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Borel equivalence and isomorphism of coanalytic sets

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Introduction

One of the problems considered as descriptive set theory developed in the first half of this century was the following: Under what conditions are two Borel sets, A and B, Borel isomorphic, that is, under what conditions does there exist a one-to-one map f of A onto B so that if E is a subset of A, then E is a Borel set relative to A if and only if $f(E)$ is a Borel set relative to B? It was shown that two Borel subsets A and B of a Polish space X are Borel isomorphic if and only if they have the same cardinality ([13], p. 451).

The corresponding problem for analytic or coanalytic sets was not solved. The reason for this, as we now know, is that this problem is intimately involved with the axioms of set theory. C. Ryll-Nardzewski and A. Maitra [15] showed that if there is a "thin" coanalytic set – an uncountable coanalytic set C which does not contain a perfect set, then C is not Borel isomorphic to a universal coanalytic set. Later the second author of this paper showed that if A is the complement of such a coanalytic set C, then A is not Borel isomorphic to $A \times A$ or to $A \times [0, 1]$ [18]. Of course, Gödel announced [7] that the existence of a thin coanalytic set follows from the Axiom of Constructibility (V = L), which he proved to be consistent with the usual axioms of set theory (denoted by ZF, for Zermelo and Fraenkel).

On the other hand, Solovay proved that if there is a measurable cardinal, then every uncountable coanalytic set has a perfect subset (see [16]). In fact, it is known that all analytic games are determined if and only if there is exactly one isomorphism class of coanalytic non-Borel sets [8]. There are various assumptions which imply that all analytic games are determined [16].

More recently, it was shown by Hrbacek [10] and others that, assuming the Axiom of Constructibility (V = L), there exists a family of continuumly many pairwise non-Borel-isomorphic coanalytic sets. One purpose of this paper is to construct several particularly nice families of non-isomorphic coanalytic sets. The following four theorems are proved in Sections 3 and 4, assuming that all reals are constructible.

THEOREM 1. *There is a coanalytic subset Q of $N^N \times N^N$ such that the projection $\pi_2(Q)$ is coanalytic, each horizontal section Q^y is clopen, each vertical section Q_x contains a perfect set and no two vertical sections are Borel isomorphic. Furthermore, if P is a relative Borel subset of Q and each P_x is thin, then no two vertical sections of $Q-P$ are Borel isomorphic.*

THEOREM 2. *There is a coanalytic subset T of $N^N \times N^N$ such that the projection $\pi_2(T)$ is a thin coanalytic set, each horizontal section of T is clopen and no two vertical sections of T are Borel isomorphic.*

In addition, we continue the study of the isomorphism properties of algebraic and set-theoretic combinations of coanalytic or analytic sets initiated in [18]. Given a family of sets $\{K_i: i \in I\}$, let $\sum \{K_i: i \in I\}$ denote the disjoint union $\bigcup \{K_i \times \{i\}: i \in I\}$; for a set K and cardinal number n, nK denotes the disjoint union of n copies of K. Finally, $\prod \{K_i: i \in I\}$ denotes the cartesian product of the family of sets.

THEOREM 3. *There is an uncountable family $\{Cu[\sigma]: \sigma < \omega_1\}$ of pairwise disjoint coanalytic sets such that for any two countable subsets $M_1 \neq M_2$ of ω_1:*

(a) *$\bigcup \{Cu[\sigma]: \sigma \in M_1\}$ and $\bigcup \{Cu[\sigma]: \sigma \in M_2\}$ are not Borel isomorphic;*

(b) *$\prod \{Cu[\sigma]: \sigma \in M_1\}$ and $\prod \{Cu[\sigma]: \sigma \in M_2\}$ are not Borel isomorphic.*

THEOREM 4. *There is a family $\{T_\alpha: \alpha < \omega_1\}$ of thin coanalytic subsets of N^N such that the sets $\sum \{n_i T_i: i \in I\}$, where I is any countable subset of ω_1 and each $n_i \leqslant \omega$, are pairwise not Borel isomorphic.*

These last two theorems answer some questions of J. P. R. Christensen [6, p. 126] and several possible generalizations.

In this paper we introduce the concept of Borel equivalence of coanalytic sets, primarily as a tool for the problem of Borel isomorphism.

One of the most important ideas of descriptive set theory is the effective decomposition of a coanalytic set into an ordered union of ω_1 disjoint Borel sets; this was first discovered by Sierpiński [22]. We introduce the notion of an admissible decomposition and say that two coanalytic sets A and B are Borel equivalent if there are two admissible decompositions $A = \bigcup A(\alpha)$ and $B = \bigcup B(\alpha)$ such that $A(\alpha)$ and $B(\alpha)$ have the same cardinality for all but countably many ordinals α.

Section one contains preliminaries concerning the effective decomposition, by inductive definition, of a coanalytic set into a union of Borel sets and the key result that Borel isomorphic sets must have similar decompositions. The related topic of countable admissible ordinals is also discussed.

The results of section one are strengthened in section two by the assumption of Godel's Axiom of Constructibility. A precise definition of 'admissible decomposition' is given and the following fundamental theorem is proved.

THEOREM 5. *Suppose that all reals are constructible. If two coanalytic sets P and Q are Borel isomorphic, then they are also Borel equivalent.*

In section five, we consider the Borel equivalence of coanalytic sets under the assumption that certain projective games are determined. As mentioned above, Steel [8] showed under this assumption that all coanalytic non-Borel sets are Borel isomorphic. Now in this situation not all reals are constructible, so that Borel isomorphism does not necessarily imply Borel equivalence. Nonetheless, we obtain the following.

THEOREM 6. *If all projective games are determined, then any two coanalytic non-Borel sets are Borel equivalent.*

Finally, in section six, some further results and open questions are considered.

We note that a result similar to Theorems 1 and 2 (although with no qualifications regarding perfect or thin sets) is essentially obtained in Hrbacek [10]. The approach taken in [10] differs substantially from ours, although the methods in both papers depend on the Boundedness Principle of Lusin and Sierpiński. Hrbacek uses some theorems from the field of admissible structures to obtain results about Kleene degrees; the existence of non-isomorphic coanalytic sets is a corollary to these results. Possible connections between Kleene degrees and Borel equivalence are discussed in section six.

The basic outline of the present paper first took shape in 1977 after some years work. Some of the results were announced at the Spring regional meeting of the Association in Houston, Texas in 1978.

1. Coanalytic sets and admissible ordinals

This section is a brief presentation of two closely related topics which are basic to our study of Borel isomorphisms of coanalytic sets. The first topic is the effective definability of coanalytic sets and the second is the family of countable admissible ordinals.

Some definitions are necessary. N will denote the set of natural numbers $\{0, 1, 2, \ldots\}$ and Seq will denote the set $\bigcup \{N^k: k \in N\}$ of finite sequences of natural numbers. The natural coding map $\#$: Seq $\to N$ is defined by $\#(\emptyset) = 1$ and $\#(m_1, \ldots, m_k) = 2^{m_1+1} 3^{m_2+1} \ldots p_k^{m_k+1}$, where $p_1, p_2, \ldots$ lists the prime numbers in increasing order. Seq has the usual Brouwer–Kleene ordering:

(1) $s = (m_0, \ldots, m_{k-1}) \leqslant (n_0, \ldots, n_{l-1}) = t$ if and only if s extends $t (s \supseteq t)$

or $(\exists j)(m_0 = n_0 \& \ldots \& m_{j-1} = n_{j-1} \& m_j < n_j)$.

For any $s = (m_0, \ldots, m_{k-1})$ and any $i \in N$, let $s * i = (m_0, \ldots, m_{k-1}, i)$. J will denote the space N^N of infinite sequences of natural numbers with the product topology. Of course J may be regarded as the space of irrational numbers between 0 and 1 via their continued fraction expansions. Throughout this paper, a real will mean an element of J. Two reals $u = (u(0), u(1), \ldots)$ and $v = (v(0), v(1), \ldots)$ may be coded together as $u * v = (u(0), v(0), u(1), v(1), \ldots)$. Any real u may be decomposed into infinitely many reals $(u)_n$, defined by $(u)_n(i) = u(p_n^{i+1})$. For any real u and any n, $u|n = (u(0), u(1), \ldots, u(n-1))$; also, write $s \subset u$ for $(\exists n)(s = u|n)$. For $s \in$ Seq,

$J(s)$ is defined to be $\{u \in J: s \subset u\}$. The family of sets $\{J(s): s \in \mathrm{Seq}\}$ forms a base for the topology on J.

J is an example of a Polish space, that is, a complete separable metric space. We will also be concerned with several other Polish spaces, such as $\mathrm{Seq} \times J$, J^i (for $i \in N$) and J^N. In any Polish space, the Borel sets compose the smallest family containing every open set and closed under complementation and countable union. A subset A of a Polish space is said to be *analytic* (or Σ_1^1) if there is a Borel subset B of $X \times J$ such that $A = \pi_1(B) = \{x: (\exists y)(x, y) \in B\}$. A subset C of X is coanalytic (or $\boldsymbol{\Pi}_1^1$) if $X - C$ is analytic. The Souslin theorem states that a set is Borel (Δ_1^1) if and only if it is both Σ_1^1 and $\boldsymbol{\Pi}_1^1$.

For two subsets B and M of a Polish space, B is said to be a *relative analytic* (respectively *coanalytic* or *Borel*) subset of M if there exists a $\boldsymbol{\Sigma}_1^1$ (resp. $\boldsymbol{\Pi}_1^1$ or $\boldsymbol{\Delta}_1^1$) set P such that $B = P \cap M$. We will call B a *pseudo-Borel* or *bianalytic subset of* M if there exist an analytic set P and a coanalytic set Q such that $P \cap M = Q \cap M = B$. Of course any relative Borel subset of M will be a pseudo-Borel subset of M and, if M is analytic, the converse is also true. In general, however, there is no relativized Souslin theorem, even for coanalytic sets.

