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Coalition Convex Preference Orders Are Almost Surely Convex

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Let E be a separable linear topological space, which admits a complete metric compatible with the topology, and $(\Omega, \mathscr{A}, \mu)$ a complete probability space. Let $\geqslant \in \mathscr{A} \otimes \mathscr{B}(E) \otimes \mathscr{B}(E)$. Then $\geqslant$ is coalition convex if and only if for almost all ω, $\geqslant_\omega$ is convex. © 1986 Academic Press, Inc.

Let E be a linear topological space which admits a complete metric compatible with the topology. Let $(\Omega, \mathscr{A}, \mu)$ be a complete probability space. Let $\mathscr{B}(E)$ be the Borel field on E. Let $\geqslant$ be an $\mathscr{A} \otimes \mathscr{B}(E) \otimes \mathscr{B}(E)$ measurable subset of $\Omega \times E \times E$. For each $\omega \in \Omega$, regard the ω-section of $\geqslant$, $\geqslant_\omega = \{(e_1, e_2) \mid (\omega, e_1, e_2) \in \geqslant\}$ as a relation on $B_\omega = \{e \mid (\omega, e, e) \in \geqslant\}$. This model of individual preferences with a measurable space Ω of individuals and E, the commodity space was initially described by Aumann [3]. For $A \in \mathscr{A}$ and f and g measurable maps of A into E, f is said to dominate g on A (symbolized by $f \geqslant_A g$) if and only if $f(\omega) \geqslant_\omega g(\omega)$ for μ-almost all $\omega \in A$. Thus, $f \geqslant_A g$ means the coalition A prefers the selection (or coalition A preference) f to the selection g. Vind [9] pursued the idea that coalitional preferences were central. This was followed by studies of Cornwall [4, 5] and Richter [8]. Debreu [6] showed that the coalitional preferences of Vind arose from individual preferences. Recently, Armstrong and Richter [1] have put Debreu's work in a more general setting. They have shown that there is a 1–1 correspondence between properties of individual preferences and properties of coalitional preferences, at least in case $E = \mathbb{R}^n$. In particular they demonstrated that almost every individual preference is monotone if and only if for each $A \in \mathscr{A}$, the set of coalition A preferences is monotone. Similar assertions hold for transitivity, asym-

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metry, and other properties. Since the fact that $\mathbb{R}^n$ is locally compact is used in these arguments, whether similar statements hold in a more general infinite dimensional setting was left open.

We say that the relation $\geqslant$ is coalition convex provided that for each $A \in \mathscr{A}$ and for each measurable map $g: A \to E$ such that for each $\omega \in A$, $g(\omega) \in B_\omega$, the set $D(g, A)$ is convex, where

$$D(g, A) = \{f \mid f: A \to E \text{ and } f \geqslant_A g\}.$$

Armstrong raised the following question at the Institute for Mathematics and Its Applications in January 1984 during the sessions concentrating on mathematical economics and discusses it in his survey [2]:

> If $\geqslant$ is coalition convex, then is it true that for almost all ω, $\leqslant_\omega$ is convex? We recall that a relation $\leqslant_\omega$ is convex means for each $x \in B_x$, $\{y \mid x \leqslant_\omega y\}$ is a convex subset of E.

We will give an affirmative answer to this question under the assumption that E is separable. The technique of proof can be used to settle the equivalence of possession, between individuals and coalitions, of a number of other properties.

THEOREM. *Let E be a separable linear topological space, which admits a complete metric compatible with the topology, and $(\Omega, \mathscr{A}, \mu)$ a complete probability space. Let $\geqslant \in \mathscr{A} \otimes \mathscr{B}(E) \otimes \mathscr{B}(E)$. Then $\geqslant$ is coalition convex if and only if for almost all ω, $\geqslant_\omega$ is convex.*

Proof. Let

$$\Gamma = \{(\omega, e_1, e_2, e_3, e_4, \alpha) \in \Omega \times E \times E \times E \times E \times [0, 1]:$$
$$e_1 \geqslant_\omega e_4, e_2 \geqslant_\omega e_4, \alpha e_1 + (1-\alpha) e_2 = e_3$$

and $e_3 \not\geqslant_\omega e_4\}$.

