduced nearly seventy years ago by Felix Hausdorff [6]. An important difference between these two is that the limits necessary to define $\dim_H$ always exist, while for $\dim_{cap}$ that is not always so. This situation leads to a question. If we have a stable algorithm, for box-counting , what happens for the case where $\dim_{cap}$ doesn't exist? To what does this "stable" procedure actually lead?

In our early experience with box-counting applied to graphs we used a simple grid to effect the count. We quickly found we could get any answer we wanted [2]. Our solution was to abandon the idea that the count has to be an integer. One simply defines the box-count in the ith ϵ-bin as $N_i(\epsilon) \equiv \epsilon^{-D}(\max_i f - \min_i f)$, where $\max_i$ and $\min_i$ denote maximum and minimum values taken by the function f in the ith bin. This is the sort of thing one might do at a first cut approximation before becoming serious about the problem; but it's actually the precisely correct thing to do. We have the immediate relationship for covers of graphs by these "real boxes"

$$N_{kf}(\epsilon) = kN_f(\epsilon) \qquad \text{(real boxes)} \quad , \tag{1}$$

for <u>all</u> ϵ. Here N_f and N_{kf} denote the counts for the cover of the graphs of f and kf, respectively, where $k \neq 0$ is a constant factor. Thus, for example, the dimension of the graph of a data-string will not depend on the choice of units since now $\log N_{kf}$ and $\log N_f$ are mere translates, by log k, of one another. Consequently, any dimension measurement procedure based on the behavior of $\log N(\epsilon)$ <u>vs</u>. $\log \epsilon$ will give the same result for both f and kf. But this critical result is gained for box-counts generated by grids only for the physically unrealizable limit, that $\epsilon \to 0$.

An interesting property of the real-box count for cover of the graph of a Levy-Mandelbrot variant of the nowhere differentiable Weierstrass-Hardy fractal function,

$$W(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{\gamma^{\alpha n}} \sin\gamma^n t, \ \gamma > 1, \ 0 < \alpha < 1 \ , \tag{2}$$

is that the self-affine scaling property, [1] $W(\gamma t) = \gamma^{\alpha} W(t)$, can be used to help show that $N(\epsilon)\ \epsilon^{-D}$, where $D = 2-\alpha$, and where now $\epsilon \to 0$ is no longer needed. More carefully, apart from a factor $1 + O(\epsilon/T)$, where T is the length of the interval over which the graph of W is covered,

$$A_1 \epsilon^{-D} < N(\epsilon) < A_2 \epsilon^{-D} \ . \tag{3}$$

The two constants, A_1, and A_2, are not equal, so the local slopes of log $N(\epsilon)$ against log ϵ, instead of being constant, can oscillate. Figure 1 shows a numerical study for a close cousin of $W(t)$, where instead of $\sin\gamma^n t$ terms in the expansion, a sawtooth function is used (Eq. (5)). The oscillations have period log γ, and happily, the answer converges to D.

Besicovitch and Ursell [7] have given an example of a class of functions whose Hausdorff dimension [6] is known to be different from $2-\alpha$. The generic Besicovitch-Ursell (BU) function is

$$f(t) = \sum b_n^{-\alpha}\,\Phi(b_n t), \quad \frac{b_{n+1}}{b_n} \geq B_1 > 1 , \tag{4}$$

where was Φ taken to be the periodic sawtooth, viz.

$$\Phi(x) = 2x, \; 0 \leq x \leq \frac{1}{2} \; ; \; \Phi(x) = \Phi(-x) = \Phi(x+1), \text{ elsewhere} . \tag{5}$$

For $b_{n+1}/b_n = B_1 \equiv \gamma$, $f(t) = W(t)$ when $\Phi \to \sin$. We refer to this as the geometric, or lacunary case. On the other hand, the class specified by two the numbers, α and d,

$$b_n = b^{\mu^{n-1}} , \; \mu \equiv \frac{1-\alpha}{\alpha}\,\frac{2-d}{d-1} , \; 1 < d < 2 - \alpha , \tag{6}$$

have graphs whose Hausdorff dimension [6] is d, and $d < 2 - \alpha$. This we refer to as the exponential, or super-lacunary case. Here again a plot of log $N(\epsilon)$ against log ϵ shows oscillations, but this time with growing period. Moreover, as may be seen from Fig. 2, attendant oscillations of log $N(\epsilon)/\log \epsilon^{-1}$ do not converge

