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CONTINUOUS ONE-TO-ONE PARAMETRIZATIONS

BY

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RÉSUMÉ. — Soient X et Y des espaces polonais. Soit F une multiapplication mesurable de X dans Y tel que (1) $\mathrm{Gr}(F)$, le graphe de F est borelien, et (2) pour tout x, $F(x)$ est une partie dense en elle-même G_δ de Y. Problème 1. Y a-t-il un isomorphisme borelien f de $X \times N^N$ sur $\mathrm{Gr}(F)$ tel que pour tout x, $f(x, .)$ est une fonction continue biunivoque de N^N sur $F(x)$? Nous avons une réponse affirmative si Y est l'espace des nombres réels. Problème 2. Si, pour tout x, $F(x)$ est 0-dimensionnel et toute partie compacte a un intérieur vide, y a-t-il un isomorphisme borélien f de $X \times N^N$ sur $\mathrm{Gr}(F)$ tel que pour tout x, $f(x, .)$ est un homéomorphisme de N^N sur $F(x)$? Nous avons une réponse affirmative si Y est 0-dimensionnel.

ABSTRACT. — Let X and Y be Polish spaces. Let F be a measurable multifunction from X into Y such that (1) the graph of F is Borel, (2) for each x, $F(x)$ is a dense-in-itself G_δ subset of Y. Problem 1. Is there a Borel isomorphism f of $X \times N^N$ onto $\mathrm{Gr}(F)$ so that for each $x, f(x, .)$ is a continuous one-to-one map of N^N onto $F(x)$? We obtain an affirmative answer in case Y is the reals. Problem 2. If for each x, $F(x)$ is 0-dimensional and has no compact relatively open subset is there a Borel isomorphism f of $X \times N^N$ onto $\mathrm{Gr}(F)$ so that for each $x, f(x, .)$ is a homeomorphism of N^N onto $F(x)$? We obtain an affirmative answer in case Y itself is 0-dimensional.

Let each of X and Y be a Polish space (= a separable topological space which admits a complete metric compatible with the topology). The Cantor set will be denoted by C and the set of irrationals by Σ. These spaces will be

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considered as $\{0, 1\}^N$ and N^N respectively, both with the product topology. A multifunction from X into Y is a map F with domain X and range a subset of the family of non-empty subsets of Y. A multifunction F is called measurable provided that for each open subset U of Y:

$$\{x \in X : F(x) \cap U \neq \emptyset\}$$

is a Borel subset of X. By the graph of F, $\mathrm{Gr}(F)$, is meant $\{(x, y) \in X \times Y : y \in F(x)\}$. By a Borel graph (=uniformization) in $X \times Y$, we mean a Borel subset Γ of $X \times Y$ such that for each x in X, $\Gamma_x \equiv \{y : (x, y) \in \Gamma\}$ consists of at most one point. By a Borel selector for a multifunction F we mean a Borel graph contained in $\mathrm{Gr}(F)$ (Note: This is usually called a partial selector for F). This paper concerns the following problem raised by Srivastava in [5] and some related ones.

PROBLEM 1. — Let F be a measurable multifunction from X into Y so that $\mathrm{Gr}(F)$ is a Borel set and for each x in X, $F(x)$ is a dense-in-itself G_δ subset of Y. Is there a Borel measurable map f from $X \times \Sigma$ onto $\mathrm{Gr}(F)$ such that for each x, $f(x, .)$ is a one-to-one continuous map of $\{x\} \times \Sigma$ onto $\{x\} \times F(x)$?

We will provide an affirmative answer to this problem in case Y is R, the space of all real numbers. The general problem remains open. Our arguments seem to depend heavily on the fact that R is one-dimensional (in the topological sense). For example it appears that our arguments do not directly extend to the case when Y is R^2.

We shall make the following conventions.