The effective analogues of analytic, coanalytic and Borel sets are the (lightface) Σ_1^1, Π_1^1 and Δ_1^1 sets. A set is (boldface) $\boldsymbol{\Sigma}_1^1$ if and only if it is (lightface) Σ_1^1 in some real parameter, and similarly for Π_1^1 and Δ_1^1. The fundamentals of effective descriptive set theory can be found in Hinman [9] and the authors' [5].

Perhaps the most important idea of descriptive set theory is the effective decomposition of a coanalytic set C into a union of ω_1 Borel sets. In its most general form, such a decomposition can be viewed as an inductive definition of the set C. Of course, any set has uncountably many such decompositions. However, all of the decompositions of a particular $\boldsymbol{\Pi}_1^1$ set will have certain properties in common. That is the key idea of this section.

The concept of inductive definability has a central role in the study of coanalytic sets. Here is a brief introduction to the theory of the inductive definability of coanalytic sets. Details can be found in [1, 4, 5].

An inductive operator Γ over a set X is a map from the power set 2^X to 2^X such that $K \subset \Gamma(K)$ for all subsets K of X. In this paper, we will assume that Γ is always monotone, that is, whenever $K \subset M$, then $\Gamma(K) \subset \Gamma(M)$. Let On denote the family of ordinals. The operator Γ constructs a transfinite sequence $\{\Gamma^\alpha: \alpha \in On\}$ as follows:

(2) $\Gamma^0 = \emptyset$; $\Gamma^{\alpha+1} = \Gamma(\Gamma^\alpha)$ for all ordinals α and $\Gamma^\lambda = \bigcup\{\Gamma^\alpha: \alpha < \lambda\}$ for limit ordinals λ.

The closure $\mathrm{Cl}(\Gamma)$ of Γ is $\bigcup\{\Gamma^\alpha: \alpha \in On\}$. The least ordinal α such that $\Gamma^{\alpha+1} = \Gamma^\alpha = \mathrm{Cl}(\Gamma)$ is $|\Gamma|$, the closure ordinal of Γ.

An important example of a Π_1^1 set with a nice inductive definition is the family of countable well-orderings, coded in the following manner. For $R \subset \mathrm{Seq}$, let $x_R \in 2^{\mathrm{Seq}}$ be defined by $x_R(s) = 1$ if and only if $s \in R$. Note that any countable linear ordering can be imbedded in Seq. The set W of countable well-orderings is defined to be $\{x_R: R \text{ is well-ordered}\} = \{x \in 2^{\mathrm{Seq}}: \forall (s_0, s_1, \ldots) \in \mathrm{Seq}^N\, (\exists n)\, x(s_n) = 0 \text{ OR } s_{n+1} \geqslant s_n\}$.

Now elements of 2^{Seq} can be regarded as reals and W can be regarded as a subset of J, as seen by the following. Recall the coding map $\#$: $\mathrm{Seq} \to N$ defined above. For $x \in 2^{\mathrm{Seq}}$, let $\theta(x)$ be the unique $y \in J$ such that $y(m) = 1$ if and only if $m = \#(s)$ and $x(s) = 1$ and $y(m) = 0$ otherwise. The map θ is a homeomorphism of 2^{Seq} onto a subset of J. It will be clear from context whether W is viewed as a subset of 2^{Seq}, 2^N or $N^N = J$. Note that for each countable infinite ordinal α, $\{x \in W: \sigma(x) = \alpha\}$ is an uncountable Borel set, where $\alpha(x_R)$ is the order type of R.

For $p \in \mathrm{Seq}$ and $R \subset \mathrm{Seq}$, let $R \restriction p$ be $\{s \in R: s < p \text{ and } p \in R\}$; for $x \in 2^{\mathrm{Seq}}$ and $s = \mathrm{Seq}$, let $x \restriction p(s) = x(s) \cdot x(p)$ if $s < p$, and 0 otherwise.

W is the closure of the inductive operator Δ, defined by:

(3) $x \in \Delta(K)$ if and only if $x \in K$ OR $(\forall s)\, x(s) = 0$ OR $(\forall p)\, x \restriction p \in K$.

In fact, for each ordinal α, $\Delta^\alpha = \{x: \sigma(x) < \alpha\}$. Thus $W = \mathrm{Cl}(\Delta)$ and $|\Delta| = \omega_1$. Note that Δ is actually a Π_1^0 operator.

The standard sieve decomposition $C = \bigcup\{C_\alpha: \alpha < \omega_1\}$ of a coanalytic set can be effected by means of a continuous function φ such that $x \in C_\alpha$ if and only if $\sigma(\varphi(x)) = \alpha$. (See [14, p. 412–414].) The more general inductive decomposition described below will simplify the analysis of the complicated non-isomorphic sets constructed in sections three and four.

For any real z, let $\varrho(z)$ be the least ordinal not recursive in z, where α is recursive in z if $\sigma(x) = \alpha$ for some x in W which is recursive in z. Sacks [20] proved that a countable ordinal α is admissible if and only if $\alpha = \varrho(z)$ for some real z. We will use this characterization as our definition of the term "admissible". Similarly, an ordinal α will be said to be *z-admissible* if $\alpha = \varrho(z * u)$ for some real u. It is clear that, for any real z, the set of z-admissible ordinals is uncountable and has least element $\varrho(z)$. Let Ad denote the set of countable ordinals which are either admissible, a limit of admissibles or zero and let Cd denote the set of x in W such that $\sigma(x)$ belongs to Ad. The following lemma is a consequence of Propositions 5.11 (p. 235) and 11.11 (p. 253) of [3].

LEMMA 1.1. Cd *is a pseudo-Borel subset of* W; *in fact*, $\{(x, z): x \in W$ *and* $\sigma(x)$ *is* z*-admissible*$\}$ *is a pseudo-Borel subset of* $W \times J$. ■

The basic results concerning the inductive definability of coanalytic sets, as developed in [1, 4, 5] are given in the following theorem. If A is a subset of a product space $X \times Y$, let $(A)_x$ denote the vertical section

$\{y: (x, y) \in A\}$ for any $x \in X$ and $(A)^y$ denote the horizontal section $\{x: (x, y) \in A\}$ for any $y \in Y$. Let X be a Polish space and let z be a real number. The terms Δ^1_1 in z, Π^1_1 in z and Σ^1_1 in z are defined in [5, p. 81].

THEOREM 1.2. (a) *For any $C \subseteq X$ which is Π^1_1 in z, there is a monotone inductive operator Δ over* Seq $\times X$ *which is Δ^1_1 in z and there is an $s \in$ Seq such that $C = (\mathrm{Cl}(\Delta))_s$.*

(b) *If Γ is a monotone operator over X which is Δ^1_1 (respectively Π^1_1) in z, then for each ordinal $\alpha < \varrho(z)$ (resp. $\leqslant \varrho(z)$), Γ^α is Δ^1_1 (resp. Π^1_1) in z.*

(c) *If Γ is a monotone operator over X which is Π^1_1 in z, then $\mathrm{Cl}(\Gamma)$ is also Π^1_1 in z.*

(d) *If Γ is a monotone operator over X which is Π^1_1 in z and A is a subset of $\mathrm{Cl}(\Gamma)$ which is Σ^1_1 in z, then $A \subseteq \Gamma^{\varrho(z)}$.* ∎

As examples of the application of Theorem 1.2, we prove the following two lemmas, which will be needed in the next section.

LEMMA 1.3. *Let R be a well-ordered subset of* Seq *which is Δ^1_1 in the real parameter z. Then the order type of R is less than $\varrho(z)$.*

Proof. Given R as described, define a monotone operator Δ by: $s \in \Delta(K)$ if and only if $(\forall t)[(t < s \;\&\; t \in R) \to t \in K]$ OR $s \in K$. Clearly, $\mathrm{Cl}(\Delta) = R$ and $|\Delta|$ is the order type of R. Now both R and Δ are Δ^1_1 in z, so by Theorem 1.2 (d), $\mathrm{Cl}(\Delta) = R \subseteq \Delta^{\varrho(z)}$. It follows that the order type of R, $|\Delta|$, is $\leqslant \varrho(z)$. Since there is no longest well-ordering Δ^1_1 in z, the inequality must be strict. ∎

LEMMA 1.4. *If the real x is Δ^1_1 in the real y, then $\varrho(x) \leqslant \varrho(y)$.*

Proof. Given that x is Δ^1_1 in y, any well-ordering R which is recursive in x will be Δ^1_1 in y. It follows from Lemma 1.3 that the order type of R is $\leqslant \varrho(y)$. Thus $\varrho(x) \leqslant \varrho(y)$. ∎

We now return to our main topic, which is the decomposition of a coanalytic set by means of an inductive definition.

Blass and the first author studied the related notion of the core of a Π^1_1 set in [1]. For any real z and set P, let $C_z(P)$ be the union of the subsets of P which are Δ^1_1 in z.

The Π^1_1 monotone operator Γ will be called *pseudo-Borel* if there exists a Σ^1_1 monotone operator Δ such that $\Gamma^\alpha = \Delta^\alpha$ for all ordinals α (equivalently, $\Gamma(M) = \Delta(M)$ for any $M \subset \mathrm{Cl}(\Gamma)$); Γ is called pseudo-Δ^1_1 in z if Γ is Π^1_1 in z and Δ is Σ^1_1 in z. The following is a direct corollary of Theorem 1.2 (b, d).

COROLLARY 1.5 *Let Δ be a monotone inductive operator which is pseudo-Δ^1_1 in the real parameter z and let $P = \mathrm{Cl}(\Delta)$. Then $C_z(P) = \Delta^{\varrho(z)}$. Similarly, if $P = (\mathrm{Cl}(\Delta))_s$, as in Theorem 1.2 (a), then $C_z(P) = (\Delta^{\varrho(z)})_s$.*

As an example, consider the set W of countable well-orderings described above. For any z, $C_z(W) = \{x: \sigma(x) < \varrho(z)\}$.

We can now see that two different decompositions of a given Π^1_1 set must coincide at uncountably many levels.

