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Mathematical Problems and Games

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INTRODUCTION

Some twenty-five years ago Stan Ulam began this new collection of problems as a sequel to his Problem Book [1]. Several years ago he invited me to become a part of his project to develop and present these problems.

Our work proceeded by bits and pieces, mostly in the summers, since I was caught up in my professorial duties during the academic year and Stan was generally busy with lecturing and other professional engagements. In addition, his work habits, at least in his later years when I knew him, made it somewhat difficult to put things together: his manner was to explore and discuss a specific problem or a vague speculation for a short time, then leave it and go on to something else. Invariably the subjects would reappear and we repeated the process, until something new would be seen and the problem or speculation became more definite. While our discussions were great fun, progress was slow. At all times, however, including those of intense concentration, our mood was lightened by Stan's frequent bursts of humor. This allowed him to be playfully creative with seemingly serious and "sacrosanct" mathematical structures.

Stan had some very general ideas concerning the edifice of mathematics and the role of mathematics in other sciences, particularly physics and biology, which he planned to develop extensively and incorporate in this collection.

We were in the process of generating specific problems to illustrate these theories and we had been planning enthusiastically to spend the summer finishing this work when our collaboration was abruptly interrupted by

Stan's sudden death, May 13, 1984, at the age of 75. I therefore have had to complete this collection alone.

Stan was a true genius and he had a tremendous talent for getting to the heart of the matter (witness in his early problems [27]), but he was loath to struggle with finishing details, and he often left that to others. Endowed with a phenomenal memory he kept most things in his mind after he had worked them out in the form of cryptic little notes scribbled on small pieces of paper.

Another characteristic of his was an uncanny feeling for relative orders of magnitude both in mathematics and in the natural sciences. This unusual ability was already apparent in his early works on the theory of cardinal numbers and continued to his conjectures in combinatorics and graph theory. This ability is well documented in the physical problems he treated in Los Alamos. At the same time it led him to become one of the originators of the use of computers as an experimental tool in physics and mathematics, and he became very adept at adjusting initial guesses and conjectures in the light of numerical evidence.

He also loved to play little games. These sometimes became transformed into seminal examples of new fascinating processes. In this connection he liked to quote two lines he had found in Shakespeare [3]:

> Things done without example
> In their issue are to be feared.

Some readers will probably recognize a number of problems; they are those Stan freely tossed around over the years in his innumerable talks and seminars and never bothered to write up. (Some of the ideas in the realm of mathematics and science which are floating around today in the public domain can be traced to his casual remarks.) Other problems we reworked together; still others we formulated jointly. Some are my own.

He meant this collection to include a gathering of all his thoughts of the last twenty-five years, but I had to abbreviate the planned sections on physics and biology for Stan's written notes concerning general schemes were unfortunately very sketchy and our interaction on these topics had not yet reached a level where it would have been proper for me to try to read his mind. Incomplete as this last section is, I hope that it may inspire some budding mathematicians to pursue and develop his train of thought, just as his ideas have inspired so many of us in the past.

Without the encouragement and assistance of Françoise Ulam it would not have been possible to complete this work. It is evident from the text that Jan Mycielski made a great contribution to this project. Among the many who graciously helped me were Paul Erdös, Ron Graham, and G.-C.

Rota. I also wish to thank the National Science Foundation, the Sloan Foundation, and Los Alamos National Laboratory for their support.

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REFERENCES

1. S. M. ULAM, "A Collection of Mathematical Problems," Interscience, New York, 1960.
2. "The Scottish Book," Birkhäuser-Boston, Boston, 1982.
3. "Henry VIII," Act I, Scene 2, lines 90–91.

I. LOGIC, SET THEORY

I.1. *Variations on Suslin's Problem*

A Suslin family of subsets of a set X is a collection S of subsets of X such that any two members of S are either disjoint or one is a subset of the other.

For each cardinal n, finite or infinite, what is the maximal size of a Suslin family of subsets of a set of cardinality n? Clearly if $\text{card}(X) = \aleph_0$, then there is a Suslin family of cardinality $2^{\aleph_0}$. For any infinite cardinal m, there is a monotone family of size m^+.

We say that two Suslin families S_1 and S_2 are "internally" isomorphic provided there is a permutation of X which takes S_1 onto S_2. We say that S_1 and S_2 are "externally" isomorphic provided S_1 and S_2 are algebraically isomorphic, i.e., there is a one-to-one map of S_1 onto S_2 which preserves disjointedness and inclusion. For each cardinal n, how many non-isomorphic Suslin families are there? Erdös comments that for $\aleph_0$, there are 2^c internally non-isomorphic Suslin families.

I.2. *Generalized Suslin—Visual Classes*

Suppose that we only "visually" distinguish between two sets. We only know which of the following three possibilities occur: they are disjoint, one is contained in the other or they have a nonempty intersection. Thus, a Suslin family is a collection of sets for which only the first two configurations are allowed.

Now according to these rules there are eleven possible visual configurations of three sets. Some estimates on the exact rate of growth in the number of configurations of n sets have been given by Lynch [1]. However, the exact rate is an open problem.

A generalized (or k) Suslin family is a family of subsets of a set X such that only certain 3 (or k) set visual configurations are allowed. For each cardinal n, finite or infinite, what is the maximal size of such a generalized Suslin family of subsets of n? For each cardinal n, how many non-isomorphic (internally or externally) Suslin families are there? What is the rate of growth in the number of these families with n? In connection with this, some estimate in the number of (internally) non-isomorphic subfamilies of the iterated power set operator have been given by Lynch [2].

I.3. *Product Space Problems*

Let $R = \{A \times B : A, B \subset [0,1]\}$. It is known that if the continuum hypothesis holds (or Martin's axiom), then every subset of $[0,1] \times [0,1]$ is in $\mathscr{B}(R)$, the Borel field generated by R. In fact every subset of $[0,1] \times [0,1]$ is in the family $R_{\sigma\delta}$. In general, if every subset of $[0,1] \times [0,1]$ is in $\mathscr{B}(R)$, then there is a countable ordinal α such that every subset of $[0,1] \times [0,1]$ is generated in α steps from R [3]. It is consistent with ZFC that this ordinal α be any countable ordinal ≥ 2 [4].

On the other hand, it is consistent that every subset of $[0,1] \times [0,1]$ be the kernel of some Suslin scheme with sets from R, and yet that not every subset be in $\mathscr{B}(R)$ [5]. What is the situation if we allow more general operations, e.g., Hausdorff operations?

If the continuum is real-valued measurable, is there a sequence of sets $\{E_n\}_{n=1}^{\infty}$ such that the Borel field generated by these sets includes all analytic sets? The reason for this question is that if the continuum is real-valued measurable, then there is a family of subsets of $[0,1]$ with cardinality $2^{\aleph_0}$ such that no sequence of sets generates this family [6]. We are seeking an explicit example of such a family.

Is there a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of $[0,1]$ which separates points and such that "all questions" are decided for the σ-projective algebra generated by this sequence? In other words, each set in this algebra is either countable or of power $2^{\aleph_0}$, each set is measurable with respect to some continuous probability measure and there is some reasonable σ-ideal of sets such that each set in the algebra has the Baire property with respect to this algebra.

I.4. *Product Space Mappings*

Is there a one-to-one map g of E^3 into E^2 such that the image of each set of the form $A \times B \times C$ is in the σ-field generated by sets of the form

$A \times B$? What is the situation when 2 and 3 are replaced by integers n and m?

Perhaps there is a theorem concerning the invariance of the combinatorial dimension of the nth product of the set of all integers E. If $n < m$, is it true that there is no mapping of E^n onto E^m such that the image of each generalized subproduct in E^n is the union of finitely many subproducts in E^m? Perhaps there is no map which takes "almost all" subproducts to such unions.

I.5. *Problems for Projective Algebra à la Ramsey*

Divide a projective algebra into two classes of elements and then add the elements 0 and 1 to each class. Must there be a big projective algebra in one of these classes? We consider this for the operations of products, unions and projections only.

An analogous problem for relation algebras occurs when we consider the operations of composition.

Choose two subsets A and B of $\mathbb{Z} \times \mathbb{Z}$ at random. (This means with respect to Haar measure on $\{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$.) What is the probability that the projective algebra generated is infinite or even dense in $\{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$? Is this probability 1?

Choose a sequence of subsets $A_1, A_2, A_2, \ldots$ of $\mathbb{Z} \times \mathbb{Z}$ independently at random. What is the probability that all the A_n's are elements of a finitely generated projective algebra?

I.6. *Infinite Sequences of* 0*'s and* 1*'s with Changes on Sets of Frequency* 0

Consider the space S of all infinite two-way sequences of 0's and 1's: $S = \{0,1\}^{\mathbb{Z}}$. We say that two sequences σ and τ are equivalent provided σ may be transformed into τ by a finite number of applications of operations of one of the following types.

Type 1: We may shift σ to the right or left one unit.

Type 2: We may erase some 0's (or 1's) in σ as long as the set of indices which are erased has frequency (or arithmetic density) zero.

Type 3: We can change some entries from 0 to 1 (or 1 to 0) provided the set of entries has frequency 0.

Type 4: We may insert 1's (or 0's) into the sequence σ as long as the set of insertions has density 0.

This clearly defines an equivalence relation on S. Is there a Borel subset of S which meets each equivalence class in exactly one point?

Mycielski [7] asks whether there is a Borel isomorphism between $P(\omega)/\text{Fin}$ and $\{0,1\}^{\mathbb{Z}}/\text{shifts}$ or even the space of equivalence classes given above? Mycielski showed $P(\omega)/\text{Fin}$ is Borel isomorphic to $\mathbb{R}/\mathbb{Q}$.

One reason for considering this space is that it models some aspects of the theory of finite sequences that arise in biology [8, 9].

I.7. *Problems in Classes of Finite Sets à la van der Waerden*

Characterize sequences of sets of integers $\{A_k\}_{k=1}^{\infty}$ such that in every division of $\mathbb{Z}$ into two disjoint classes at least one must contain infinitely many of the A_k's. If a permutation of $\mathbb{Z}$ is given, must there then be infinitely many A_k's going into A_i's?

Erdös comments that there is probably no reasonable answer to the first question. The second question raises some interesting problems. First, a negative example. If $A_1, A_2, A_3, \ldots$ is a list of all arithmetic progressions of length t, then the answer is no provided t is chosen such that any subset of size $t/2$ of an arithmetic progression of size t contains an arithmetic progression of length 3. To see this, write $N = A \cup B$, where B contains no three term arithmetic progressions and $A = N \setminus B$. Let φ be a permutation of N which maps A onto B. Now, suppose $x_1, \ldots, x_t$ is an arithmetic progression of length t. At least $t/2$ of the x's lie in A. Otherwise, the x's in B would contain an arithmetic progression of length 3 and B has none. Let $y_i = \varphi(x_i)$. If $y_1, \ldots, y_t$ formed an arithmetic progression of length t, then, again, by Roth's theorem, the y's in B would contain an arithmetic progression of length 3. The situation when the A's list all three or four term progressions remains open. Davis, Entringer, Graham, and Simmons [10] have considered similar problems.

I.8. *Internal and External Boolean Operations on Classes of Sets (Joint with Rota)*

Let E be a set—finite, infinite or uncountable. We consider families of subsets of E. If $\mathscr{A}$ and $\mathscr{B}$ are two such families, then $\mathscr{A} \cup \mathscr{B}$, the usual "external" union, is the family of all sets which are elements of $\mathscr{A}$ or $\mathscr{B}$. The "internal" union of $\mathscr{A}$ and $\mathscr{B}$ consists of all sets C which can be expressed as the union of two sets, one from $\mathscr{A}$ and the other from $\mathscr{B}$. Note that an internal union of a family with itself leads, in general, to a new family in contrast to an external or Boolean such union. Similarly, we consider the internal intersection of two families and the internal complement of a family.

One problem concerns the existence of a basis for families of sets. One way to construct such a basis is to find an analog of the Rademacher–Walsh sequence of sets which through the Boolean operations give all possible subsets. Given a fixed basis one may define the complexity of a family by the smallest number of operations leading from the basis to the given family. The allowed operations could be not only the internal and the external Boolean operations but also the operation of direct product, projection and perhaps fixed allowed transformations of a given initial set E into itself.

As in the study of subsets themselves one could perhaps allow the formation of the operation of subsets—in our case of families of sets, the formation of all subfamilies.

A number of questions concern the character set theoretically or topologically, of families of sets; for example, the idea of a Borel family or projective family, etc.

One would like to generalize problems like that of Suslin for existence of families of sets with given properties by considering subsets of the set of all integers or the set of all real numbers. The idea is to define a class of properties of families generalizing the property of Suslin and then attempt to prove that the set of all undecidable families is of measure one in the class of all properties defined initially.

This could perhaps be attempted through picturing families of sets as a hyper-graph (a set of the family being a vertex of the hyper-graph) enumerated by real numbers. In this way, the meaning of measurable or a Borel set of families will be made precise.

Let E be a set with n elements. What is the probability that if kn classes of subsets of E are chosen at random the union of the sets in these classes covers E? Finally, how many non-isomorphic "external" rings of classes of subsets of E are there? Rota also had in mind problems concerning these Boolean operations in an entirely different direction.

I.9. *Product Isomorphisms*

Let E be a set and A and B be subsets of $E \times E$. We say that A and B are product isomorphic provided there is a one-to-one map τ of E onto E such that $\tau \times \tau(A) = B$, where $\tau \times \tau(x, y) = (\tau(x), \tau(y))$. If $|E| = n$, how many classes of product isomorphic sets are there? Of course, this number is $\geq 2^{n^2}/n!$. Note that if $E = \{1, \ldots, n\}$ and $A = \{(i, j) | i \leq j\}$, then the images of A under product isomorphisms are distinct. We say that A and B are k-weakly product isomorphic provided A can be partitioned into sets $A_1, \ldots, A_k$ and B into sets $B_1, \ldots, B_k$ such that for each i, A_i is product isomorphic to B_i.

For A and B to be k-weakly isomorphic for some k, is it necessary and sufficient that $|A| = |B|$? Call $D(A, B)$ the minimal k for such A and B. What is the expected $D(A, B)$ taken over all A and B with cardinality n?

One can of course consider these problems for subsets of $E \times E \times E$, etc.

I.10. *Problems on Matrices of Abstract Sets*

Erdös and Ulam have raised the following problem [11]: Let $\bar{k} = \langle k_n : n \in \omega \rangle$ with $2 \le k_1 < k_2 < \cdots$ and let κ be a cardinal. Let PART$(\bar{k}, \kappa)$ say: there exist partitions of κ into k_n disjoint pieces $A_0^n, A_1^n, \ldots, A_{k_n-1}^n$ such that for all $f \in \pi k_n$,

$$|\kappa \setminus \bigcup A_{f(n)}^n| \le \omega.$$

So, PART$(\bar{k}, \omega)$ is true. PART$(\bar{k}, c^+)$ is false; moreover, $\kappa < \lambda$ and PART$(\bar{k}, \lambda)$ implies PART$(\bar{k}, \kappa)$.

Erdös and Ulam showed that for any $\omega \le \kappa$ and partitions $A_0^n, \ldots, A_{k_n-1}^n$, $n = 1, 2, 3, \ldots$, there is always some p_n, $0 \le p_n < k_n$, $n = 1, 2, 3, \ldots$, such that

$$|\kappa \setminus \bigcup A_{p_n}^n| \ge \omega.$$

They noted that if $c = \omega_1$, then PART$(\bar{k}, c)$ is true. They speculated that whether PART$(\bar{k}, c)$ is true could not be answered without some assumption about the power of the continuum. Ken Kunen has shown that this is indeed the case as follows.

To get c big and PART$(\bar{k}, c)$: Let GCH hold in V, and let $\kappa > \omega_1$ be regular. Let $V[G]$ be V with κ Cohen reals added. So, $V[G] \models c = \kappa$. Think of G as adding ω mutually generic functions, φ_n: $\kappa \to k_n$, and let $A_i^n = \varphi_n^{-1}\{i\}$. Then for each $f \in \pi k_n$, $\bigcup A_{f(n)}^n$ contains all $\alpha < \kappa$ except possibly for those (countably many) such that $\lambda_n \cdot \varphi(\alpha)$ is not Cohen generic over $V[f]$.

On the other hand, $MA + {}^\neg CH \Rightarrow {}^\neg$PART$(\bar{k}, \omega_1)$. To see this: Given the partitions, for each α let $\varphi_\alpha \in A_{\varphi_\alpha(n)}^n$. By MA, choose $f \in \pi k_n$ which is eventually different from all φ_α. Then for each $\alpha < \omega_1$, there is an $f' \simeq f$ (mod finite) with $\alpha \notin \bigcup A_{f'(n)}^n$. Thus, $\omega_1 = \bigcup_{f' \simeq f}[\omega_1 \setminus \bigcup A_{f'(n)}^n]$. So, for some $f' \simeq f$,

$$|\omega_1 \setminus \bigcup A_{f'(n)}^n| = \omega_1.$$

Among the questions raised by Erdös and Ulam is this. If PART$(\bar{k}_0, c)$ holds for a given $\bar{k}_0$, then does PART$(\bar{k}, c)$ hold for all $\bar{k}$?

I.11. *Universal Sets*

Is there an analytic subset E of $[0,1] \times [0,1]$ which is universal for all Borel sets with positive measure? This means that for each $x \in [0,1]$, E_x is a Borel subset of $[0,1]$ with positive measure and if B is such, then there is some x such that $B = B_x = \{y | (x, y) \in E\}$. There is a coanalytic set with these properties [12]. Is there such an E with the property that for each B of positive measure there is a unique x with $B = E_x$?

