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ON THE BOREL CLASS OF THE DERIVED SET OPERATOR

BY

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RÉSUMÉ. — Soit X une espace topologique métrisable compact, 2^X l'espace topologique des compacts de X avec la topologie de Hausdorff et soit D la dérivation de Cantor. Kuratowski a démontré que D est borélienne et précisément de la deuxième classe, et a posé le problème de trouver la classe précise des dérivés successifs D^α. Nous démontrons que les classes précises sont non bornées dans ω_1; D^α n'est pas de la classe α et si λ est un ordinal de seconde espèce, alors D^λ est précisément de la classe $\lambda + 1$.

ABSTRACT. — KURATOWSKI showed that the derived set operator D, acting on the space 2^X of closed subsets of a metric space X, is a Borel map of class exactly two and posed the problem of determining the precise classes of the higher order derivatives D^α. We show that the exact classes of the higher derivatives D^α are unbounded in ω_1. In particular, we show that D^α is not of class α and that, for limit ordinals λ D^λ is of Borel class exactly $\lambda + 1$. The proof involves the construction of a sub-lattice $\mathcal{N}$ of the space of closed subsets of 2^N on which (1) both the union and intersection maps are continuous lattice homomorphisms, (2) D is a lattice homomorphism, and (3) the derived set order map is a lattice homomorphism, and (3) the derived set order map is a lattice homomorphism into ω_1.

Introduction

In this paper, we consider the Borel class of the derived set operator D and its transfinite iterates D^α, acting on the space 2^X of closed subsets of a metric space X. The study of this operator seems to have been initiated by KURATOWSKI [5]. In section one, we recount his result that the operator D is exactly of class two. Many years later, KURATOWSKI [8] posed the problem of determining the precise classes of the operators D^α (also known as the

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derivatives of higher order). We obtain in section one simple upper bounds for the classes of these operators.

The remainder of the paper deals primarily with the more difficult problem of finding some lower bounds on the complexity of these operators. In section two, we demonstrate that the exact classes of the operators D^α are unbounded in ω_1.

In sections four and six, we prove that, for each countable ordinal α, the iterated derived set operator D^α is not of Borel class α. Combined with results from section two, this shows that for limit ordinals λ, D^λ is exactly of class $\lambda + 1$. Section four contains the finite case and section six the infinite case; the two cases require slightly different methods.

We actually show, for each α, that the family $(D^\alpha)^{-1}(\{\varnothing\})$ of closed subsets F of the Cantor set (2^N) such that $D^\alpha(F) = \varnothing$ is not both of additive and multiplicative class α. This follows from the construction, for each subset A of N^N of additive class α (if α is even) or multiplicative class α (if α is odd), of a continuous function H mapping N^N into the space of closed subsets of 2^N such that $A = H^{-1}((D^\alpha)^{-1}(\{\varnothing\}))$. The argument outlined here is easily accomplished for $\alpha = 1$. The proof then proceeds by transfinite induction on α. The induction step requires that the continuous mappings H_n constructed for sets A_n be nicely "stitched together" into mappings which will serve for $\cup A_n$ and $\cap A_n$.

Difficulties arise both in assuring the continuity of the stitched function H and in controlling the derived set order of the images $H(x)$. These difficulties are primarily due to two unfortunate facts : (1) The intersection map from $2^X \times 2^X$ into 2^X is not continuous; (2) The derived set operator D is not a lattice homomorphism on the lattice of closed subsets of X -- $D(F \cap G)$ does not always equal $D(F) \cap D(G)$.

To overcome these difficulties, we describe in section five a sublattice $\mathcal{N}$ of the space of closed subsets of 2^N where the behavior of various operators is more cooperative. In particular, both the union and intersection maps will be continuous lattice homomorphisms, the derived set operator D will be a lattice homomorphism and the derived set order map will be a lattice homomorphism from $\mathcal{N}$ into ω_1. In addition, a stitching operator from $\mathcal{N}^N$ into $\mathcal{N}$ will be defined which is continuous and which commutes with D. In effect, the stitching operator builds sets of higher derived set order and the operator D serves to unstitch the set constructed. In section six, we use this machinery to obtain lower bounds on the Borel classes of the operators D^α.

In section four, a slightly different stitching operator with similar properties is used to obtain better lower bounds on the Borel classes of the finitely iterated derived set operators D^n. The needed machinery is developed in section three.

Some open problems are stated in section seven.

It should be mentioned that the derived set operator has been studied recently as an important example of derivation [2, 3], as an inductive operator [1, p. 61] and as a classical operator [4, 11]. It has also played a useful role in selection theory [12].

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1. The Borel class of the derived set operator

Let (X, ρ) be an uncountable compact metric space. The space 2^X of closed subsets of X, provided with the exponential topology [2, p. 45] has a subbase of open sets of two types, for any open $V \subset X$:

$$C(V) = \{ F : F \subset V \}$$

and

$$I(V) = \{ F : F \cap V \neq \varnothing \}.$$

Note that each $C(V)$ is of the form $\bigcup_n I(V_n)^c$ and also each $I(V)$ is of the form $\bigcup_n C(V_n)^c$. It is easily seen that the space 2^X is also compact and metrizable.

The Borel class of a set or mapping may be defined as follows. Open sets are of additive class zero or $\sum_1^0$; closed sets are of multiplicative class zero or $\prod_1^0$. For any ordinal α, a set is of additive class α or $\sum_{\alpha+1}^0$ if it is a countable union of sets of Borel class $< \alpha$; similarly, a set is of multiplicative class α or $\prod_{\alpha+1}^0$ if it is a countable intersection of sets of class $< \alpha$. For limit ordinals λ, a set is $\sum_\lambda^0$ if it is $\sum_\alpha^0$ for some $\alpha < \lambda$. This differs from modern usage, where $\sum_\lambda^0 =$ our $\sum_{\lambda+1}^0$. Our notation is designed to agree with the definition of a map of class α. A mapping H is of Borel class α if $H^{-1}(V)$ is $\sum_{\alpha+1}^0$ for any open set V. A set or mapping is Borel class exactly α if it is of class α but not of any class $< \alpha$.

The derived set operator D maps 2^X into 2^X and is defined by

$$D(F) = F' = \{ x \colon x \in Cl(F - \{ x \}).$$

Note that $D(F)$ is also $\{x\colon F \cap V$ is infinite for any open V containing $x\}$. In this section we recall Kuratowski's theorem [5] that the operator D is of Borel class exactly two.

This means that (1) for any open $M \subset 2^X$, $D^{-1}(M)$ is a $G_{\delta\sigma}$ set and (2) for some open $M \subset 2^X$, $D^{-1}(M)$ is not an F_σ set.

To prove part (1), it suffices to consider only subbasic open sets.

LEMMA 1.1. — *For any open $V \subset X$, $\{F\colon F \cap V$ is finite$\}$ is an F_σ subset of 2^X.*

Proof. — Since $\{F\colon F \cap V \text{ is finite}\} = \bigcup_m \{F\colon |F \cap V| < m\}$, it is sufficient to show that $\{F\colon |F \cap V| \geqslant m\}$ is open for each m. Suppose now that $|F_0 \cap V| \geqslant m$; then are disjoint open subsets $V_1, \ldots, V_m$ of V such that $F_0 \cap V_i \neq \varnothing$ for each i. Let M be the open set $I(V_1) \cap I(V_2) \cap \ldots \cap I(V_m)$; then $F_0 \in M \subset \{F\colon |F \cap V| \geqslant m\}$, proving that the latter set is open. □

Now fix an open subset V of X and, for each n, let

$$U_n = \{x\colon \rho(x, X - V) < l/n\} \quad \text{and let}$$
$$V_n = \{x\colon \rho(x, X - V) > l/n\}.$$

Then

$$D^{-1}(C(V)) = \{F\colon F' \subset V\} = \bigcup_n \{F\colon F \cap U_n \text{ is finite}\}$$

and is an F_σ set by Lemma 1.1.

$$D^{-1}(I(V)) = \{F\colon F' \cap V \neq \varnothing\sum = \bigcup_n \{F\colon F \cap V_n \text{ is infinite}\}$$

and is therefore a $G_{\delta\sigma}$ set.