It can be checked $\Gamma \in \mathscr{A} \otimes \mathscr{B}(E) \otimes \mathscr{B}(E) \otimes \mathscr{B}(E) \otimes \mathscr{B}(E) \otimes \mathscr{B}([0, 1])$. Let $S = \pi_\Omega(\Gamma)$. The claim of the theorem is that $\mu(S) = 0$. Let $G = \pi_{\Omega \times [0,1]}(\Gamma)$.

We note that it follows from known theorems that $S \in \mathscr{A}$ and G is $\mu \times \lambda$-measurable, where λ is Lebesgue measure on $[0, 1]$ [7, p. 44]. Also, note that $\Pi_\Omega(G) = S$.

LEMMA . *If $\omega \in S$, then $\lambda(G_\omega) > 0$.*

Proof. Suppose $\omega \in S$ and $\lambda(G_\omega) = 0$. Choose e_1, e_2, e_3, e_4, and α such that

$$(\omega, e_1, e_2, e_3, e_4, \alpha) \in \Gamma.$$

This means

$$e_1 \geqslant_\omega e_4, \qquad e_2 \geqslant_\omega e_4$$

and

$$\alpha e_1 + (1-\alpha)e_2 = e_3 \not\geqslant_\omega e_4.$$

But, since $\lambda(G_\omega) = 0$, there are numbers β, γ, and τ in $[0, 1]\backslash G_\omega$ such that

$$e_3 = \tau(\beta e_1 + (1-\beta)e_2) + (1-\tau)(\gamma e_1 + (1-\gamma)e_2).$$

Now, $\beta \notin G_\omega$ and, again, $e_1 \geqslant_\omega e_4$, $e_2 \geqslant_\omega e_4$. So, if $\beta e_1 + (1-\beta)e_2 \not\geqslant_\omega e_4$, then β would be in G_ω. Thus,

$$e_1' = \beta e_1 + (1-\beta)e_2 \geqslant_\omega e_4.$$

Similarly, $e_2' = \gamma e_1 + (1-\gamma)e_2 \geqslant_\omega e_4$.

Now, for the same reason, $\tau e_1' + (1-\tau)e_2' \geqslant_\omega e_4$. But $\tau e_1' + (1-\tau)e_2' = e_3$. This is a contradiction. Therefore, if $\omega \in S$, then $\lambda(D_\omega) > 0$. Since G is $(\mu \times \lambda)$-measurable, $(\mu \times \lambda)(G) = \int_S \lambda(G_\omega)\, d\mu(\omega) > 0$. On the other hand, by Fubini's theorem,

$$(\mu \times \lambda)(G) = \int_{[0,1]} \mu(G^\alpha)\, d\lambda(\alpha),$$

where $G^\alpha = \{\omega \mid (\omega, \alpha) \in G\}$. This means there is some α such that $G^\alpha \in \mathscr{A}$ and $\mu(G^\alpha) > 0$.

Let $A = G^\alpha$. Let $M = \Gamma \cap (A \times E \times E \times E \times E \times \{\alpha\})$. Let ϕ be a map of A into $E \times E \times E \times E \times \{\alpha\}$ which is a $(\Sigma, \mathscr{B}(E) \otimes \mathscr{B}(E) \otimes \mathscr{B}(E) \otimes \mathscr{B}(E) \otimes \mathscr{B}([0, 1]))$-measurable selection for M [7, p. 54]. Thus, $\phi(\omega) = (f_1(\omega), f_2(\omega), f_3(\omega), f_4(\omega), \alpha)$, for each ω in A, where the maps $f_i(\omega)$ are $(\Sigma, \mathscr{B}(E))$-measurable. But, for each $\omega \in A$, $g(\omega) = f_4(\omega) \in B_\omega$. Also, f_1 and f_2 are in $D(g, A)$, but $f_1 + (1-\alpha)f_2 = f_3 \notin D(g, A)$. This contradiction establishes the theorem.

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