I.12. *Mauldin Problem: Analytic Sets and the Borel Hierarchy*

Let E be a subset of $[0,1]$ and let $\mathscr{B}$ be the family of Borel subsets of E. Suppose there is a subset A of E such that A is the kernel of a Suslin scheme,

$$A = \bigcup_{\sigma \in \mathbf{N}^{\mathbf{N}}} \bigcap_{n=1}^{\infty} G(\sigma | n),$$

where each $G(s_1, s_2, \ldots, s_n)$ is open relative to E, and A is not a Borel set with respect to E. Is it true that for every $\alpha < \omega_1$, there is a Borel subset of E which is not generated in α steps from the open sets by the operations of countable unions and intersections?

I.13. *Problems and Theorems from Ulam's Master's Thesis of 1932*

Jan Mycielski has gone through Ulam's unpublished Master's Thesis and has translated and reformulated some of the problems and theorems in it, for "this may have some historical interest, indicating what a very good student in Lwów around 1930 was thinking about."

Ulam in his autobiography [13] describes how he worked on it for a week and wrote it up in one night!

(1) What are the cardinalities of the subgroups of S_n? This problem is open.

(2) Let G be a subgroup of S_n. Is there a binary relation $R \subseteq n \times n$ such that G is the automorphism group of the structure $\langle n, R \rangle$? If not, is there a ternary relation with that property?

(3) How many isomorphism types of n-ary relations on a set of cardinality n are there? This problem was apparently first solved by Oberschelp [14].

(4) A and B finite groups, $A^n \cong B^n$. Is $A \cong B$? This problem was solved by L. Lovasz in the affirmative (*Acta. Math. Acad. Sci. Hungar.* **18** (1967), 321–328). His theorem yields this result for any finite algebras A and B.

(5) THEOREM. *If for all A, $B \in \mathbf{C}$ ($\mathbf{C}$ is a class of structures closed under direct products) $A^2 \cong B^2 \Rightarrow A \cong B$ and for some $D \in \mathbf{C}$, $D^n \cong D$ for all $n = 2, 3, 4, \ldots$, then $D^n \not\cong D^m$ for all $n \neq m$.*

(6) Let X be a Frechet space, i.e., a set R of ω-type sequences of elements of X, i.e., $R \subseteq X^{\omega}$, is distinguished and a function $L\colon R \to X$ is given such that

(A_1) If $\sigma \in R$ and σ_0 is a subsequence of σ, then $\sigma_0 \in R$ and $L(\sigma_0) = L(\sigma)$.

(A_2) $(a, a, \ldots) \in R$, for each $a \in X$, and $L(a, a, \ldots) = a$.

(A_3) If $\sigma = (\sigma_0, \sigma_1, \ldots) \in R$, and $\tau = (\tau_0, \tau_1, \ldots) \in R$, and $L(\sigma) = L(\tau)$, then $L(\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots) = L(\sigma)$.

THEOREM. *$\langle X, R, L\rangle$ and $\langle Y, S, Q\rangle$ are homeomorphic Frechet spaces if and only if the structures $\langle X, R\rangle$ and $\langle X, S\rangle$ are isomorphic.*

Proof. If one knows $\langle X, R\rangle$, L can be reconstructed. Namely, $L(a_0, a_1, \ldots) = a$ if and only if $(a_0, a, a_1, a, a_2, a, \ldots) \in R$.

II. COMBINATORICS, GAME THEORY, NUMBER THEORY, COMPLEXITY

II.1. *Games with Sets as a Result*

A game between two players on the set $X = \prod_{i=1}^{\infty} X_i$ is played where $X_i = \{1, 2, \ldots, 3^i\}$. A Borel measurable subset E of X is given. Player I chooses $n_1 \in X_1$, Player II chooses $n_2 \in X_2$, etc. Consider the set $G = \{x \in X: x(i) \neq n_i;\ i = 1, 2, 3, \ldots\}$. The set G is closed in X and has measure $1/2$, with respect to the standard product measure m. If the measure of $G \cap E$ is greater than the measure of $E - G$, then player I wins. Is it true that one of the players has a winning strategy? What if we declare that player I wins in case the measure of $G \cap E$ is greater than $G \cap (X - E)$? What if we declare that player I wins in case $m(G \cap E)/m(E) > m(G \cap E')/m(X \setminus E)$? Comment: it follows from Martin's [15] theorem that all Borel games are determined that the answer to both of these questions is yes.

Consider a game on X between three players. A partition of X into sets A, B and C is given. Each player in turn chooses an integer n_i. Consider the set G as before. Player I wins if $m(G \cap A)/m(A)$ is greater than the other two ratios or relative measure. Does one of the players have a winning strategy? We assume that no coalition is allowed.

II.2. *Some Problems Connected with Playing Solitaires*

We shall give a few examples of mathematical problems based on playing combinatorial schemata extracted from playing card solitaires and phrased in probabilistic connections and with investigations of strategies which are optimal for winning such games.

These games are played against a passive player, so to say, who does not participate actively in the game. —"Nature" might play such a role, and understanding some physical problems could possibly be of that sort.—

We start with examples of the simplest games played on a finite set; we shall generalize these to, say, countably infinite ones.

EXAMPLE 1. Suppose a permutation is given of the first N integers which is written down on a line as follows: We first write zero, leave one place next to it on the right vacant, then write the given permutation of the integers from 1 to N. The game proceeds in this fashion: We put in the vacant place on the right of 0 the number 1, which is written somewhere in the permutation. This will leave a vacant place where the number 1 was located. The place immediately to the left of this vacant place is occupied by some integer. Call it n_1. We look for the integer $n_1 + 1$ and put it in this vacant place. This will leave a vacancy at the place from where it was taken. Let n_2 be the integer occupying the place immediately to the left of the new vacancy. We continue this procedure as long as we can. The solitaire is won if we obtain the permutation which is identity, that is, gives $0, 1, 2, \ldots, N$.

Clearly the chance of winning is very small, because if N is sizeable, the chance is very large of getting, during the process, a vacancy such that N is the leftmost preceding it. This situation ends the game since there is no longer any way to move. In fact, it is easy to see that the chance of winning decreases exponentially with N. There is no question of strategy in this game; it is completely deterministic.

EXAMPLE 2. Suppose we now have a different arrangement: We write 0 and then we put two vacancies next to it on the right and then a given permutation of the N integers. Now at every move there are two choices of putting the numbers into the vacancies. There will be in general, in the beginning, at least two different vacancies, and the outcome of the game depends on the strategy of the order in which we fill these vacancies. The problem arises: What are the probabilities of winning using the best strategy for a given N? The chance of winning is now larger.

We could, of course, have more than two holes. One little mathematical question that arises would be, e.g., if the number of vacancies is, say, about $\sqrt{N}$, is there a sizeable chance of winning, i.e., not tending to 0 with N tending to infinity?

This scheme is reminiscent of the Canfield solitaire. We have considered, so to say, a linear or one-dimensional game. One could imagine more generally that we have a number of permutations of integers which are of k different colors. We now write each given permutation of the first N integers in a row, forming k rows one above the other. In each row we start with 0 and have a vacancy after it. The game is now played by trying to obtain the identity permutation in each row, but in each row the colors must be the same. Of course, in each move of the game we can still fill a vacancy behind some integer n with any of the k integers $n + 1$ regardless of the colors. Obviously there are many choices in each step and the game is highly non-commutative.

The mathematical problem now is how to estimate the probability of winning the game if k is some (small) fraction of N. With $k = 4$ and $N = 13$, this closely resembles the actual Canfield solitaire.

EXAMPLE 3. We consider another solitaire analogous again to one played with a deck of cards, but we shall of course schematize it and at the same time generalize it into a combinatorial question.

Imagine the integers from one to N^2 ordered arbitrarily into a square array. We may remove any two numbers in the same row or in the same column if they add up to a number belonging to a prescribed class of integers. For example, if they add up to a prime (or in another game if they add up to a square, etc.). The question could be what is the probability that you could remove by suitable choices all or "almost" all the integers? Speaking precisely for how many orderings of the N^2 into an array can it be done?

EXAMPLE 4. An infinite solitaire is played as follows. First, suppose the positive integers are listed at random in a square array. If two numbers in the same column or row sum to a prime (or square), then they can be removed. What is the probability of removing a set of positive density?

We can list the positive integers at random according to the following rule. First one chooses a number z in $[0,1]$ at random. Then consider the continued fraction expansion of z. The entries of the bottom row in the array are the first, third, fifth, etc. integers which appear in the expansion of z. The second row consists of the second, sixth, tenth, etc.

II.3. *Is Possession of a Winning Strategy Independent of Base for Expansion?*

We consider Ulam's modification of Mazur's game (Problem 43 in "The Scottish Book"). For a given subset E of $[0,1]$ players A and B give in turn the digits 0 or 1. Player A wins if the number whose infinite dyadic expansion consists of these digits taken in order belongs to E.

We also consider the same game where players A and B give in turn the digits 0, 1, or 2 and so on.

Name a set E which is a winning set for the first player in the binary development but not in the ternary development. Is it true that for most sets E, player I wins with asymptotic density $1/2$?

II.4. *Games by Teams Playing Each Other*

Let $E \subset [0,1] \times [0,1]$. Consider two teams of two players each. Player A_1 selects the first digit in the binary expansion of the x coordinate; then player A_2 selects the y coordinate without the knowledge of his partner's selection. After this, players B_1 and B_2 have their turn and the game continues. Team A wins provided the final point determined by the play is in E. In which cases can there be an arrangement of strategies between the partners so that they have a winning strategy even though the winning strategy is not unique? (Otherwise, trivially each will select a move with the knowledge that his partner has made the best selection which is unique; if the number of winning moves at some stage is greater than one or even infinite, there has to be a convention between the players on how to select.) Games between teams of course, are very general and common, e.g., "armies" without a "general"—games between organisms and groups of animals or groups of cells, etc.

II.5. *Complexity of Integers*

Let us define the arithmetic complexity $|n|$ of a positive integer n to be the smallest number of operations of addition, multiplication and exponentiation which combine 1's to form n. Thus $|1| = 0$, $|2| = 1$ and $|5| = 4$ since $5 = (1 + 1) + (1 + 1) + 1$ and no fewer operations with 1's will form 5. This notion has been studied and some tables of the behavior of $|n|$ are given by Beyer, Stein, and Ulam [16].

A number n is said to be complicated if $|n| > |m|$ for all $1 \leq m < n$. Clearly there are infinitely many complicated numbers. Is it true that there is an integer k such that if $n > k$ and n is complicated, then n is prime? Is it true that if n is sufficiently large, then n is the sum of $< \log_2(n)$ complicated integers? Is there a number c such that $|n| < c + \sqrt{\log_2 n}$, for n sufficiently large? For each n, let $k(n)$ be the average complexity of all integers $\leq n$. What is the order of growth of $k(n)$? Let $f(n)$ be the number of integers with complexity exactly n. A straightforward calculation shows that $f(n)$ is bounded above by the nth Catalan number. What is the order of growth of $f(n)$?

For each x in $[0,1]$ with binary expansion $x = .a_1a_2a_3\ldots$ let $c(x) = \overline{\lim}_{n\to\infty}(a_1 2^n + \cdots + a_n)/k(2^{n+1} - 1)$. What is the expected value of $c(x)$?

Suppose we are given points $x_1, \ldots, x_N$ of a set E and binary operations $R_i(x, y)$, $i = 1, 2, \ldots, k$ on E. Consider the structure S generated by these relations. We define the complexity of an element of S according to the rules (1) $|x_1| = \cdots = |x_N| = 1$; (2) if $z = R_i(x, y)$, then $|z| \leq |x| + |y| + 1$; and (3) for each z in S there are points x and y and some i such that $z = R_i(x, y)$ and $|z| = |x| + |y| + 1$. In particular, what happens if $A_1, \ldots, A_N$ are subsets of $E = Y \times Y$ and the binary operations are intersection, union, projection and direct product?[1]

[1]*Note added in proof.* Mycielski asks: Let T and S be two theories containing PA and let $T \vdash \mathrm{Con}(S)$ (S is recursively axiomatizable). Let $d_T(n)$ and $d_S(n)$ be the lengths of the shortest definitions of n in T and in S, respectively, such that those theories can prove that those definitions define n. Is it true that for all recursive functions r there are arbitrarily large n such that $r(d_T(n)) < d_S(n)$?

II.6. *Winning Almost Always and Change of Roles and Rules*

For what subsets E of $[0, 1]$ is it true that one of the players almost always wins (almost always means in the sense of measure or in the sense of category in the space of all sequences of choices).

For what subsets E of $[0, 1]$ is it true that one of the players has a winning strategy when the rule of the game is that if either of the players has lost the usual binary game after a finite number of moves, that player can "take back a move," i.e., may change the preceding move or, say, a previous move on a specified level of moves beforehand.

II.7. *Ramsey and van der Waerden Problems*

Let E be a set of n elements. Divide the class of all subsets of E into two parts such that a set is in one part if and only if its complement is in that same part. How big is, of necessity, the biggest Boolean algebra contained in one of the two parts?

Consider the family of all subsets of a given set and the group structure on it provided by the operation of symmetric differences. Again we divide the subsets of this set into two classes. How big is, of necessity, the symmetric difference group of sets which will be contained in one or the other parts of the division of the class of all subsets?

If E is taken to be countably infinite and the class of all subsets is divided as above into two parts, must one of these parts contain arbitrarily large finite Boolean algebras or groups?

Let E be a countably infinite Abelian group. Partition E into two sets, A and B. Must one of the parts contain arbitrarily long arithmetic progressions: for each n there are elements $x_1, \dots, x_n$ all in A or all in B such that $x_{i+1} - x_i = x_{i+2} - x_{i+1}$, $i = 1, \dots, n-2$. In particular, what about the weak direct product of Z_2's?

Finally, let E be a group. Partition E into two sets such that x is in one set if and only if x^{-1} is in the same set. How big is a subgroup or coset which one can find in one or the other part? What is the situation in which each element of E is of finite order?

II.8. *Games and Number Theory*

Let $|E| = N$. Two players choose in turn points of $E \times E$. The player with the largest product set $A \times B$ in their final set wins; if both players' largest product set have the same size, it is a draw. What is the largest product the first player can achieve? (Erdös comments surely $(c \log n) \times (c \log n)$.) Who has a winning strategy? (Erdös believes it is a draw.)

Now suppose $E = \{1, \dots, n\}$. Two players choose in turn elements of E. The player with the largest subset such that the sum of two distinct elements of this set are all different wins; if both players' sets are of the same size it is a draw. What is the maximal guaranteed size subset of the first player's set such that the sums of two distinct elements of this set are all different? Or such that no sum of any two numbers is equal to another number of the set?

Erdös notes that in the first case there is a "Sidon" set of size $(1 + o(1))\sqrt{n}$. So, each player can make sets of size at least $(1 + o(1))\sqrt{n}/2$. Can a bigger set always be obtained? Erdös also suggests some variations. Two players in turn choose an edge of a complete graph. Player I wins if he has the larger complete subgraph, otherwise he loses. Erdös conjectures if $n > 2$, player II always wins. One can modify this game by declaring that I wins if he has more complete maximal subgraphs than II [17]. Another modification [18] would be that I wins if the degree is bigger—perhaps neither player gets more than $n/2 + c \log n$ in this case?

II.9. *Feebly Isomorphic Structures*

Two graphs G and G' are called feebly isomorphic provided there is a partition $\{E_1, \ldots, E_n\}$ of $E(G)$, the edge set of G, and a partition $\{E_1', \ldots, E_n'\}$ of $E(G')$ such that for each i, E_i and E_i' are isomorphic as graphs. If G and G' are feebly isomorphic, then G and G' have the same number of edges. Also, if G and G' have the same number of edges, then G and G' are feebly isomorphic. Let $U_2(G, G')$ be the smallest value of n such that there are pairwise isomorphic partitions of $E(G)$ and $E(G')$. For each positive integer n, set

$$U_2(n) = \max_{G, G'} U_2(G, G'),$$

where G and G' each have n vertices and the same number of edges.

Chung *et al.* [19] have shown that $U_2(n) = 2n/3 + O(n)$ and have raised a number of interesting questions. Yao has shown that the question "Is $U_2(G, G') = 2$?" is NP-complete [20].

One can extend the notion of feeble isomorphism to three or more graphs. In [21] it is shown that

$$U_k(n) = 3n/4 + O_k(n), \qquad \text{for each } k \geq 3.$$

Of course, $U_k(n)$ is the largest possible value $U(G_1, \ldots, G_k)$ can assume where $G_1, \ldots, G_k$ are graphs, all having n vertices and e edges and $U(G_1, \ldots, G_k)$ is the smallest possible value of r such that there are partitions $\{E_{ij}\}_{j=1}^{r}$ of $E(G_i)$ such that for each j the graphs E_{ij}, $i = 1, \ldots, k$, are isomorphic. Some of the results are surveyed in [22].

The notion of feeble isomorphism is extended to hypergraphs by Chung and Erdös [23] and they have focused on the major techniques used in these works [24].

II.10. *The Reconstruction Conjecture for Graphs*

Let G and G' be graphs each with n vertices. Suppose that for each subset A of the vertex set of G with cardinality $n - 1$ there is a subset B of the vertex set G' such that A and B are isomorphic as subgraphs. Is it true that the graphs G and G' are isomorphic? This problem was formulated many years ago by P. J. Kelly and Ulam. Some of the first results concerning this problem were obtained by Kelly [25], who proved the truth of the conjecture for trees. This is one of the central unsolved problems today in graph theory [26, 27].

II.11. *Random Partitions*

One can generate an ordered partition of N, the set of all positive integers, at random as follows. First, choose a number p_1 in the interval $[0, 1]$ at random and form the first set A_1 in the partition by declaring that n is in A_1 with probability p_1 independent of the other choices. Second, choose a number p_2 in the interval $[0, 1]$ at random and form the second set A_2 in the ordered partition by declaring that an integer n which is not in A_1 is in A_2 with probability p_2. Continue this process through all the positive integers forming $A_1, A_2, A_3, \ldots$. We note that with probability one each of the sets A_n is infinite and with probability one $\{A_1, A_2, A_3, \ldots\}$ is a partition of N.