THEOREM 1.6. *Let Δ and Γ be two monotone inductive operators which are pseudo-Δ^1_1 in the real parameter z and which have the same closure P. Then for every z-admissible ordinal α, $\Gamma^\alpha = \Delta^\alpha$. Similarly, if $P = (\mathrm{Cl}(\Delta))_s = (\mathrm{Cl}(\Gamma))_t$, then $(\Delta^\alpha)_s = (\Gamma^\alpha)_t$ for every z-admissible ordinal α.*

Proof. Let z, P, Δ, Γ and α be given as described. Choose a real u so that $\alpha = \varrho(z * u)$; thus Δ and Γ are pseudo-Δ^1_1 in $z * u$. The conclusion now follows from Corollary 1.5.

We now wish to consider properties of a decomposition which are invariant under Borel isomorphism. There are several notions of Borel isomorphism, two of which are fairly standard. Subsets P and Q of X are said to be *intrinsically Borel isomorphic* if there is a one-to-one map θ of P onto Q such that a subset E of P is a relative Borel subset of P if and only if $\theta(E)$ is a relative Borel subset of Q. Subsets P and Q of X are said to be *(extrinsically) Borel isomorphic* if there is a Borel isomorphism τ of X onto X such that $\tau(P) = Q$. For coanalytic sets, these two notions coincide.

THEOREM 1.7. *Let P and Q be coanalytic, non-Borel subsets of a Polish space X. The sets P and Q are intrinsically Borel isomorphic if and only if they are extrinsically Borel isomorphic.*

Proof. Clearly if P and Q are extrinsically Borel isomorphic, then P and Q are intrinsically isomorphic. Now suppose that τ is a relative Borel isomorphism of P onto Q. According to a theorem of Kuratowski [13, p. 436], there are Borel sets E and F in X and a Borel isomorphism T of E onto F so that $P \subseteq E, Q \subseteq F$ and $T|P = \tau$. Let M be a Cantor set lying in the uncountable analytic set $E-P$. Then $T|E-M$ is a Borel isomorphism of $E-M$ onto $F-T(M)$. Let G be a Borel isomorphism of the uncountable Borel set $M \cup (X-E)$ onto the uncountable Borel set $T(M) \cup (X-F)$. Let H be the Borel isomorphism of X onto X which agrees with T and G on their domains. Clearly, $H(P) = Q$. ∎

Throughout the remainder of this paper, Borel isomorphic will mean extrinsically Borel isomorphic.

Just as any Borel subset B of a Polish space is actually Δ^1_1 in some real parameter, any Borel mapping H of a Polish space X onto itself is actually Δ^1_1 in some real parameter z, meaning that $H^{-1}(B)$ is Δ^1_1 in z for any set B which is Δ^1_1.

THEOREM 1.8. *Let Δ and Γ be two monotone inductive operators which are pseudo-Δ^1_1 in the real parameter z; let $P = \mathrm{Cl}(\Delta)$, $Q = \mathrm{Cl}(\Gamma)$ and let H be a Borel isomorphism with $H(P) = Q$ which is Δ^1_1 in z. Then for every z-admissible ordinal α, $\Gamma^\alpha = H(\Delta^\alpha)$. Similarly, if $P = (\mathrm{Cl}(\Delta))_s$ and $Q = (\mathrm{Cl}(\Gamma))_t$, then $(\Gamma^\alpha)_t = H((\Delta^\alpha)_s)$ for every z-admissible ordinal α.*

Proof. Let z, Δ, Γ, P, Q and H be as described. Consider the inductive operator Σ, defined by $\Sigma(K) = H^{-1}(\Gamma(H(K)))$. It can be seen by induction that $H(\Sigma^\alpha) = \Gamma^\alpha$ for all ordinals α, so that $P = \mathrm{Cl}(\Sigma)$. Now it follows from Theorem 1.6 that $\Sigma^\alpha = \Delta^\alpha$ for any z-admissible ordinal α. Therefore, $\Gamma^\alpha = H(\Sigma^\alpha) = H(\Delta^\alpha)$ for every z-admissible ordinal α. The proof is similar in the case where P and Q are reducible to the closures of Δ and Γ. ∎

The implication of Theorem 1.7 is that Borel isomorphic sets P and Q must have similar inductive structures. For example, if P adds uncountably many elements at each inductive level, while Q adds only countably many, then P and Q cannot be Borel isomorphic. The existence of sets of the latter type turns out to depend on set-theoretic axioms such as (V = L).

2. The hypothesis of constructibility

The constructible universe L was defined by Gödel [7] by a sequence $\{C(\sigma): \sigma \in On\}$, where each set $C(\sigma)$ is constructed from certain previous sets $C(\tau)$ in a manner encoded by σ. For $\alpha \in \mathrm{Ad}$, L_α will denote $\{C(\sigma): \sigma < \alpha\}$; thus $L = \bigcup \{L_\alpha: \alpha \in \mathrm{Ad}\}$. Gödel showed that L satisfies the usual axioms of Zermelo–Fraenkel set theory plus the axiom of choice and the continuum hypothesis. The proof of the continuum hypothesis consists of showing that every constructible real is actually $C(\sigma)$ for some countable ordinal σ. The following is an exercise in Shoenfield [21, p. 318].

LEMMA 2.1. *Let $D = \{(u, x): u \in W \mathbin{\&} x = C(\sigma(u))\}$; then D is a pseudo-Borel subset of $W \times J$.*

We remark that the set D and its complement can both be given by Δ^1_1 monotone inductive definitions. Furthermore, the real x belongs to L_α if and only if there is a Δ^1_1 relation $B \subseteq J \times J$ and an ordinal $\tau < \alpha$ such that, for any u with $\sigma(u) = \tau$, x is the unique element of $\{y: B(y, u)\}$. Lemma 2.1 implies that the set of constructible reals is Σ^1_2, since $L \cap J = \{x: (\exists u)[u \in W \mathbin{\&} D(u, x)]\}$.

LEMMA 2.2. *If $x = C(\tau)$, then x is Δ^1_1 in any u with $\sigma(u) \geqslant \tau$.*

Proof. Suppose that $x = C(\tau)$ and $\sigma(u) \geqslant \tau$. Then, for some p, $\sigma(u \restriction p) = \tau$ and x is the unique element of $\{z: D(u \restriction p, z)\}$.

We conclude this sequence of facts about the constructible hierarchy with the following.

PROPOSITION 2.3. *If the real $x = C(\tau)$, then every admissible ordinal $\alpha > \tau$ is x-admissible.*

Proof. Given $x = C(\tau)$ and admissible $\alpha > \tau$, let $\alpha = \varrho(y)$. Now there is some real u recursive in y with $\sigma(u) = \tau$. By Lemma 2.2, x is Δ^1_1 in u and therefore Δ^1_1 in y. Thus $x * y$ is Δ^1_1 in y; of course y is also Δ^1_1 in $x * y$. It follows by Lemma 1.4 that $\varrho(y) = \varrho(x * y) = \alpha$, so that α is x-admissible. ∎

This proposition allows us to remove the parameter z from Theorems 1.6 and 1.7 to obtain boldface versions of these theorems under the hypothesis of constructibility (V = L).

THEOREM 2.4. *Suppose that all reals are constructible. Let Δ and Γ be pseudo-Δ^1_1 monotone operators having the same closure. Then for some countable ordinal σ and all admissible $\alpha > \sigma$, $\Delta^\alpha = \Gamma^\alpha$. Similarly, if $(\mathrm{Cl}(\Delta))_s = (\mathrm{Cl}(\Gamma))_t$, then $(\Delta^\alpha)_s = (\Gamma^\alpha)_t$ for all admissible $\alpha > \sigma$.*

Proof. Since Δ, Γ are pseudo-Δ^1_1, there is some real parameter z such that Δ and Γ are both pseudo-Δ^1_1-in-z. By Theorem 1.6, $\Delta^\alpha = \Gamma^\alpha$ for every z-admissible ordinal α. Now since all reals are constructible, $z = C(\sigma)$ for some countable ordinal σ. By Proposition 2.3, every $\alpha > \sigma$ is z-admissible. It follows that for every admissible $\alpha > \sigma$, $\Delta^\alpha = \Gamma^\alpha$. ∎

THEOREM 2.5. *Suppose that all reals are constructible. Let Δ and Γ be pseudo-Δ^1_1 monotone inductive operators; let $P = \mathrm{Cl}(\Delta)$, $Q = \mathrm{Cl}(\Gamma)$ and let H be a Borel isomorphism with $H(P) = Q$. Then for some countable ordinal σ and all admissible $\alpha > \sigma$, $\Gamma^\alpha = H(\Delta^\alpha)$. Similarly, if $P = (\mathrm{Cl}(\Delta))_s$ and $Q = (\mathrm{Cl}(\Gamma))_t$, then $(\Gamma^\alpha)_s = H((\Delta^\alpha)_t)$ for all admissible $\alpha > \sigma$.* ∎

Proof. Since Δ, Γ, H are pseudo-Δ^1_1, there is some real parameter z such that Δ, H are pseudo-Δ^1_1 in z. By Theorem 1.8, $\Gamma^\alpha = H(\Delta^\alpha)$ for every z-admissible ordinal α. As in the proof of Theorem 2.4, we conclude that $\Gamma^\alpha = H(\Delta^\alpha)$ for every admissible $\alpha > \sigma$, where $z = C(\sigma)$.

For any ordinal α, α^+ is the least admissible ordinal greater than α. If $P = (\mathrm{Cl}(\Gamma))_s$, where Γ is some pseudo-Borel operator over $\mathrm{Seq} \times X$, then Γ decomposes P into ω_1 disjoint *admissible constituents* $P(\alpha) = (\Gamma^{\alpha^+} - \Gamma^\alpha)_s$, for ordinals $\alpha \in \mathrm{Ad}$. It follows from Theorem 1.2 (b) that each admissible constituent is a Borel set.

Recall that a set P is said to be *thin* if it has no perfect subset. We note that a coanalytic set P is thin if and only if each admissible constituent of P is countable.