Set $\mathrm{Seq} = \bigcup \{N^k : k = 1, 2, 3, \ldots\}$. If $s = \langle s_1, \ldots, s_k \rangle \in \mathrm{Seq}$, then $lh(s) = k$ and if $i \in N$, $s \star i \equiv \langle s_1, \ldots, s_k, i \rangle$. If $\langle d_1, \ldots, d_p \rangle \in \{0, 1\}^p$, then:

$$C(\langle d_1, \ldots, d_p \rangle) \equiv \{\sigma \in C : \sigma | p = \langle d_1, \ldots, d_p \rangle\}.$$

We shall also make one further convention which is actually an abuse of notation. From this point on if f is a map from $X \times Z$ onto $B \subseteq X \times Y$, when we say $f(x, .)$ maps Z onto B_x we will actually mean that f maps $\{x\} \times Z$ onto $\{x\} \times B_x$.

Our first Theorem is a "parametrized" version of the characterization of those subsets of C which are homeomorphic to Σ.

THEOREM 1. — *Let X be a Polish space and C the Cantor set. Let $F : X \to C$ be a measurable multifunction such that* (1) *for all x, $F(x)$ is a dense-in-itself G_δ subset of C which has no compact, relatively open subsets and* (2)

$\mathrm{Gr}(F)$ *is Borel. Then there is a Borel isomorphism f from $X \times \Sigma$ onto* $\mathrm{Gr}(F)$ *such that for all x, $f(x, .)$ is a homeomorphism of Σ onto $F(x)$.*

Proof. – Let $G = \mathrm{Gr}(F)$. By [2], we know there is a decreasing sequence of Borel subsets of $X \times C$, $\{G_n\}_{n=1}^{\infty}$ each of which has open X-sections such that $\bigcap G_n = G$. For each positive integer k, let $\{V^k(n)\}_{n=1}^{\infty}$ be an enumeration of the clopen basis:

$$\{C(\langle d_1, \ldots, d_{k+p}\rangle) : \langle d_1, \ldots, d_{k+p}\rangle \in \{0, 1\}^{k+p} \text{ and } p \geqslant 0\}$$

in such a way that if $\langle d_1, \ldots, d_{k+n}\rangle$ properly extends $\langle d_1, \ldots, d_{k+m}\rangle$ then $C(\langle d_1, \ldots, d_{k+m}\rangle)$ is listed before $C(\langle d_1, \ldots, d_{k+n}\rangle)$. Thus, for each k, n, and m, if $m < n$, then either $V^k(n) \subseteq V^k(m)$ or $V^k(n) \cap V^k(m) = \emptyset$.

We will define, for each $s \in \mathrm{Seq}$, a function $f_s : X \to N$ such that

(1) f_s is Borel measurable;

(2) if $lh(s) = k$ and x is in X, $V^k(f_s(x)) \cap G_x \neq \emptyset$;

(3) for each k and x,

$$G_x \subseteq \bigcup \{V^k(f_s(x)) : 1h(s) = k\} \subseteq G_{kx};$$

(4) if $lh(s) = k = 1h(r)$, $s \neq r$, and $x \in X$, then $V^k(f_s(x)) \cap V^k(f_r(x)) = \emptyset$ and for each positive integer i, $V^{k+1}(f_{s*i}(x)) \subseteq V^k(f_s(x))$.

Assuming the functions f_s have been given, define $f : X \times \Sigma \to \mathrm{Gr}(F)$ by setting $f(x, \sigma) = (x, y)$ where:

$$\{y\} = \bigcap \{V^k(f_{\sigma|k}(x)) : k = 1, 2, 3, \ldots\}.$$

It can be checked that the function f satisfies the conclusion of Theorem 1.

Now, we proceed to construct the function f_s by induction (recursion).

Let

$$\begin{aligned} D_n^1 = \{x : \forall m, \forall (p_1, \ldots, p_m)\ [&V^1(p_1) \cup \ldots \cup V^1(p_m) \\ &\subseteq G_{nx} \text{ and } V^1(p_1) \cap G_x \neq \emptyset, \ldots, \\ &V^1(p_m) \cap G_x \neq \emptyset] \to \exists s [V^1(s) \cap (V^1(p_1) \cup \ldots \cup V^1(p_m)) = \emptyset, \\ &V^1(s) \subseteq G_{nx}, \text{ and } V^1(s) \cap G_x \neq \emptyset\}. \end{aligned}$$

In other words $x \in D_n^1$ if and only if G_x cannot be covered by finitely many basic clopen sets from $\{V^1(m) : m \geqslant 1\}$ each of which is contained in G_{nx}. Notice that $\bigcup \{D_n^1 : n \geqslant 1\} = X$. (If not, then there would be some x so that for each n, $G_x \subset K_n \subset G_{nx}$, where K_n is a finite union of clopen sets. This would imply $\bigcap K_n = G_x$ and G_x would be a compact subset of C.)