One can alter this process to obtain partitions where some of the sets are finite. The first set is formed as before. Next, let n_1 be the first integer not in A_1 and let F_1 consist of the first n_1 integers not in A_1. Now choose q_1 and take $[q_1 n_1]$ integers from F_1 to form a finite set B_1 which is in the partition. Form the infinite set A_2 as before, but only examining the integers not in $A_1 \cup F_1$. Let n_2 be the first integer not in $A_1 \cup F_1 \cup A_2$. Let F_2 consist of the first n_2 integers which are not in this union. Choose q_2 and form B_2 by taking $[q_2(n_1 - [q_1 n_1])]$ of the integers in $F_1 \setminus B_1$. Also form B_3 by taking $[q_2 n_2]$ of the integers in F_2. Finally form the infinite set A_3 by examining the integers not in $A_1 \cup F_1 \cup A_2 \cup F_2$. Continue this process.

We can consider the process just described for the partition of F_1 as a process for generating a partition of a set with n elements at random. What is the expected number of sets in such a partition and what is the distribution of the sizes of the sets in the partition?

II.12. *Permutations*

Two players build two permutations $\pi_1 = (k_1, \ldots, k_n)$ and $\pi_2 = (m_1, \ldots, m_n)$ of the first n integers as follows. First, player I chooses k_1, then player II chooses m_1. Next, player I chooses m_2 and then player I chooses k_2, etc. Consider the subgroup G of S_n generated by π_1 and π_2. What is the size of the subgroup that player I can always achieve?

Is there a permutation P of $\{1, \ldots, n\}$ such that for all $i \neq j$,

$$|P_{(i)} - P_{(j)}| \neq |i - j|$$

or even,

$$||P_{(i)} - P_{(j)}| - |i-j|| \geq c\sqrt{|i-j|},$$

for some positive constant c? The first problem is, of course, the old problem of whether n Queens can be placed on an $n \times n$ chess board so that no two attack each other [28].

II.13. *Approximation of Primes by Product of Three Fractions*

Let p be a prime. If a_1, a_2, b_1 and b_2 are integers less than p then the minimum of

$$d_2 = \left| \frac{a_1}{b_1} \cdot \frac{a_2}{b_2} - p \right|$$

is obtained for $a_1 = a_2 = p - 1$, $b_1 = 1$ and $b_2 = p - 2$. Suppose we want to minimize

$$d_3 = \left| \frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} - p \right|,$$

for a_i and b_i less than p. If one chooses $a_1 = a_2 = p - 1$, $a_3 = p - 4$, $b_1 = 1$, and $b_2 = b_3 = p - 3$, then $d_3 = 4/(p-3)^2$. However, this is not the best choice for all p. For example, if $p = 13$, the above formula yields $d_3 = 1/25$. If one chooses $a_1 = a_2 = a_3 = 10$, $b_1 = 1$, $b_2 = 7$ and $b_3 = 11$, then $d_3 = 1/77$. Does the formula yield the best approximation for all but finitely many primes? In the Introduction to "Problems in Modern Mathematics" it is conjectured that $\min d_3 \to 1/p^2$ as $p \to \infty$.

II.14. *Quasi Primes*

We consider a probabilistic method of generating a sequence which may mimic the sequence of primes in its various asymptotic properties.

Begin with the sequence $d_1 = 2$, $d_2 = 2$, and $d_3 = 4$ and define d_{n+1} by first choosing j, $1 \leq j \leq n$, at random and setting d_{n+1} equal to the d_j or else to $d_j + 2$ with equal probability. As a sample asymptotic property note that with probability one infinitely many of the d_i's are equal to 2. Thus, with probability one there are infinitely many "twin" quasi primes.

Is it true that with probability one

$$p_n := 1 + \sum_{i=1}^{a} d_i \approx n \log n?$$

A heuristic for this conjecture is the fact that the conjecture is true for the sequence $e_1, e_2, e_3, \ldots$ of expected values $d_1, d_2, d_3, \ldots$. In fact, we have the general recursion $e_{n+1} = e_n + 1/n$ from which logarithmic growth follows.

Is it true that with probability one, the integers p_n which are equal to 1 modulo 4 have density $1/2$?

Occasionally, rarely, the difference between consecutive primes exceeds any previous difference by more than two. So, we could modify our procedure by choosing d_{n+1} as follows. First, select j, $1 \leq j \leq n$, at random. Then with probability $1/2$, d_{n+1} is d_j, with probability $1/2 - 1/2^n$, $d_{n+1} = d_j + 2$, and finally, with probability $1/2^n$, $d_{n+1} = d_j + [\sqrt{j}]$.

One can refine this random recursion to mimic the sequence of differences of primes even more closely. For example, one can introduce the rule that there must not be two consecutive 2's or 4's in the sequence $d_3, d_4, d_5, \ldots$. Even with these refinements computer studies by Tony Warnock indicate that the p_n's grow as $n \log n$.

We also consider the same problem for "deterministic" methods of generating pseudoprimes. For example, as before, set $d_1 = 2$, $d_2 = 2$, $d_3 = 4$. Now, if n is even set $d_{n+1} = d_{[2n/5]} + 2$ and if n is odd, set $d_{n+1} = d_{[3n/5]}$:

$$2, 2, 4, 2, 4, 4, 4, 2, 6, \ldots .$$

II.15. *Sequences of Integers Defined by Unique Sums*

Beginning with a list of two distinct integers, add an integer to the list if it is uniquely expressible as the sum of two distinct integers already in the list. If we begin with 1 and 2, does the sequence have density zero? Studies of this problem were made by Queneau [29].

II.16. *Sets of Integers from Integers*

We make correspond to every positive integer n a set of positive integers as follows: If n is developed in a binary sequence with 0's and 1's, we attach to this integer the set of all indices where in the development of n we have a 1. We may now consider all the sets of integers corresponding to primes. On this class of sets of integers we may now perform some Boolean operations. As noticed by Erdös, one obtains all finite sets of integers as differences of such. What does one obtain using only sums, i.e., unions? What are the sets of integers which we obtain starting from the class of integers corresponding to squares or cubes, etc?

Similarly for a number x in the unit interval, let $A_x = \{n \mid a_n = 1\}$, where $.a_1 a_2 a_3 \ldots$ is the infinite dyadic expansion of x. Now define $x + y$ as $A_x \cup A_y$, etc. What do the rational, quadratic irrational or algebraic numbers generate in this algebraic system?

II.17. *Generation of Primes*

Starting with two primes, say $p_1 = 2$ and $p_2 = 3$, put $p_{n+1} = p_n + p_k - 1$, where $k \leq n$ is to be chosen every time. Can one obtain an infinite sequence consisting of primes only or containing infinitely many primes? One cannot, of course, obtain all primes because there exists a prime p_s such that the next prime p_{s+1} is such that $p_{s+1} - p_s = 8$. Thus 97 is the smallest prime one cannot get in order. Erdös believes the density of primes one obtains is 0, but almost certainly infinite. He comments that if $p_{n+1} - p_n$ is not a prime then you cannot get p_{n+1} from the sequence.

II.18. *Little Number Theoretical Curiosities concerning Primes*

Which fractions can be obtained in the form $p_1/q_1 + p_2/q_2$, where p_1 and p_2 are primes?

Suppose we enumerate all rational fractions by lattice points in the plane putting the fraction p/q (in reduced form) at (x, y), where $x = p$, $y = q$, and make an Erastothenes sieve on a spiral. Consider those fractions which correspond to primes in the sieve. Which fractions can one get by sums of two "prime fractions"?

Consider the expansion $p = 1 + \alpha_1 2 + \alpha_2 2^2 + \cdots + \alpha_n 2^n$, $\alpha_i = 1$ or 0. Are there, asymptotically, as many primes with an even number as with an odd number of α_i's corresponding to $\alpha_i = 1$? Is it true that for most primes the difference between the number of α_i's $= 1$ and the number of α_i's $= 0$ is $\leq \sqrt{n}$, etc.?

II.19. *An Old Conundrum*

Let a_0 and a_1 be given positive integers. For each $n > 1$, set $a_{n+1} = (a_n + a_{n-1}) \pmod{n}$.

For each positive integer k consider the subsequence of $a_0, a_1, a_2, \ldots$ consisting of all a_j's $\leq k$. Is it true that each integer i, $1 \leq i \leq k$, appears in this subsequence with frequency $1/k$?

Let $a_n = 2^n \pmod{n}$, for $n = 1, 2, 3, \ldots$. For each positive integer $k \geq 2$ consider the subsequence of this sequence consisting of all a_j's from 2 through k. Is it true that each integer in this subsequence appears with frequency $1/k - 2$?

II.20. *Problems on Neighbors of Sets*

Suppose $N \times N$ is covered by sets of uniformly bounded cardinality. We say two points are neighbors if the distance between them is no more than $\sqrt{2}$. It follows from a topological theorem on covers of the plane by uniformly bounded regions that there must be some set in the partition which has six neighbors. Two sets are neighbors means each contains a point such that the two points are neighbors.

Suppose neighbor is defined by distance ≤ 1. Cenzer and Howorka [30] have shown that any such cover of lattice points as before must contain at least one set with at least four neighbors. They also reference a number of related results and questions.

In three dimensions, if we define neighbor by distance ≤ 1, the conjecture is that there is at least one set with 6 neighbors. If we define neighbor by distance $\leq \sqrt{3}$, the conjecture is that there is at least one set with at least 14 neighbors. If neighbors is defined by distance $\leq \sqrt{2}$, what can be said about guaranteed neighbors of at least one set?

In n dimensions, with distance $\leq \sqrt{n}$ defining neighbors, the conjecture concerning neighbors of a set is $2^{n+1} - 2$.

Presumably, the theorem should assert not merely the existence of one such set, but that the number of such sets has positive asymptotic density: If we consider concentric spheres with radii $1, 2, 3, \ldots,$ the number of sets with the guaranteed number of neighbors which are subsets of the sphere of radius n divided by the number of all sets in the cover which are subsets of the sphere of radius n has a positive lower limit as n goes to infinity.

One might ask about neighbors of neighbors in the above sense. What is the minimum guaranteed number of such?

Finally the whole problem may be formulated for graphs which are infinite, but with each vertex having a fixed finite number of edges; the idea of neighbor being two vertices joined by an edge.

Given a set of N elements we may consider two subsets as being neighbors if the symmetric difference in cardinality is less than k. Now consider two classes of subsets as neighbors if each class contains a set so that these two sets are neighbors. Estimate the number of classes which must have a given number of neighbors.

Analogous problems arise for subsets of the set of integers when two sets are neighbors if the Hausdorff distance between sets is less than a given integer k.

Determine $x = x(n, j)$ or $x = x(n, j, k)$ such that if the class of all subsets of a set E with cardinality 2^n is partitioned into subclasses each of power n (or 2^{n-1} or 2^{n-k}) then there must be one of them which has x neighbors. Again two sets are neighbors if their symmetric difference is in cardinality no more than j.

II.21. *Problems by Erdös*

1. Sierpinski proved $c \not\to (\aleph_1, \aleph_1)^2$, i.e., one can color the edges of a complete graph of power c by two colors so that every complete subgraph of power $\aleph_1$ contains edges of both colors. Assuming $c = \aleph_1$, Hajnal,

Rado, and I proved that one can color the edges of a complete graph of power c by c colors such that every subgraph of power c contains all the colors.

Let us now assume no hypothesis about c. Is it true that $c \to [\aleph_1]^2_3$, i.e., can one color the edges of a complete graph of power c by three colors so that every complete subgraph of power $\aleph_1$ contains all three colors? We made no progress with this problem, which quite possibly is independent.[2]

Galvin and Shelah [31] showed that $\aleph_1 \not\to [\aleph_1]^2_4$ and $c \not\to [c]^2_{\aleph_0}$.

Similar problems are stated in my paper with Hajnal [32]. Finally, Todorevic states that $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$, i.e., one can color the edges of a complete graph of power $\aleph_1$ with $\aleph_1$ colors so that every square of size $\aleph_1$ contains all the colors.

2. Join two points in the plane if their distance is j. Hadwiger and Nelson asked, What is the chromatic number of this graph? It is known to be between 4 and 7 and is probably greater than 4. Now join two points whose distance is one of the numbers r_i, $1 \le i \le k$. Denote the chromatic number by $f(r_1, \ldots, r_k)$ and put

$$\max_{r_1, \ldots, r_k} f(r_1, \ldots, r_k) = F(k).$$

Is $F(k)$ polynomial, exponential, or something in between? Is it true that $F(k)/k \to \infty$? (The answer is yes for lattice points in the plane.)

II.22. *Problems by Ronald Graham*

1. Is it true that if $\{a_1, a_2, \ldots, a_n\}$ is a set of n positive integers, then for some i and j,

$$\frac{a_i}{\text{g.c.d.}(a_i, a_j)} \ge n?$$

Comments. This problem has had an active history (some of which is mentioned in the little monograph with Erdös (pp. 78–79)) [33]. In the case that a_k are square free, it is equivalent to the (true) theorem that for any family of n (distinct) sets $A_1, \ldots, A_n$, there are always at least n distinct differences $A_i - A_j$. M. Szegedy has just found a very ingenious proof. Graham has paid (see the end of this section).

2. For a finite set X of points in the Euclidean plane, let $L(X)$ denote the shortest total length a tree connecting together the points of X can have. Is it true that if $Y \subset X$, then

$$\frac{L(X)}{L(Y)} \ge \frac{\sqrt{3}}{2}?$$

[2] *Note added in proof.* Shelah has shown $c \to [\aleph_1]^2_3$ is consistent.

Comments. By a tree we mean a collection of edges between various pairs of points of X which is connected and has no cycles. Thus, $L(X)$ is the length of the so-called minimum spanning tree for X. At first, it is not obvious that it is possible to have $Y \subset X$ and $L(Y) > L(X)$. However, the set $X = \{0, 1, \exp(2\pi i/3), \exp(4\pi i/3)\}$ and $Y = X - \{0\}$ give $L(X) = 3$ and $L(Y) = 2\sqrt{3}$. The conjecture (due to Gilbert and Pollak and now more than 25 years old) is that this is as small as the ratio ever gets. There have been a series of increasing lower bounds given on this ratio over the years, the most recent due to Fan Chung and myself [83] which asserts that

$$\frac{L(X)}{L(Y)} \geq 0.8214\ldots$$

(the RHS is a root of an irreducible 12th degree polynomial equation). If instead of Euclidean distance we use the l_1 metric

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

then the corresponding ratio is known to be bounded below by $2/3$ which is in fact the best possible lower bound.

3. It is true that for any integer $k \neq 1$ there are infinitely many n such that n divides $2^n - k$?

Comments. It is easy to show that $2^n \not\equiv 1 \pmod{n}$ for $n > 1$. Of course, $2^n \equiv 2 \pmod{n}$ quite often. There does not seem to be any good reason why $2^n \pmod{n}$ cannot be anything besides 1. The Lehmers did a computer search some years ago and finally found solutions for all k, $2 \leq k \leq 100$. The last holdout was $k = 3$, for which the least (and only known) solution is $n = 4700063497$. A related question is whether for all k there are infinitely many n such that $\phi(n)$ divides $n + k$ (where $\phi(n)$ is the Euler phi-function of n). This has not been looked at much but it is clean.

4. What is the maximum area an octagon of unit diameter can have?

Comments. In general, one can ask for the largest area possible for the convex hull of a set of n points of diameter 1. For *odd n*, it has been known for more than 50 years that it is given uniquely by the regular n-gon of diameter 1. For $n = 4$, a square of side $1/\sqrt{2}$ (and diagonal 1) gives the maximum possible area for $n = 4$ but it is not the only such figure. In fact, there are infinitely many, all having an orthogonal pair of diagonals of length 1. The case $n = 6$ was open since the 1920s until I settled it about 10 years ago [34]. The maximizing hexagon is unique and *not* the regular hexagon.

5. Is it true that if the integers

$$\{1,\ldots,t(n)\},$$

where $t(n)$ is the tower function of powers of 2:

$$t(n) = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}\text{ height } n$$

are partitioned into n subsets, then at least one of the subsets contains an arithmetic progression of n terms?

Comments. The exact form of this bound for van der Waerden's theorem is not crucial. *Any* primitive recursive function would do and would be of great interest. Whether such bounds actually hold is not as clear as it once seemed because of the well-known work of Paris and Harrington (and recently, Friedman), who showed that for some similar-looking problems, no primitive recursive bound can hold. I still believe the answer to this question is yes, however.

6. Show that for infinitely many n,

$$\text{g.c.d.}\left(\binom{2n}{n}, 105\right) = 1.$$

Comments. This, in spite of appearances, is a very tough problem. It is equivalent to asking if there are infinitely many n which when expanded in bases 3, 5 and 7, have digits all less than half the base. The answer is yes for any *two* distinct prime bases p and q [84].

A related problem: Show that only finitely many n satisfy

$$\text{g.c.d.}\left(\binom{2n}{n}, 3\cdot 5\cdot 7\cdot 11\right) = 1.$$

In fact, it is known that this holds for $n = 3160$ but no n satisfying $3160 < n < 10^{10,000}$. Probably $n = 3160$ is the last one.

As a token of my estimate of the difficulty of the preceding problems, I am willing to part with the following amounts (for proofs or counterexamples).

1. \$ 100[3]
2. \$ 500
3. \$ 100
4. \$ 50
5. \$1000
6. \$ 100

Erdös comments that it has been conjectured by Erdös, Graham, and Straus that for $n > 4$, $\binom{2n}{n}$ is never squarefree. Sárközy proved this if $n > n_0$. His proof will appear in the *Journal of Number Theory*.

