A SURVEY OF RESULTS AND PROBLEMS CONCERNING ORTHOGONAL TRANSITION KERNELS

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Recently a number of notions of orthogonality for transition kernels have been studied not only for intrinsic interest in the structure of orthogonal transition kernels but also because these types of kernels appear from other considerations.

We will now give the definitions of some of these notions of orthogonality. Our general setting is the following. Let X and Y be Polish spaces and (μ_x) a transition kernel from X to Y. Thus, for each x in X, μ_x is a probability measure defined on the Borel subsets of Y and for each Borel subset E of Y the function $x \to \mu_x(E)$ is a Borel measurable map of X into $I = [0,1]$.

The kernel (μ_x) is said to be pairwise orthogonal (p.o.) provided that if $x \neq x'$, then μ_x and $\mu_{x'}$ are mutually singular (or orthogonal).

The kernel is said to be uniformly orthogonal (u.o.) provided there is a Borel subset B of $X \times Y$ such that for each x, $\mu_x(B_x) = 1$ and if $x \neq x'$, then $\mu_{x'}(B_x) = 0$; (μ_x) is said to be modified uniformly orthogonal (m.u.o.) if the set B is not required to be a Borel set.

A kernel (μ_x) is said to be completely orthogonal (c.o.) provided there is a Borel set B in $X \times Y$ such that for each x in X, $\mu_x(B_x) = 1$ and if $x \neq x'$, then $B_x \cap B_{x'} = \emptyset$. The set B is said to

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completely separate (μ_x). The kernel (μ_x) is modified completely orthogonal (m.c.o.) if the requirement that B be a Borel set is dropped.

Certainly, we have the following implications. (c.o.) → (µ.o.) → (p.o.); (m.c.o.) → (m.u.o.) → (p.o.); (c.o.) → (m.c.o.) and (m.u.o.) → (p.o.).

We also know that some of the possible converses do not hold. R. J. Gardner [1] gave an example of a transition kernel (μ_x) from I to I x I which is uniformly orthogonal and such that if E is a Borel subset of X and B is a Borel set in E x (I x I) which completely separates $(\mu_x)_{x \in E}$, then the Lebesgue measure of E = 0. Thus, this transition kernel is (µ.o.) but its restriction to any Borel subset E of X = I with positive Lebesgue measure fails to be (c.o.). Fremlin [2] has shown that it is consistent with ZF that there be a uniformly orthogonal kernel which is not modified completely orthogonal.

Our first unsolved problems are the following.

PROBLEM 1. Is a modified uniformly orthogonal kernel necessarily a uniformly orthogonal kernel?

PROBLEM 2. Is a modified completely orthogonal kernel a completely orthogonal kernel?

PROBLEM 3. Let M be a maximal set of pairwise orthogonal atomless probability measures on I. Can M be an analytic set? Can M be (modified) uniformly orthogonal? (The second question had been asked independently by D. Maharam at the conference).

Concerning Problem 3, we consider the space P of probability measures to be provided with the narrow topology. The idea behind Problem 3 is that since one must seemingly use some form of the axiom of choice to obtain such a set M, M cannot be analytic.

R. J. Gardner's example also shows that the following "Cantorian" theorem of Burgess and Mauldin [3] is the best possible.

THEOREM 1. Let (μ_x) be a transition kernel from X to Y. Then either

(1) every subset K of X such that $\{\mu_x = x \in K\}$ is pairwise orthogonal is countable, or

(2) there is a compact perfect subset M of X such that $\{\mu_x = x \in M\}$ is completely orthogonal.

Thus, the Cantorian theorem states that if a kernel is p.o. and X is uncountable, then this kernel restricted to some compact perfect set is c.o.. Gardner's example shows that the set M cannot be taken to be very large.

We would also like to point out that the Cantorian theorem was demonstrated in response to a problem of D. Maharam. Maharam in studying a problem of Tweedie, used CH to construct a pairwise orthogonal family of probability measures on I x I, $\{\mu_x : x \in I\}$, such that no uncountable subfamily is uniformly orthogonal. Her family was not a transition kernel and she asked whether one could find a transition kernel with these properties [4]. The Cantorian theorem shows that the answer is no.

There is exactly one isomorphism class of atomless transition kernels. A canonical example of such a kernel follows. Let λ be Lebesgue measure on I and let $\nu : I \times B(I \times I) \to [0,1]$ by $\nu_t(E) = (E_t \times \lambda)(E) = \lambda(E_t)$. The transition kernel ν is completely separated by the Borel set $B = \{(t,(t,s)) \in I \times I \times I = (t,s) \in I \times I\}$.

