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Ulam's Redistribution of Energy Problem: Collision Transformations★

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Abstract. Ulam conjectured that for each given law of redistribution of energy, D, there corresponds a limiting distribution, $C(D)$, the 'collision transform' of the given law such that if X is an initial distribution of energy, then the distributions of the iterates of X under redistribution, converge to $C(D)$. We give examples of this behaviour and prove that Ulam's conjecture is correct in case all moments of X exists.

Several years ago, Stan Ulam with his usual insight proposed a problem which has a simple physical interpretation and, at the same time, is amenable to computer simulations. The problem has evolved and it, together with some variants, forms a part of a manuscript of problems which Mauldin and Ulam were preparing for publication.

The problem may be stated as follows:

Consider a vast number of particles and let us redistribute the energy of these particles as follows. First, pair the particles at random. Second, for each pair, redistribute the total energy of the pair between these particles according to some given fixed probability law of redistribution. Is it true that under iteration the distributions of energy converge to some distribution which is independent of the initial distribution of energy (but, of course, is dependent upon the law of redistribution)?

Ulam believed that the answer is 'yes' and called the final limiting distribution the 'collision transformation'. We will give one of several possible interpretations of this problem, and demonstrate that he was essentially correct. We will also indicate some of the unsolved problems.

DEFINITION. A redistribution of energy law is a distribution function D which is supported on $[0, 1]$, symmetric about $\frac{1}{2}$ and has second moment less than $\frac{1}{2}$.

The assumption that D is symmetric about $\frac{1}{2}$ means that the particles are indistinguishable. For any symmetric distribution on $[0, 1]$ the second moment is $\leqslant \frac{1}{2}$ and it is $\frac{1}{2}$ if and only if D assigns probability $\frac{1}{2}$ to 0 and to 1. In our model, this would

★ In memory of Stan Ulam.

correspond to always giving one of the particles all of the energy – one of the particles always disappears. We rule out this possibility.

If X is a random variable giving the original distribution of energy, then the new distribution of energy will be distributed as the random variable $TX = U \cdot (X_1 + X_2)$, where U, X_1, and X_2 form an independent family of random variables, U has distribution D and X_1 and X_2 are both distributed as X. We will assume that $X > 0$ and that the total energy has been normalized, i.e., $E(X) = 1$.

Ulam's problem concerns the behaviour of the iterates TX, $T^2(X), \ldots$.

Consider the following simple example: Whenever the particles are paired, the total energy is shared equally. Each particle receives one-half of the total energy. Intuitively, it should be true that under iteration all the particles will be at the same energy level. Under our interpretation, in this case, we have D is point mass at $\frac{1}{2}$. Thus, the new distribution of energy will be the distribution of $TX = \frac{1}{2}(X_1 + X_2)$. Since $T^n(X) = (X_1 + \cdots + X_{2^n})/2^n$, it follows from the strong law of large numbers that $T^n(x) \to 1$ a.s. Thus, Ulam's conjecture is true and $C(D)$, the collision transformation, is point mass at 1.

Let $u_0 = 1$, $u_1 = \frac{1}{2}$, $u_2, u_3, \ldots$ be the moments of D.

THEOREM. *Let X be an initial distribution of energy with finite moments m_n. The T^nX converges in distribution to the distribution $C(D)$, which is the unique distribution with moments $a_0, a_1, a_2, \ldots$ given recursively by*:

$$a_0 = 1, \qquad a_1 = 1, \quad \textit{and for } n \geqslant 1,$$

$$a_{n+1} = \frac{u_{n+1}}{1 - 2u_{n+1}} \sum_{j=1}^{n} \binom{n+1}{j} a_j a_{n+1-j}. \tag{1}$$

In particular, $C(D)$ is the unique fixed distribution among all the distributions with finite moments of all orders.

Proof. Let $m_0 = 1$, $m_1 = 1$, $m_2, m_3, \ldots$ be the moments of X. Let $T^k(n)$ be the nth moment of $T^k(X)$. We have

$$T^1(n) = E[(U \cdot (X_1 + X_2))^n] = E(U^n)E[(X_1 + X_2)^n],$$

or

$$T^1(n) = u_n \sum_{j=0}^{n} \binom{n}{j} m_j m_{n-j}. \tag{2}$$

Of course,

$$T^k(1) = 1 \quad \text{and} \quad T^k(2) = (2u_2)^k m_2 + \sum_{i=1}^{k} (2u_2)^i, \quad \text{for all } k.$$

Since

$$0 \leqslant u_2 < \tfrac{1}{2}, \qquad \lim_{k \to \infty} T^k(2) = a_2 = (2u_2)/1 - 2u_2.$$

From (2), we find

$$T^{k+1}(n+1) = u_{n+1}\left[2T^k(n+1) + \sum_{j=1}^{n} \binom{n+1}{j} T^k(j) T^k(n+1-j)\right]. \tag{3}$$

Thus,

$$\left[\frac{1-2u_{n+1}}{u_{n+1}}\right]T^{k+1}(n+1)$$
$$= -2[T^{k+1}(n+1) - T^k(n+1)] + \sum_{j=1}^{n}\binom{n+1}{j}T^k(j)T^k(n+1-j). \tag{4}$$

We wish to show by recursion that $\lim_{k\to\infty} T^k(n) = a_n$, for $n = 0, 1, 2, \ldots$. This is true for $n \leqslant 2$ and if it is so for all $j \leqslant n$, then, according to (1) and (4), this limit will hold for $n + 1$, provided $\lim_{k\to\infty} d_k = 0$, where $d_k = T^{k+1}(n+1) - T^k(n+1)$.