III. Algebra

III.1. *The Infinite Permutation Group S_∞*

S_∞ is the group of all permutations π of the set of all integers. The results about this group contained in the papers by Ulam and Schreier [35] can be obtained, if necessary mutatis mutandis, for certain subgroups of it. The problems concerning such groups refer to the existence or non-existence of normal subgroups, their automorphisms; the existence of a finite base in the sense of approximating all permutations of the subgroup by means of composition of a finite number, etc.

S_b is the group of all π that move each n a bounded distance: $|\pi(n) - n| < k$, where k is independent of n but, of course dependent on π. S_{bf} denotes the group of all π for which, except for a set of frequency 0, all the displacements are uniformly bounded. Clearly, one can consider other "naturally defined" subgroups of S_∞, among them the group of permutations which are recursive in the sense of logic. S_d is the group of all π which preserve the density of every subset of the integers. In particular, Erdös showed that S_b is not isomorphic to S_∞. Prove S_d is not isomorphic to S_∞.

The reader can generalize these questions to the case where we consider S_∞ as the group of permutations of Z^2, the lattice points in the plane. Thus, one could consider the group of all π such that $|\pi(n) - n|$ is bounded by a polynomial in n. Or one can consider the permutations which are of a specified type horizontally and of a different specific type vertically and their compositions. To increase the complications one could even allow switches (i.e., add to the permutations a change by 90° of the vertical into the horizontal). This would be in a way analogous to the consideration of ideals of sets in the plane (e.g., those sets which are of the first category on every horizontal line and of measure 0 on every vertical).

[3]*Note added in proof*. This \$100 has already been collected.

Finally one can consider the problem of defining measures on these subgroups with various "natural" properties as discussed in problem V.5.

III.2. *Groups Isomorphic via Decomposition*

Are there non-isomorphic groups G and H such that G and H can be decomposed into two parts: $G = G_1 \cup G_2$ and $H = H_1 \cup H_2$ with $G_1 \sim H_1$ and $G_2 \sim H_2$. The notation $A \sim B$, where $A \subset G$ and $B \subset H$, means there is a one-to-one map f of A onto B such that if x, y, and xy are all in A, then $f(xy) = f(x)f(y)$.

III.3. "*Funny Addition*"

What are the possible additions on the integers which distribute with respect to the usual multiplication? For example, let T be a permutation of the primes. Extend T to all positive integers via their decomposition into products of primes. Now define $a \oplus b = T^{-1}(Ta + Tb)$. This defines continuumly many such additions. Consider the same problem for the rationals. Note that for each such addition $\oplus$ on the integers, the addition $a/b \oplus c/d = ad \oplus bc/bd$ on the rationals distributes with respect to multiplication [36].

III.4. *Composition of Relations in Three Variables*

If R and S are binary relations on a set X, it is standard to define $R \circ S$, the composition of R and S, as $\{(x, y) |$ there is some z with (x, z) in R and (z, y) in $S\}$. However, if R and S are ternary relations on a set X, there are several distinct possibilities which present themselves. For example, we could set

$$R \circ S = \bigcup_{x_1} (R_{x_1} \circ S_{x_1}) \bigcup_{x_2} (R_{x_2} \circ S_{x_2}) \bigcup_{x_3} (R_{x_3} \circ S_{x_3}),$$

where
$R_{x_1} \circ S_{x_1} = \{(x_1, y, z) |$ there is some u with $(x_1, y, u) \in R$ and $(x_1, u, z) \in S\}$ and similarly for $R_{x_2} \circ S_{x_2}$ and $R_{x_3} \circ S_{x_3}$.

It is known that if $\{R_n\}$ is a sequence of binary relations on a countably infinite set, then there exist two binary relations S and T such that the semigroup generated by them contains all the R_n's. Is this true for ternary relations?

III.5. *Maximal Group Structure on a Countable Set*

Is there a group structure on a countable set such that the group of endomorphisms is maximal?

III.6. *Possible Generalizations of the Fröbenius–Perron Theorem*

This theorem, which has important applications in probability theory—specifically in the study of branching processes—asserts the convergence of iterates of a linear transformation of the n-dimensional space into itself, provided all the coefficients in the matrix representing the transformation are positive, to a unique characteristic vector in the positive "octant" of the space. Actually, the condition can be weakened; it suffices that some power of the transformation has, in matrix form, all terms positive. From this theorem one concludes, in certain stochastic processes, the expected variants for ratios of the population tend to stabilize—speaking loosely one can say that the steady state, or an equilibrium, is approached.

Among the applications of this theorem, when generalized to infinitely many dimensions, one can assert that for equations of the type of diffusion and multiplication by a potential, e.g., in the Schrödinger equation, the highest energy value is the characteristic multiplicand of the state represented by the invariant vector.

A number of generalizations will be proposed for the case where the transformation is not linear but, e.g., quadratic, and instead of asserting the convergence to an eigenstate, we will merely desire convergence in the mean or rather the validity of an ergodic behavior, that is to say, the existence of limits of the ratios of the time spent by iterates of the transformation in certain regions, e.g., cones. Clearly the convergence to a line, in direction, guarantees the existence of the time of sojourn—in each cone the limit of the time spent there is equal to 1, provided it contains the invariant direction, and it is 0 otherwise.

We should also consider the fact that in the other, so to say, extreme, the existence of the limit is guaranteed. If the given linear transformation is a rotation, the theorem of Weyl guarantees the equipartition of iterated points, i.e., the existence of the "time" limit, in this case equal to the space measure. It is of interest to consider from this point of view more general linear transformations of Euclidean n-space.

Suppose we have a linear or quadratic transformation, T, of Euclidean n-space and let C be a cone of directions issuing from the origin. We are interested in the behavior of

$$\lim_{k\to\infty} 1 \Big/ k \sum_{p=0}^{k-1} 1_C(T^p(x)).$$

We want to know how the value of the limit depends on the initial vector x. Our conjecture is that given such a cone, the number of possible values for the limit is finite. In other words, the space of the initial vectors can be divided into a finite number of "regions" in which each initial point gives rise to the same limit of average time.

IV. Topology

IV.1. *Groups of Homeomorphisms*

Ulam and von Neumann [37] noted that the identity component of the homeomorphism group of the two-dimensional sphere is simple. Anderson [38] extended this result to various groups of homeomorphisms of other spaces. The simplicity of the identity component of the homeomorphism group of various manifolds, in particular, S^4, remains open.

Similarly, every automorphism of the homeomorphism group of S^4 is inner. For what topological spaces does this result hold?

Finally, Everett and Ulam [39] noted that the topology of the real line is uniquely determined by the group of homeomorphisms in the usual topology. For what spaces X is there a unique topology on X in which X is a complete separable metric space? When is X determined by its homeomorphism group? As noted in [39], these problems in an algebraic formulation originated in conversations of Everett and Ulam with Teller and von Neumann. Recently Kallman [40] has given some general conditions which answer these questions for various spaces X. The problem seems to remain open for the universal curve of Sierpinski.

IV.2. *Isometric Squares*

Let A and B be two topological spaces such that $A \times A$ and $B \times B$ are homeomorphic. Is then the space A homeomorphic to B? This is Problem 77b formulated by Ulam in the Scottish Book. The commentary accompanying that problem details some of the tremendous amount of work concerning it. The answer in general is no. R. H. Fox showed that the answer is yes if A and B are two-dimensional compact manifolds with or without boundary. In fact, Rosicki [41] has given an affirmative answer in case A and B are compact, two-dimensional polyhedra. The situation concerning compact, three-dimensional manifolds remains open.

If A and B are metric spaces and $A \times A$ is isometric to $B \times B$, then is A isometric (or homeomorphic) to B? This question has a negative answer in general [42]. However, it remains open in case A is a complete or compact metric space. Some sufficient conditions are given by Kelly [43] in case A is finite.

IV.3. *Transformations Preserving Various Properties*

Let E and F be two topological (e.g., metric) spaces. Let T be a transformation of E into F such that if A, $B \subset E$ are homeomorphic, then $T(A)$ and $T(B)$ are homeomorphic in F. Must T be a homeomorphism (e.g., for E a Euclidean space!)?

Variations on this theme: Suppose that if B is a continuous image of A, then $T(B)$ is a continuous image of $T(A)$. Does it follow that T is continuous? etc. Or suppose the map T has the property that if $A \approx B$, then $T(A) \approx T(B)$. Does it follow that $T(A) \approx A$? Here $\approx$ means isometric, linear similarity or congruence in a general sense or perhaps A and B have the same measure.

IV.4. *Natural Measures on Homeomorphism Groups*

In many attempts to model various phenomena, one finds that there is a space of objects which is being acted upon by some group of transformations—a group of similarity maps, diffeomorphisms, C^∞ maps, etc. If one wants to make some statements about average or expected behavior of this system, some sort of "natural" measure is needed on this group. If the group is locally compact, then left Haar measure is available as a first choice. However, if the group is large, then there is no such measure. Our idea here is that one could manufacture some measure on the group which is appropriate for the problem under consideration. We illustrate this idea with the space of objects being I, the unit interval, and the group being $H(I)$, the homeomorphism group of I [44].

It is known that there is no σ-finite measure on $H(I)$ such that this measure is left invariant or even, more generally, left quasi-invariant. However, there may be measures on this space which are "natural" in some sense. For example, if one is asked to produce a homeomorphism h of $[0,1]$ onto itself such that $h(0) = 0$ and $h(1) = 1$, one could proceed by indicating the values of h at particular points. Thus, $h(1/2)$ is somewhere between 0 and 1. Not having any additional information, let us say that $h(1/2)$ is uniformly distributed on $[0,1]$. Now, given $h(1/2)$ what is $h(1/4)$? We know $h(1/4)$ is somewhere between 0 and $h(1/2)$, let us say $h(1/4)$ is uniformly distributed over this interval. Continue this process. It turns out that this process produces a homeomorphism of $[0,1]$ onto itself with probability one [45]. Thus we have a natural measure P on $H(I)$. Of course, one could think of the random homeomorphisms as random distributions, and indeed, this process was first introduced and studied from this point of view by Dubins and Freedman [46].

There are many interesting properties of these random homeomorphisms. For example, with probability one, there are a finite odd number of fixed points which alternate between repellent and attractive.

We have indicated a method of generating a "natural" measure P on the homeomorphism group of the unit interval. By identifying end points, the measure P may be regarded as a probability measure on $H_0(T)$, the group of homeomorphisms of T, the unit circle, which leave the complex number $1 = e^0 = e^{i2\pi}$ fixed. Now choose one of these homeomorphisms at random with respect to P and follow this by a rotation of the circle chosen at random with respect to u, Lebesgue measure (or Haar measure) on the unit circle. This yields a probability measure Q on $H(T)$, the homeomorphism group of T. Formally, $Q(B) = u \times P(\pi^{-1}(B))$, where π maps $T \times H_0$ into H by $\pi(s, f) = s \circ f$. The measure Q has the natural property that Q is invariant under the left action of the isometry group of T acting on $H(T)$. It can be shown that the Q measure of the homeomorphisms with a fixed point is positive and the Q measure of the homeomorphisms with no fixed point is positive. Is the Q measure of the homeomorphisms with no periodic points positive? If so, is the measure of the "Denjoy rotations" positive? These are the homeomorphisms with no periodic points but which are not conjugate to irrational rotations. These are the homeomorphisms of T which leave some Cantor set invariant and permute the complementary intervals in a fashion such that there are no periodic points.

Another way to produce a homeomorphism of T at random is as follows. First one chooses at random the images a', b', and c' of the cube roots of unity $a = 1$, $b = (-1 + \sqrt{3}\,i)/2$ and $c = (-1 - \sqrt{3}\,i)/2$. Next, let d be the point on T which divides the arc from a to b which does not contain c in half. Then choose the image d' of this point on the arc $a'b'$ which does not contain c' according to the uniform distribution on this arc. Continue in this fashion. Computer studies indicate that with probability one there is a periodic point. In fact, it seems that there is a point of period ≤ 5 with a very high probability. There is no proof of this conjecture yet.

Of course, one could carry out this process for homeomorphisms of the unit square by first generating at random those which leave vertical fibers invariant, and then those which leave horizontal fibers invariant. Eggleston [47] has shown that the group generated by these is dense in the group of all homeomorphisms which fix the boundary. Thus, one could generate such a homeomorphism by choosing some number of those which leave fibers invariant and then composing them. If one wanted to move the boundary, one could follow this by some homeomorphism of the boundary which has been extended radially. There are many questions here. For example, what is the expected number of fixed points? One can extend this process to higher-dimensional cubes. Whether the probability measure so generated would give each open set positive measure rests on the following apparently unresolved question in topology:

For $2 < n$, let G_n be the group of all homeomorphisms of $H(I^n)$ generated by those homeomorphisms h such that h leaves all $(n-1)$-

dimensional cubes parallel to some coordinate axis invariant and fix the boundary. Is this group dense in the space of all homeomorphisms of I^n which fix the boundary?

One could ask the same question for the group generated by all homeomorphisms h which leave invariant all intervals I parallel to some axis.

Now, one could generate homeomorphisms of a planar annulus $A = \{(r, 0) : R_1 \leq r \leq R_2\}$ at random as follows. First, one builds a homeomorphism of A which leaves each radial interval $\theta = \theta_0$ invariant. This is done by choosing at random a continuous map of $\{\theta : 0 \leq \theta \leq 2\pi\}$ into $H(I)$, the homeomorphism group of the unit interval which has the same value of 0 and 2π. Next one chooses a homeomorphism of A which leaves the circles $r = r_0$ invariant, or equivalently choosing at random a continuous map of $\{r : R_1 \leq r \leq R_2\}$ into $H(T)$. Next one chooses a positive integer n with probability, say 2^{-n}. Then one chooses n homeomorphisms of the first type, $f_1, \ldots, f_n$, and n homeomorphisms of the second type, $g_1, \ldots, g_n$. Finally, set $h = f_1 \circ g_1 \circ f_2 \circ \cdots \circ f_n \circ g_n$. This will generate a probability measure on the space of all homeomorphisms of A. This measure will give each nonempty open set of homeomorphisms positive measure, provided the group generated by the set of all homeomorphisms of the first or second type is dense. But, this is true. One could generate twist homeomorphisms of the annulus in this fashion. These homeomorphisms either have a fixed point or else move some essential closed curve inside itself. What is the probability of each case?

It would be very interesting to obtain similar results for the production of measure preserving homeomorphisms at random.

For example, on S_2, for each latitude choose a rotation of that latitude in a continuous fashion. This of course yields a measure preserving homeomorphism of S_2. Consider the set M of all such homeomorphisms together with some isometry T of S_2. Is the group generated by $M \cup \{T\}$ dense in the group of all measure preserving homeomorphisms of S_2? In particular, what happens in case T has axis of rotation perpendicular to the north–south axis and angle of rotation 90°? What is the situation if we consider the group generated by M together with all isometries?

How can the measure preserving twist homeomorphisms of a planar annulus be generated?

IV.5. *Random Cantor Sets and Continua*

Let us describe two methods of constructing a Cantor subset—a perfect, nowhere dense subset—of $[0, 1]$.

For the first method one first chooses an open subinterval of $[0, 1]$ at random, which is then deleted. In each of the two remaining closed subintervals one chooses an open subinterval at random which are then deleted. Continue this process.

For the second method, one first chooses a point x_1 in $[0,1]$ which is not to be in the final Cantor set. Next, one chooses at random a_1, $0 < a_1 < x_1$, and b_1, $x_1 < b_1 < 1$. The open interval (a_1, b_1) is deleted. Now, choose x_2 in $[0, a_1]$ and x_3 in $[b_1, 1]$ at random. Then one chooses open intervals about x_2 and x_3 to be deleted. What is the distribution of the measure of these sets? What is the expected Hausdorff dimension? These processes have been investigated by Kahane and Peyrière [48] and more generally, by Mauldin and Williams [49].

IV.6. *Random Topological Objects*

Consider the unit square $[0,1] \times [0,1]$. Choose at random an open rectangle, $(a, b) \times (c, d)$, lying in the square which is then removed. Extend the edges of this rectangle dividing the square into eight remaining rectangles. In each of these, choose an open rectangle at random, each of which is removed. Continue this process. What are the topological and measure theoretical properties of the sets which remain? In the plane, do we get a Sierpinski curve and, in R^3 a Menger curve?

One can alter this process in many ways. For example, first let us choose an open ball at random which is then removed. At the nth stage, choose an open ball at random, but disjoint from those already chosen. Of course, what remains is a continuum.

One could generate Cantor sets at random by the following procedure: First, fix a base of open balls $B_1, B_2, B_3, \dots$. In each ball choose at random an open spherical shell in the ball with the same center as the ball. With probability one, the set that remains after the shells are removed is a Cantor set. What is the probability that the Cantor set is wild for this construction carried out in dimension ≥ 3?

IV.7. *Continuous Maps as Projections of Homeomorphisms*

We consider the problem of reconstructing a homeomorphism of "phase" space from the knowledge of one or more of its "projections." For example, let f be a continuous map of $[0,1]$ into itself. Under what conditions is there a homeomorphism h of $[0,1] \times [0,1]$ such that for each $x \in [0,1]$ there is a point p in the square such that for each n, $f^n(x) = \mathrm{proj}_1(h^n(p))$? Can one find such an h if the desired equation is to hold for one fixed x given in advance; or, for a dense set of x's, etc.? Of course, one could consider this problem for general spaces, $X \times Y$. In particular, what is the situation for the plane?

Finally, when can h be constructed on $X_1 \times \cdots \times X_n$ if "projections" f_i on X_i are given? We seek h such that if $x_i \in X_i$, then there is a point p such that for each n, $f^n(x_i) = \text{proj}_i(h^n(p))$.