Let us say that two subsets P and Q of the Polish space X are *Borel equivalent* if there exist admissible decompositions $\{P(\alpha): \alpha \in \mathrm{Ad}\}$ of P and $\{Q(\alpha): \alpha \in \mathrm{Ad}\}$ of Q such that $P(\alpha)$ and $Q(\alpha)$ have the same cardinality for all but countably many ordinals. For example, the set W has admissible constituents $W(\alpha) = \{x: \alpha \leqslant \sigma(x) < \alpha^+\}$; the set Cd has admissible constituents $\mathrm{Cd}(\alpha) = \{x: \sigma(x) = \alpha\}$. Since, for each infinite ordinal α, there are continuumly many reals x with $\sigma(x) = \alpha$, $W(\alpha)$ and $\mathrm{Cd}(\alpha)$ have the same cardinality for all ordinals $\alpha \in \mathrm{Ad}$. It follows that W and Cd are Borel equivalent. It is also clear that these two sets are not Borel equivalent to any thin coanalytic set. The following result will be useful in showing that given sets are not Borel equivalent.

THEOREM 2.6. *Suppose that all reals are constructible. Then coanalytic subsets P and Q of a Polish space X are Borel equivalent if and only if* *or any admissible decompositions* $\{P(\alpha): \alpha \in \mathrm{Ad}\}$ *of* P *and* $\{Q(\alpha): \alpha \in \mathrm{Ad}\}$ *of* Q, $P(\alpha)$ *and* $Q(\alpha)$ *have the same cardinality for all but countably many ordinals* α.

Proof. It clearly suffices to show that for any two admissible decompo-itions $P_1(\alpha)$ and $P_2(\alpha)$ of a single coanalytic set P, $P_1(\alpha)$ and $P_2(\alpha)$ have he same cardinality for all but countably many ordinals α. But it follows rom Theorem 2.4 that $P_1(\alpha) = P_2(\alpha)$ for all but countably many ordinals. ∎

The existence of coanalytic sets which are not Borel is well known. Such sets can be characterized by the following.

THEOREM 2.7. *Let* P *be a coanalytic subset of a Polish space* X *and et* Γ *be a pseudo-Borel monotone inductive operator which decomposes* P *into idmissible constituents* $\{P(\alpha): \alpha \in \mathrm{Ad}\}$. *Then* P *is Borel if and only if* $P(\alpha) = \emptyset$ *or all but countably many ordinals* $\alpha \in \mathrm{Ad}$.

Proof. ($\rightarrow$) Let $P = (\mathrm{Cl}(\Gamma))_s$. If P is Borel, then $\{s\} \times P$ is an analytic subset of $\mathrm{Cl}(\Gamma)$. It follows from Theorem 2.1 (d) that $\{s\} \times P \subset \Gamma^{\sigma}$ for some countable ordinal σ; now $P(\alpha) = \emptyset$ for all $\alpha > \sigma$. ($\leftarrow$) If $P(\alpha) = \emptyset$ for all $\alpha > \sigma$, then $P = \bigcup \{P(\alpha): \alpha \leqslant \sigma\}$ gives P as a countable union of Borel sets: thus P is also Borel.

The examples of non-Borel-isomorphic sets which we will construct in the next section will be pseudo-Borel subsets of the set W of countable well-orderings, such as the set Cd defined above. The following lemma enables us to determine the Borel equivalence of such sets.

LEMMA 2.8. *Let* P *be a pseudo-Borel subset of the coanalytic set* Q, *where* Q *has admissible decomposition* $\{Q(\alpha): \alpha \in \mathrm{Ad}\}$. *Then* P *has an admissible decomposition given by* $P(\alpha) = Q(\alpha) \cap P$. ∎

Proof. Let Γ be a pseudo-Borel monotone inductive operator over $\mathrm{Seq} \times X$ such that $Q = (\mathrm{Cl}(\Gamma))_0$. Define the operator Δ by

$$\Delta(K) = \{(\langle 0\rangle * s, x): (s, x) \in \Gamma(\{(s, x): (\langle 0\rangle * s, x) \in K\})\} \cup$$
$$\cup \{(\emptyset, x): (\langle 0\rangle, x) \in \Delta(K) \,\&\, x \in P\}.$$

It can be seen by transfinite induction that for any countable ordinal α,

$$\Delta^{\alpha} = \{(\langle 0\rangle * s, x): (s, x) \in \Gamma^{\alpha}\} \cup \{(\emptyset, x): (\emptyset, x) \in \Gamma^{\alpha} \,\&\, x \in P\}.$$

It follows that the admissible decomposition of P corresponding to Δ has the desired property.

This lemma can be used to obtain admissible decompositions for a special type of subset of a coanalytic set. We first need to consider a version of the Prewellordering Theorem.

Suppose that $P = (\mathrm{Cl}(\Gamma))_s$ and $Q = (\mathrm{Cl}(\Delta))_t$, where Γ and Δ are pseudo-Borel operators over $\mathrm{Seq} \times X$ and $\mathrm{Seq} \times Y$ and s and t are in Seq. A mapping from X into the set of countable ordinals together with infinity (∞) is

defined by $|x|_\Gamma$ = the least α such that $(s, x) \in \Gamma^{\alpha+1}$ if $x \in P$, and $|x|_\Gamma = \infty$ otherwise. A similar definition can be given for $|y|_\Delta$. The Prewellordering Theorem for pseudo-Borel operators is given in the following; the proof is immediate from the discussion in [5], p. 68.

THEOREM 2.9. *Let* P, Q, Γ *and* Δ *be as described above. Then* $\{(x, y): |x|_\Gamma < |y|_\Delta \,\&\, y \in Q\}$ *and* $\{(x, y): |x|_\Gamma \leqslant |y|_\Delta \,\&\, y \in Q\}$ *are both pseudo-Borel subsets of* $X \times Q$. ∎

Applying this result to the set W of countable well-orderings, we obtain the following.

COROLLARY 2.10. $\{(u, v) \in W \times W: \sigma(u) < \sigma(v)\}$ *and* $\{(u, v) \in W \times W: \sigma(u) \leqslant \sigma(v)\}$ *are both pseudo-Borel subsets of* $J \times W$. ∎

The next two lemmas are needed in section five; the first is a refinement of Lemma 2.8. A subset B of W is said to be *saturated* if, for any $u, v \in W$, $u \in B$ and $\sigma(u) = \sigma(v)$ imply $v \in B$.

LEMMA 2.11. *Let* Q *be a* Π^1_1 *subset of the Polish space* X, *let* $s \in \mathrm{Seq}$ *and let* Γ *be a pseudo-*Δ^1_1 *operator over* $\mathrm{Seq} \times X$ *such that* $Q = (\mathrm{Cl}(\Gamma))_s$. *Let* B *be a saturated pseudo-Borel subset of* W *and let* $P = \bigcup \{(\Gamma^{\alpha})_s: \alpha \in \sigma(B)\}$. *Then* P *is coanalytic and there is a pseudo-Borel operator* Δ *such that, for each ordinal* α, $(\Delta^{\alpha})_{\emptyset} = (\Gamma^{\alpha})_s \cap P$.

Proof. It clearly suffices, if we use the method of Lemma 2.8, to show that P is a pseudo-Borel subset of Q. Now the set P can be defined in two ways

$$P = \{x \in Q: (\forall u)\, \sigma(u) = |x|_\Gamma \rightarrow u \in B\}$$
$$= \{x \in Q: (\exists u)\, \sigma(u) = |x|_\Gamma \,\&\, u \in B\}.$$

LEMMA 2.12. *Let* B *a saturated pseudo-*Δ^1_1 *subset of* W *with* $\sigma(B)$ *uncountable and let* η *be an order isomorphism of* $\aleph_1$ *onto* $\sigma(B)$. *Then* $\{(u, v): \eta(\sigma(u)) \leqslant \sigma(v)\}$ *and* $\{(u, v): \eta(\sigma(u)) < \sigma(v)\}$ *are both pseudo-Borel subsets of* $W \times W$.

Proof. For $v, w \in 2^{\mathrm{Seq}}$, define the pseudo-Borel subset M of $W \times W$ by $M(v, w)$ if and only if $(\forall s)(w(s) = 1 \leftrightarrow v(s) = 1 \,\&\, v \restriction s \in B)$. Then $M(v, w)$ implies that $\eta(\sigma(w)) = \sigma(v)$ and, for each v, there is a unique w such that $M(v, w)$. Now we can define $\{(u, v): \eta(\sigma(u)) \leqslant \sigma(v)\}$ in two ways:

$$\eta(\sigma(u)) \leqslant \sigma(v) \leftrightarrow (\exists w)(M(v, w) \,\&\, \sigma(u) \leqslant \sigma(w))$$
$$\leftrightarrow (\forall w)(M(v, w) \rightarrow \sigma(u) \leqslant \sigma(w)).$$

(The argument is similar for "<".)

We conclude this section with the theorem which will be our basic tool for proving that two coanalytic sets are not Borel isomorphic. This theorem indicates the close relation between Borel equivalence and Borel isomorphism. We do not know if the hypothesis of constructibility can be removed.

THEOREM 2.13. *Suppose that all reals are constructible. If two* Π_1^1 *sets* P *and* Q *are Borel isomorphic, then they are also Borel equivalent.*

Proof. Let H be a Borel isomorphism of J onto J with $H(P) = Q$ and suppose that $P = (\mathrm{Cl}(\Delta))_s$ and $Q = (\mathrm{Cl}(\Gamma))_t$. Let $P(\alpha) = (\Delta^{\alpha^+} - \Delta^\alpha)_s$ and $Q(\alpha) = (\Gamma^{\alpha^+} - \Gamma^\alpha)_t$ be the corresponding admissible decompositions of P and Q. By Theorem 2.5, $(\Gamma^\gamma)_t = H((\Delta^\gamma)_s)$ for all but countably many ordinals $\alpha \in \mathrm{Ad}$. Since H is an isomorphism, it follows that $H(P(\alpha)) = Q(\alpha)$ for almost all $\alpha \in \mathrm{Ad}$. ■

3. Ordinal partitions and non-isomorphic sets

In this section we show that, assuming all reals are constructible, there exists an uncountable family of coanalytic, non-Borel sets which are pairwise non-Borel isomorphic.