From (3), we find

$$d_k = 2u_{n+1}d_{k-1} + \sum_{j=1}^{n+1}\binom{n+1}{j}[T^k(j)T^k(n+1-j) - T^{k-1}(j)T^{k-1}(n+1-j)]. \tag{5}$$

Set

$$e_k = \left|\sum_{j=1}^{n}\binom{n+1}{j}T^k(j)T^k(n+1-j) - T^{k-1}(j)T^{k-1}(n+1-j)\right|. \tag{6}$$

By our induction hypothesis $\lim_{k\to\infty} e_k = 0$. Also,

$$d_k \leqslant 2u_{n+1}d_{k-1} + e_k, \tag{7}$$

and by recursion

$$d_{k+1} \leqslant (2u_{n+1})^k d_1 + (2u_{n+1})^{k-1}e_2 + (2u_{n+1})^{k-2}e_3 + \cdots + e_{k+1}. \tag{8}$$

Since $u_{n+1} \leqslant u_2 < \frac{1}{2}$, the first term on the r.h.s. of (8) goes to 0 as $k \to \infty$. The sum of the other terms can be shown to converge to 0 by splitting it into two parts and using the facts $e_k \to 0$ and $\Sigma(2u_{n+1})^k < \infty$.

Thus, for $n = 0, 1, 2, \ldots$

$$\lim_{k\to\infty} T^k(n) = a_n. \tag{9}$$

Next, we claim

$$\sum_{n=1}^{\infty}(a_n)^{-1/2n} < \infty. \tag{10}$$

Then, according to Carleman's theorem [1], there is a unique distribution function, $C(D)$, on $[0, +\infty]$ having nth moment a_n. Also, $T^k(X)$ converges in distribution to $C(D)$ [2].

To demonstrate (10), set $A = u_2/(1 - 2u_2) < \infty$. Since $u_{n+1} \leqslant u_2$, we have $u_{n+1}/1 - 2u_{n+1} \leqslant A$, for $n = 1, 2, 3, \ldots$ Set $b_1 = 1$, and for $n \geqslant 1$,

$$b_{n+1} = \sum_{j=1}^{n} b_j b_{n+1-j}. \tag{11}$$

We claim that for all $n \geqslant 0$,

$$a_{n+1} \leqslant A^n(n+1)!\,b_{n+1}. \tag{12}$$

Clearly, (12) is true if $n = 0$, 1, or 2. If (12) holds for all $k \leqslant n$, then (12) follows for $n + 1$ via substitution into (1).

We need the upper estimates: For all $n \geqslant 1$,

$$b_n \leqslant n! \tag{13}$$

If (13) holds for all $k \leqslant n$, then, from (11)

$$b_{n+1} \leqslant \sum_{j=1}^{n} j!\,(n+1-j)! \leqslant (n+1)! \sum_{j=1}^{n} 1\bigg/\binom{n+1}{j}. \tag{14}$$

But,

$$\sum_{j=1}^{n} 1\bigg/\binom{n+1}{j} \leqslant 1 .$$

Thus, (13) follows and we have for all n,

$$a_n \leqslant A^{n-1}(n!)^2 . \tag{15}$$

According to Stirling's formula,

$$A^{n-1}(n!)^2 \leqslant A^{n-1}(2\pi n)n^{2n}\,\mathrm{e}^{-2n}\,\mathrm{e}^{1/6n} . \tag{16}$$

Or,

$$c_n = \frac{1}{e\sqrt{A}}\left(\frac{A}{2\pi}\right)^{1/2n} \mathrm{e}^{-1/12n^2}\left(\frac{1}{n^{1+1/2n}}\right) \leqslant [A^{n-1}(n!)^2]^{-1/2n} . \tag{17}$$

But, $c_n \sim 1/n$. Thus, $\Sigma c_n = +\infty$ and (10) follows. □

EXAMPLE. Let D be the uniform distribution on $[0, 1]$. It is easy to check that for each n, $a_n = n!$ Therefore, the collision transformation in this case is the standard exponential distribution.

We note that studying this iteration from the viewpoint of characteristic functions, we have

$$(T\phi)(x) = \int_0^1 \phi^2(tx)\,\mathrm{d}D(t)\,, \tag{18}$$

where ϕ is the characteristic function of X and $T\phi$ is the characteristic function of TX. Thus, for our last example, one can check that the characteristic function, ϕ which is the solution of

$$\phi(x) = \int_0^1 \phi^2(tx)\,\mathrm{d}t \tag{19}$$

is $\phi(x) = (1 - ix)^{-1}$, the characteristic function of the exponential distribution.

We also note that our first simple example is the only case when the final limiting distribution is bounded.

THEOREM. *If D is not point mass at $\frac{1}{2}$, then $C(D)$ is unbounded.*

Proof. If X has distribution $C(D)$, then $T(X)$ has distribution $C(D)$. Choose $c > \frac{1}{2}$ such that $P(U > c) > 0$ and set $\gamma = (2c - 1)$. Suppose $P(X > M) > 0$. Then

$$
\begin{aligned}
&P(TX > M + M\gamma) \\
&\quad = P(U \cdot (X_1 + X_2) > M + M\gamma) \\
&\quad \geqslant P(U > c)P(X_1 + X_2 > M(1 + \gamma)/c) \\
&\quad \geqslant P(U > c)P(X > M(1 + \gamma/2c)) \\
&\quad = P(U > c)P(X > M) > 0 . \qquad \square
\end{aligned}
$$

What happens to the distributions of $T^n X$ in case all the moments of X do not exist?

Is the distribution $C(D)$ given in the main theorem the only distribution of energy which is fixed under T?

Under what conditions on D, is it true that $C(D)$ has a density? We do know that if $C(D)$ has a density, then D is continuous at 1.

References

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2. Billingsley, P., *Probability and Measure*, Wiley, New York, 1979, p. 344.