IV.8. *Problem by Mycielski and Ryll-Nardzewski*

Let A and B be compact metric spaces and f a continuous map from $A \times B$ into C, the Cantor set. A game is played as follows. A subset E of C is given. Player I chooses a point a of A and then player II chooses a point b of B without any information about the choice made by I. Player I wins if and only if $f(a, b) \in E$. Suppose there is a winning strategy for one of the players whenever E is both open and closed in C. Is there then a winning strategy for one of the players if E is a Borel set?

V. Real Variables, Functional Analysis, Measure Theory

V.1. *Set Valued Measures*

Is there a non-trivial set valued map F defined on the family of all classes of subsets of N, the natural numbers, such that each finite or even countably infinite class is mapped to the empty set, and, if $R_1 \cap R_2 = 0$, then both $F(R_1) \cap F(R_2) = 0$ and $F(R_1 \cup R_2) = F(R_1) \cup F(R_2)$? One can consider many variations of this problem including the degree of additivity of F, e.g., can F be made countably additive?

V.2. *Relative Measure*

It can be shown by the axiom of choice that for every set A of measure zero in the unit interval, there is a countably additive measure μ_A defined on the family of Borel subsets of A so that

1. $\mu_A(A) = 1$;
2. $\mu_A(\{p\}) = 0$ for all points p if A is uncountable;
3. if $A \subset B \subset C$, then $\mu_C(A) = \mu_B(A)\mu_C(B)$.

Problem. Let 2^I be the space of all closed subsets of $I = [0,1]$ and $B(I)$, the Borel field on I. Is there a map μ: $2^I \times B(I) \to [0,1]$ so that

1. for each $F \in 2^I$, $\mu(F, \cdot)$ is a probability measure on $B(I)$ and $\mu(F,F) = 1$;
2. for each $B \in B(I)$, $\mu(\cdot, B)$ is a Borel measurable map of 2^I into I;
3. if $A \subseteq B \subseteq C$, then $\mu(C, A) = \mu(B, A)\mu(C, B)$?

In this problem we regard the space 2^I provided with the Hausdorff distance.

V.3. *Invariant Measures in the Hilbert Cube*

Is there a measure m defined on all Borel subsets of the Hilbert cube such that if A and B are isometric, then $m(A) = m(B)$? Does the usual product measure have this property? What is the situation for I^n? The best results to date have been obtained by J. Fickett, *Studia Math.* **72** (1982), 37–46.

V.4. *Problems on ϵ-Theorems and Theories*

The problems considered here concern the notion of stability in a wide sense. Broadly speaking we consider the following situation: Given a theorem with its hypothesis and conclusion, you might inquire whether a "small change" in the assumptions will allow a statement with a small change in the conclusion. In other words, whether a change in the assumption will produce a correspondingly small change in the thesis of the theorem. The problems surrounding stability arose at first, naturally, in problems of mechanics. Mathematically speaking, they concern the dependence in a solution of a problem on initial parameters or conditions. Often, one might desire continuity in bounded time, or, in some cases, arbitrarily long times. Similar problems arise of course in other branches of physics—in mechanics of continua, in field theories, etc.

We shall not try to formulate a most general problem involving such ideas of stability but will instead proceed by a sequence of concrete problems starting with some in pure mathematics—analysis or even indiscrete algebraic situations—in many cases without regard to physical interpretations or consequences. Perhaps the most elementary such problem concerned the stability in the above sense of a classic functional equation. A problem formulated by Ulam around 1941 was as follows:

The functional equation

$$f(x) + f(y) = f(x + y) \tag{1}$$

can be replaced by the inequality

$$|f(x) + f(y) - f(x + y)| < \epsilon. \tag{2}$$

The question arose whether a solution of this inequality must of necessity be close to the solution of the strict functional equation above. Hyers' result asserts that this is so for continuous or measurable functions [50]. More generally, even without the assumption of measurability, any solution of (2) will be close to certain solutions—perhaps a non-measurable one of the Hamel type. The result is true more generally in that the variables x, y need not be real numbers. They can be elements of a Banach space.

More generally, one may ask, at first for compact continuous groups, whether an "almost automorphism" or endomorphism is necessarily close to a strict such transformation. This does not seem obvious even for some classical metric groups [51].

Or one may inquire about almost additive set functions—functions f which satisfy (2) when x and y are disjoint sets in some algebra or satisfy

$$|f(x) + f(y) - f(x \wedge y) - f(x \vee y)| < \epsilon$$

for x and y elements of some lattice.

In a different type of generalization, one could study functional equations from this point of view. For example, one could ask whether functions satisfying an "almost algebraic" addition formula are close to the famous analytic functions which satisfy it exactly.

In problems of geometry, one could, taking a rather special example at random, ask whether the theorems of Pascal and Brianchon on hexagons lying on conics remain true, if the hypotheses on the intersections of three diametric diagonals in the same point replaced by an assumption that the three intersections be close together imply that the points are located on a circle which is "close" to a conic.

Hyers and Ulam have studied the stability problem for isometric transformations. If we introduce a notion of "almost isometry" one might ask whether such a transformation is necessarily close to a true isometry. A theorem stating the "stability" of this notion is contained in the papers of Hyers and Ulam [52, 53]. Partial results and a short survey were given by Gruber [54]. The problem was solved by Gervirtz [55] and extensions were given by Lindenstrauss and Szankowski [56].[4]

Let us consider still another question of stability of a more geometric nature. Let $\epsilon > 0$. Suppose we are given two surfaces which can be mapped into each other in such a way that the curvatures and the inverses of curvature at corresponding points differ by less than ϵ. Are there then, surfaces within $C(\epsilon)$ of the given ones that are strictly isometric in the sense of internal geometry?

V.5. *Natural Measures, Metrics and Pairs*

Let (X, τ) be a topological space. We can place stronger and stronger conditions for a measure μ to be a "natural" measure.

A measure μ is natural provided there is a metric p_μ compatible with the topology and a base $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$ such that the sets in $\mathscr{B}_n$ are congruent open sets of equal measure.

We give the following example on S_∞. It is known that S_∞ does not possess a σ-finite measure which is invariant or even quasi-invariant under the group action [57]. However, S_∞ does possess a "natural measure" in the above sense. Note that S_∞ is a complete separable metric space in which

[4] *Note added in proof.* Related results have recently been obtained by J. Rassias, *J. Approx. Theory*, to appear.

every compact set is nowhere dense. Thus there is a homeomorphism h of the set of all irrational numbers, J, onto S_∞.

Now define a new metric on S_∞ by setting $p(\sigma, \tau) = |h^{-1}(\sigma) - h^{-1}(\tau)|$. This metric is compatible with the topology of S_∞. We set

$$\mathscr{B}_n = \left\{ h\left(\left[\frac{i}{n}, \frac{i+1}{n}\right]\right) \cap J : 0 \leq i \leq n-1 \right\}.$$

Certainly, the sets in $\mathscr{B}_n$ are open and congruent. Also $\mathscr{B} = \bigcup \mathscr{B}_n$ is a base for the topology. Define $\mu(E) = \lambda(h^{-1}(E))$, where λ is Lebesgue measure. Then μ is a natural measure since the sets in $\mathscr{B}_n$ have equal measure.

Is there a natural measure μ on S_∞ so that the corresponding metric is complete? Note that we can construct a natural measure μ on S_∞ and corresponding metric ρ such that the group of rational translations is the group of isometries; can one do this so that the isometry group is uncountable?

What groups can be realized as the group of isometries of S_∞ for some ρ_μ where μ is a natural measure on S_∞?

A natural pair for a topological space (X, τ) is a pair (μ, ρ) such that μ is a natural measure with corresponding metric ρ such that if ν is a natural measure for X with corresponding metric d, the isometry group of X under the metric d is isomorphic to a subgroup of the isometry group under metric ρ.

Does S_∞ possess a natural pair? What about R^n?

One could say that a measure μ on a topological space S is natural provided the group of measure preserving homeomorphisms is maximal or the semigroup of continuous similarity maps. A map f is a similarity means if $\mu(A) = \mu(B)$, then $\mu(f(A)) = \mu(f(B))$. For example, it is easy to see that Lebesgue measure on $[0,1]$ is natural in this sense. Is Lebesgue measure on $[0,1] \times [0,1]$ natural in this sense?

V.6. *Premeasures*

Can one prove in ZFC that the power set of the unit interval can be partitioned in families of sets, ξ_α, $0 \leq \alpha \leq 1$, with the following properties:

(1) If A is the disjoint union of $\{A_i\}_{i=1}^\infty$, B is the disjoint union of $\{B_i\}_{i=1}^\infty$ and, for each i, A_i and B_i are in the same class ξ_α, then A and B are in the same class,

(2) If $A \in \xi_\alpha$, then for each β, $0 \leq \beta \leq \alpha$, there is a subset C of A such that $C \in \xi_\beta$,

(3) If A and B are in ξ_α and $A \subset C \subset B$, then $C \in \xi_\alpha$?

One could continue in this manner toward a measure, e.g.,

(4) If $N \in \xi_\alpha$ and $A \in \xi_\alpha$, then $A - N$ and $A \nabla N$ are in ξ_α.

What is the situation for other cardinal combinations?

V.7. *Curiosities*

Is there an infinite array of 0's and 1's such that the frequencies of 0's and 1's are equal along every line? What is the situation if we allow some of the lattice points to be unmarked?

V.8. *A Problem by Ronald Graham*

What is the length of the shortest curve which cannot be enclosed in an open unit equilateral triangle?

COMMENTS. It is perhaps surprising that this length is strictly less than 1. When I first raised the problem at the 1963 Boulder Number Theory Conference, I had examples of polygonal curves of length 0.99 which would not fit into a unit triangle. In contrast, for the circle and square, any curve having length less than the diameter will fit. A few papers have appeared on this question, including one by Besicovitch [58], but it is still wide open. Straus solved the problem for the square but it has never been published.

I offer $100 for the solution.

V.9. *Problems by Erdös*

1. Is it true that if $x_n \to 0$ there is always a set of positive measure which contains no subset similar (in the Euclidean sense) to $x_1, x_2, x_3, \ldots$? For finite sequences of course this is true. Komjath [59] has shown this is true if x_n converges slowly to zero. The problem remains open.

2. Is it true that there is an absolute constant c so that every set in the plane of area $> c$ contains three points x_1, x_2, and x_3 so that the area of the triangle determined by x_1, x_2, and x_3 is 1? the best value of c may be given by the area of the circle whose inscribed equilateral triangle has area 1.

Perhaps the following more general result is true. Denote by c_k the area of a circle so that the inscribed regular k-gon in such a circle has area 1. Is it then true that if a measurable set S in the plane has area $> c_k$, then one can find k points $x_1, \ldots, x_k$ in S so that the area of the convex polygon $x_1, \ldots, x_k$ is 1? These problems can of course be generalized to higher dimensions.

3. Let S be a planar set of positive measure. Is it true that there is a point x (x is in S) so that the set of distances $d(x, y)$, $y \in S$, contains an interval [60]?

4. Let G be a denumerable complete graph. To every edge (x, y) of G there corresponds a subset $M(x, y)$ of $(0,1)$ of measure $> \epsilon$. Prove (or disprove) that there exists an infinite path so that the corresponding sets have a nonempty intersection. Several related problems stated in [61] have been solved by Fremlin.

5. E. Klein asked: Is there an $f(k)$ so that every set of $f(k)$ points in the plane no three of which are on a line has a convex subset of k points? Szekeres and I proved

$$2^{k-1} + 1 \leq f(k) \leq \binom{2k-4}{k-2}.$$

Probably $f(k) = 2^{k-2} + 1$.

Is it true that there is an $F(k)$ for which any set of $F(k)$ points, no three on a line, contains k points which form a convex set which has none of the other points in the interior? $F(4) = 5$ is simple. $F(5) = 10$ has been proved by Harborth [62]. It is not known if $F(6)$ exists. Horton [63] proved $F(7)$ and $F(n)$, $n \geq 7$, do not exist.

6. Is it true that to every $c > 0$ there is an $r_0(c)$ such that for every $r \geq r_0(c)$ and for every set S of measure $> cr^2$ in the circle of radius r, S contains the vertices of an equilateral triangle of side > 1? Straus added that it perhaps suffices to have the measure of $S > cr$ if c is a sufficiently large constant. Is it true that to every c there is a k so that there are integers $1 = r_1 < r_2 < \cdots < r_k$ ($r_i = 2^i$ should be OK) so that if S is a set in $|z| < R$ and $m(S) > cR^2$, then S contains the vertices of an equilateral triangle of side r_i for some i, $1 \leq i \leq k$?

The reason for the conjecture is that having no equilateral triangle of a certain size decreases the measure by a constant factor and if the sizes are sufficiently different they seem to be independent.

VI. Metric Spaces, Geometry

VI.1 *Metrics on Spaces of Algebraic Objects of a Given Type*

What we wish to do is place a metric on the set of all finite objects of a given algebraic structure. Actually our metric will be on the space of all isomorphism classes of these objects. For example, a metric can be placed on the space of graphs or relations as follows. First consider two subsets A

and B of $\{1, \ldots, N\} \times \{1, \ldots, N\} = N^2$, where N^2 is imagined to consist of all lattice points in the Euclidean plane with each coordinate a positive integer $\leq N$. Compute the Hausdorff distance between A and B, $d_{\mathrm{H}}(A, B)$. Now, this is not the distance between the relations or graphs that A and B represent since this present number depends on an arbitrary numbering of the points of the graph or given relation. Thus to obtain a distance we set

$$d(G, G') = \min d_{\mathrm{H}}(A, B),$$

where the minimum is taken over all pairs (A, B) such that the graph or relation determined by A is isomorphic to G and similarly for B. What is the expected distance between two graphs with n vertices? Set $d_n = \max d(G, G')$, where the maximum is taken over all G and G' with n vertices. What is the rate of growth of d_n?

Note that this distance is independent of N so we have defined a metric on the space of all finite binary relations or graphs. We note that this metric will make random graphs rather close to each other with great probability.

A metric of the same type can be placed on the space of partially ordered sets, groups, lattices or geometries. On which of these spaces is there a natural metric?

We can define a metric on the space of all metric spaces of the same cardinality as follows: Let (X, d) and (X', d') be finite metric spaces with $|X| = |X'|$. For each bijection f of X onto X' set

$$\epsilon_f(X, X') = \tfrac{1}{2} \sum_{x, y \in X} |d(x, y) - d'(f(x), f(y))|$$

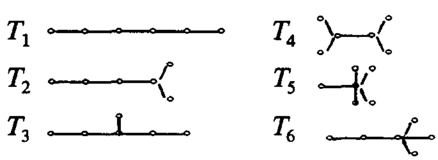
and define the distance $\rho(X, X')$ by

$$\rho(X, X') = \min \epsilon_f(X, X'),$$

where f runs over all bijections of X onto X'. One can ask for the expected distance between two metric spaces of cardinality n, the rate of growth of this expected distance or of the maximal distance with n, etc.

One can specialize this distance function and various questions to special classes of metric spaces. For example, let us consider T_n, the space of all isomorphism classes of trees having n vertices. We define an "internal" distance on each isomorphism class by the "minimum path length" between vertices.

For example we consider the six trees on six vertices:



Bob Schrandt produced the following tables.

1. The distance table for T_4.

	T_1	T_2	T_3	T_4	T_5	T_6
T_1	0	5	8	10	16	11
T_2		0	5	5	11	8
T_3			0	6	10	5
T_4				0	8	5
T_5					0	5
T_6						0

2. A table which gives for each pair of trees the number of permutations (out of 720) which yield the minimum distance.

	T_1	T_2	T_3	T_4	T_5	T_6
T_1		3	11	15	479	12
T_2			4	7	120	24
T_3				16	120	12
T_4					240	24
T_5						120
T_6						

3. A table which gives for each pair of trees the maximum value of ϵ_f.

	T_1	T_2	T_3	T_4	T_5	T_6
T_1		25	24	22	18	21
T_2			23	21	15	20
T_3				20	14	19
T_4					12	17
T_5						11

Again one can inquire about the rate of growth of the expected distance in T_n, etc. Also, the computation of these distances between elements of T_n quickly requires more power than our present computers have. Thus we are interested in obtaining bounds on some "Monte Carlo" techniques of estimating distances. For example, given two trees on n vertices, let us compute all the distances between two subsets of k vertices in the two trees where $k \leq n$ and let us compute the average distance. How does this average distance compare with the defined distance? Of course, another random method would be to choose two subsets of size k from the trees at random and calculate the distance between these two subsets.

VI.2. *Natural Metrics*

Let X be a topological space. A metric d on X which is compatible with the topology of X is maximal means that the isometry group $I(d)$ is maximal. This means that the isometry group of X under any other metric on X is isomorphic to a subgroup of $I(d)$. Kallman and Mauldin [64] have shown the following:

1. There is a maximal metric on the Cantor set K or $\{0,1\}^{\mathbb{N}}$ with the product topology and under such a metric K cannot be isometrically embedded in $\mathbb{R}^n$ for any n.

2. There is a maximal metric on K under which K can be embedded in Hilbert space.

3. With a little work one can prove that the usual metric $|x-y|$ is maximal for the unit interval on $\mathbb{R}$.

4. The usual Euclidean metric is maximal on S^2, the unit sphere in $\mathbb{R}^3$.

5. On the other hand, there are two metrics ρ_1 and ρ_2 on $\mathbb{R}^2$ such that for any metric ρ on $\mathbb{R}^2$ yielding the usual topology, $I(\rho)$ is isomorphic to a subgroup of $I(\rho_1)$ or $I(\rho_2)$.