Now, we say that two transition kernels $\mu : X \times \mathcal{B}(Y) \to [0,1]$ and $\eta : X' \times \mathcal{B}(Y') \to [0,1]$ are isomorphic provided there are Borel isomorphisms f of X onto X′ and g of Y into Y′ such that $\mu_x(B) = \eta_{f(x)}(g(B))$. The v. Neumann type isomorphism theorem is the following.

THEOREM 2 [1]. Let X and Y be uncountable Polish spaces. Let (μ_x) be an atomless completely orthogonal transition kernel from X to Y. Let B be a Borel subset of X×Y which completely separates (μ_x). Then for each Borel isomorphism ψ of X onto I there is a Borel isomorphism ϕ of $\pi_Y(B)$ onto I×I such that for each x in X the set B_x is mapped under ϕ onto $\{\psi(x)\} \times I$ and $\varepsilon_{\psi(x)} \otimes \lambda$ is the image measure of μ_x under ϕ.

There are yet a number of other notions of orthogonality. Perhaps the most important of these is that of being "orthogonality preserving." A kernel (μ_x) is said to orthogonality preserving provided that if p and q are orthogonal probability measures on B(X), then the mixture μ^p of p is orthogonal to the mixture μ^q of q where, for any probability measure m or B(X), μ^m is a probability measure on B(Y) defined by:

$$\mu^m(E) = \int_X \mu_x(E)\, dm(x),$$

It is easy to see that if (μ_x) is completely orthogonal then (μ_x) is orthogonality preserving. However, an example is given in [1] to show that the converse does not hold. The construction of this example is very involved. We formulate the following problem.

PROBLEM 4. Every pairwise orthogonal measurable family of translates of Weiner measure on C([0,1]) forms an orthogonality preserving kernel. Is such a kernel necessarily completely orthogonal?

On the other hand, Mokobodzki has shown that if (μ_x) is orthogonality preserving, then (μ_x) is "universally measurably" completely separated, at least under certain set theoretical assumptions.

THEOREM 3[1]. Let (μ_x) be orthogonality preserving. If a medial limit exists, then there is a universally measurable map $\tau : Y \to X$ such that (μ_x) is separated by the fibers of τ.

PROBLEM 5. Is Mokobodzki's result true without any additional set theoretical assumptions?

It is also known that being orthogonality preserving is stronger than being uniformly orthogonal.

THEOREM 4[1] If (μ_x) is orthogonality preserving, then (μ_x) is uniformly orthogonal.

The proof of Theorem 4 involves the problem of separating two sets of probability measures on X, M and N, such that if $\mu \in M$ and $\nu \in N$, then μ and ν are orthogonal. The general problem is this. Given M and N, is there a Borel subset B of X such that if $\mu \in M$ and $\nu \in N$, $\mu(B) = 1 = \nu(X - B)$? The separation theorem used to prove Theorem 4 is the following.

THEOREM 5 [1]. Let M and N be two convexly analytic sets of probability measures on X such that $\mu \perp \nu$ whenever $\mu \in M$ and $\nu \in N$. If one of the two sets is a countable union of compact convex sets, then there is a Borel set S in X such that $\mu(S) = 1$, for all $\mu \in M$ and $\nu(S) = 0$ for all $\nu \in N$.

We recall that a subset M of P(X) is said to be convexly analytic provided there is an upper semicontinuous correspondence ϕ of a Polish space T onto M such that for each compact set C of T, there is a compact convex set K such that $\phi(C) \subset K \subset M$.

The first example of orthogonal Borel subsets M and N of P(X) which cannot be separated by Borel sets seems to have been given by Dubins and Freedman[8]. We formulate the following problem.

PROBLEM 6. Can each pair of orthogonal measurable measure convex sets of probability measures on [0,1] be separated by a set from the smallest σ-algebra containing the Borel sets and closed under operation A (these sets are known as the C-sets of Selivanovskii)?

Finally, one can formulate a number of other notions of orthogonality. Some of these are discussed in [1] and in [5]. One of the more interesting questions concerning these notions is the following.

PROBLEM.6. Is there any analytical condition for a kernel to be completely orthogonal. For example, does the following property (G) imply complete orthogonality? Or, is it equivalent to orthogonality preserving?

(G) For every pair A, B of disjoint Borel subsets of X, the condition

$$\nu \leq \sup_{x \in A} \mu_x \qquad \text{and} \qquad \lambda \leq \sup_{x \in B} \mu_x$$

implies $\nu \perp \lambda$.

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