This shows that D is of Borel class 2; we next show that D is not of Borel class one.

First of all, notice that $\{\varnothing\} = \{F\colon F \subset \varnothing\} = \{F\colon F \cap X = \varnothing\}$ is both open and closed. If D were of Borel class 1 then $D^{-1}(\{\varnothing\})$ would have to be both F_σ and G_δ. Now $D^{-1}(\{\varnothing\}) = \{F\colon F \text{ is finite}\}$ and is therefore an F_σ set by Lemma 1.1. Also, $D^{-1}(\{\varnothing\})$ is dense (each nonempty $C(V)$ and $I(V)$ clearly contain finite sets).

Now suppose that X is perfect, that is, $D(X) = X$. Then

$$2^X - D^{-1}(\{\varnothing\}) = \{F\colon F \text{ is infinite}\}$$

is also dense (each $I(V)$ contains X and each $C(V)$ contains some closed ball); this set is G_δ. If $D^{-1}(\varnothing)$ were also G_δ, then we would have disjoint dense G_δ sets, which is impossible in a compact space.

Finally, any uncountable space $X = P \cup S$ for some perfect P and countable S. The argument above leads now to disjoint dense G_δ sets in the closed subspace 2^P, which is again impossible.

This completes the proof of the following.

THEOREM 1.2. — *For any uncountable, compact metric space X, the derived set operator is of Borel class exactly two.*

The $\alpha'th$ iterate D^α of the derived set operator D may be defined for all ordinals α by letting $D^{\alpha+1}(F) = D(D^\alpha(F))$ for all α and $D^\lambda(F) = \cap \{ D^\alpha(F) \colon \alpha < \lambda \}$ for limit ordinals λ. (Of course $D^0(F) = F$.) One direction of Theorem 1.2 has a natural extension to all the iterates of D.

THEOREM 1.3. — *For any finite n* and any limit ordinal λ:

(a) *D^n is of Borel class $2n$;*

(b) *D^λ is of class $\lambda + 1$;*

(c) *$D^{\lambda+n}$ is of class $\lambda + 2n + 1$.*

Proof. — For $n = 1$, this is simply part (1) of Theorem 1.2. The remainder of the proof proceeds by transfinite induction. There are two cases.

(Successor): Let U be an open subset of 2^X. For any ordinal α,

$$(D^{\alpha+1})^{-1}(U) = (D^\alpha)^{-1}(D^{-1}(U)).$$

By the $(n = 1)$ case, $D^{-1}(U) = \bigcup_n \bigcap_m V(n, m)$ for some open sets $V(n, m)$. Then $(D^{\alpha+1})^{-1}(U) = \bigcup_n \bigcap_m (D^\alpha)^{-1}(V(n, m))$. Now if $\alpha = n$ and D^n is of Borel class $2n$, then each $(D^n)^{-1}(V(n, m))$ is of Borel class $2n$, so $(D^{n+1})^{-1}(U)$ is of Borel class $2(n + 1)$. Since U was arbitrary, the operator D^{n+1} is of Borel class $2(n + 1)$, which completes the proof of part (a). The prof of part (c) is similarly obtained when $\alpha = \lambda + n$.

(Limit): Let λ be a countable limit ordinal. By the induction hypothesis, we may assume that, for all $\alpha < \lambda$, D^α is of Borel class $< \lambda$. Recall that $D^\lambda(K) = \cap \{ D^\alpha(K) \colon \alpha < \lambda \}$, so that for any closed subset K of X and any open subset U of 2^X, $K \in (D^\lambda)^{-1}(U)$ if and only if $\cap \{ D^\alpha(K) \colon \alpha < \lambda \} \in U$. To show that D^λ is of Borel class $\lambda + 1$, it clearly suffices to show that $(D^\lambda)^{-1}(I(V))$ and $(D^\lambda)^{-1}(C(V))$ are both $\sum^0_{\lambda+1}$ for any open subset V of X. Now by compactness, $\cap \{ D^\alpha(K) \colon \alpha < \lambda \} \in C(V)$ if and only if $D^\alpha(K) \subset V$ for some $\alpha < \lambda$. So, $(D^\lambda)^{-1}(C(V)) = \cup \{ (D^\alpha)^{-1}(C(V)) \colon \alpha < \lambda \}$ and is of additive Borel class λ. Let $V = \bigcup_n M_n$, where each M_n is closed.

Then

$$K \in (D^\lambda)^{-1}(I(V)) \leftrightarrow (\exists n)\ D^\lambda(K) \cap M_n \neq \varnothing$$
$$\leftrightarrow (\exists n)\ (\forall \alpha < \lambda) \quad D^\alpha(K) \cap M_n \neq \varnothing.$$

The second equivalence follows from the compactness of X. Restating, we have

$$(D^\lambda)^{-1}(I(V)) = \bigcup_n \bigcap_{\alpha<\lambda} (D^\alpha)^{-1}(I(M_n)).$$

Thus, $(D^\lambda)^{-1}(I(V))$ is of additive Borel class $\lambda + 1$. Therefore, D^λ is a mapping of class $\lambda + 1$.

The remainder of this paper is devoted primarily to finding lower for the Borel classes of the mappings D^α.

2. The mappings D^α are of unboundel Borel class

In this section, we prove that when X is the Polish space 2^N there is no countable ordinal β such that each mapping D^α is of Borel class β.

LEMMA 2.1. — *(Sierpinski-Mazurkiewicz) For any analytic subset A of a Polish space X, there is a closed subset M of $X \times 2^N$ such that, for all x, $x \in A$ if and only if M_x is uncountable.* ($M_x = \{ y\colon (x, y) \in M \}$.) □

Let A be an analytic of a Polish space X and let the closed subset M of $X \times 2^N$ be given by Lemma 2.1. Define the upper semi-continuous map ψ from X into the space of closed subsets of 2^N by $\psi(x) = M_x$ [6, p. 58]. Since any closed set F is countable if and only if $D^\alpha(F) = \varnothing$ for some countable ordinal α, we now have:

$$(*) \qquad X - A = \bigcup_{\alpha<\omega_1} \psi^{-1}(D^\alpha)^{-1}(\{\varnothing\}).$$

LEMMA 2.2. — *The decomposition (*) satisfies the Boundedness Principle, that is, for any analytic subset E of $X - A$, there is a countable ordinal β such that $E \subset \psi^{-1}(D^\beta)^{-1}(\{\varnothing\})$.*

Proof. — Let $T = (E \times 2^N) \cap M$. Then T is analytic and, for all x, either $T_x = \varnothing$ or $T_x = M_x$ and $x \in X - A$, so that T_x is closed and countable. Thus for each x, $D^\beta(T_x) = \varnothing$ for some countable ordinal β_x, that is, T_x is scattered. Now by a theorem of the second author (Theorem L of [10]), there is a countable ordinal β such that $D^\beta(T_x) = \varnothing$ for all $x \in X$.

THEOREM 2.3. — *There is no countable ordinal* β *such that each mapping* D^α *is of Borel class* β.

Proof. — Suppose by way of contradiction that the Borel classes of the mappings D^α were bounded by the countable ordinal β. Let A be an analytic subset of $X = 2^N \times 2^N$ which is universal for the analytic subsets of 2^N. Then the sets $\psi^{-1}(D^\alpha)^{-1}(\{\varnothing\})$ in the decomposition (*) of $X - A$ would all be of Borel class $\beta + 1$. But this would now imply, since A is universal, that every Borel subset of 2^N is of Borel class $\beta + 1$, which is of course false. (This argument is given in Theorem 3 of [10]).