The existence of a maximal metric on the space of irrational numbers seems to be related to the question posed by Schreier and Ulam in Problem 95 of the Scottish Book, which essentially asks whether every complete separable metric group can be abstractly embedded in S_∞. If the answer to this question is yes, then one can obtain a maximal metric on the space of irrationals (or equivalent S_∞) by simply taking a left invariant metric d on S_∞. This metric is maximal. This follows from the fact that S_∞ is a subgroup of $I(d)$. If r is any compatible metric on S_∞, then $I(r)$ is a subgroup of the isometry group of the metric completion of S_∞ under the metric r, and the isometry group of a complete separable metric space is a complete separable metric group.

Is there a natural metric on the nth symmetric product of the interval? Certainly this space is metrizable [65].

As a matter of fact, Kallman and Mauldin noted that there is a natural metric on the space of all closed subsets of $[0,1]$ provided with the Vietoris topology. According to a result of Curtis and Shori [66] we are asking whether there is a natural metric on the Hilbert cube. Let $X = \Pi G_n$, where each G_n is the unitary group on a complex n-dimensional Hilbert space and each such group is some G_n. Thus x is a compact metric group such that every compact metric group is isomorphic to a subgroup of X. Let ρ be a left invariant metric on X. Then X is a subgroup of $I(\rho)$. Since the isometry group of a compact metric space is a compact metric group, ρ is a maximal metric on X. But since each G_n is a Peano continuum, X is homeomorphic to the Hilbert cube.

Another notion of maximality of a metric d is that there be no compatible metric r such that $I(d)$ is isomorphic to a proper subgroup of $I(r)$. How is this notion of maximality related to the first one?

Finally, let us mention two other notions of maximality. A metric d could be said to be maximal provided the semigroup $S(d)$ of all similarity maps of X into X is maximal. We consider then two notions of maximality of $S(d)$. First, if r is compatible metric, then $S(r)$ is algebraically isomorphic to a subsemigroup of $S(d)$. Second, there is no compatible metric r such that $S(d)$ is isomorphic to a proper subsemigroup of $S(r)$. How are these four notions of maximality related?

VI.3. *Distances or Metrics in the Space of All Graphs or Relations*

One sensible such metric is obtainable as follows. Given two graphs, i.e., subsets of all pairs of integers from one to N located on the lattice points, i.e., points with integer valued coordinates ranging from one to N, we may compute the Hausdorff distance between them. This of course depends on an arbitrary numbering of the points of the graph or the given relation.

We now get rid of the arbitrariness by considering not just the given sets A and B representing the two relations, but also two classes consisting of all possible relations of the two sets by all possible numberings of the given points.

We now use the Hausdorff idea iterated twice. This will make two random graphs rather close to each other with great probability. We can consider similarly a definition of distance between two families of subsets of a given set. Erdös comments:

Let $|S| = n$, and $A_1, A_2, \ldots, A_t$ be a maximal family of sets so that $d(A_i, A_j) > n/10$. It is well known and easy to see that

$$2^{n(1-c_2)} > t > 2^{n(1-c_1)}. \tag{1}$$

No doubt

$$t = 2^{n(1-c+o(1))}$$

for a certain c, but I think this is not yet known. Thus from (1) there are at least $2^{2^{n(1-c_1)}}$ families of sets so that the distance between any two is $\geq n/10$. Since if A_1 occurs in one of the families and not in the other, their distance is at least $n/10$. Now that we have a, not too bad, lower bound let us try to get an upper bound. Suppose that there are $f(n)$ families any two of which have distance $> n/10$. This means that one of the families contains a set A so that all the sets of the other family have distance $> n/10$ from A. Now apply this to all pairs of our families. Then it easily follows that there is a set A_1 and more than $f(n)/2^n$ families so that none of these families contain a set whose distance is $\leq n/10$ from A_1. Henceforth, we only consider these families. Repeat this process k times. Then we get k sets A_i, $1 \leq i \leq k$, every two of which have distance $> n/10$ and there are $\geq f(n)/2^{kn}$ families so that every set occurring in these families has distance $> n/10$ from all of the A's. Thus by (1) if we repeat this process $2^{n(1-c_2)}$ times we arrive at a contradiction with (1). But this means that

$$f(n) \leq 2^{2^{n(1-c_2)}n},$$

which is a reasonably good upper bound since we have

$$2^{2^{n(1-c_1)}} < f(n) < 2^{2^{n(1-c_2)}}.$$

VI.4. *Natural Metric for Lorentz Group*

Let E and T be metrizable topological spaces. For each metric σ on E and metric τ on T, consider the set of all cones in $E \times T$. Each point $(x_0, t_0) \in E \times T$ defines a cone by

$$C(x_0, t_0) = \{(x, t) : \sigma(x, x_0) = \tau(t, t_0)\}.$$

The Lorentz group associated with the metric σ and τ is the group of all transformations f of $E \times T$ into itself which takes cones to cones and such that if $(s_i', t_i') = f(s_i, t_i)$, for $i = 1, 2$, then

$$\sigma^2(s_1, s_2) - \tau^2(t_1, t_2) = \sigma^2(s_1', s_2') - \tau^2(t_1', t_2').$$

Do there exist natural metrics σ and τ on each E and T, respectively, for the Lorentz group $L(\sigma, \tau)$? In other words, metrics σ and τ such that $L(\sigma, \tau)$ is maximal in one of the senses given in problem VI.2.

If σ and τ are maximal for the Lorentz group, are σ and τ maximal metrics on E and T?

In particular, what is the situation when T is $\mathbb{R}^1$ and E is either $\mathbb{R}^2$ or some n-dimensional torus?

VI.5. *Borsuk's Intrinsic Metric* [68]

Let X be a metric space with metric ρ such that for every two points x and y there exist arcs in X of finite length joining x to y. Let $\rho_X(x, y)$ be the lower bound of the lengths of all such arcs. Borsuk introduced this metric. If for every sequence $\{y_k\}_{k=1}^{\infty}$ converging to x, $\lim_{k\to\infty}\rho_X(x, y_k) = 0$, then the topology induced by ρ_X in X coincides with the topology induced by ρ. Is it true that ρ_X is a natural metric on X? This is true if X is the reals, the circle or S_2, the two-dimensional sphere.

VI.6. *Analytic Sets and the Visual Hull*

It seems that the existence of many objects can only be proved with the aid of the axiom of choice. This is perhaps best illustrated by the results of Solovay [69] concerning the existence of non-measurable sets. However we can also estimate the complexity of a construction by the lowest possible descriptive character of such an object. Thus a non-measurable set cannot be an analytic or coanalytic set. This in itself indicates that the construction of non-measurable sets cannot be very explicit. We raise a number of questions concerning this notion.

There is a subset of the Euclidean plane which meets each straight line in exactly two points. Can such a set be analytic? It is known that such a set cannot be the union of countably many compact sets [70]. One can prove that every such set is totally disconnected.

Let M be a maximal set of pairwise orthogonal probability measures on $[0, 1]$. Can M be an analytic set in the weak topology on the space of measures?

Let $E \subset \mathbb{R}^n$. For each set $\mathcal{M}$ of subspaces of $\mathbb{R}^n$. Let $H_{\mathcal{M}}(E)$ be the largest subset of $\mathbb{R}^n$ which includes E and such that $H_{\mathcal{M}}(E)$ has the same projection as E does for each $M \in \mathcal{M}$. Thus $H_{\mathcal{M}}(E)$ is a visual hull of E [71]. If E and $\mathcal{M}$ are analytic (or Borel) sets, must $H_{\mathcal{M}}(E)$ be analytic or Borel? What if $\mathcal{M}$ is σ-compact? Some partial results are given by Larman and Mani [72].

VI.7. *Random Metric Spaces*

We will define a metric d of diameter ≤ 1 on the set of all positive integers.

Let $d(1, 2)$ be a number chosen at random from the open interval $(0, 1)$. Suppose n is a positive integer and the numbers $d(i, j)$, for $1 \leq i,\ j \leq n$, have been determined such that the numbers obey all the axioms of a metric whenever all the quantities in question have been defined. Choose a permutation π of $\{1, \dots, n\}$ at random and a number $d(\pi(1), n + 1)$ in the open unit interval at random. Suppose $1 < j \leq n$ and the function d has been extended to all pairs $(\pi(i), n + 1)$, for $1 \leq i < j$, and suppose that d again satisfies all the axioms of a metric whenever all the quantities in question have been defined.

To define $d(\pi(j), n + 1)$, we need the inequalities

$$d(\pi(j), n+1) \leq d(\pi(i), \pi(j)) + d(\pi(i), n+1), \qquad \text{for } 1 \leq i \leq j, \tag{1}$$

$$d(\pi(i), n+1) \leq d(\pi(i), \pi(j)) + d(\pi(j), n+1), \qquad \text{for } 1 \leq i \leq j, \tag{2}$$

and

$$d(\pi(i), \pi(j)) \leq d(\pi(i), n+1) + d(\pi(j), n+1), \qquad \text{for } 1 \leq i < j, \tag{3}$$

to all hold simultaneously. For these inequalities to be satisfied, it is enough to have for all $i_0, i_1, 1 \leq i_0, i_1 < j$,

$$d(\pi(i_0), n+1) - d(\pi(i_0), \pi(j)) \leq d(\pi(i_1), j) + d(\pi(i_1), n+1) \tag{4}$$

and

$$d(\pi(i_0), \pi(j)) - d(\pi(i_0), n+1) \leq d(\pi(i_1), \pi(j)) + d(\pi(i_1), n+1). \tag{5}$$

We can then choose $d(\pi(j), n + 1)$ at random from the interval with left-hand end point the maximum of all quantities on the left-hand side of (4) and (5) and with right-hand end point the minimum of the number one and all quantities on the right-hand side of (4) and (5). Now, to see that (4) holds, we already have

$$\begin{aligned} d(\pi(i_0), n+1) &\leq d(\pi(i_0), \pi(i_1)) + d(\pi(i_1), n+1) \\ &\leq d(\pi(i_0), \pi(j)) + d(\pi(i_1), \pi(j)) + d(\pi(i_1), n+1), \end{aligned}$$

since d is defined for all quantities which appear. But, rearranging we have (4). Inequality (5) can be obtained similarly.

Let (X, d) be the separable space obtained by completing the metric d. We note that with probability one this metric space is dense-in-itself. What is the probability the space is connected or infinite dimensional? There are of course any number of questions concerning these spaces.

VI.8. *The Copernicus Problem*

The general idea is as follows: We consider properties of a number of curves in Euclidean space viewed from a moving system of coordinates. An example will indicate the type of mathematical questions which could be studied. Suppose several curves are given, considered as trajectories described in time by points moving on these curves. Is it possible by viewing these trajectories from a moving system of coordinates to make them "simpler"? Indeed, the Ptolemaic motions on epicycloidal trajectories as viewed from the earth will appear simpler if one assumes a motion of the observer around the sun. An example of the question could be: Given say, four closed curves, described by uniformly moving points, can one find a motion of the system of coordinates, by which we mean an arbitrary motion of the origin, and an arbitrary rotation of the axis in time so that in this system all the given curves would appear to be convex? More generally, what are the invariants of a system of n trajectories for an arbitrary motion of the system of coordinates? It is clear that for $n = 3$ there will be no interesting properties since we may assume the origin of the coordinate system to be located on one of these points and the rotation of the axes such as to have one of them always pass through the second point. In this system one point will be at rest, the second will move up and down on one of the axes and the motion of the third one may appear to be quite simple by suitable rotation of the system of axes.

We should assume that the motion of the coordinate system should be continuous but perhaps otherwise arbitrary. What are the properties of surfaces, or more generally sets other than trajectories with respect to a general group of transformations of the coordinate system? We require, of course, that the coordinate system be in each instant of time a Euclidean rigid form.

What is a space-time formulation generalizing this mathematical puzzle above?

VII. Physics and Biology

This section is based upon some sketches and general speculations of Ulam. Some of the topics planned to be discussed here were presented in [73]. Some of his general ideas are also mentioned in [9, 74]. Here are some of his general thoughts for the readers' perusal:

Is an "experimentum crucis" possible—an experiment to decide the finiteness or the infinite nature of a true physical model? Or, is this question undecidable? Which is the more convenient thing to assume—the role of "convenience" and simplicity for a perhaps only temporary description; cf. the discussion in Poincaré of the invariance or conservation of energy? Is it however possible that, for example, one could see a mirror type or isomorphism of a physical system in a part of itself which is similar or a constituent? Would that be a hint then of a true infinity?

Discuss the differences between mathematics and theoretical physics. The inverse of mathematical procedures: in physics, given "facts," find their "bases" or "laws." In mathematics we assume a base, that is to say, axioms, and we derive facts, i.e., theorems. That is an inverse procedure.

Develop games for various statistics—the Planck distribution, Fermi–Dirac statistics, etc., and statistical mechanics in general. Investigate the fundamental role of the "identity" of different particles, namely which is the primitive notion: analog of points or sets of points, that is to say, unordered collections.

Can one attempt to axiomatize the idea of experiments (*sic*!)? More generally, how to define heuristic approaches in physical sciences or "observations" of theoretical experiments or "experiments in theory."

VII.1. *Pair Production with Conservation, Collision Transformation*

Consider a large number of particles to each of which has been assigned a positive number, for example, energy. Pair the particles at random. Each pair produces a new pair and each particle of the new pair is assigned new numbers at random with the constraint that the sum of the new numbers must be the sum of the numbers assigned to the old pair.

Iterate this process. It seems that for a number of fixed random methods U of redistributing the numbers (the energies) there is a limiting distribution $G(U)$ such that for each initial distribution P of the numbers, the iterates $T(P)$, $T(T(P)), \ldots$ converge to the distribution $G(U)$.

We consider two examples of redistribution. For the first method we assign to each of the new particles one-half the sum of the energies of the "parents." For the second method, the new distribution is generated by choosing a number α from $[0, 1]$ according to the uniform distribution, and then assigning to the first new particle α times the sum of the old numbers and $(1 - \alpha)$ times the sum to the second particle.

For the first method, it is clear on physical grounds that the limiting distribution $G(U)$ should be point mass at 1. This follows from the strong law of large numbers: Since if u is the distribution on R_+ with first moment 1 (normalized total energy), then $T(\mu)$ is the distribution of $(X_1 + X_2)/2$, where X_1 and X_2 are independent with distribution μ. Thus,

$T^n(\mu)$ is the distribution of $(X_1 + \cdots + X_{2^n})/2^n$, where $X_1, \ldots, X_{2^n}$ are independent with distribution μ. Therefore, $T^n(\mu)$ converges to point mass at 1.

For the second method Ulam [75] conjectured on the basis of some computer studies that the limiting distribution $G(U)$ is the exponential distribution. It can be seen that the exponential distribution is the only distribution of energy which is fixed under this method of redistribution and $T^n(P)$ converges to this distribution at least for all distributions P on R_+ such that all moments exist [77].

Ulam also believed there is a fixed limiting distribution in case the law of redistribution is given by always assigning $\sin^2 u$ times the sum to the first new particle and the remainder to the other. Blackwell and Mauldin [77] proved this.

For each given law of redistribution, L, is it true that there is a distribution $G(L)$ such that for any initial distribution of energy P, the iterates $T(P)$, $T(T(P)), \ldots$ converge to $G(L)$? In case $G(L)$ exists, what is it? In case $G(L)$ exists, we will call it the collision transformation of L. ($G(L)$ exists and is an attractive fixed point [77].)

More generally, we have associated with each particle a point in a space X. X could be the space of pairs of energy and momentum, for example. We are given an invariant of the system—a map g: $X \times X \to X$. The particles are paired at random. Each of the new particles is assigned at random new labels in X with the constraint that the g-value of the pair of new labels must be the g-value of the pair of old labels. Again, for each given law of redistribution of labels L, does the collision transform $G(L)$ exist?

VII.2. *Natural Metric in Lorentz Space*

Some problems on the "natural" group of isometries in the four-dimensional time-space Euclidean version:

One can consider straight lines in this space by presenting life lines of points moving with constant velocities. The question arises of a metric in the space of such lines restricted, in the relativistic case, to velocities less than c. In the relativistic case one could take as a distance between points the usual Minkowski expression.

In the Lorentz space we would like the distance between life lines to be invariant under the usual method and perhaps more.

In general, there seems to be no good simple generalization of the Lorentz method for a wide class of "special relativity" type geometries.

But, even in the ordinary three-dimensional space the problem is to define a metric which would be invariant under ordinary isometries of the point-space period. And perhaps under an even larger group?

One might try to define the Hausdorff formula using a stereographic projection Hausdorff distance of acquiring a compact space which underlies the space of objects for which we want to define a distance.

VII.3. *Random Walks*

Consider a random walk in one dimension with step n of size $\pm n^2$ with equal probability. It is possible that the walk passes through every integer. In fact, with probability 1 the walk passes through every integer infinitely often. But, what are the average relative frequencies between k and ℓ,

$$\lim_{N\to\infty} \frac{k(N)}{\ell(N)},$$

where $j(N)$ is the average number of times the walk is at integer j in the first N steps? Also, consider this problem with step $\pm n^3$ or $\pm f(n)$.

If we consider the walk with step n equal to 0 or n^2 Graham has shown that the walk can pass through every integer ≥ 128. In other words every integer ≥ 128 is expressible as the sum of squares such that no square is used more than once. He also showed that for cubes, every sufficiently large integer is so expressible.

For $f(n) = \pm n^3$, we have $-(n+1)^3 + (n+2)^3 + (n+3)^3 - (n+4)^3 + (n+5)^3 - (n+6)^3 - (n+7)^3 + (n+8)^3 = 48$. So, if the integer k occurs, then every integer of the form $k + 48j$, $j = 1, 2, 3, \ldots$, also can occur. All we need check is that the integers from 0 to 47 occur.

VII.4. *Random Walks Again*

Consider a random walk of a point on the line starting at 0 and with step n of size $\pm 1/2^n$. Then the final position of the particle is uniformly distributed over $[-1, 1]$.