Out first requirement is a partition of the set of countable ordinals into infinitely many uncountable subsets. Now any ordinal α can be written uniquely in the form $\lambda + n$, where λ is either a limit ordinal or zero and $n \in N$. For each n, let $A(n) = \{\alpha: \alpha = \lambda + n$ with λ either a limit or zero$\}$. This partitions the family of countable ordinals into uncountable sets as desired. See Kuratowski and Mostowski [14] for basic facts about ordinal numbers.

Recall the set Ad of ordinals which are either admissible, the limit of admissibles or zero. Enumerate Ad as $\{\alpha(\tau): \tau < \omega_1\}$ and let $\mathrm{Ad}[n] = \{\alpha(\tau): \tau \in A(n)\}$ for $n \in N$. Similarly, $\mathrm{Cd}[n] = \{x \in \mathrm{Cd}: \sigma(x) \in \mathrm{Ad}[n]\}$. Note that each set $\mathrm{Cd}[n]$ contains a perfect subset since, as noted above, $\{x: \sigma(x) = \alpha\}$ is an uncountable Borel set for each infinite ordinal α.

LEMMA 3.1. *For each* n, $\mathrm{Cd}[n]$ *is a pseudo-Borel subset of the set* W *of countable well-orderings.*

Proof. $\mathrm{Cd}[0] = \{x \in \mathrm{Cd}: (\forall s)[x \restriction s \in \mathrm{Cd} \rightarrow (\exists t > s) x \restriction t \in \mathrm{Cd}]\}$. For each n, $\mathrm{Cd}[n+1] = \{x \in \mathrm{Cd}: (\exists s)[x \restriction s \in \mathrm{Cd}[n] \;\&\; (\forall t > s) x \restriction t \notin \mathrm{Cd}]\}$. Since Cd is known, by Lemma 1.1, to be a pseudo-Borel subset of W, it follows that each $\mathrm{Cd}[n]$ is also. ■

Combining this result with Lemma 2.8, we see that $\mathrm{Cd}[n]$ has admissible constituents

$$\mathrm{Cd}[n](\alpha) = \begin{cases} \{x: \sigma(x) = \alpha\}, & \text{if } \alpha \in \mathrm{Ad}[n]; \\ \emptyset & \text{otherwise.} \end{cases} \tag{4}$$

THEOREM 3.2 *Suppose that all reals are constructible. Then for any* $m \neq n$, $\mathrm{Cd}[m]$ *and* $\mathrm{Cd}[n]$ *are not Borel isomorphic.*

Proof. It is clear that for $m \neq n$, the natural admissible decompositions given by (4) for $\mathrm{Cd}[m]$ and $\mathrm{Cd}[n]$ disagree for every admissible ordinal α, so that $\mathrm{Cd}[m]$ and $\mathrm{Cd}[n]$ are not Borel equivalent. Now, by Theorem 2.13, $\mathrm{Cd}[m]$ and $\mathrm{Cd}[n]$ can not be Borel isomorphic. ■

In fact, we can prove more. For any set $M \subseteq N$, let $\mathrm{Ad}[M] = \bigcup\{\mathrm{Ad}[n]: n \in M\}$ and $\mathrm{Cd}[M] = \bigcup\{\mathrm{Cd}[n]: n \in M\}$.

LEMMA 3.3. *Suppose* $K = \bigcup\{K_n: n \in N\}$ *and that each set* K_n *has admissible constituents* $\{K_n(\alpha): \alpha \in \mathrm{Ad}\}$. *Then* K *has admissible constituents* $K(\alpha) = \bigcup\{K_n(\alpha): n \in N\}$.

Proof. Suppose $K_n = (\mathrm{Cl}(\Delta_n))_\emptyset$ for each n. Define an inductive operator Δ by putting $((n) * s, x) \in \Delta(K)$ if and only if $(s, x) \in \Delta_n((K)_{(n)}, s)$ and $(\emptyset, x) \in \Delta(K)$ if and only if $(\exists n)((n), x) \in \Delta(K)$. It can be seen by transfinite induction that $(\Delta^\alpha)_\emptyset = (\bigcup \Delta_n^\alpha)_\emptyset$ for each ordinal α. It follows that $K(\alpha) = \bigcup\{K_n(\alpha): n \in N\}$ for the admissible decomposition of K corresponding to Δ. ■

THEOREM 3.4. *Suppose that all reals are constructible. Then for any two subsets* $M_1 \neq M_2$ *of* N, $\mathrm{Cd}[M_1]$ *and* $\mathrm{Cd}[M_2]$ *are not Borel isomorphic. Furthermore, for any two thin sets* B *and* C, B *a relative Borel subset of* $\mathrm{Cd}[M_1]$ *and* C *a relative Borel subset of* $\mathrm{Cd}[M_2]$, $\mathrm{Cd}[M_1] - B$ *is not Borel isomorphic to* $\mathrm{Cd}[M_2] - C$.

Proof. By formula (4) and Lemma 3.3, $\mathrm{Cd}[M_i]$ has admissible constituents $\mathrm{Cd}[M_i](\alpha)$ such that $\mathrm{Cd}[M_i](\sigma(x))$ is non-empty if and only if $x \in \mathrm{Cd}[M_i]$ for $i = 1, 2$. If $M_1 \neq M_2$, choose $m \in (M_2 - M_1) \cup (M_1 - M_2)$; $\mathrm{Cd}[M_1](\alpha)$ and $\mathrm{Cd}[M_2](\alpha)$ will have different cardinality for every ordinal α belonging to the uncountable set $\mathrm{Ad}[m]$. It follows that $\mathrm{Cd}[M_1]$ and $\mathrm{Cd}[M_2]$ are not Borel equivalent and therefore, by Theorem 2.13, are not Borel isomorphic. Since the non-empty admissible constituents of any $\mathrm{Cd}[M]$ are all uncountable, the removal of a thin set, which has all constituents countable, does not effect the Borel equivalence class of $\mathrm{Cd}[M]$. This proves the second part of the theorem. ■

Remark. Given any two thin coanalytic sets B and C and perfect sets P and Q such that $B \cap P = \emptyset = C \cap Q$, it can be seen that B is Borel isomorphic to C if and only if $B \cup P$ is Borel isomorphic to $C \cup Q$. Thus if B and C are not Borel isomorphic, one can trivially "fatten" them up to non-isomorphic coanalytic sets which contain perfect sets. The thrust of the last sentence of Theorem 3.4 is that the sets $\mathrm{Cd}[M]$ cannot be obtained in this fashion.

We now have a family of continuumly many coanalytic subsets of Cd, no two of which are Borel isomorphic. This family of subsets of J can be realized as the family of vertical sections of a coanalytic subset of $J \times J$, as shown in the following.

THEOREM 3.5. *Suppose that all reals are constructible. Then there is a coanalytic subset* Q *of* $J \times J$ *such that the projection* $\pi_2(Q) = \mathrm{Cd}$, *each horizontal section* Q^y *is clopen, each vertical section* Q_x *contains a perfect set and no two vertical sections are Borel isomorphic. Furthermore, if* P *is a relative Borel subset of* Q *and each* P_x *is thin, then no two vertical sections of* $Q - P$ *are Borel isomorphic.*

Proof. For any real x, let $M(x) = \{n+x(0)+ \ldots +x(n): n \in N\}$. It is lear that M is a one-to-one map from J onto the family of infinite ubsets of N. Define the coanalytic subset Q of $J \times J$ to be $\{(x, y): \in \mathrm{Cd}[M(x)]\}$. Then, for any $x \in J$, $Q_x = \mathrm{Cd}[M(x)]$. It follows that for $_1 \neq x_2$, $M(x_1) \neq M(x_2)$ and, by Theorem 3.4, Q_{x_1} is not Borel isomorphic o Q_{x_2}. The last part of the theorem follows from the last part of Theorem 3.4. ∎

The sets considered in the last two theorems were constructed from the artition $\{\mathrm{Cd}[n]: n \in N\}$ of the set Cd by the operation of union. We next uild a family of sets using the operation of direct product. For any ubset M of N, let $P[M]$ be the direct product of the collection $\mathrm{Cd}[m]: m \in M\}$.

LEMMA 3.6. *Let M be a subset of the natural numbers.*

(a) *If M is finite, then $P[M]$ is Borel equivalent to* $\mathrm{Cd}[M]$.

(b) *If M is infinite, then $P[M]$ is Borel equivalent to* $\mathrm{Cd}[M \cup \{0\}]$.

Proof. Let $M = \{m_0, m_1, \ldots\}$, let $B = P[M]$ and let $C = \mathrm{Cd}[M]$. Let $\mathrm{Cd}[m_i] = (\mathrm{Cl}(\Delta_i))_{i+1}$ for all i and let $\Delta(K) = \bigcup \{\{i+1\} \times \Delta_i((K)_{i+1}): i \in N\} \cup \{0\} \times \prod \{\Delta_i((K)_{i+1}): i \in N\}$. This natural inductive lefinition of B produces admissible constituents $B(\alpha) = \{(x_0, x_1, \ldots): (\forall i)(x_i \in \mathrm{Cd}[m_i]) \,\&\, \alpha = \sup\{\sigma(x_i): i \in N\}\}$. On the other hand, C has admissible constituents $C(\alpha) = \{x: (\exists i)(x \in \mathrm{Cd}[m_i] \,\&\, \alpha = \sigma(x))\}$. Notice that, in both cases, each constituent is either empty or has the cardinality of the continuum. Suppose first that $B(\alpha) \neq \emptyset$; then there is some $(x_0, x_1, \ldots)$ with each $x_i \in \mathrm{Cd}[m_i]$ and $\sup\{\sigma(x_i): i = 0, 1, \ldots\} = \alpha$. If M is finite or if $\alpha \notin \mathrm{Ad}[0]$, then $\alpha = \sigma(x_i)$ for some i, which implies that $C(\alpha) \neq \emptyset$. The other direction requires a little more work. For each i, let σ_i be the least ordinal in $\mathrm{Ad}[m_i]$ and choose $x_i \in \mathrm{Cd}[m_i]$ with $\sigma(x_i) = \sigma_i$; let $\sigma = \sup\{\sigma_i: i \in N\}$. Now suppose that $C(\alpha) \neq \emptyset$ for some $\alpha > \sigma$. Then for some i and some $x \in \mathrm{Cd}[m_i]$, $\sigma(x) = \alpha$; let $i = 0$ without loss of generality. Then $\sup\{\sigma(x), \sigma(x_1), \sigma(x_2), \ldots\} = \sigma(x) = \alpha$, so that $(x, x_1, x_2, \ldots) \in B(\alpha)$ and $B(\alpha) \neq \emptyset$. Recall that $\mathrm{Ad}[0]$ is the set of ordinals which can be given as the limit of a sequence of admissible ordinals. Thus if the ordinal α belongs to $\mathrm{Ad}[0]$ and is greater than σ and if M is infinite, then α can be expressed as $\sup\{\alpha_i: i \in N\}$ with each $\alpha_i \in \mathrm{Ad}[M]$. It follows that $B(\alpha) \neq \emptyset$ for all but countably many $\alpha \in \mathrm{Ad}[0]$. ∎