3. The first stitching operator

In this section, we study the action of the derived set operator D on the space $\mathscr{K}$ of closed subsets of 2^N. A needed characterization of the set of continuous maps from an arbitrary space into $\mathscr{K}$ is given. A countable subset S of 2^N is defined and the action of D on $\mathscr{K} \cap P(S)$ is described, where $P(S)$ is the family of subsets of S. A continuous stitching operator Φ is defined for sequences of sets from $P(S)$ and it is shown how the derived set order of the resulting stitched set may be determined from the orders of the components. (The derived set order $o(K)$ of a scattered set K is the least ordinal α such that $D^{\alpha+1}(K) = \varnothing$).

Recall that the space 2^N has a countable basis of clopen sets of the form $B(s) = \{x: (\forall i < k) \quad x(i) = e_i\}$, where $s = (e_0, e_1, \ldots, e_{k-1})$ is a finite sequence of 0s and 1s.

Let V be an open subset of 2^N; then $V = \cup\{B(s_n): n \in N\}$ for some sequence $\{s_n: n \in N\}$. Recall that the space $\mathscr{K}$ has a subbase of sets of the two forms $C(V) = \{F: F \subset V\}$ and $I(V) = \{F: F \cap V \neq \varnothing\}$. Now it follows from compactness that

$$C(V) = \cup\{C(B(s_0) \cup B(s_1) \cup \ldots \cup B(s_n)): n \in N\}. \tag{3.1}$$

Of course it will always be true that

$$I(V) = \cup\{I(B(s_n)): n \in N\}. \tag{3.2}$$

Thus in fact $\mathscr{K}$ has a subbase of sets of the form $I(V)$ and $C(V)$, where V is *clopen*. Also, since the sets $C(V)$ and $I(2^N - V)$ are complements, these subbasic open sets are actually clopen.

Now let V be a clopen subset of 2^N and let

$$2^N - V = B(s_1) \cup B(s_2) \cup \ldots \cup B(s_n).$$

Since $F \subset V$ if and only if $F \cap (2^N - V) = \varnothing$, we have $C(V) = \mathscr{K} - I(2^N - V)$. That is,

$$\mathscr{K} - C(V) = I(B(s_1)) \cup \ldots \cup I(B(s_n)). \tag{3.3}$$

Equations (3.1, 2 and 3) can now be combined to yield

LEMMA 3.4. — *Let H map the topological space X into $\mathscr{K}$. Then H is continuous if and only if $H^{-1}(I(B(s)))$ is clopen for every finite sequence s of "0"s and "1"s.*

For any (finite or infinite) sequence x of "0"s and "1"s, let $\{m_0, m_0 + m_1 + 1, m_0 + m_1 + m_2 + 2, \ldots\}$ enumerate $\{n: x(n) = 1\}$; then x will be coded by the sequence $Qn(x) = \langle m_0, m_2, m_2, \ldots \rangle$. (Slanted brackets "$\langle \ldots \rangle$" will always indicate such a code.) The coded sequence

$$u = \langle u(0), u(1), \ldots \rangle$$

is said to be a subsequence of $v = \langle v(0), v(1), \ldots \rangle$ if there is an increasing function f mapping the domain of u into the domain of v such that $u(n) = v(f(n))$ for all n; this is written $u < v$.

Define the countable subset S of 2^N to be

$$\{x: (\exists m)(\forall n > m)\ x(n) = 0\}.$$

Then for $x \in 2^N$, $Qn(x)$ is finite if and only if $x \in S$. *An element of S will usually be identified with its code.* If $x(n) = 0$ for all n, then $Qn(x) = \langle\,\rangle$; x is also denoted by 0. For any $s = \langle m_0, m_1, \ldots, m_{k-1} \rangle$ and

$$t = \langle n_0, n_1, \ldots, n_{l-1} \rangle, \text{ let } s * t = st = \langle m_0, \ldots, m_{k-1}, n_0, \ldots, n_{l-1} \rangle;$$

the length $l(S) = k$. For and $F \subset S$ and any $s \in S$, let $F[s] = \{t: st \in F\}$. It should be noted that the subsequence ordering $<$ on S does agree with the usual Kleene-Brouwer order.

The action of the derived set operator on $\mathscr{K} \cap P(S)$ is described by the following.

LEMMA 3.5. — *For any closed subset F of 2^N which is included in S, any $s, t \in S$ and any countable ordinal α:*

(a) $s \in D(F) \leftrightarrow (\forall m)(\exists n > m) F[s\langle n \rangle] \neq \varnothing$;

(b) $st \in D^\alpha(F) \leftrightarrow t \in D^\alpha(F[s])$.

Proof. — Part (*a*) just restates the definition of the derived set in terms of 2^N. Part (*b*) is proved induction on α. If $\alpha = 0$ or is a limit ordinal, (*b*) is obvious. Now suppose that (*b*) holds for α. Then

$$\begin{aligned} st \in D^{\alpha+1} &\leftrightarrow (\forall m)(\exists n > m)\ D^{\alpha}(F)[st\langle n\rangle] \neq \varnothing \\ &\leftrightarrow (\forall m)(\exists n > m)\ (D^{\alpha}(F[s]))[t\langle n\rangle] \neq \varnothing \\ &\leftrightarrow t \in D^{\alpha+1}(F[s]). \end{aligned}$$

The first and last equivalences come from part (*a*); the middle equivalence is by the induction hypothesis.

For each $n \in N \cup \{-1\}$, there is a canonical subset C_n of S having derived set order n: $C_n = \{s \in S\colon l(s) \leqslant n\}$. It can be seen that, for each n, C_n is closed, $D(C_{n+1}) = C_n$ and $o(C_n) = n$. (These last two follow from Lemma 3.7 below.) Also, $\cup\{C_n\colon n \in N\} = S$; S of course is not closed, since it is dense in 2^N.

Notice that, for each n, $C_{n+1} = \{\langle m\rangle s\colon m \in N \text{ and } s \in C_n\} \cup \{0\}$. We define the first stitching operator with this in mind.

DEFINITION 3.6. — *For any sequence* $\hat{F} = (F_0, F_1, \ldots)$ *of sets from* $\mathscr{K} \cap P(S)$, $\Phi(\hat{F}) = \{0\} \cup \{\langle n\rangle s\colon s \in F_n \text{ and } n \in N]$.

LEMMA 3.7. — *For any sequence* $\hat{F} = (F_0, F_1, \ldots)$ *of sets from* $\mathscr{K} \cap P(S)$ *and any natural number* k:

$$D^{k+1}(\Phi(\hat{F})) = \begin{cases} \Phi(D^{k+1}(F_0), D^{k+1}(F_1), \ldots) \text{ if } (\forall m)(\exists n > m)\ D^k(F_n) \neq \varnothing; \\ \Phi(D^{k+1}(F_0), D^{k+1}(F_1), \ldots) - \{0\} \text{ otherwise.} \end{cases}$$

Proof. — Let $F = \Phi(\hat{F})$ and note that $F[\langle n\rangle] = F_n$ for all n. We need to prove that $\langle n\rangle t \in D^{k+1}(F)$ if and only if $t \in D^{k+1}(F_n)$ and that $0 \in D^{k+1}(F)$ if and only if infinitely many $D^k(F_n)$ are nonempty. The first equivalence follows from Lemma 3.5 (*b*), since

$$\begin{aligned} \langle n\rangle t \in D^{k+1}(F) &\leftrightarrow 0 \in D^{k+1}(F[\langle n\rangle t]) \\ &\ 0 \in D^{k+1}(F_n[t]) \\ &\ t \in D^{k+1}(F_n). \end{aligned}$$

Restating, we now have

$$D^k(F)[\langle n\rangle] = D^k(F_n).$$

The second equivalence now follows from Lemma 3.5 (*a*), since

$$0 \in D^{k+1}(F) = D(D^k(F)) \leftrightarrow (\forall m)(\exists n > m)\ D^k(F)[\langle n \rangle] \neq \varnothing$$
$$\leftrightarrow (\forall m)(\exists n > m)\ D^k(F_n) \neq \varnothing. \quad \square$$

It follows from this lemma that if each F_n has finite derived set order and if $F_0 \subset F_1 \subset F_2 \subset \ldots$, then

$$\circ(\Phi(F_0, F_1, \ldots)) = \sup\{\circ(F_n) + 1 : n \in N\}.$$

This fact and the Lemma above could be extended into the transfinite; however, we are only interested in the finite case.