Next, consider a random walk of a point on the line starting at 0 and with step n of size $X_n = \pm 1/n$ with equal probability. Now, with probability one, ΣX_n exists and the limiting distribution has variance $\pi^2/6$. What is the limiting distribution?

Consider a random walk with $X_0 = 0$ and $X_n = \pm n$ with equal probability. Is it true that with probability one the walk goes through every integer infinitely often? Is it true that the distribution of $S_n = X_1 + \cdots + X_n$ is approximately uniformly distributed over $[-1/\sqrt{n}, 1/\sqrt{n}]$?

In general, consider a random walk $(X_n\}_{n=1}^{\infty}$, with $X_n = \pm f(n)$ with equal probability. Then for almost every walk is it true that

(1) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i(w)$ exists?

Also, does

$$(2)\quad \lim_{n\to\infty} \frac{\#(\{i \le n : S_i(w) = k\})}{\#(\{i \le n : S_j(w) = 1\})} \text{ exist?}$$

Next, consider a random walk in R^2 with $X_0 = (0,0)$ and $X_n = (0, 1/2n)$, $(0, -1/2n)$, $(1/2n, 0)$ or $(-1/2n, 0)$ with equal probability. Then the final position of the particle is uniformly distributed over the ℓ_1 unit ball $\{(x, y) : |x| + |y| \le 1\}$.

More generally, consider a random walk in the plane where step n has magnitude $f(n)$ and the direction is chosen according to a given distribution function on the directions—on the unit circle. Is it true that for almost every walk

(a) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i(w)$ exists?

(b) the average amount of time spent in a sector exists?

Consider the relative motion of two or more random walks with red particles and black particles starting at different points in some dimension, with each obeying one of the rules given earlier in this section with some additional rules concerning their interaction. For example, a red and a black particle annihilate each other if they come within distance $\epsilon \ge 0$. Or more generally, some rule is given governing the production of elements when two particles come within a specified distance.

Finally suppose we begin with a distribution of points at time $t = 0$ on the unit sphere. Consider various specified walks which the particles perform simultaneously with various rules governing possible constraints, e.g.,

Rule A: A particle cannot move if all possible positions are occupied.

Rule B: The particle moves, but there may be more than one particle occupying a given system.

What are time n distributions of density of particles and the limiting behaviour?

Let $g = R^2 \to R^+$ and suppose we are given k angles $\theta_1, \ldots, \theta_k$, $0 \le \theta_1 < 2\pi$. Fix $\epsilon > 0$. For points p and q of R^2 we define a "deterministic" distance $\rho_{\pi,\epsilon}(p)$ between p and q by setting the distance equal to the smallest value of k such that $T_{i_k} \cdots T_{i_1}(p)$ is within ϵ of q (in the Euclidean norm) where $T_i(x) = x + g(x)e^{i\theta_i}$.

We also define a "probabilistic" distance $\rho_{d,\epsilon}(p, q)$ by setting it equal to the positive integer n such that random walk starting at p has greatest probability of being within ϵ of q after n steps. The walk proceeds by choosing one of the transformations T_i at random at each step.

Conjecture. There is a map f_ϵ: $N \to N$ such that $\rho_{\pi,\epsilon}(p, q) = f_\epsilon(\rho_{d,\epsilon}(p, q))$.

VII.5. *Symbolic Dynamics of Billiards*

Consider a closed convex curve in R^2 and a light ray emerging from some point on the curve into the interior of the curve. We associate with this motion a sequence of the symbols R, L and C as follows: At each time that the ray hits the curve we look into the region from the outward normal at the point. We write down an R if the light ray leaves the point going to the right. Similarly we write down an L if it leaves going to the left. We write down C if it leaves along the inward normal. What are the possible sequences? What are the possible periodic sequences?

One can of course model the system by points on a planar annulus [78]. With each point of the annulus we have associated a symbolic sequence. It would be interesting to study the behaviour of the sequences as one moves around the annulus.

It is true that for each point the arithmetic density of the number of R's exists? What is the behaviour of this density? What is the expected density?

Consider a rectangle with sides of lengths a and b. For each angle α, $0 < \alpha < \pi/2$, shoot a light ray out from one fixed corner at angle α and associate with α the symbolic sequence of R's and L's. How does the frequency of R's vary with α?

VII.6. *Some Notes about the Brain (Synopsis of Peripheral Items)*

Notions of memory, not the physiological memory which is not at all known, but the schematic or combinatorial nature of memory—ideas of holographic or multistored items. The duality between holographic or Fournier series type of notation and point notation, whatever the points are —individual sets of points, graphs or trees, general multigraphs.

1. Recognition: mention and description of report with Schrandt about recognition of letters, speculation on generalizing it [82, pp. 150–151].

2. Building of a network or interaction between elements of memory in cells or in sets of cells.

3. Analogies with existing computers and their codes. Outline of scheme for parallel work [82].

4. Molecules, autonomous nervous system recognition of immunology.

5. Access to memory, various schemata, i.e., astronomers find new stars by flicking sequences of photographs of the sky.

6. Codes or distances between encephalographic data. Detailed discussion of distances between pictures.

a. Distances between codes, one-dimensional pictures.

b. Hausdorff distance between pictures on screens, in a plane or even three-dimensional space by the sense of touch.

VII.7. *Hypotheses concerning the Brain*

(From Ulam's remarks at Rockefeller Neurosciences Institute/August, November 1983)

One fragment of the problem of recognition, e.g., specifically visual impressions or "pictures," is their various codings in the brain. Everything that follows is relative to a preferred size and orientation.

Hypothesis 1 The brain performs deformations of the retinal input and compares the deformed schemata with a given storage or several storages contained in the memory. These processes must be performed extremely fast. In fact one can try to estimate the speed of the neuronal operations from the time taken by selection of one shape among a very great given number or say finding a subpattern in a complicated maze.

Hypothesis 2 is an evolutionary or individual "micro evolution" in the brain by adapting its functioning to the set of previous experiences with a "Darwinian" process of establishing habits of inferences by a probabilistic or statistical mechanism from its history.

Hypothesis 3 is that a picture could be thought of as an analog of a "photograph," that is to say a set of states of groups of neurons or of impulses. How many could be stored? There are about 10^{14} connections. In addition, sets of such connections corresponding to impressions need not be disjoint, i.e., a neuron may initiate in connection with some other neurons different sets of impulses. A temporary or very short memory may initiate many comparisons. They may be organized in clusters. A class of impressions we may call "concept." A collection of classes of concepts we may call "ideas," etc.

Hypothesis 4 deals with the notion of distance between pictures or their codes in the memory. We hypothesize that there are dozens of metrics between these coded pictures. For example:

1. A Hausdorff distance.

2. Hausdorff $\pm$ a number of deletions or additions of elements to the two pictures which we compare, say the square root of their number is added to the Hausdorff distances.

3. A distance is obtained by subdivision of the screen on which the pictures are compared by a weighted system of coordinates, e.g., the one used by Schrandt and myself.

4. The minimum number of "errors" necessary to bring one picture into the other.

5. A distance between codes analogous to the distance I have defined for linear sequences, DNA codes, etc.

6. The distance between "features," e.g., we treat lines as elements or, even more generally, a number of given types of curves as points and use one of the above distances between those.

7. Unions of pictures, that is, the Boolean union of sets, e.g., just the decomposition of each of the two pictures into two patterns which are individually compared, etc.

Hypothesis 5 is that there is a physiological or anatomical mechanism to measure such distances, i.e., at least to scrutinize and decide whether they are sufficiently "small." With a discernment that none of them is sufficiently small the new picture or impression is put into the memory(?).

VII.8. *Genealogical Distance*

We begin with an initial population consisting of n asexual individuals. This population mates and changes at time $t = 0, 1, \ldots$ according to the following rules:

1. A fixed proportion p of the population which is unmated at time t mates and forms a couple at another time t.

2. There is a fixed distribution function for all couples having $0, 1, 2, \ldots$ offspring. This distribution function has a finite expectation. Also there is a fixed distribution for the first offspring to appear at time n units after the formation of the couple, and for the second offspring at time m units after the appearance of the first, etc.

3. Once a couple forms, it exists until the last offspring is produced.

What is the expected length of ancestry of an individual who exists at time n? What is the expected difference in the lengths?

One can form a number of interesting metrics on this population. Some of these and their properties are given in papers by Mycielski and Ulam [79] and by Kahane and Marr [80]. One could inquire about the properties of these metrics in the framework.

VII.9. *Items from Discussions with Mycielski on the Building of Memory in the Brain or the Nervous System*

1. Distances between sets (i.e., pictures on a screen—the retina) the class of "equivalent" sets obtained by enlargement, by small rotations and

small "wigglings." The distance between such, obtained by either Hausdorff metric, Steinhaus metric, and/or the distance between such obtained by Rademacher–Walsh coefficients and by a "dual" method that is using the first few coefficients in a series development *and* a small number of individual points.

2. "An electronic way" to compute such distances, perhaps by addition of points modulo 2, and registering total luminosity, in the case of $S(?)$. Distance—find the way of realizing the other distances through the parallel boards behind the retina, processing it.

3. The role of composing transformations or, more generally, relations on sets. The idea is to use a small number of given functions or relations to obtain approximations to "arbitrary" pictures. This is an economic way to memorize a great number of sets, by remembering the sequence of approximating transformations. Is there a mechanism, in the nervous system or in the brain to effect such?

4. Classes of such collections of pictures. How to devise distances between them perhaps by coding linearly each set—and iterating the Hausdorff distance?

5. The operation of "gluing" a number of pictures, say four, and remembering their addresses, that is to say coding by simple logical or Boolean operations.

6. The problem of taking representative examples from classes of sets and having those in the memory—either at random, as Mycielski likes, or from the most economical finite number approximating the class or defining the "center" of mass of such collections. The problem of what the most economical procedures for such are.

7. The operator of "consciousness" that is in a subconscious branching process of search and analogy finding, selecting a one-parameter path through these which, on the one hand, is determined by the branching process and on the other hand, directs it by a conscious decision sequence.

8. The method of search through a tree analogous to the Mycielski–Ehrenfeucht procedure for "learning."

9. The mathematical problem of the analogy to measure theory: instead of attaching real numbers to sets, we want to attach single sets to various classes of sets so that the empty set would be attached to classes composed of a single set. The correspondence should be additive, that is to say, to a union of classes there should correspond the union of the corresponding sets and finally, if T is an arbitrary point transformation on

the fundamental space, then the generated transformation of classes of sets should have the generated individual sets correspond to them.

VII.10. *Notes on Organization of Memory and the Tree of Associations*

The number of neurons in a human brain seems to be of the order of 10^{11}; the number of connections from each to the others, of the order of 10^5. Very roughly the number of connections then, from an average element, is of the order of $\sqrt{N}$, where N is the number of elements. These elements seem to be much more than mere "on/off" or similar switches.

It seems manifest that any given impression or "thought" is deposited in a considerable number of different places. This seems particularly apparent when one tries consciously to remember the name of an object or person. The conscious search, for example, tries to find the name under a beginning letter or by recollecting whether the name is short or long, or whether it is in English or some other language, or it searches the number of syllables, or the appearance of consonants in the middle. Often remembering the first name leads to the recollection of the last name, etc.

One can hypothesize that a metric, given an idea of proximity or similarity or analogy, is used for locating the deposited information. Indeed this measure of similarity must be akin to the metric which we used for the space of DNA codes.

Roughly speaking in a very general situation the distance is expressed by the number of steps leading from one mental picture to another. These steps for the sequence of symbols in the DNA code consisted of replacing one symbol by another, erasures, or insertions, or, more generally, the application of some deformation stated in advance.

An interesting feature of the organization in the nervous system is, however, that a number (20?) of metrics are used, corresponding to the different classes of location in which the sought for impression is located. The organization then would conjecturally involve a number of different metrics in the same network. One could speculate about the way in which the neural system can produce, working simultaneously in parallel, a great number of stored pictures together with a number of programmed small deformations in order to find the location for a new impression or a "question."

We can take as an example the problem of how to encode a great number, say 10^4, of numbers which are written in a binary (or decimal) system if we imagine each to be written on the circumference of a circle so that there is no clear beginning or end to arrange them lexicographically. This is so because forgetting the initial digit would be completely misleading. In fact the similarity of two numbers each of forty or fifty digits would be of a global nature defined by a metric involving the "morphology" of the sequence.

Presumably the connections are not completely randomly arranged but are largely between clusters of receptors and emitters and, in the building up of a memory of new impressions or experiences, they are placed in the clusters according to the metric, or distance proximity, with other parts of the cluster.

We shall enumerate now some of the possible metrics of which, as conjectured earlier, there might be a sizeable number of different ones.

(a) The metric used for linear codes in DNA. In the simplest case depending on the number of erasures and substitutions to make two linear sequences of symbols (i.e., digits), on a circular arrangement. A similar one can be defined for two-dimensional pictures consisting of symbols of a number of letters or colored dots. This refers to a visual distance or similarity.

(b) A number of α-tuples of the same symbols consecutively where we imagine α to be a small integer, say 3, or 4, or 5 ....

(c) For linear arrays of 0's and 1's we may imagine them to be coding integers in a binary development and the distance between two of them can be the relative complexity as discussed elsewhere. One can generalize this to two-dimensional rectangular arrays of 0's and 1's.

(d) The simplest distance between sequences could be the comparison of the number of occurrences of the same symbols irrespective of position.

(e) We may imagine that we count the number of identical first differences between successive symbols or even second differences and compare these sequences according to some of the indicated metrics. This would perhaps play a role in the auditory memory of sequence of tones or sounds.

(f) We can consider for two sequences the number of motions of each symbol into other places to make the two identical. One may count not only the number but the sum of all the distances in those motions operating on one or both sequences. Similarly for two-dimensional pictures consisting of a number of dots of several colors. This distance is in a way analogous to the definition of the minimal work in the problem of "déblais and remblais."

(g) Possibly, in a simultaneous or parallel scheme, a new picture or arrangement is presented to several clusters and placed where its distance from other members of this cluster is small. If none of this takes place we have a "new" element in a new potential cluster in vacant places. Whether this is registered permanently would depend on repetition of this or very similar first impressions. For such a scheme to function, one would need to have an arrangement which in some way plays the role of a counter.

VII.11. *On the Evolutionary Importance of Mathematics as a Biological Force as a Way to Perfect the Brain of Mankind and on Understanding some Mental Processes*

This is a philosophical view which Ulam often mentioned but which we did not develop.

VII.12. *A Distance between Two-Dimensional Codes*

Suppose we are given an $n \times n$ matrix A of 0's and 1's and an $m \times m$ matrix B of 0's and 1's. We are allowed to change an entry in such a matrix to a blank. If we have altered A by such changes to A' and B to B', we then calculate a number $p_1(A', B')$ as follows.

First we form the subset α_1 of $[1, n+1] \times [1, n+1]$, which is the union of all squares $[i, i+1] \times [j, j+1]$ such that $a'_{ij} = 1$. Similarly, we form a subset B_1 of $[1, m+1] \times [1, m+1]$.

Set $p_1(A', B') = \min\{p_H(T(\alpha_1), B_1)\}$, where p_H is the Hausdorff distance between $T(\alpha_1)$ and B, and T runs through all translations of the plane which take lattice points to lattice points. We also calculate $p_0(A', B') = \min\{p_H(T(\alpha_0), B_0)\}$, where α_0(resp. B_0) consists of the union of all intervals $[i, i+1] \times [j, j+1]$ such that $a'_{ij} = 0$(resp. $b'_{ij} = 0$). Finally, we define the distance between A and B to be the minimum of $k_1 + k_2 + p_0(A', B') + p_1(A', B')$, where k_1 is the number of blanks in the matrix A' and k_2 is the number of blanks in the matrix B'.

VII.13. *Quadratic Transformations and Pair Production*

Consider a large population consisting of individuals of two types. Let us denote the proportions of these types by x and y, so,

$$x + y = 1.$$

Let us suppose that these individuals are paired at random and each pair produces a new pair according to the following pair production rules:

$$(x, x) \to (x, y)$$
$$(x, y) \to (x, x)$$
$$(y, y) \to (y, y).$$

What is the expected behaviour of this population under iteration of mixing and pair production? Letting x' and y' be the expected proportions on the next level we have

$$x' = (1/2)x^2 + 2xy$$
$$y' = (1/2)x^2 + y^2$$

or

$$f(x) = x' = 2x - (3/2)x^2.$$

For this map of the interval, 2/3 is an attractive fixed point for all initial values of x; i.e., if $0 \leq x \leq 1$, then $x, f(x), f(f(x)), \ldots$ converges to 2/3.

If we use the pair production rules

$$\begin{aligned} (x, x) &\to (x, x) \\ (x, y) &\to (x, x) \\ (y, y) &\to (y, y), \end{aligned}$$

then 0 is an unstable fixed point and if $0 < x \leq 1$, then the iterates converge to 1, the stable fixed point.

The situation becomes more complicated when one has more than two types of individuals and even more so if, instead of binary reactions, triple collisions, etc., are allowed. Years ago Menzel, Stein, and Ulam [81] carried out numerical studies of pair production with particles of three types. A number of interesting conjectures were made in that report, and later studies led Stein and Ulam to believe that some "strange" invariant sets could be obtained from these systems. We will consider one example and show how one could possibly incorporate the notion of time into such a system. Of course, there are many variations on this method.