THEOREM 3.7. *Suppose that all reals are constructible. Then for any two subsets $M_1 \neq M_2$ of the positive integers, $P[M_1]$ and $P[M_2]$ are not Borel isomorphic.*

Proof. By Lemma 3.6, $P[M_i]$ is Borel equivalent to $\mathrm{Cd}[M_i]$ for $i = 1, 2$. It was shown in the proof of Theorem 3.4 that $\mathrm{Cd}[M_1]$ and $\mathrm{Cd}[M_2]$ are not Borel equivalent. It follows that $P[M_1]$ and $P[M_2]$ are

not Borel equivalent. Now by Theorem 2.13, $P[M_1]$ and $P[M_2]$ cannot be Borel isomorphic. ∎

Theorems 3.2, 3.4 and 3.7 can be improved slightly by starting with a partition of the set Ad into ω_1 disjoint uncountable sets. To this end, we define a map $F: \omega_1 \times \omega_1 \to \omega_1$ by transfinite induction as follows:

$$F(0, 0) = 0; \; F(0, \tau+1) = F(0, \tau)+\tau+1 \text{ for all } \tau;$$

$$(5) \qquad F(0, \lambda) = \sup\{F(0, \tau): \tau < \lambda\} \text{ for limit ordinals } \lambda;$$

$$F(\sigma, \tau) = F(0, \sigma+\tau)+\sigma \text{ for all ordinals } \sigma > 0.$$

Thus $F(0, 1) = 1$, $F(1, 0) = 2$, $F(0, 2) = 3$, $F(1, 1) = 4$, $F(2, 0) = 5$ and so on.

Now enumerate as $\{\beta(0), \beta(1), \ldots\}$ the set of ordinals in Ad which are not limits of admissibles; those limits are deleted in order to simplify the direct product theorem. For each countable ordinal σ, let $\mathrm{Au}[\sigma] = \{F(\sigma, \beta(\tau)): \tau < \omega_1\}$ and let $\mathrm{Cu}[\sigma] = \{x \in \mathrm{Cd}: \sigma(x) \in \mathrm{Au}[\sigma]\}$.

LEMMA 3.8. (a) *For each countable ordinal σ, $\mathrm{Au}[\sigma]$ is an uncountable subset of* Ad.

(b) *For any two countable ordinals $\sigma \neq \tau$, $\mathrm{Au}[\sigma]$ and $\mathrm{Au}[\tau]$ are disjoint.*

(c) *For any countable ordinal σ, $\mathrm{Cu}[\sigma]$ is a pseudo-Borel subset of* Cd. ∎

Remark. Parts (a) and (b) follow immediately from definition (5). Part (c) is proved in the manner of Lemma 3.1. The details are left to the reader.

The sets $\mathrm{Cu}[\sigma]$ can now be combined using the operations of union and direct product. The proof of the following result is similar to those of Theorems 3.4 and 3.7.

THEOREM 3.9. *Suppose that all reals are constructible. Then there is an uncountable family $\{\mathrm{Cu}[\sigma]: \sigma < \omega_1\}$ of pairwise disjoint coanalytic sets such that for any two countable subsets $M_1 \neq M_2$ of ω_1;*

(a) $\bigcup \{\mathrm{Cu}[\sigma]: \sigma \in M_1\}$ *and* $\bigcup \{\mathrm{Cu}[\sigma]: \sigma \in M_2\}$ *are not Borel isomorphic;*

(b) $\prod \{\mathrm{Cu}[\sigma]: \sigma \in M_1\}$ *and* $\prod \{\mathrm{Cu}[\sigma]: \sigma \in M_2\}$ *are not Borel isomorphic.*

In both cases, the sets remain non-isomorphic if thin relative Borel subsets are removed from each. ∎

4. Thin non-isomorphic sets

The basic non-isomorphic sets $\mathrm{Cd}[n]$ defined in the previous section were all large in that each had a perfect subset. It is a classical result that any uncountable analytic set has a perfect subset. However, if all reals are constructible, then an uncountable thin coanalytic set exists, as discussed in the introduction; we describe such a set below. In this section, we construct an uncountable family of pairwise non-Borel isomorphic thin coanalytic sets.

The collection of thin coanalytic sets is characterized by the following result, noted above in section two.

Theorem 4.1. *Let Q be a coanalytic set with admissible constituents $\{Q(\alpha)\colon \alpha\in\mathrm{Ad}\}$. Then Q is thin if and only if each $Q(\alpha)$ is countable.*

Proof. ($\rightarrow$) The constituents of Q are all analytic. So by the classical result stated above, if any $Q(\alpha)$ is uncountable, then $Q(\alpha)$ has a perfect subset. Thus that Q is not thin.

($\leftarrow$) Suppose Q has a perfect subset B. Then B is analytic, so by Theorem 1.2 (d), $B\subseteq\bigcup\{Q(\alpha)\colon \alpha<\beta\}$ for some countable ordinal β. Since B is uncountable, some admissible constituent $C(\alpha)$ must also be uncountable. ■

For each ordinal $\alpha\in\mathrm{Ad}$, let $L(\alpha)=(L_{\alpha^+}-L_\alpha)\cap J$, where L_α is the αth level of the constructible universe as described above in section two. It should be noted that uncountably many of the $L(\alpha)$ are empty. Each $L(\alpha)$ is a countable Borel set, but $\{L(\alpha)\colon \alpha\in\mathrm{Ad}\}$ is not the set of constituents of a coanalytic set if (V = L), since $\bigcup\{L(\alpha)\colon \alpha\in\mathrm{Ad}\}=L\cap J$ would not be thin.

However, we can define an uncountable coanalytic set T such that, for each $\alpha\in\mathrm{Ad}$, $T(\alpha)\subseteq L(\alpha)$: such a set must be thin by Theorem 4.1. Before doing so, we consider what such a set must be like.

Suppose now that $x\in L(\alpha)$. Then by Proposition 2.3, α^+ is x-admissible, so that $\varrho(x)\leqslant\alpha^+$. On the other hand, if $x\in T(\alpha)$, then by Theorem 1.2 (d), $\varrho(x)\geqslant\alpha^+$. Combining these, we have the following result.

Proposition 4.2. *Let Q be a coanalytic set, $x\in J$ and $\alpha\in\mathrm{Ad}$. If $x\in Q(\alpha)\cap\cap L(\alpha)$, then $\varrho(x)=\alpha^+$ and $L_{\varrho(x)}$. If $Q(\alpha)\subseteq L(\alpha)$ for all $\alpha\in\mathrm{Ad}$, then $Q\subseteq\{x\colon x\in L_{\varrho(x)}\}$.* ■

The well-known largest thin Π^1_1 set C_1 is defined by:

$$C_1=\{x\colon x\in L_{\varrho(x)}\}. \tag{6}$$

It is a consequence of Lemma 2.1 that C_1 is Π^1_1. By Proposition 4.2, any coanalytic set C with $C(\alpha)\subseteq L(\alpha)$ for all $\alpha\in\mathrm{Ad}$ is included in C_1. It also follows that a real x is in C_1 if and only if, for some countable ordinal α, $x\in L(\alpha)$ and $\varrho(x)=\alpha^+$. It is not hard to see that C_1 has admissible constituents $C_1(\alpha)=L(\alpha)\cap\{x\colon \varrho(x)=\alpha^+\}$. Thus each $C_1(\alpha)$ is countable and, by Theorem 4.1, C_1 is thin.

It remains to be seen that the set C_1 is uncountable if (V = L). The following facts are taken from Kechris [11]; a detailed discussion of the set C_1 can be found there.

Theorem 4.3. (a) *For each non-empty $L(\alpha)$, there is a real $u\in\mathrm{Cd}\cap C_1$ with $\sigma(u)=\alpha$.*

(b) *Any thin Π^1_1 set C is included in C_1.*

(c) *If all reals are constructible, then C_1 is an uncountable thin Π^1_1 set.* ■

Remark. Part (a) is essentially due to Boolos and Putnam [2]. Part (c) follows from part (a), since if all reals are constructible, then uncountably many $L(\alpha)$ are non-empty. That $\{x\colon x\in L_{\varrho(x)}\}$ is the largest thin Π^1_1 set is due independently to G. Sacks and D. Guaspari.

Keeping part (a) above in mind, let Ld denote the set of ordinals $\alpha\in\mathrm{Ad}$ such that $L(\alpha)$ is non-empty and let Td denote the set of reals $x\in\mathrm{Cd}\cap C_1$ such that no real $y\in\mathrm{Cd}\cap C_1$ with $\sigma(y)=\sigma(x)$ is constructed before x. It can be seen that Td is a relative Δ^1_1 subset of Cd and that the admissible constituent $\mathrm{Td}(\alpha)$ is empty if $\alpha\notin\mathrm{Ld}$ and is a singleton $\{x\}$ with $\sigma(x)=\alpha$ if $\alpha\in\mathrm{Ld}$.