The continuity of the first stitching operator is given by

LEMMA 3.8. — *Let* $(H_0, H_1, \ldots)$ *be a sequence of continuous functions mapping a topological space* X *into the space of closed subsets of* 2^N *such that each* $H_n(x) \subset S$. *Then the function* H, *defined by*

$$H(x) = \Phi((H_0(x), H_1(x), \ldots)),$$

is also continuous.

Proof. — Recall from Lemma 3.4 that H is continuous if and only if $H^{-1}(I(B(s)))$ is clopen for any finite sequence s of 0s and 1s. Thus we may assume that each $H_n^{-1}(I(B(s)))$ is clopen. There are two cases.

(i) If $s = 0^n$ for some n, then $H^{-1}(I(B(s))) = X$.

(ii) If $s = 0^n 1 t$ for some n and t, then

$$H^{-1}(I(B(s))) = H_n^{-1}(I(B(t))).$$

It follows that H is continuous.

4. D^n is not of Borel class n

Recall from the proof of Theorem 1.2 that for any uncountable compact metric space X, the family $D^{-1}(\{\varnothing\})$ of finite subsets of X is an F_σ but not a G_δ subset of the space 2^X of closed subsets of X. If X is the Cantor set 2^N, then $D^{-1}(\{\varnothing\}) \cap S = C_0$, where S and C_0 are defined above in section three. In this section, we show that $D^{-1}(\{\varnothing\})$ (and C_0) are *universal* F_σ sets, that is, for any F_σ subset B of N^N, there is a continuous function H mapping N^N into $\mathscr{K}$ such that $B = H^{-1}(D^{-1}(\{\varnothing\}))$ — B is said to be *reducible* to $D^{-1}(\{\varnothing\})$. Similarly, we show that every set of additive class $k+1$ (if k is even) or multiplicative class $k+1$ (if k is odd) is reducible to $(D^{k+1})^{-1}(\{\varnothing\})$. It will follow from this result that D^{k+1} cannot be of Borel class $k+1$.

PROPOSITION 4.1. — *For any F_σ subset B of N^N, there is a continuous function H mapping N^N into $\mathscr{K}$ such that, for all x, $x \in B$ if and only if $H(x)$ is finite; furthermore, each $H(x) \subset C_1$.*

Proof. — Suppose that $x \in B \leftrightarrow (\exists m)(\forall n)\, R_{m,n}(x)$, with each $R_{m,n}$ clopen; we assume without loss of generality that each $R_{m,n} \subset R_{m+1,n}$. Let

$$H(x) = \{\langle 2^m 3^n \rangle \colon \neg\, R_{m,n}(x) \text{ and } (\forall i < n) R_{m,i}(x)\}.$$

It is clear that $H(x) \subset C_1$ and that, for each m, at most one $\langle 2^m 3^n \rangle$ belong to $H(x)$. Now suppose first that $x \in B$ and choose m such that $(\forall n)\, R_{m,n}(x)$. Then for any $p \geqslant m$, no $\langle 2^p 3^n \rangle$ can belong to $H(x)$. It follows that $H(x)$ contains at most m element and is therefore finite. Suppose now that $x \notin B$; for each m, let $n(m)$ be the least n such that $\neg\, R_{m,n}(x)$. Then $H(x) = \{2^m 3^{n(m)} \colon m \in N]$ and is infinite.

It remains to be seen that H is continuous. Recall from Lemma 3.4 that H is continuous provided that each $H^{-1}(I(B(s)))$ is clopen. There are three cases. If $s = 0^p$ for some p, then $H^{-1}(I(B((s))) = N^N$. If $s = 0^p 1$, where $p = 2^m 3^n$, then $H^{-1}(I(B(s))) = (R_{m,0} \cap R_{m,1} \cap \ldots \cap R_{m,n-1}) - R_{m,n}$. Otherwise, $H^{-1}(I(B(s))) = \varnothing$. $\square$

Restating the conclusion of Proposition 4.1, we have

$$B = H^{-1}(D^{-1}(\{\varnothing\})).$$

Now if D were of Borel class one, then, since $\{\varnothing\}$ is clopen, B would have to be both F_σ and G_δ. However, it is well know that there exist subsets B of N^N which are F_σ but not G_δ (see KURATOWSKI and MOSTOWSKI [9], p. 425). This is an alternative proof that D is not of Borel class one. More generally, we need the following from [9].

PROPOSITION 4.2. — *For any countable ordinal α, there exists a subset of N^N which is of additive Borel class α but not of multiplicative class α.* $\square$

THEOREM 4.3. — *For any natural number k and any subset A of N^N which is $\sum^0_{k+2}$ (if k is even) or $\prod^0_{k+2}$ (if k is odd), there is a continuous function H mapping N^N into the space of closed subsets of 2^N such that, for all x,*

(a) $H(x) \subseteq C_{k+1}$;

(b) $x \in A$ *if and only if* $D^{k+1}(H(x)) = \varnothing$

(c) $x \notin A$ *if and only if* $D^{k+1}(H(x)) = \{0\}$.

Proof. — For $k = 0$, this is just a restatement of Proposition 4.1. The proof now proceeds by induction on k. Suppose therefore that the result holds for $k - 1$ and that either

(i) k is even and $A = \bigcup_n A_n$, where each $A_n \subset A_{n+1}$ and is $\prod_{k+1}^0$; or

(ii) k is odd and $A = \bigcap_n A_n$, where each $A_n \supset A_{n+1}$ and is $\sum_{k+1}^0$. In either case, we have

$$x \in A \leftrightarrow \{n\colon x \notin A_n\} \text{ is finite.}$$

By the induction hypothesis, there are continuous maps H_n such that each $H_n(x) \subset C_k$, and

$$x \in A_n \leftrightarrow D^k(H_n(x)) = \varnothing$$

and

$$x \notin A_n \leftrightarrow D^k(H_n(x)) = \{0\}.$$

Now let $H(x) = \Phi((H_0(x), H_1(x), \ldots))$ for all $x \in N^N$. It follows from Definition 3.6 that $H(x) \subset C_{k+1}$; this implies that $D^{k+1}(H(x))$ is either $\{0\}$ or $\varnothing$.

Suppose now that $x \in A$. Then $\{n\colon x \notin A_n\}$ is finite, so that $D^k(H_n(x)) \neq \varnothing$ for only finitely many n. It follows from Lemma 3.7 that

$$D^{k+1}(H(x)) = \Phi((\varnothing, \varnothing, \varnothing, \ldots)) - \{0\} = \varnothing.$$

Suppose next that $x \notin A$. Then infinitely many $D^k(H_n(x)) \neq \varnothing$, so by Lemma 3.7

$$D^{k+1}(H(x)) = \Phi((\varnothing, \varnothing, \varnothing, \ldots) = \{0\}.$$

Finally, Lemma 3.8 implies that the map H is continuous. □

COROLLARY 4.4 — For all finite $k > 0$, the iterated derived set operator D^k is not of Borel class k.

Proof. — Suppose that k is odd and let A be a subset of N^N which is $\sum_{k+1}^0$ but not $\prod_{k+1}^0$, as given by Proposition 4.2. By Theorem 4.3, there is a continuous H such that

$$A = H^{-1}((D^k)^{-1}(\{\varnothing\})).$$

If D^k were of Borel class k, then A would have to be both $\sum_{k+1}^0$ and $\prod_{k+1}^0$. This contradiction establishes the fact that D^k is not of Borel class k. If k is even, the proof is the same except for replacing A with $N^N - A$, which is $\prod_{k+1}^0$ but not $\sum_{k+1}^0$.

Unfortunately, the methods of sections three and four cannot be directly extended to the transfinite. The proof that D^α is not of Borel class α comes in section six and depends on techniques developed in section five.