Since we will make a number of similar constructions, we will check in detail that each set D_n^1 is a Borel set. We have $X - D_n^1 = \bigcup B(m, p_1, \ldots, p_m)$, where for each $(m, p_1, \ldots, p_m)$

$$B(m, p_1, \ldots, p_m) = \bigcap \{ B(m, p_1, p_2, \ldots, p_m, s) : s \in N \},$$

where for each $(m, p_1, p_2, \ldots, p_m, s)$:

$$x \in B(m, p_1, \ldots, p_m, s) \leftrightarrow \begin{cases} (V^1(p_1) \cup \ldots \cup V^1(p_m)) \subseteq G_{nx} \\ \text{and:} \\ V^1(p_1) \cap G_x \neq \emptyset, \ \ldots, \ V^1(p_m) \cap G_x \neq \emptyset, \\ \text{and} \\ \text{either } (\exists i)\ [i \leqslant m \text{ and } V^1(p_i) \cap V^1(s) \neq \emptyset] \\ \text{or } V^1(s) \nsubseteq G_{nx} \quad \text{or} \quad V^1(s) \cap G_x = \emptyset. \end{cases}$$

Let $E(j) = \{ x : V^1(j) \subseteq G_{nx} \}$. We have

$$X - E(j) = \pi_1([(X \times C) - G_n] \cap (X \times V)).$$

Thus, $X - E(j)$ is the projection of a Borel subset of $X \times C$ each of whose sections is compact. Therefore, $E(j)$ is a Borel set [2]. Since F is measurable, $F(j) = \{ x : V^1(j) \cap G_x \neq \emptyset \}$ is a Borel set. Thus,

$$B(m, p_1, \ldots, p_m, s) = E(p_1) \cap \ldots \cap E(p_m) \cap F(p_1) \cap \ldots \cap F(p_s),$$

if there is some $i \leqslant m$ so that $V^1(p_i) \cap V^1(s) \neq \emptyset$ and

$$\begin{aligned} B(m, p_1, \ldots, p_m, s) \\ = [E(p_1) \cap \ldots \cap E(p_m) \cap F(p_1) \cap \ldots \cap F(p_m)] \\ \cap [\{ x : V^1(s) \nsubseteq G_{1x} \} \cup (X - F(s))], \end{aligned}$$

otherwise. In either case $B(m, p_1, \ldots, p_m, s)$ is a Borel set. Thus, each set D_n^1 is a Borel set.

Let

$$D_1 = D_1^1 \quad \text{and} \quad D_n = D_n^1 - \bigcup \{ D_m^1 : m < n \}, \quad \text{if } n > 1.$$

Let

$$H_1 = \bigcup \{ (D_n \times C) \cap G_n : n \in N \}.$$

We now proceed to define $f_{\langle m\rangle}$ for $m=1,2,3,\ldots$, by induction on m. For each n, let

$$T^1(n)=\{x : V^1(n)\subset H_{1x} \text{ and } V^1(n)\cap G_x\neq\emptyset\}.$$

Let

$$T(1)=T^1(1) \quad\text{and}\quad T(n)=T^1(n)-\bigcup\{T^1(m) : m<n\},\quad \text{if } n>1.$$

Set $f_{\langle 1\rangle}(x)=n$ if and only if $x\in T(n)$. Now suppose $f_{\langle k\rangle}$ has been defined for all $k<m$. Set

$$T^1(m, n)=\{x : V^1(n)\subseteq H_{1x}-\bigcup\{V^1(f_{\langle k\rangle}(x)) : k<m\}$$

and

$$V^1(n)\cap G_x\neq\emptyset\}.$$

Let

$$T(m, 1)=T^1(m, 1)$$

and

$$T(m, n)=T^1(m, n)-\bigcup\{T^1(m, p) : p<n\},\qquad \text{if}\quad n>1.$$

Note that $\bigcup\{T(m, n) : n\geqslant 1\}=X$ and each $T(m, n)$ is a Borel set. Set $f_{\langle m\rangle}(x)=n$ if and only if $x\in T(m, n)$. By recursion, there is a sequence $\{f_{\langle m\rangle} : m\in N\}$, satisfying conditions (1)-(4) when appropriate.