As was done in section three for the set Ad, the set Ld can be partitioned into either countably many or uncountably many disjoint subsets $\{\mathrm{Ld}[n]\colon n\in N\}$ or $\{\mathrm{Lu}[\sigma]\colon \sigma<\omega_1\}$ with corresponding partitions $\mathrm{Td}[n]=\{x\in\mathrm{Td}\colon \sigma(x)\in\mathrm{Ld}[n]\}$ and $\mathrm{Tu}[\sigma]=\{x\in\mathrm{Td}\colon \sigma(x)\in\mathrm{Lu}[\sigma]\}$ of the Π^1_1 set Td. These sets can be combined by union and direct product as in Theorems 3.4, 3.7 and 3.9 and can be parametrized as in Theorem 3.5. We obtain an improvement in Theorem 3.9 (a) due to the fact that the non-empty constituents $\mathrm{Td}(\alpha)$ are singletons.

Lemma 4.4. *Suppose that all reals are constructible. Let T be a coanalytic set such that uncountably many admissible constituents $T(\alpha)$ are finite and non-empty. Then the family $\{nT\colon n\leqslant\omega\}$ is pairwise not Borel isomorphic.*

Proof. It is clear that nT has admissible constituents $nT(\alpha)$ for each $n\leqslant\omega$ and $\alpha\in\mathrm{Ad}$. Now if $T(\alpha)$ is finite and non-empty, then each $nT(\alpha)$ has a different cardinality. Since uncountably many $T(\alpha)$ are finite and non-empty, the sets nT are pairwise not Borel equivalent. It follows from Theorem 2.13 that they are also pairwise not Borel isomorphic. ■

Since the proofs of the following results differ little from those given in section three for Theorems 3.4, 3.5 and 3.7, they are omitted here.

Theorem 4.5. *Suppose that all reals are constructible. Then there is a family $\{T_\alpha\colon \alpha<\omega_1\}$ of thin coanalytic sets such that (a) the sets $\sum\{n_i T_i\colon i\in I\}$, where I is any countable subset of ω_1 and each $n_i\leqslant\omega$, are pairwise not Borel isomorphic; (b) the sets $\prod\{T_i\colon i\in I\}$, where I is any countable subset of ω_1, are pairwise not Borel isomorphic.*

(In each case, the set I is assumed to be listed in increasing order.) ■

Theorem 4.6. *Suppose that all reals are constructible. Then there is a coanalytic subset T of $J\times J$ such that the projection $\pi_2(T)$ is a thin coanalytic set, each horizontal section of T is clopen and no two vertical sections of T are Borel isomorphic.* ■

5. The hypothesis of projective determinacy

An infinite game G_A of perfect information can be associated with each subset A of the continuum J, as follows. Players I and II alternately select natural numbers $x(0), x(1), x(2), \ldots$, resulting a play of the game, a real x. A strategy for one of the players is a function from Seq into N. Player I follows the strategy ξ in the play x provided that, for all n, $x(2n) = \xi(x|2n)$; II follows ξ provided that $x(2n+1) = \xi(x|2n+1)$ for all n.

If player I plays $x(2n) = u(n)$ and II follows ξ, the resulting play $v(n) = x(2n+1)$ is denoted by $P(u, \xi)$; similarly $P(\xi, v)$ results when II plays v and I follows ξ.

The above definition of following a strategy leads directly to the following lemma.

LEMMA 5.1. *For any strategy ξ, $\{u * P(u, \xi): u \in J\}$ and $\{P(\xi, v) * v: v \in J\}$ – the set of plays resulting when one players follows ξ – are closed; $\{P(u, \xi): u \in J\}$ and $\{P(\xi, v): v \in J\}$ – the set of responses dictated by the strategy – are analytic. Also, the functions $P(-, \xi)$, $P(\xi, -)$ are continuous.* ∎

A strategy ξ is said to be *winning* for a player if he wins every play of the game in which he follows ξ. The game G_A (and the set A) is said to be *determined* if one of the two players has a winning strategy for G_A; of course at most one player could have a winning strategy. There are many interesting consequences when a game G_A is determined. See Mycielski [19] for some examples. The following in particular will prove useful.

THEOREM 5.2. *If all coanalytic games are determined, then any thin coanalytic set is countable.* ∎

It follows from Theorem 4.3 that if all reals are constructible, then not all Π^1_1 games are determined. Martin has proved that all Borel games are determined [17] and that if there is a measurable cardinal then all Π^1_1 games are determined [16]. The Axiom of Projective Determinacy (PD) states that all projective games are determined. Of course the Axiom of Choice implies that some games are not determined.

Combining Theorems 4.1 and 5.2, we obtain

COROLLARY 5.3. *Suppose that all Π^1_1 games are determined. Let Q be a coanalytic non-Borel set with admissible constituents $\{Q(\alpha): \alpha \in \mathrm{Ad}\}$. Then, for uncountably many ordinals α, $Q(\alpha)$ is uncountable.*

Proof. Suppose by way of contradiction that only countably many $Q(\alpha)$ are uncountable; choose a countable ordinal β such that $Q(\alpha)$ is countable for all $\alpha > \beta$. Now let $B = \{u \in \mathrm{Cd}: \sigma(u) > \beta\}$; B is clearly a pseudo-Borel subset of Cd. By Lemma 2.11, $P = \bigcup\{Q(\sigma(u)): u \in B\}$ is a coanalytic set with admissible decomposition $\{P(\alpha): \alpha \in \mathrm{Ad}\}$ such that every $P(\alpha)$ is countable. P is non-Borel since it differs from the non-Borel set Q by the Borel set $\bigcup\{Q(\alpha): \alpha \leqslant \beta\}$. However, P is now thin by Theorem 4.1 and therefore countable by Theorem 5.2. Thus P is Borel. This contradiction proves the corollary. ∎

By assuming a little more determinacy, we can improve this result to show that Q has an admissible decomposition with every $Q(\alpha)$ uncountable.

Let us call a family Σ of relations *nice* if it includes the Π^1_1 relations and is closed under recursive substitution and number quantification. Examples include the family of Δ^1_{n+1} relations and the family $B(\Pi^1_n)$ of Boolean combinations of Π^1_n relations, for $n \geqslant 1$. Recall that a subset B of W is saturated if, for any $u, v \in W$, $u \in B$ and $\sigma(u) = \sigma(v)$ imply $v \in B$.

The following is essentially due to Solovay.

THEOREM 5.4. *Let Σ be a nice family of relations such that every game in $B(\Sigma)$ is determined. Then every saturated subset B of W which is in Σ is actually a pseudo-Borel subset of W.*

Proof. Given B and Σ as described, let $B^* = \{\sigma(u): u \in B\}$ and let G_A be the Solovay game given by

$$A = \{u * v: v \in W \to B^* \cap (\sigma(v)+1) \subset \{\sigma((u)_n): n \in N\} \subset B^*\}.$$

The first inclusion in the definition of A can be written

$$\big(v \in B \to (\exists n)\,\sigma(v) = \sigma((u)_n)\big) \,\&\, (\forall p)\big(v \upharpoonright p \in B \to (\exists n)\,\sigma(v \upharpoonright p) = \sigma((u)_n)\big)$$

and the second can be written $(\forall n)\,(u)_n \in B$. Thus, if Σ is nice, then A is in $B(\Sigma)$. Thus either player I or player II must have a winning strategy.

The idea of the game is that player II must play a real v from W and I must respond with a u which codes up a subset of B^* including any ordinal in B^* which is $\leqslant \sigma(v)$. Now if player II had a winning strategy ξ then by Lemma 5.1 his set of responses would be an analytic subset of W. Then by the Boundedness Principle (Theorem 1.2 (d)), $\{\sigma(P(u, \xi)): u \in J\}$ is bounded above by some countable ordinal β. But player I can now defeat the strategy ξ by playing some real u which codes up $B^* \cap (\beta+1)$.

It follows that player I must have a winning strategy ξ. Now $B = \{v \in W: (\exists n)\sigma(v) = \sigma(P(\xi, v))\}$ and is therefore a pseudo-Borel subset of W by Corollary 2.10. ∎

We want to apply this result to $\{u: (\Gamma^{\sigma(u)})_s$ is countable$\}$, where the coanalytic non-Borel set $Q = (\mathrm{Cl}(\Gamma))_s$. The next result determines the appropriate family Σ of relations in this case.

THEOREM 5.5. *For any pseudo-Borel monotone inductive operator Γ over $\mathrm{Seq} \times J$ and any $s \in \mathrm{Seq}$, $\{u: (\Gamma^{\sigma(u)+1} - \Gamma^{\sigma(u)})_s$ is countable$\}$ is Π^1_1.*

Proof. It follows from Theorem 2.9 that $\{(u, x): (s, x) \in \Gamma^{\sigma(u)+1} - \Gamma^{\sigma(u)}\}$ is a relative Σ^1_1 subset of $W \times J$. A more general result of Kechris [13], p. 378, now completes the proof.

We are now approaching the proof of the main theorem of the chapter

(Theorem 6 of the introduction), that if all $B(\Pi_1^1)$ games are determined, then any two coanalytic non-Borel sets are Borel equivalent. Fix now a coanalytic non-Borel subset Q of J, a pseudo-Borel monotone inductive operator Γ over Seq $\times J$ and an $s \in$ Seq such that $Q = (\mathrm{Cl}(\Gamma))_s$. Also, assume that all $B(\Pi_1^1)$ games are determined.

Let $B = \{u\colon (\Gamma^{\sigma(u)+1} - \Gamma^{\sigma(u)})_s$ is uncountable$\}$. Combining Theorems 5.4 and 5.5, we obtain that B is a pseudo-Borel subset of W. B is unbounded by Corollary 5.3. Now let η be an order isomorphism of $\aleph_1$ onto $\sigma(B)$. Our goal is to obtain a pseudo-Borel operator Δ such that, for all α,

(7) $$(\Delta^{\alpha+1})_\emptyset = (\Gamma^{\eta(\alpha)+1})_s.$$

It follows from (7) that $(\Delta^{\alpha+1} - \Delta^\alpha)_\emptyset$ is uncountable for every countable ordinal α. Clearly then, every constituent of the admissible decomposition of Q corresponding to Δ is uncountable. If the same thing can be done for every coanalytic non-Borel set Q, then of course any two such sets will be Borel equivalent.