5. Normal sets

In this section, the family $\mathcal{N}$ of normal subsets of 2^N is defined and studied. It is shown that $\mathcal{N}$ is a sublattice of under union and intersection and that the derived set order map o is a lattice homomorphism from $\mathcal{N}$ onto ω_1. A stitching operator θ is defined for sequences of normal sets and it is shown how the derived set order of the resulting stitched set may be determined from the orders of the components. The sequence of canonical sets C_n of derived set order n is extended into the transfinite. A characterization of the set of continuous maps from an arbitrary space into $\mathcal{N}$ is given and is used to show that the union, intersection and stitching operations are all continuous over $\mathcal{N}$.

DEFINITION 5.1. — *A subset F of 2^N is said to be* normal *provided that F is closed, $F \subset S$ and for all $s, t \in S$ and all ordinals α:*

(1) *whenever $s < t$ and $t \in D^\alpha(F)$, then $s \in D^\alpha(F)$;*

(2) *whenever $s \in D^{\alpha+1}(F)$, then $(\exists m)(\forall n > m)\ s\langle n \rangle \in D^\alpha(F)$.*

The sets C_n defined in section three are all normal. Note that if F is normal, then $D^\alpha(F)$ is normal for all α and $0 \in D^{o(F)}(F)$.

LEMMA 5.2. — *If F and G are normal, then $F \cup G$ and $F \cap G$ are also normal; in addition, $o(F \cup G) = \max(o(F), o(G))$ and $o(F \cap G) = \min(o(F), o(G))$.*

Proof. — Suppose that F and G are normal. Then, in fact, for each ordinal α, we can show:

$$D^\alpha(F \cup G) = D^\alpha(F) \cup D^\alpha(G) \quad \text{and}$$
$$D^\alpha(F \cap G) = D^\alpha(F) \cap D^\alpha(G).$$

The lemma follows easily from these equalities, which are proven by induction on α. As usual, the argument is obvious when $\alpha = 0$ or α is a limit ordinal. Consider next the case $\alpha = 1$. Now for any sets F and G in any topological space, $D(F \cup G) = D(F) \cup D(G)$ and $D(F \cap G) \subset D(F) \cap D(G)$. Suppose now that $s \in D(F) \cap D(G)$. Since F and G are normal, there exist m_1 and m_2

such that $n > m_1$ implies $s\langle n\rangle \in F$ and $n > m_2$ implies $s\langle n\rangle \in G$. Let $m = \max(m_1, m_2)$; then $n > m$ implies $s\langle n\rangle \in F \cap G$. It follows that $s \in D(F \cap G)$.

Finally, consider the successor case. Suppose the desired equalities hold for the ordinal α. Then

$$\begin{aligned} D^{\alpha+1}(F \cap G) &= D(D^{\alpha}(F \cap G)) \\ &= D(D^{\alpha}(F) \cap D^{\alpha}(G)) \quad \text{(by the induction hypothesis)} \\ &= D^{\alpha+1}(F) \cap D^{\alpha+1}(G) \quad \text{(by case } \alpha = 1\text{).} \end{aligned}$$

The union argument here is similar. □

The second stitching operator acts on the infinite sequence $\hat{F} = (F_0, F_1, F_2, \ldots)$ of subsets of S according to

DEFINITION 5.3. — $\theta(\hat{F}) = \{x \in 2^N : (\forall p \in N)(\forall s \in S)(\langle p\rangle s < Qn(x) \rightarrow s \in F_p)\}$.

(*It is immediate that* $\Theta(\hat{F}) \subset \Phi(\hat{F})$ *for all* $\hat{F}$.)

Note that $0 \in \theta(\hat{F})$ for any $\hat{F}$, that $\langle m\rangle \in \theta(\hat{F})$ if and only if $0 \in F_m$ and that $\langle m, n\rangle \in \theta(\hat{F})$ if and only if $\langle n\rangle \in F_m$ and both $\langle m\rangle$ and $\langle n\rangle$ are in $\theta(\hat{F})$. In general, $\theta(\hat{F})$ is closed under subsequences and, if $\langle m_0, \ldots, m_{k-1}\rangle \in \theta(\hat{F})$, then $0 \in F_{m_i}$ for all $i < k$.

It is easy to see that, for all n, $\theta((C_n, C_n, \ldots)) = C_{n+1}$. Now let $C_\omega = \theta((C_0, C_1, C_2, \ldots))$; then

$$C_\omega = \{\langle m_0, \ldots, m_{k-1}\rangle : (\forall i < k)\ k - i - 1 \leqslant m_i\}.$$

It will follow from Proposition 5.8 below that C_ω has derived set order ω.

We will also show in Proposition 5.8 that if $\hat{F}$ is a sequence of normal sets, then $\theta(\hat{F})$ is also normal. We begin with the following.

LEMMA 5.4. — *For any sequence* $\hat{F} = (F_0, F_1, \ldots)$ *of sets from* $\mathscr{K} \cap P(S)$, $\theta(\hat{F})$ *also belongs to* $\mathscr{K} \cap P(S)$.

Proof. — $\theta(\hat{F}) = \bigcap_n \bigcap_{s \in S - F_n} \{x : \neg(\langle n\rangle s < Qn(x))\}$; since, for any t, $[x : t < Qn(x)\}$ is open, $\theta(\hat{F})$ is the intersection of closed sets and is therefore closed, even if the F_ns are not.

To see that $\theta(\hat{F}) \subset S$, suppose by way of contradiction that $x \in \theta(\hat{F})$ and $Qn(x) = \langle m_0, m_1, m_2, \ldots\rangle$ is infinite. Then for all k, $\langle m_1, m_2, \ldots, m_k\rangle \in F_{m_0}$; now since F_{m_0} is closed, we have $(Qn)^{-1}(\langle m_1, m_2, m_3, \ldots\rangle) \in F_{m_m}$. But this contradicts $F_{m_0} \subset S$. □

The usefulness of the operator θ lies in its connection with the properties that the sequence $(F_0, F_1, \ldots)$ possesses "in the limit". This is indicated by the following.

LEMMA 5.5 — *Let $\hat{F} = (F_0, F_1, \ldots)$ be a sequence from $\mathscr{K} \cap P(s)$ and suppose that only finitely many F_n are nonempty.*

Then $\theta(\hat{F})$ is finite, that is, $D(\theta(\hat{F})) = \varnothing$.

Proof. — Suppose that F_n is empty for all $n \geqslant m$. Then, for any $s = \langle m_0, m_1, \ldots, m_{k-1} \rangle \in \theta(\hat{F})$, each $m_i < m$ (since $0 \in F_{m_i}$). It follows from Lemma 3.5 (a) that $D(\theta(\hat{F})) = \varnothing$. □

The next lemma gives us control over $\theta(\hat{F})$ for more complicated sequences of normal sets.

LEMMA 5.6. — *Let $\hat{F} = (F_0, F_1, \ldots)$ be a sequence of normal sets such that infinitely many F_n are nonempty. Then $D(\theta(\hat{F})) = \theta(D(\hat{F}))$, where $D(\hat{F}) = (D(F_0), D(F_1), \ldots)$.*

Proof. — There are two directions.

$(\subset)$: Suppose $t \in D(\theta(\hat{F}))$. Then by Lemma 3.5

$$(\forall m)(\exists n > m)\theta(\hat{F})[t\langle n \rangle] \neq \varnothing.$$

Now by the definition of θ and its closure under subsequences, there are infinitely many n such that

$$(\forall p)(\forall s \in S)\,(\langle p \rangle s < t\langle n \rangle \rightarrow s \in F_p).$$

Now for any such n, any p and any $s \in S$, $\langle p \rangle s < t$ implies $\langle p \rangle s \langle n \rangle < t \langle n \rangle$, which implies $s\langle n \rangle \in F_p$. It follows that

$$(\forall p)(\forall s \in S)\,(\langle p \rangle s < t \rightarrow s \in D(F_p)).$$

Thus $t \in \theta(D(\hat{F}))$.