Let us note that one can see that for each x,

$$G_x\subset K_x=\bigcup\{V^1(f_{\langle m\rangle}(x)) : m\in N\}$$

as follows. Suppose $\delta\in G_x-K_x$. Let k be the first positive integer so that $\delta\in V^1(k)$ and $V^1(k)\subset H_{1x}$. Since $\{f_{\langle m\rangle}(x)\}_{m=1}^{\infty}$ is an increasing sequence, there is some m so that $f_{\langle m-1\rangle}(x)<k<f_{\langle m\rangle}(x)$. Now because of the manner in which $V^1(k)$ is listed, $V^1(k)\cap V^1(f_{\langle j\rangle}(x))=\emptyset$ for $j<m$. This would imply that $f_{\langle m\rangle}(x)\leqslant k$.

Now suppose $1<k$, m is a positive integer, and for all $s\in\text{Seq}$ with $lh(s)<k$ and for all $i<m$; f_s and f_{s*i} have been defined so that the conditions (1)-(4) are satisfied when appropriate. Let $s\in\text{Seq}$ with $lh(s)=k-1$. Let

$$A(s)=\bigcup_n\{x : f_s(x)=n\}\times V^{k-1}(n)\}.$$

Thus, $A(s)$ is a Borel subset of $X\times C$ with open sections. Let

$$K_n^1(s)=\{x : \forall m\ \forall(p_1, \ldots, p_m)([V^k(p_1)\cup\ldots\cup V^k(p_m)\subseteq(A(s)\cap G_{k+n-1})_x$$

$$\text{and}\qquad V^k(p_1)\cap G_x\neq\emptyset, \ldots, V^k(p_m)\cap G_x\neq\emptyset]$$

$$\Rightarrow\ (\exists t)[V^k(t)\cap(V^k(p_1)\cup\ldots\cup V^k(p_m)=\emptyset,\ V^k(t)\subseteq(A(s)\cap G_{k+n-1})_x$$

$$\text{and}\qquad V^k(t)\cap G_x\neq\emptyset])\}.$$

Again, since $(A(s)\cap G)_x$ is not compact for any x, $\bigcup D_n^1(s)=X$. Let:

$$D_n(s)=D_n^1(s)-\bigcup\{D_m^1(s) : m<n\}.$$

Let

$$G(s, k)=\bigcup_n (D_n(s)\times C)\cap G_{k+n-1}\cap A(s).$$

Then

$$G\cap A(s)\subseteq G(s, k)\subseteq G_k\cap A(s).$$

Let

$$B^1(s, m, n)=\{x : V^k(n)\subseteq(G(s, k))_x - \bigcup\{V^k(f_{s*i}(x)) : i<m \text{ and } V^k(n)\cap G_x\neq\emptyset\}.$$

Let:

$$B(s, m, n)=B^1(s, m, n)-\bigcup\{B^1(s, m, j) : j<n\}.$$

Notice that $\bigcup B(s, m, n)=X$ and each set $B(s, m, n)$ is a Borel set. Set $f_{s*i}(x)=n$ if and only if $x\in B(s, m, n)$. It can be checked that the maps f_s and f_{s*i} where $lh(s)<k$ and $i\leqslant m$, satisfy the conditions (1)-(4) when appropriate. Thus, by recursion, there is a family of maps f_s where $s\in\text{Seq}$ which satisfy (1)-(4).

Q.E.D.

Before proceeding with futher results, we would like to point out an obvious generalization of Theorem 1. We do not know the answer to the following problem.