We consider example I.2.m from the report. We have three types of particles and the rule governing the system is

$$(\mathrm{R}_0) \qquad \begin{cases} (x, x) & \to (y, y) \\ (x, y) & \to (x, x) \\ (x, z) & \to (z, z) \\ (y, y) & \to (x, x) \\ (y, z) & \to (y, y) \\ (z, z) & \to (x, x). \end{cases}$$

Thus

$$(\dagger) \qquad \begin{cases} x' = y^2 + z^2 + 2xy \\ y' = x^2 + 2yz \\ z' = 2xz. \end{cases}$$

Since $x + y + z = 1$, we have the transformation

$$(*) \qquad \begin{cases} x' = 2y^2 + 4xy + x^2 - 2y - 2x + 1 \\ y' = x^2 + 2y + - 2xy - 2y^2, \end{cases}$$

which takes the triangle region $\Delta = \{(x, y) : x, y \geq 0, x + y \leq 1\}$ into itself. The transformation (†) has one fixed point $x_0 = 1/2$, $y_0 = 1/\sqrt{8}$,

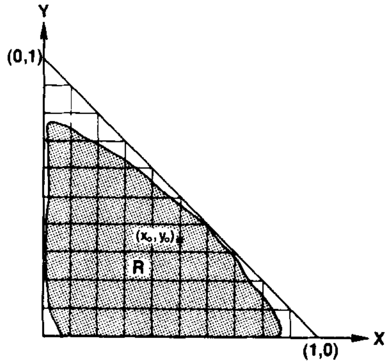


FIGURE 1

and $z_0 = (2 - \sqrt{2})/4$. The point (x_0, y_0) is an attractive fixed point for $(*)$ since the maximum modulus of an eigenvalue of the Jacobian matrix of $(*)$ at (x_0, y_0) is less than 1.

There is one fixed point of $(*)$ on the boundary of the triangular region since $(*)$ takes the line $x + y = 1$ to itself. This point is $((\sqrt{5} - 1)/2, (3 - \sqrt{5})/2)$ and it is a repellent fixed point.

The points $(1,0)$ and $(0,1)$ are exchanged under $(*)$ and form an attractive two period.

This example is one of many studied by Menzel, Stein and Ulam. They conducted numerical studies which show that the region Δ is partitioned as shown in Fig. 1. The iterates of an initial point in the region R are attracted to the internal fixed point $(1/2, 1/\sqrt{8})$. The iterates of an initial point in $\Delta \setminus R$ are attracted to the period two orbit $\{(1,0), (0,1)\}$.

Now we would like to modify this example to demonstrate how one can use iterations of maps to code the passage of time until some qualitative change in the behaviour of the system takes place.

We introduce additional variables $w_1, w_2, \ldots, w_n$ and add to the rule (R_0) the additional transforms:

$(w_1, a_1) \to (w_2, w_2)$,	where a_1 is any type,
$(w_2, a_2) \to (w_3, w_3)$,	where a_2 is any type other than w_1,
$(w_3, a_3) \to (w_4, w_4)$,	where a_3 is any type other than w_1 and w_2,
$\vdots$	
$(w_{n-1}, a_{n-1}) \to (w_n, w_n)$,	where a_{n-1} is x, y, z, w_{n-1} or w_n,
$(w_n, a) \to (x, x)$,	where a_n is x, y, z, or w_n.

Thus (†) now becomes

$$(\dagger\dagger)\qquad \begin{cases} x' &= y^2 + z^2 + 2xy + w_n^2 + 2w_n(w_n + x + y + z) \\ y' &= x^2 + 2yz \\ z' &= 2xz \\ w_1' &= 0 \\ w'_2 &= w_1^2 + 2w_1(w_2 + \cdots + w_n + x + y + z) \\ &\vdots \\ w'_n &= w_{n-1}^2 + 2w_{n-1}(x + y + z). \end{cases}$$

We iterate the transformation (††) but at each stage we regard the w's as fictitious quantities which only act as counters or catalysts. Thus, at each stage we examine only the proportions $x/(x + y + z)$, $y/(x + y + z)$ and $z/(x + y + z)$.

If we begin with an initial distribution of x, y, and z and $w_1 > 0$ but $w_2 = \cdots = w_n = 0$, then we see that the transformation (††) iterates x, y, and z similar to the way in which (†) does until n iterates have occurred. Then in the $(n + 1)$st iterate there is a tremendous increase in the proportion of x. After this iterate, there are no w's and the transformation (††) is (†). For example, if one takes $n = 10$ and the initial distribution $x = 0.3$, $y = 0.5$, $z = 0.1$, $w_1 = 0.1$. Then the first ten iterates of x and y converge quickly toward the internal fixed point. However, the 11th is close to the point $(1, 0)$ in the periodic attractor and of course the system converges toward the periodic attractor from "then on."

References

1. J. F. Lynch, The visually distinct configurations of k sets, *Discrete Math.* **33** (1981), 281–287.
2. J. F. Lynch, An asymptotic formula for the number of classes of sets of n indistinguishable elements, *J. Combin. Theory Ser. A.* **19** (1975), 109–112.
3. R. H. Bing, W. W. Bledsoe, and R. D. Mauldin, On sets generated by rectangles, *Pacific J. Math.* **51** (1974), 27–36.
4. A. Miller, On the length of Borel hierarchies, *Ann. of Math. Logic* **16** (1979), 233–267.
5. A. Miller, Generic Suslin sets, *Pacific J. Math.* **97** (1981), 171–181.
6. R. D. Mauldin, Countably generated families, *Proc. Amer. Math. Soc.* **54** (1976), 291–298.
7. J. Mycielski, *Amer. Math. Monthly* **82**, No. 3 (1975), 308–309.
8. D. Sankoff and J. B. Kurskal (eds.), "Time Warps, Strings Edits, and Macromolecules: The Theory of Practice of Sequence Comparison," Addison–Wesley, Reading, MA, 1983.
9. W. A. Beyer, P. H. Sellers, and M. S. Waterman, Stanislaw M. Ulam's contribution to theoretical theory, *Lett. Math. Phys.* **10** (1985), 231–242.
10. J. A. Davis, R. C. Entringer, R. L. Graham, and G. J. Simmons, On permutations containing no long arithmetic progressions, *Acta. Arith.* **34** (1977–1978), 81–90.

11. P. ERDÖS AND S. ULAM, On equations with sets as unknowns, *Proc. Nat. Acad. Sci. U.S.A.* **60** (1968), 1189–1195.
12. D. CENZER AND R. D. MAULDIN, Inductive definability: Measure and category, *Adv. in Math.* **38** (1980), 55–90.
13. S. ULAM, "Adventures of a Mathematician," p. 46, Scribner's, New York, 1976.
14. R. FAGIN, The number of finite relational structures, *Discrete Math.* **19** (1977), 17–21.
15. D. A. MARTIN, Borel determinancy, *Ann. of Math.* **102** (1975), 363–371.
16. W. A. BEYER, M. L. STEIN, AND S. M. ULAM, "The Notion of Complexity," Los Alamos Report LA-4822, 1971.
17. J. BECK, Van der Waerden and Ramsey type games, *Combinatorica* **1**, No. 2 (1981), 103–116.
18. J. BECK AND L. CAIRMAZ, Variations on a game, *J. Combin. Theory Ser. A* **33**, No. 3 (1982), 297–315.
19. F. R. K. CHUNG, P. ERDÖS, R. L. GRAHAM, S. M. ULAM, AND F. F. YAO, Minimal decompositions of two graphs into pairwise isomorphic subgraphs (Proceedings, 10th S.E. Conf. Combinatorics, Graph Theory and Computing), *Congr. Numer.* **23** (1979), 3–18.
20. F. F. YAO, Graph 2-isomorphism is NP-complete, *Inform. Process. Lett.* **9** (1979), 68–72.
21. F. R. K. CHUNG, P. ERDÖS, AND R. L. GRAHAM, Minimal decomposition of graphs into mutually isomorphic subgraphs, *J. Combin.* **1** (1981), 13–24.
22. F. R. K. CHUNG AND R. L. GRAHAM, Recent results in graph decompositions, *in* "Proceedings, 8th British Comb. Conference, 1981," pp. 103–123.
23. F. R. K. CHUNG, P. ERDÖS, AND R. L. GRAHAM, Minimal decompositions of hypergraphs into mutually isomorphic subhypergraphs, *J. Combin. Theory Ser. A* **32** (1982), 241–251.
24. F. R. K. CHUNG AND P. ERDÖS, On unavoidable graphs, *Combinatorica* **3** (1983), 167–176.
25. P. J. KELLY, A congruence theorem for trees, *Pacific J. Math.* **7** (1957), 961–968.
26. J. A. BONDY AND R. L. HEMMINGER, Graph reconstruction—A survey, *J. Graph Theory* **1** (1977), 227–268.
27. W. T. TUTTE, Graph theory, *in* "Encyclopedia of Mathematics and Its Applications, Vol. 21," Addison–Wesley, Reading, MA, 1984.
28. A. BRUEN AND R. DIXON, The *n*-queens problem, *Discrete Math.* **12** (1975), 393–395.
29. R. QUENEAU, Sur les suites *s*-additives, *J. Combin. Theory Ser. A* **12** (1972), 31–71.
30. D. CENZER AND E. HOWORKA, On vertex *k*-partitions of certain infinite graphs, *Discrete Math.* **23** (1979), 105–113.
31. F. GALVIN AND S. SHELAH, Some counterexamples in the partition calculus, *J. Combin. Theory Ser. A* **15** (1973), 167–174.
32. P. ERDÖS AND A. HAJNAL, Unsolved problems in set theory, *in* "Axiomatic Set Theory" (Proceedings, Sympos. Pure Math., Vol. XIII, Pt. 1, University of California, Los Angeles, CA, 1967), pp. 17–48.
33. P. ERDÖS AND R. L. GRAHAM, "Mongraphie No. 28 de L'Enseignement Mathématique," L'Enseignement Mathématique Université de Genève, Imprimerie Kundig, Geneva, 1980.
34. R. GRAHAM, The largest small hexagon, *J. Combin. Theory Ser. A* **18** (1975), 165–170.
35. STANISLAW ULAM, "Sets, Numbers, and Universes: Selected Works," MIT Press, Cambridge, MA, 1974.
36. C. J. EVERETT AND S. M. ULAM, On the problem of determination of mathematical structures by their endomorphisms, *Bull. Amer. Math. Soc.* **54** (1948), 646 (abstract).
37. S. M. ULAM AND JOHN VON NEUMANN, On the group of homeomorphisms of the surface of the sphere, *Bull. Amer. Math. Soc.* **53** (1947), 506 (abstract).
38. R. D. ANDERSON, The algebraic simplicity of certain groups of homeomorphisms, *Amer. J. Math.* **80** (1958), 955–963.
39. C. J. EVERETT AND S. M. ULAM, On the problem of determination of mathematical structures by their endomorphisms, *Bull. Amer. Math. Soc.* **54** (1948), 646 (abstract).

40. R. R. KALLMAN, Uniqueness results for homeomorphism groups, *Trans. Amer. Math. Soc.* **295** (1986), 389–396.
41. W. ROSICKI, On a problem of S. Ulam concerning Cartesian squares of 2-dimensional polyhedra, *in* "Proceedings, International Conference on Geometric Topology," Polish Scientific Publishers, Warsaw, 1980.
42. G. FOURNIER, On a problem of S. Ulam, *Proc. Amer. Math. Soc.* **29** (1971), 622.
43. P. J. KELLY, On isometries of square sets. *Bull. Amer. Math. Soc.* **51** (1945), 960–963.
44. S. M. ULAM, Transformations, iterations and mixing flows, *in* "Dynamical Systems II" (A. R. Bednarek and L. Cesari, Eds.), pp. 419–426, Academic Press, Orlando, FL, 1982.
45. S. GRAF, R. D. MAULDIN AND S. WILLIAMS, Random homeomorphisms, *Adv. in Math.* **60** (1986), 239–359.
46. L. E. DUBINS AND D. A. FREEDMAN, Random distribution functions, *in* "Proceedings, Fifth Berkeley Symp. on Math. Statistics and Probability" (L. M. LeCam and J. Neyman, Eds.), pp. 183–214, Univ. of California Press, Berkeley/Los Angeles, 1967.
47. H. G. EGGLESTON, A property of plane homeomorphisms, *Fund. Math.* **42** (1955), 61–74.
48. J.-P. KAHANE AND J. PEYRIÈRE, Sur certaines martingales de Benoit Mandelbrot, *Adv. in Math.* **22** (1976), 131–145.
49. R. D. MAULDIN AND S. WILLIAMS, Random recursive constructions: Asymptotic geometric and topological properties, *Trans. Amer. Math. Soc.* **295**, No. 1 (1986), 325–346.
50. D. H. HYERS, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222–224.
51. D. CENZER, The stability problem for transformations of the circle, *Proc. Roy. Soc. Edinburgh Sect. A* **84** (1979), 279–281.
52. D. H. HYERS AND S. M. ULAM, On approximate isometries, *Bull. Amer. Math. Soc.* **51** (1945), 288–292.
53. D. H. HYERS AND S. M. ULAM, Approximate isometries of continuous functions, *Ann. of Math.* **49** (1947), 285–289.
54. P. GRUBER, Stability of isometries, *Trans. Amer. Math. Soc.* **245** (1978), 263–277.
55. J. GERVIRTZ, Stability of isometries on Banach spaces, *Proc. Amer. Math. Soc.* **89** (1983), 633–636.
56. J. LINDENSTRAUSS AND A. SZANKOWSKI, Non-linear perturbations of isometries, in "Colloquium in Honor of Laurent Schwartz, Vol. I," Palaiseau, 1985.
57. R. R. KALLMAN, A uniqueness result for the infinite symmetric group, *in* "Studies in Analysis" (G-C. Rota, Ed.), pp. 321–322, Advances in Mathematics Supplementing Studies, Vol. 4, Academic Press, Orlando, FL, 1979.
58. BESICOVITCH, On arcs that cannot be covered by an open equilateral triangle of side 1, *Math. Gaz.* **49** (1965), 286–288.
59. P. KOMJATH, Large sets not containing images of a given sequence, *Canad. Math. Bull.* **26** (1983), 41–43.
60. P. ERDÖS, "The Art of Counting, Selecting Writings," MIT Press, Cambridge, MA, 1973.
61. P. ERDÖS AND A. HAJNAL, Some remarks on set theory. IX. Combinatorial problems in measure theory and set theory, *Michigan Math. J.* **11** (1964), 107–127.
62. H. HARBORTH, Konvexe Fünfecke in ebenen Punktmengen, *Elem. Math.* **33** (1978), 116–118.
63. J. D. HORTON, Sets with no empty convex 7-gons, *Canad. Math. Bull.* **26** (1983), 482–484.
64. R. R. KALLMAN AND R. D. MAULDIN, Maximal metrics, *Abstracts Amer. Math. Soc.* **3** (1982), 542.
65. K. BORSUK AND S. M. ULAM, On symmetric products of topological spaces, *Bull. Amer. Math. Soc.* **37** (1931), 875–882.
66. D. CURTIS AND R. M. SHORI, 2^X and $C(X)$ are homeomorphic to the Hilbert cube, *Bull. Amer. Math. Soc.* **80** (1974), 927–931.

67. J. E. WEST, Infinite products which are Hilbert cubes, *Trans. Amer. Math. Soc.* **150** (1970), 1–25.
68. K. BORSUK, On intrinsic isometries, *Bull. Acad. Polon. Sci.* **29**, (1981), 83–90.
69. R. M. SOLOVAY, A model of set-theory in which every set of reals is Lebesgue measurable, *Ann. of Math.* **92** (1970), 1–56.
70. D. G. LARMAN, A problem of incidence, *J. London Math. Soc.* **43** (1968), 407–409.
71. G. H. MEISTERS AND S. ULAM, On visual hulls of sets, *Proc. Nat. Acad. Sci. U.S.A.* **57** (1967), 1172–1174.
72. D. G. LARMAN AND P. MANI, On visual hulls, *Pacific J. Math.* **22** (1970), 157–171.
73. S. ULAM, Speculations on some possible mathematical frameworks for the foundations of certain physical theories, *Lett. Math. Phys.* **10**, Nos. 2, 3 (1985), 101–106.
74. L. H. FINKEL AND G. M. EDELMAN, Stanislaw Ulam: The warmth and brilliance of an eclectic mind, *Ibid.*, 243–245.
75. S. M. ULAM, On the operations of pair production, transmutations and generalized random walk, *Adv. in Appl. Math.* **1** (1980), 7–21.
76. C. J. EVERETT AND S. M. ULAM, "The Entropy of Interacting Populations," Los Alamos Scientific Laboratory Report LA-4256, Los Alamos, October 1969.
77. D. BLACKWELL AND R. D. MAULDIN, Ulam's redistribution of energy problem: Collision transformations, *Lett. Math. Phys.* **10** (1985), 149–153.
78. G. D. BIRKHOFF, "Dynamical Systems," *Amer. Math. Soc. Colloq. Publ.*, Vol. 9, Amer. Math. Soc., Providence, RI, 1927.
79. J. MYCIELSKI AND S. M. ULAM, On the pairing process and the notion of genealogical distance, *J. Combin. Theory* **6** (1969), 227–234.
80. J. KAHANE AND R. MARR, On a class of stochastic pairing processes and the Mycielski–Ulam notions of genealogical distance," *J. Combin. Theory Ser. A* **13** (1973), 383–400.
81. M. T. MENZEL, P. R. STEIN, AND S. M. ULAM, "Quadratic Transformations, Part I," Los Alamos Report LA-2305, 1959.
82. S. M. ULAM, "Science, Computers and People from the Tree of Mathematics," Birkhauser, Boston, 1986.
83. R. L. GRAHAM, P. ERDÖS, I. RUZSA, AND E. G. STRAUS, On the prime factors of $\binom{2n}{n}$, *Math. Comp.* **29** (1975), 83–92.
84. F. R. K. CHUNG AND R. L. GRAHAM, A new bound on Euclidean–Steiner minimal trees, *in* Discrete Geometry and Complexity. Vol. 440, New York Acad. Sci., New York, 1985.