$(\supset)$: Suppose $t \in \theta(D(\hat{F}))$. First of all, since infinitely many F_n are nonempty, infinitely many $\langle n \rangle \in \theta(\hat{F})$, so that $0 \in D(\theta(\hat{F}))$. Thus we may assume that $t \neq 0$. Now by Definition 5.3,

$$(\forall p)(\forall s \in S)\,(\langle p \rangle s < t \rightarrow s \in D(F_p)).$$

Then by (2) of Definition 5.1, we obtain

$$(\forall p)(\forall s \in S)(\exists m)(\forall n > m)\,(\langle p \rangle s < t \rightarrow s\langle n \rangle \in F_p).$$

Since there are only finitely many subsequences $\langle p \rangle s$ of t, we can take the maximum of the required "m" and obtain

$$(\exists m)(\forall n > m)(\forall p)(\forall s \in S)\ (\langle p \rangle s < t \rightarrow s \langle n \rangle \in F_p).$$

Now fix n, m, p and $s \in S$ and suppose that $\langle p \rangle s < t \langle n \rangle$. There are two cases: (This is where $t \neq 0$ is used.)

(1) $s = s' \langle n \rangle$ and $\langle p \rangle s' < t$, in which case $s = s' \langle n \rangle \in F_p$ follows directly.

(2) $\langle p \rangle s < t$, in which case $s \langle n \rangle \in F_p$, so that $s \in F_p$ by normality. We have now shown

$$(\exists m)(\forall n > m)(\forall p)(\forall s \in S)\ (\langle p \rangle s < t \langle n \rangle \rightarrow s \in F_p).$$

By Definition 5.3, this implies

$$(\exists m)(\forall n > m)\ t \langle n \rangle \in \theta(\hat{F})$$

Finally, by Lemma 3.5, this implies that $t \in D(\theta(\hat{F}))$. □

LEMMA 5.7. — *Let $\hat{F} = (F_0, F_1, \ldots)$ be a sequence of normal sets and let $\gamma = \lim_{n \to \infty} \sup_{m > n} (o(F_m) + 1)$. Then for all ordinals $\alpha \leqslant \gamma$: $D^\alpha(\theta(\hat{F})) = \theta(D^\alpha(\hat{F}))$. Furthermore, $D^{\gamma+1}(\theta(\hat{F})) = \varnothing$.*

Proof. — The proof is by induction on α. There are three cases.

($\alpha = 1$): $\gamma \geqslant 1$ implies that infinitely many F_n are nonempty; thus Lemma 5.6 applies.

($\alpha + 1$): Suppose that $D^\alpha(\theta(\hat{F})) = \theta(D^\alpha(\hat{F}))$ and that $\alpha + 1 \leqslant \gamma$.

Then infinitely many $D^\alpha(F_n)$ are nonempty, so by Lemma 5.6

$$\theta(D(D^\alpha(\hat{F}))) = D(\theta(D^\alpha(\hat{F}))),$$

which equals $D^{\alpha+1}(\theta(\hat{F}))$ by the induction hypothesis.

(limit): Suppose that λ is a limit ordinal $\leqslant \gamma$ and that the equality holds for all $\alpha < \lambda$. There are two directions.

($\supset$): Suppose $t \in \theta(D^\lambda(\hat{F}))$. Now for all $\alpha < \lambda$, $D^\alpha(\hat{F}) \supset D^\lambda(\hat{F})$, so that $t \in \theta(D^\alpha(\hat{F}))$. Then, by the induction hypothesis, $t \in D^\alpha(\theta(\hat{F}))$ for all $\alpha < \lambda$. It follows that $t \in D^\lambda(\theta(\hat{F}))$.

($\subset$): Suppose $t \in D^\lambda(\theta(\hat{F}))$. Then for all $\alpha < \lambda$, $t \in D^\alpha(\theta(\hat{F}))$ by the induction hypothesis. By Definition 5.3, this means

$$(\forall \alpha < \lambda)(\forall p)(\forall s \in S)\ (\langle p \rangle s < t \rightarrow s \in D^\alpha(F_p)).$$

But then

$$(\forall p)(\forall s \in S)\,(\langle p \rangle s < t \rightarrow s \in D^{\lambda}(F_p)).$$

It follows that $t \in \theta(D^{\lambda}(\hat{F}))$.

Finally, notice that only finitely many $o(F_n) \geqslant \gamma$, so that only finitely many $D^{\gamma}(F_n) \neq \emptyset$. By Lemma 5.5, $\theta(D^{\gamma}(\hat{F}))$ is finite. But we have just shown that $\theta(D^{\gamma}(\hat{F})) = D^{\gamma}(\theta(\hat{F}))$. It follows that $D^{\gamma+1}(\theta(\hat{F})) = \emptyset$ as desired. □

PROPOSITION 5.8. — *Let $\hat{F} = (F_0, F_1, \ldots)$ be a sequence of normal sets and let $\gamma = \lim_{n\to\infty} \sup_{m>n} (o(F_m) + 1)$. Then $\theta(\hat{F})$ is normal and $o(\theta(\hat{F})) = \gamma$.*

Proof. — By Definition 5.3, two things are required for $\theta(\hat{F})$ to be normal: for all $r, t \in S$ and all ordinals α:

(1) whenever $r < t$ and $t \in D^{\alpha}(\theta(\hat{F}))$, then $r \in D^{\alpha}(\hat{F}))$;

(2) whenever $t \in D^{\alpha+1}(\theta(\hat{F}))$, then $(\exists m)(\forall n > m)\ t \langle n \rangle \in D^{\alpha}(\theta(F))$.

By Lemma 5.7, it suffices to prove these for $\alpha = 0$.

(1) Suppose $r < t \in \theta(\hat{F})$. By Definition 5.3, we have

$$(\forall p)(\forall s \in S)\,(\langle p \rangle s < t \rightarrow s \in F_p).$$

But $\langle p \rangle s < r$ implies $\langle p \rangle s < t$, so the same statement is true with "r" in place of "t". Against by Definition 5.3, $r \in \theta(\hat{F})$.

(2) Suppose $t \in D(\theta(\hat{F}))$. It follows from Lemma 5.5 that infinitely many F_n are nonempty. Then, by Lemma 5.6, $t \in \theta(D(\hat{F}))$. The desired conclusion now follows as in the proof of the second inclusion ($\supset$) of Lemma 5.6. □

We can now extend the family of canonical sets C_n of derived set order n into the transfinite. Recall that $C_{\omega} = \theta((C_0, C_1, C_2, \ldots))$. Now fix for each countable limit ordinal $\lambda > \omega$ an increasing sequence $\{\alpha_n : n \in N\}$ with $\sup\{\alpha_n : n \in N\} = \lambda$ and each $\alpha_n > \omega$. The sets C_{α} can now be defined uniformly by

DEFINITION 5.9. — (*a*) $C_{-1} = \emptyset$;

(b) *for any* α, $C_{\alpha+1} = \theta((C_{\alpha}, C_{\alpha}, C_{\alpha}, \ldots))$;

(c) *for any limit ordinal* λ, $C_{\lambda} = \theta((C_{\alpha_0}, C_{\alpha_1}, C_{\alpha_2}, \ldots))$, *where* $(\alpha_0, \alpha_1, \ldots)$ *is the fixed sequence corresponding to* λ.

The exact composition of the sets C_{α} depends on the particular family of sequences $(\alpha_0, \alpha_1, \ldots)$. However, the important properties of these sets do not so depend. The following is an easy application of Proposition 5.8.

PROPOSITION 5.10. — *For all countable ordinals* α, C_α *is normal and* $o(C_\alpha) = \alpha$. $\square$

We next consider the continuity of mappings into $\mathcal{N}$.