PROBLEM 2. — Let X and Y be Polish spaces and F a measurable multifunction from X into Y such that (1) for all x, $F(x)$ is a 0-dimensional dense-in-itself G_δ subset of Y which has no compact relatively open subset and (2) $\text{Gr}(F)$ is Borel. Is there a Borel isomorphism f from $X\times\Sigma$ onto $\text{Gr}(F)$ such that for all x, $f(x, .)$ is a homeomorphism of Σ onto $F(x)$?

We note that Theorem 1 provides a positive solution to this problem if Y is itself 0-dimensional simply because Y is homeomorphic to a subset of C.

COROLLARY 2. — *Let G be a Borel subset of $X\times\Sigma$ with non-empty open sections. Then there is a Borel isomorphism f of $X\times\Sigma$ onto G such that for each x, $f(x, .)$ is a homeomorphism onto G_x.*

Proof. – Let $F(x)=G_x$. We only need to show that F is measurable. This corollary will then follow from Theorem 1. Let U be a non-empty open subset of Σ. Let $\{\sigma_n\}_{n=1}^{\infty}$ be a countable dense subset of U. For each n, $\pi_X(G\cap(X\times\{\sigma_n\}))=M_n$ is a Borel subset of X since π_X is one-to-one when restricted to $G\cap X\times\{\sigma_n\}$. Since

$$\{x : F(x)\cap U\neq\emptyset\}=\bigcup\{M_n : n\in N\},$$

F is measurable.

Q.E.D.

COROLLARY 3. – *Let $F : X\to\Sigma$ be a measurable multifunction whose values are dense-in-themselves G_δ sets. Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a countable family of Borel selectors for F such that for all x, $F(x)\subseteq\overline{\bigcup\{\Gamma_{nx} : n\in N\}}$. Then there is a Borel isomorphism $f : X\times\Sigma\to \mathrm{Gr}(F)-\bigcup\Gamma_{nx}$.*

Proof. – Let $H(x)=F(x)-\bigcup\Gamma_{nx}$. For each x, $H(x)$ is a dense-in-itself G_δ subset of Σ. Notice that $H(x)$ has no compact relatively open subsets. So, this corollary will follow from Theorem 1 provided H is measurable. Let U be a non-empty open subset of Σ. Notice that:

$$\{x : H(x)\cap U\neq\emptyset\}=\{x : F(x)\cap U\neq\emptyset\}=\bigcup\{x : \Gamma_{nx}\cap U\neq\emptyset\}.$$

Since $\{x : \Gamma_{nx}\cap U\neq\emptyset\}$ is a Borel set, H is measurable.

Q.E.D.

LEMMA 4. – *If $B\subseteq Y$ is one-to-one continuous image of Σ and $p\in\overline{B}-B$, then $B\bigcup\{p\}$ is a one-to-one continuous image of Σ.*

Proof. – Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of distinct elements of B converging to p. For each n, let U_n be an open set in Y such that $p_n\in U_n$ and if $m\neq n$, $\overline{U}_n\cap\overline{U}_m=\emptyset$ and diam $(U_n)<2^{-n}$. For each n, let $\sigma_n=f^{-1}(p_n)$ and let V_n be a clopen set such that $\sigma_n\in V_n$ and $V_n\subseteq f^{-1}(U_n)$. Let $V_0=\Sigma-\bigcup V_n$. Note that we can (and do) choose the V_n's so that $V_0\neq\emptyset$. Also, note that V_0 is open for if not, let $\{x_n\}$ be a sequence in $\Sigma-V_0$ converging to some $x\in V_0$. Note that $\{x_n\}\not\subseteq\bigcup\{V_n : n\leqslant m\}$, for any n. Hence, without loss of generality, let $x_n\in V_{m_n}$ with $m_1<m_2<\dots$ Then $f(x_n)\in U_{m_n}$ so that $f(x_n)$ converges to p. Since f is continuous, $f(x_n)$ converges to $f(x)\in B$. Contradiction.

Thus, V_n, $n=0, 1, 2, \dots$ are homeomorphic to Σ. Now let μ_0, μ_2, μ_4, $\dots$; μ_1, μ_3, μ_5, $\dots$ be rational numbers with $\mu_{2n}\uparrow\sqrt{2}$ and $\mu_{2n+1}\downarrow\sqrt{2}$.