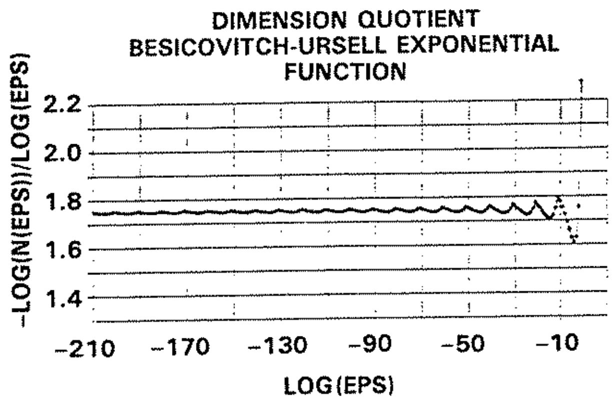

FIGURE 1 $\log_2 N(\varepsilon)/\log_2 \varepsilon^{-1}$ for the cover of the graph of W(t) on [0,1], for $\alpha = 0.25$ and $\gamma = 2^{10}$. Theorem (BU): dim $\leq 2 - \alpha = 1.75$.

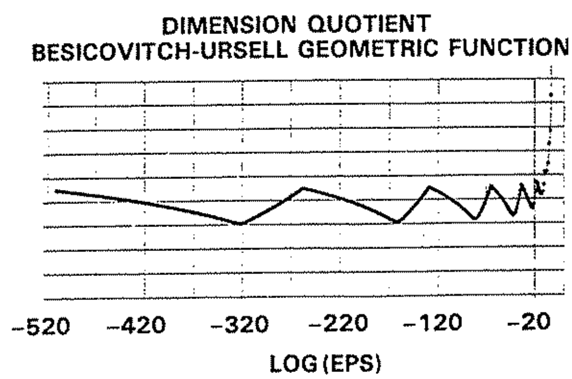

FIGURE 2 $\log_2 N(\varepsilon)/\log_2 \varepsilon^{-1}$ for the cover of the graph of f(t) on [0,1], for $\alpha = 0.25$ and b = 2, $\mu = 2$, which gives d = 1.6. Theorem (BU): dim = d = 1.6 $< 2 - \alpha = 1.75$.

as $\epsilon \to 0$, but instead, as can be proven, bounce indefinitely between d and $2 - \alpha$: the limit for the box-counting dimension simply doesn't exist; it's not as general a concept as the Hausdorff dimension.

The mathematical situation is summarized by the following:

Theorem (BU). Let f be a BU series. Then

$\dim_H \leq 2-\alpha$ (lacunary case), $\quad \dim_H = d < 2 - \alpha$ (super-lacunary case).

Remark. The Hausdorff dimension of the graph of f(t) against t for the geometric, or lacunary case is still an open question.

Theorem. Let f be a continuous, non-constant function on [0,1]. If any one of the limits,

$$d_g = \lim_{\epsilon\to 0} \frac{\log N_g(\epsilon)}{-\log \epsilon}, \quad d_i = \lim_{\epsilon\to 0} \frac{\log N_i(\epsilon)}{-\log \epsilon}, \quad d_r = \lim_{\epsilon\to 0} \frac{\log N_r(\epsilon)}{-\log \epsilon}, \qquad (7)$$

exists, then the other two exist and are the same. In the theorem, N_g is the box-count obtained by laying a square grid of mesh size ϵ over the graph of f, N_i is the minimal count using square (integer) boxes, not necessarily arranged as a grid, and N_r is the real-boxes count described in the previous section, and employed in previous data analyses [2,3].

Theorem. Let f be a BU function. Then

$\dim_{cap} = 2 - \alpha$ (lacunary case, γ:integer)

$\overline{\lim}_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log \epsilon^{-1}} = 2 - \alpha$; $\underline{\lim}_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log \epsilon^{-1}} = d$ (super-lacunary case).

Remark. If Φ is replaced by a C^1 periodic function in Eq. (4), $\dim_{cap} = 2 - \alpha$ was proved in [9]. It is probably true for Eq. (4) for all γ, also, but we don't have a proof at present.

The forgoing results have been gained for the deterministic case. Randomized versions of BU series probably have similar properties. The result gained from dimension measurements based on $\log N(\epsilon)$ vs. $\log \epsilon$, behavior for the super-lacunary case will depend on how one specifies D from the plots; any value from d to $2 - \alpha$ may be gotten. However, if the super-lacunary case can be ruled out, e.g., by examination of the psd, then within the BU class, Eq. (7) quarantees that the real-box count limit gives $\dim_{cap}$.

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