LEMMA 5.11. — *Let H map the topological space X into the family* $\mathcal{N}$ *of normal subsets of* 2^N. *Then H is continuous if and only if, for all* $t \in S$, $\{x\colon t \in H(x)\}$ *is clopen.*

Proof. — Recall from Lemma 3.4 that H is continuous if and only if $H^{-1}(I(B(s)))$ is clopen for every finite sequence s of "0"s and "1"s. Let $t = (t_0, t_1, \ldots, t_{k-1}, 0, 0, \ldots)$ be a typical element of S and let $s = (t_0, t_1, \ldots, t_{k-1})$. We claim that, for any normal set F,

$$t \in F \leftrightarrow F \cap B(s) \neq \varnothing.$$

The direction $(\rightarrow)$ is immediate, since $t \in B(s)$. For the other direction, suppose $r \in F \cap B(s)$; then $t < r$, so $t \in F$ by normality. It follows that $\{F\colon t \in F\} = I(B(s))$ and therefore $\{x\colon t \in H(x)\} = H^{-1}(I(B(s))$. Thus the family of sets of the form $\{x\colon t \in H(x)]$ and the family of sets of the form $H^{-1}(I(B(s))$ are identical, which completes the proof. $\square$

For any compact metric space X, Kuratowski showed that the union map from $2^X \times 2^X$ into 2^X is continuous [6; p. 166] and that the intersection map is upper semicontinuous [6; p. 180].

LEMMA 5.12. — *The intersection map from* $\mathcal{N} \times \mathcal{N}$ *into* $\mathcal{N}$ *is continuous.*

Proof. — By Lemma 5.11, it suffices to show that $\{(F, G)\colon t \in F \cap G]$ is clopen for all $t \in S$. But this set equals $(\{F\colon t \in F] \times \mathcal{N}) \cap (\mathcal{N} \times \{F\colon t \in F\})$ and is clopen by Lemma 5.11. $\square$

LEMMA 5.13. — *The stitching operator* θ *from* $\mathcal{N}^N$ *into* $\mathcal{N}$ *is continuous.*

Proof. — By Lemma 5.11, it suffices to show that

$$\{\hat{F} = (F_0, F_1, \ldots)\colon t \in \theta(\hat{F})\}$$

is clopen. But, by Definition 5.3, this set is the finite intersection over those $p \in N$ and $s \in S$ such that $\langle p \rangle s < t$ of the clopen sets $\{F\colon s \in F_p\}$.

Some remarks are probably in orders as to the necessity of different methods of proof for the finite and infinite iterations of the derived set operator.

First of all, we can show that the results of section four cannot be obtained using normal sets. In fact, as we will now demonstrate, even Proposition 4.1

fails if we require H to map into the family of normal sets. To see this, consider $S = \{x: (\exists m)(\forall n > m)\ x(n) = 0\}$ as a subset of N^N and suppose that H maps N^N continuously into $\mathcal{N}$ such that $x \in S$ if and only if $H(x)$ is finite. Now $x_0 = 0 \in S$, so $H(x_0)$ is finite; choose p_0 so that $\langle p_0 \rangle$ is not in $H(x_0)$. Then $H(x_0) \subset 2^N - \{\langle p_0 \rangle\}$; by continuity, there is some n_0 such that $B_0 = B(0^{n_0}) \subset H^{-1}(C(2^N - \{\langle p_0 \rangle\}))$; let $x_1 = \langle n_0 \rangle$. Suppose now that we have constructed $x_k = \langle n_0, n_1, \ldots, n_{k-1} \rangle$ and found $p_0 < p_1 < \ldots < p_{k-1}$ such that

$$B(0^{n_0}10^{n_1}1 \ldots 0^{n_{k-1}}) \subset H^{-1}(C(2^N - \{\langle p_0 \rangle, \langle p_1 \rangle, \ldots, \langle p_{k-1} \rangle\})).$$

Once again $x_k \in S$, so that $H(x_k)$ is finite; choose $p_k > p_{k-1}$ such that $\langle p_k \rangle$ is not in $H(x_k)$. By continuity, there is some n_k such that $B(0^{n_0}10^{n_1}1 \ldots 0^{n_k}) = B_k \subset H^{-1}(C(2^N - \{\langle p_0 \rangle, \langle p_1 \rangle, \ldots, \langle p_k \rangle\}))$. Finally, let $x = \lim_{k \to \infty} (x_k)$; by construction x is not in S and therefore $H(x)$ is infinite and so nonempty. Since $H(x)$ is normal, it follows that $0 \in D(H(x)$ and that all but finitely many $\langle p \rangle$ belong to $H(x)$. On the other hand, for all k, $x \in B_k$ and therefore $\langle p_k \rangle$ is not in $H(x)$. This contradiction establishes the original claim.

Here is an illustration of the difficulties which arise if one tries to apply the methods of sections three and four to infinite iterations of the derived set operator. Let $A_0, A_1, \ldots$ be an increasing sequence of subsets of N^N and let $\hat{F} = (F_0, F_1, F_2, \ldots)$ be a sequence of closed subsets of 2^N such that, for all n, $x \in A_n$ if an only if $o(F_n) = k$ and $o(F_n) = \omega$ otherwise. Then

$$o(\Phi(\hat{F})) = \begin{cases} k + 1, & \text{if } (\forall n)\ x \in A_n \\ \omega + 1, & \text{if } (\forall m)(\exists n > m)\ x \notin A_n \\ \omega, & \text{otherwise.} \end{cases}$$

Thus, if $x \in \cup \{A_n: n \in N\}$, then $o(\Phi(\hat{F}))$ could be either k or ω. This and other dichotomies prevent the easy extension of Theorem 4.3 into the transfinite. Thus we are led to the family of normal sets and the methods of this section.

6. The universality of the mapping D^α; the infinite case

In this section, we extend results (4.3) and (4.4) to infinite iterations of the derived set operator. This requires that only normal sets be used in the range of the continuous function H. As noted in section five, this leads to a weaker result for the finite levels. However, some improvements are also gained.

THEOREM 6.1. — *For any natural number k, any ordinal $\beta > k$ and any subset A of N^N which is $\sum_k^0$ or $\prod_k^0$, there is a continuous function H mapping N^N into the space of closed subsets of 2^N such that, for all x,*

(a) *$H(x) \subset C_\beta$;*

(b) *$H(x)$ is normal;*

(c) *$x \in A$ if and only if $o(H(x)) = k - 1$;*

(d) *$x \notin A$ if and only if $o(H(x)) = \beta$.*

Proof. — The proof is by induction on k. There are two cases. ($k = 0$) Given a clopen set A and an ordinal α, let

$$\mathrm{H(x)} = \begin{cases} C_\beta & \text{if } x \in A; \\ \varnothing & \text{if } x \notin A. \end{cases}$$

H has the desired properties by Proposition 5.10: H is continuous since, for any family V of closed subsets of 2^N, $H^{-1}(V)$ is either $\varnothing$, A, $N^N - A$ or N^N.

($k + 1$). Suppose that the result holds for k and that either

(i) $A = \bigcup_n A_n$, where each $A_n \subset A_{n+1}$ and is $\prod_k^0$, or

(ii) $A = \bigcap_n A_n$, where each $A_n \supset A_{n+1}$ and is $\sum_k^0$.

In either event, we have

$$x \in A \leftrightarrow \{ n\colon x \notin A_n \} \quad \text{is finite.}$$

There are two sub-cases.

(β a successor). In this sub-case, there are continuous maps H_n such that each $H_n(x)$ is normal and a subset of $C_{\beta-1}$ and such that $o(H_n(x)) = k - 1$ if $x \in A_n$ and $o(H_n(x)) = \beta - 1$ if $x \notin A_n$. Let H be defined by

$$(*) \qquad H(x) = \Theta((H_0(x), H_1(x), H_2(x), \ldots).$$

H is continuous by Proposition 5.13; each $H(x)$ is included in C_β by Definition 5.9 and is normal by Proposition 5.8. Suppose now that $x \in A$. Then $\{ n\colon x \notin A_n \}$ is finite, so that $o(H_n(x)) = k - 1$ for all but finitely many n. It follows that $\lim_{n\to\infty} \sup_{m>n} (o(H_n(x)) + 1) = k$. Thus $o(H(x)) = k$ by Proposition 5.8. Suppose on the other hand that $x \notin A$. It follows that $\lim_{n\to\infty} \sup_{m>n} (o(H_n(x)) + 1) = \beta$, so that $o(H(x)) = \beta$.