Let $W_0=\{\sigma\in\Sigma:\sigma<\mu_0 \text{ or } \mu_1<\sigma\}$. At this point we are considering the points of Σ as irrational numbers *via* their standard continued fraction expansion. For $n>0$, let $W_n=\{\sigma\in\Sigma:\mu_{2n-2}<\sigma<\mu_{2n}$ or $\mu_{2n+1}<\sigma<\mu_{2n-1}\}$. For each n, W_n is homeomorphic to Σ and hence to V_n. Let φ_n be a homeomorphism of W_n onto V_n. Define φ on Σ by:

$$\varphi(\sigma)=\begin{cases} f(\varphi_n(\sigma)) & \text{if} \quad \sigma\in W_n, \\ p & \text{if} \quad \sigma=\sqrt{2}. \end{cases}$$

It can be checked that φ is a one-to-one continuous map of Σ onto $B\cup\{p\}$.

Q.E.D.

Our next Theorem is a parametrized version of Lemma 4.

THEOREM 5. — *Let X and Y be Polish spaces, let B be a Borel subset of $X\times Y$ and Γ a Borel graph such that for all x, if $\{y\}=\Gamma_x$, then $y\in\overline{B}_x-B_x$. Further let f be a Borel isomorphism of $X\times\Sigma$ onto B such that for each x, $f(x, .)$ is a continuous one-to-one map of Σ onto B_x. Then there is a Borel isomorphism h of $X\times\Sigma$ onto $B\cup\Gamma$ such that for each x, $h(x, .)$ is a continuous one-to-one map of Σ onto $(B\cup\Gamma)_x$.*

Proof. — Without loss of generality, we can (and do) take $\{x:\Gamma_x\neq\emptyset\}=Z$. Let g be the Borel measurable map of X into Y whose graph is Γ. It is easy to construct Borel measurable functions $g_n: X\to Y$ such that for all x and n and m, $g_n(x)\in B_x$, and $g_n(x)\neq g_m(x)$, $n\neq m$ and such that $\{g_n(x)\}_{n=1}^{\infty}$ converges pointwise to $g(x)$. We construct a sequence $\{U_n\}_{n=1}^{\infty}$ of disjoint Borel subsets of B such that for all n, $g_n(x)\in U_{nx}$ is open in B_x and $\operatorname{diam}(U_{nx})<2^{-n}$ and $\overline{U_{nx}}\cap\overline{U_{mx}}=\emptyset$ if $n\neq m$. The method of construction is standard and we omit it. For each n, let C_n be a Borel subset of $f^{-1}(U_n)$ with clopen sections such that for all x, $\Sigma-(\cup C_n)_x\neq\emptyset$. Let $C_0=X\times\Sigma-(\cup C_n)$. By Corollary 1, there is a Borel isomorphism h_n of $X\times\Sigma$ onto C_n such that for all x, $h_n(x, .)$ is a homeomorphism of $\{x\}\times\Sigma$ onto C_{nx}.

Now, imitating the proof of Lemma 4, we get the desired result.

Q.E.D.

THEOREM 6 (The main Theorem). — *Let X be a Polish space and F a G_δ valued measurable multifunction into R such that for each x, $F(x)$ is dense-in-itself and such that $\operatorname{Gr}(F)$ is Borel. Then there is a Borel isomorphism f of $X\times\Sigma$ onto $\operatorname{Gr}(F)$ such that for each x, $f(x, .)$ is a one-to-one continuous map of Σ onto $F(x)$.*

Proof. – Let $G_1 = \mathrm{Gr}(F) - \bigcup \{ X \times \{r\} : r \text{ is rational} \}$. Let $\{T_n\}_{n=1}^{\infty}$ be a family of pairwise disjoint Borel graphs contained in G_1 such that for all x, $(G_1)_x \subseteq \overline{\bigcup \{(T_n)_x : n \in N\}}$. The existence of such a family of graphs follows from the results of Srivastava [4]. Let $G = G_1 - \bigcup \{T_n : n \in N\}$. According to Corollary 2, there is a Borel isomorphism g of $X \times \Sigma$ onto G such that for all x, $g(x, .)$ is a homeomorphism of Σ onto G_x.

Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint, uncountable, dense-in-itself, Borel subsets of Σ such that each H_n is dense in Σ and $\bigcup H_n = \Sigma$. Then there exist one-to-one continuous functions f_n on Σ such that $H_n = f_n(\Sigma) \cup R_n$, where each R_n is countable and $f_n(\Sigma) \cap R_n = \emptyset$ [1]. Let $\bigcup R_n = \{p_n : n \in N\}$.

Let $\{\Gamma_n\}_{n=1}^{\infty}$ enumerate the following countable family of partial Borel graphs which are contained in $\mathrm{Gr}(F)$:

$$\{T_n : n \in N\} \cup \{g(X \times \{p_n\}) : n \in N\} \cup \{X \times \{r\} \cap \mathrm{Gr}(F) : r \text{ is rational}\}.$$

Define α_n on $X \times \Sigma$ by $\alpha_n(x, \sigma) = g(x, f_n(\sigma))$. Then α_n is a Borel isomorphism of $X \times \Sigma$ into $\mathrm{Gr}(F)$ such that for each x, $\alpha_n(x, .)$ is a continuous one-to-one map of $\{x\} \times \Sigma$ into $\{x\} \times F_x$. Also, notice that $\alpha_n(\{x\} \times \Sigma)$ is dense-in-itself and dense in $\{x\} \times F(x)$ for all x. Thus, if $\Gamma_{nx} = \{y\}$, then $y \in \overline{(\varphi_n(X \times \Sigma))_x} - (\varphi_n(X \times \Sigma))_x$. According to Theorem 5, there is a Borel isomorphism ψ_n of $X \times \Sigma$ into $\mathrm{Gr}(F)$ such that for each x, ψ_n is a one-to-one continuous map of $\{x\} \times \Sigma$ onto $\{x\} \times (\alpha_n(X \times \Sigma) \cup \Gamma_n)_x$.

Now, define f on $X \times \Sigma$ by setting $f(x, \sigma) = \psi_n(x, \sigma^*)$, where $\sigma(1) = n$ and $\sigma^*(n) = \sigma(n+1)$, for all m. The map f meets all our requirements since the family of sets $\{\psi_n(X \times \Sigma)\}_{n=1}^{\infty}$ are pairwise disjoint.

Q.E.D.

We should like to point out that as a corollary of Lemma 4, we obtain the following theorem. This theorem is credited by KURATOWSKI to SIERPINSKI ([1], p. 477). However, the paper of SIERPINSKI [3] referred to by KURATOWSKI proves the theorem only for subsets of R. Surely this theorem is known but we do not know of any proof in the literature.

THEOREM 7. – *Let B be a Borel subset of a Polish space X. Then B is a continuous one-to-one image of Σ if and only if each point of B is a condensation point of B.*

Proof. — Clearly, if B is a continuous one-to-one image of Σ, then every point of B is a condensation point of B.

Next, suppose each point of B is a condensation point of B. Let M be a closed subset of Σ and f a continuous one-to-one map of M onto B. Let K be the dense-in-itself kernel of M and let D be a countable dense subset of K. Thus, $K-D$ is homeomorphic to Σ. So, we have $B=E\cup F$, where $E\cap F=\emptyset$, F is countable and infinite, and a continuous one-to-one map g of Σ onto E. Partition Σ into Borel sets J_n, $n=1, 2, 3, \ldots$ where each J_n is condensed-in-itself and dense in Σ. For each n, set $J_n=K_n\cup D_n$ where $K_n\cap D_n=\emptyset$, D_n is countable and K_n is a continuous one-to-one image of Σ. For each n, let $B_n=g(K_n)$. Each set B_n is a continuous one-to-one image of Σ and B_n is dense in B. Let $\{p_n : n\in N\}$ be an enumeration of $B-\bigcup\{B_n : n=1, 2, \ldots\}$. According to Lemma 4, there is a continuous one-to-one map of Σ onto $B_n\cup\{p_n\}$. Since the union of countably many disjoint copies of Σ is homeomorphic to Σ, there is a continuous one-to-one map of Σ onto

$$\bigcup\{B_n\cup\{p_n\} : n=1, 2, 3, \ldots\}=B.$$

Q.E.D.

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