We now turn to the construction of the desired operator Δ. One more game-theoretic lemma is needed.

LEMMA 5.6. *Let B be a saturated pseudo-Borel subset of W such that $\sigma(B)$ is uncountable and let η be an order isomorphism of $\aleph_1$ onto $\sigma(B)$. Suppose that all $B(\Pi_1^1)$ games are determined. Then there is a continuous function $f\colon J \to J$ such that, for all $u \in W$, $\sigma(f(u)) > \eta(\sigma(u))$.*

Proof. Let B and η be given as above and let G_A be the Solovay game given by

$$A = \{u * v\colon v \in W \to (u \in W \,\&\, \sigma(u) > \eta(\sigma(v)))\}.$$

It follows from Lemma 2.12 that A is a Boolean combination of Π_1^1 sets. Therefore either player I or player II must have a winning strategy. Now if player II follows a strategy ξ, then by Lemma 5.1 his set of responses would be an analytic subset of W. Then by the Boundedness Principle, $\{\sigma(P(u, \xi))\colon u \in J\}$ is bounded above by some countable ordinal β. Player I can now defeat the strategy by playing some fixed real $u \in W$ such that $\sigma(u) > \eta(\beta)$. It follows that player I must have a winning strategy ξ. The continuous function f may now be defined by $f(v) = P(\xi, v)$. ■

We are now ready to define the desired pseudo-Borel monotone inductive operator Δ over Seq $\times J$ satisfying (7).

$$(\langle 0\rangle, u) \in \Delta(K) \leftrightarrow (\langle 0\rangle, u) \in K \text{ OR } (\forall s)\, u(s) = 0 \text{ OR } (\forall p)\, (\langle 0\rangle, u \restriction p) \in K$$

(8) $$(\emptyset, x) \in \Delta(K) \leftrightarrow x \in Q \,\&\, (\forall u)\, \big(\sigma(u) = |x|_\Gamma \to (\exists p)\, ((\langle 0\rangle, u \restriction p) \in \Delta(K) \,\&\, \eta(\sigma(u \restriction p)) \geqslant \sigma(u))\big)$$
$$\leftrightarrow (\exists v)\, (\exists q)\big((\langle 0\rangle, v) \in \Delta(K) \,\&\, |x|_\Gamma \leqslant \sigma(f(v) \restriction q) \,\&\, \&\, \eta(\sigma(v)) = \sigma(f(v) \restriction q)\big).$$

It can be seen by induction that for each countable ordinal α,

$$\Delta^\alpha = \{(\langle 0\rangle, u)\colon u \in W \,\&\, \sigma(u) < \alpha\} \cup \{(\emptyset, x)\colon |x|_\Gamma < \eta(\alpha)\}.$$

Equation (7) follows immediately from the above. The fact that Δ is pseudo-Borel follows from results 2.9, 2.10, 2.12 and 5.6 along with remarks preceding (7). This completes the proof of the following theorem.

THEOREM 5.7. *Suppose that all $B(\Pi_1^1)$ games are determined. Then for any coanalytic non-Borel subset Q of J, there is a pseudo-Borel monotone inductive operator Δ over* Seq $\times J$ *such that $Q = (\mathrm{Cl}(\Delta))_\emptyset$ and, for all countable ordinals α, $(\Delta^{\alpha+1} - \Delta^\alpha)_\emptyset$ is uncountable.*

COROLLARY 5.8. *Suppose that all $B(\Pi_1^1)$ games are determined. Then any two coanalytic non-Borel subsets of J are Borel equivalent.*

We remark that these last results imply only that every coanalytic set has *some* nice admissible decomposition. Of course, even if all games are determined, the sets Cd $[n]$ defined in section three will still have their usual admissible decompositions, differing from each other at every level. However, by combining the techniques of Corollary 5.3 and Theorem 5.7, we can obtain the following.

THEOREM 5.9. *Suppose that all $B(\Pi_1^1)$ games are determined. Then for any admissible decomposition $\{Q(\alpha)\colon \alpha \in \mathrm{Ad}\}$ of a Π_1^1 subset Q of J, $\{\alpha\colon Q(\alpha)$ is countable and nonempty$\}$ is countable.*

6. Further results and open questions

We first consider those results which can be obtained if the hypothesis that all reals are constructible is weakened or removed. Of course, the results of section two can be relativized to the hypothesis that all reals are constructible from a single fixed real (that is, V = L$[x]$). The results in sections three and four then go through unchanged.

If the hypothesis of constructibility is removed entirely, an extremely weakened version of Theorem 2.13 can still be proved. Let us call a coanalytic set C *thick* if, under some inductive decomposition, there are uncountably many admissible constituents $C(\alpha)$ which are uncountable; call C *mixed* if it is neither thin nor thick.

THEOREM 6.1. *If A, B and C are coanalytic sets such that A is thin, B is thick and C is mixed, then A, B and C are pairwise not Borel isomorphic.*

Proof. Let $C(\alpha)$ be an uncountable admissible constituent of C. If H is a Borel isomorphism of C onto A, then $H(C(\alpha))$ is an uncountable Borel subset of A, which implies that A has a perfect subset. The same argument proves that B is not Borel isomorphic to A. Now suppose that H is a Borel isomorphism of B onto C. Let α be a countable ordinal such that for all

$\geqslant \alpha$, $C(\beta)$ is countable and let $D = \bigcup \{C(\sigma): \sigma < \alpha\}$. Then $H^{-1}(D)$ is Borel subset of B and is therefore included in $\bigcup \{B(\sigma): \sigma < \tau\}$ for some untable ordinal τ. Now choose $\gamma > \tau$ so that $B(\gamma)$ is uncountable. $H(B(\gamma))$ now an uncountable Borel subset of $C-D$ and is therefore included in $\{C(\beta): \alpha \leqslant \beta < \lambda\}$ for some countable ordinal λ. Since this is a countable nion, there is some $\beta \geqslant \alpha$ with $C(\beta)$ uncountable, contradicting our ioice of α. ∎

Since any set is Borel isomorphic to itself, Theorem 5.1 implies the llowing invariance result.

COROLLARY 6.2. *A coanalytic set C is thick if and only if, under any iductive decomposition, there are uncountably many uncountable admissible onsituents; C is thin if and only if, under any inductive decomposition, there re no uncountable admissible constituents; C is mixed if and only if, under ny inductive decomposition, there are countably many (but some) uncountable dmissible consituents.* ∎

Now suppose that an uncountable thin coanalytic set T exists. Then the isjoint union $T+J$ is mixed. The set W is of course thick. The set $T\times J$ s also thick but cannot be Borel isomorphic to W for the following reason: ach admissible constituent of $T\times J$ is an F_σ set, that is, the countable nion of closed sets, whereas the admissible constituents of W have arbitrarily igh Borel class. That two such sets are not Borel isomorphic follows rom an argument of the second author [18, p. 243].

We have now proved the following.

THEOREM 6.3. *Suppose that an uncountable thin coanalytic set exists. Then here is a family of four coanalytic, non-Borel sets which are pairwise not Borel isomorphic.*

The number "four" here can presumably be improved by further analysis of the four types of sets considered. Of course, one could also look for conditions under which any two sets of a particular type would have to be isomorphic. It should be noted that the thin set T is Borel equivalent to the mixed set $T+J$. This example shows that Borel equivalence does not necessarily imply Borel isomorphism.

Recall that the thin set Td defined in section four had the special property that each nonempty consituent Td (α) was a singleton. It is not hard to see that the admissible constituents of $\mathrm{Td}\times\mathrm{Td}$ are almost all countably infinite. Thus Td and Td^2 are not Borel equivalent or Borel isomorphic if V = L is assumed. However, for each m and $n \geqslant 2$, it is clear that the sets Td^m and Td^n are Borel equivalent; it can be shown that in fact Td^m and Td^n are Borel isomorphic.

CONJECTURE. Let T be a thin coanalytic set. Then for any m and $n \geqslant 2$, T^m and T^n are Borel isomorphic.

The concepts of thin, mixed and thick sets allow us to compare the relative sizes of coanalytic sets. Let C denote the equivalence class of the coanalytic set C under the relation of Borel equivalence. If we assume that all reals are constructible, then the family of equivalence classes possesses a natural partial ordering, defined by $[C] \leqslant [D]$ if and only if there exist admissible decompositions $\{C(\alpha): \alpha\in \mathrm{Ad}\}$ of C and $\{D(\alpha): \alpha\in\mathrm{Ad}\}$ of D such that $\mathrm{card}\,(C(\alpha)) \leqslant \mathrm{card}\,(D(\alpha))$ for all but countably many ordinals $\alpha\in\mathrm{Ad}$. The class $[\emptyset]$ of Borel sets is clearly the least element in this ordering and the class [Ad] is the largest. There are chains of length ω_1 and longer. For example, take the sets $\mathrm{Cu}\,[\sigma]$ from Theorem 3.9 and let $C_\alpha = \bigcup \{\mathrm{Cu}\,[\sigma]: \sigma < \alpha\}$ for all $\alpha \leqslant \omega_1$; then $[C_\alpha] < [C_\beta]$ for all $\alpha < \beta \leqslant \omega_1$. Note that the class $[C]$ of a thin set C is not necessarily less than the class $[D]$ of a thick set.

Hrbacek [10] describes another ordering, that of "boldface relative recursiveness", which also has interesting properties if (V = L) is assumed. As with Borel equivalence, two sets with different positions in the ordering cannot be Borel isomorphic if (V = L). However, the two orderings are in general quite different. For example, let T be uncountable but thin. Then T and $T\times J$ have the same (Kleene) degree under relative boldface recursiveness, but are clearly not Borel equivalent. It can be seen that the sets Ad and W, which are Borel equivalent, will not have the same Kleene degree if V = L.

Finally, we consider the problem of Borel isomorphisms between analytic sets. It is clear that two analytic sets A and B are extrinsically Borel isomorphic if and only if their complements are extrinsically Borel isomorphic. Thus, if all reals are constructible, then there exist nice families of non-extrinsically-Borel isomorphic analytic sets. In fact, it can be shown that these analytic sets are also not intrinsically isomorphic.

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