(β a limit) First of all, let $(\alpha_0, \alpha_1, \ldots)$ be the fixed sequence of ordinals with supremum β. For $\alpha_n \leqslant k$, let $H_n(x) = \varnothing$ for all x. For each $\alpha_n > k$, there is a continuous map H_n such that each $H_n(x)$ is normal and included in C_{α_n} and such that $o(H_n(x)) = k$ if $x \in A_n$ and $o(H_n(x)) = \alpha_n$ if $x \notin A_n$. Once again the map H is defined by (*). For $x \in A$, the argument is the same. For $x \notin A$, it follows that $o(H_n(x)) = \alpha_n$ for all but finitely many n. Thus $o(H(x)) = \sup (\alpha_n \dot{+} 1) = \beta$. □

We turn now to the infinite analogue of the preceding theorem, which returns to the alternating form of Theorem 4.3.

THEOREM 6.2. — *For any countable limit ordinal* λ, *any natural number* k, *any countable ordinal* $\beta > \lambda + k$ *and any subset* A *of* N^N *which is* $\sum^0_{\lambda+k+1}$ *(if* k *is even) or* $\prod^0_{\lambda+k+1}$ *(if* k *is odd), there is a continuous function* H *mapping* N^N *into the space of closed subsets of* 2^N *such that, for all* x,

(a) $H(x) \subset C_\beta$;

(b) $H(x)$ *is normal;*

(c) $x \in A$ *if and only if* $o(H(x)) < \lambda + k$;

(d) $x \notin A$ *if and only if* $o(H(x)) = \beta$.

Proof. — The proof is by induction on $\lambda + k$. There are three cases: $k = 0$ and $\lambda = \omega$, $k = 0$ and $\lambda > \omega$ and k a successor. The proof of the successor case is virtually identical with the proof of that case in Theorem 6.1. The details are left to the reader. We now present the proofs of the other two cases.

($\lambda = \omega$). Suppose now that A is $\sum^0_{\omega+1}$ and that $\beta > \omega$. Then, without loss of generality, $A = \bigcup_n A_n$, where, for each n, A_n is $\prod^0_n$ and $A_n \subset A_{n+1}$. As in Theorem 6.1, there are two subcases.

Suppose first that β is a successor. Then by Theorem 6.1 there are continuous maps H_n such that each $H_n(x)$ is normal and a subset of $C_{\beta-1}$ and such that

$$o(H_n(x)) = \begin{cases} n-1 & \text{if } x \in A_n \\ \beta - 1 & \text{if } x \notin A_n. \end{cases}$$

For each n, let

$$I_n(x) = \bigcap\nolimits_{m \leq n} H_m(x)$$

and let

$$H(x) = \theta((I_0(x), I_1(x), \ldots)).$$

It follows from results (5.12) and (5.13) that H is continuous. Each $H(x)$ is included in C_β by Definition 5.9 and is normal by results (5.2) and (5.8).

Suppose now that $x \in A$ and let m be the least integer such that $x \in A_m$. Then, using Lemma 5.2, $o(I_n(x)) = m - 1$ for all $n \geqslant m$. It follows from Proposition 5.8 that $o(H(x)) = m < \omega$.

Suppose on the other hand that $x \notin A$. Then, again using Lemma 5.2, $o(I_n(x)) = \beta - 1$ for all n, so that $o(H(x)) = \beta$.

Now suppose that β is a limit and let $(\beta_0, \beta_1, \ldots)$ be the fixed sequence of ordinals with supremum β; recall that each $\beta_n > \omega$. Let H_n be given by the successor argument so that

$$o(H_n(x)) = \begin{cases} m & \text{if } m \text{ is least such that } x \in A_m. \\ \beta_n + 1 & \text{if } x \notin A. \end{cases}$$

Let $H(x) = \theta((H_0(x), H_1(x), \ldots))$. As above, H is continuous and each $H(x)$ is a normal subset of C_β. For $x \in A$, $o(H_n(x)) = m$ for all but finitely many n, where m is least such that $x \in A_m$; thus $o(H(x)) = m + 1$. For $x \notin A$, each $o(H_n(x)) = \beta_n + 1$, so that $o(H(x)) = \beta$. This completes the proof for the case $(\lambda = \omega)$.

$(\lambda > \omega)$. Suppose that $\beta > \lambda > \omega$, that A is $\sum_{\lambda+1}^0$, and that the theorem is true for all $\lambda' + k < \lambda$. Let $(\alpha_0, \alpha_1, \ldots)$ be the fixed sequence of ordinals with supremum λ. Then, without loss of generality, $A = \bigcup_n A_n$, where each A_n is $\sum_{\alpha_n+1}^0$ (if α_n is even) or $\prod_{\alpha_n+1}^0$ (if α_n is odd). Again there are two subcases. When β is a successor, the proof is the same as for $\lambda = \omega$, except that "$n - 1$" becomes "α_n", "$m - 1$" becomes "α_m" and "$o(H(x)) = m < \omega$" becomes "$o(H(x)) = \alpha_m < \lambda$".

Suppose now that β is a limit and let $(\beta_0, \beta_1, \ldots)$ be the fixed sequence with supremum β. Since $\beta > \lambda$, there is some k such that $\beta_n > \alpha_n$ for all $n > k$. For $n \leqslant k$, let $H_n(x) = \varnothing$ for all x. For $n > k$, let H_n be given by the successor argument so that

$$o(H_n(x)) = \begin{cases} \alpha_m & \text{if } m \text{ is least such that } x \in A_m; \\ \beta_n + 1 & \text{if } x \notin A. \end{cases}$$

Once again $H(x) = \theta((H_0(x), H_1(x), \ldots))$, H is continuous and each $H(x)$ is a normal subset of C_β. For $x \in A$, let m be least such that $x \in A_m$. Then $o(H_n(x)) = \alpha_m$ for all but finitely many n, so that $o(H(x)) = \alpha_m + 1 < \lambda$. For $x \notin A$, each $o(H_n(x)) = \beta_n + 1$, so that $o(H(x)) = \sup(\beta_n + 2) = \beta$. This completes the proof of Theorem 6.2. $\square$

COROLLARY 6.3. — *For any ordinal $\alpha > 0$, the iterated derived set operator D^α is not of Borel class α.*

Proof. — For finite α, this is given by Corollary 4.4. For infinite α, let $\alpha = \lambda + k$, where λ is a limit and k is finite and let $\beta = \alpha + 1$. Suppose k is even and let A be a subset of N^N which is $\sum^0_{\alpha+1}$ but not $\prod^0_{\alpha+1}$, as given by Proposition 4.2. By Theorem 6.2, there is a continuous H such that

$$A = H^{-1}((D^\alpha)^{-1}(\varnothing)).$$

The rest of the proof follows that of Corollary 4.4.

Combining this result with Theorem 1.3, we have the following.

COROLLARY 6.4. — *For all limit ordinals λ, the iterated derived set operator D is of Borel class exactly $\lambda + 1$.*

7. Some open questions

We would like to leave the reader with two problems connected with the above results.

PROBLEM 1. — *(Kuratowski). What is the exact class of the iterated derived set operator D^α when $\alpha > 1$ and is not a limit?*

PROBLEM 2. — *Is there a Borel operator D on the space of closed subsets of 2^N such that*

(1) *$D(F) \subset F$ for all F;*

(2) *for each F, there is a countable ordinal α such that $D^{\alpha+1}(F) = D^\alpha(F)$;*

(3) *for each countable α, there is an F such that $D^{\alpha+1}(F) \neq D^\alpha(F)$;*

(4) *the iterated operators D^α are of bounded Borel class?*

Note that (2) *follows from* (1).

Added in proof. We have recently refined the methods of this paper to show that D^n is of Borel class exactly 2^n and that $D^{\lambda+n}$ is of class exactly $\lambda + 2^n + 1$.

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