Some examples of σ-ideals and related Baire systems

by

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In [4], the author gave some characterizations of the Baire system of functions generated by the collection of all functions continuous almost everywhere with respect to a σ-ideal R of a metric space S.

In this paper are given some examples of σ-ideals, and some theorems which are connected with them [6]. The first example is the σ-ideal of all first category sets. Kuratowski has analysed the Baire system related to this σ-ideal [2]. In case S is complete, separable and uncountable, this Baire system does not generate all the real functions on S. Theorem 1 gives necessary and sufficient conditions on a σ-ideal R so that the related Baire system is the set of all real functions on S. In Theorem 2 a characterization of scattered or dispersed subsets of complete and separable metric spaces is given as a consequence of Theorem 1.

A second example is the σ-ideal of all countable subsets of R. The final example is the σ-ideal of all sets of measure 0 where μ is a complete, regular, σ-finite measure on S, $\mu(S)\neq 0$. In Theorem 3, it is shown that the related Baire system should be the set of all μ-measurable functions is equivalent to each of the following three conditions: 1) the Baire process of taking pointwise limits should end in one step, 2) there be a scattered (dispersed) subset M of S such that $\mu(S-M)=0$ and 3) the Baire system be the collection of all real functions on S.

The notation of this paper is the same as in [4].

Example 1. Suppose S is a separable and complete metric spaces and R is the σ-ideal of all first category subsets of S. Then $B_0(G)=G$ is the collection of all functions continuous almost everywhere with respect to R. Kuratowski has shown that a function f is in $B_1(G)$ if and only if there is a subset E of S such that $S-E$ is of the first category and f_E is continuous [2]. Moreover, Kuratowski showed that in this case $B(G) = B_1(G)$. This means that the Baire system connected with sets of the first category is generated by taking limits one time. This contrasts with the fact that in case S contains a perfect set, the Baire system, generated by the continuous functions is not generated by countably many iterations of the process of taking limits.

Even though G is a more numerous collection than C, the continuous functions on S, in general $B(G)$ is still a proper subset of $F(S)$, the set of all real functions on S. In fact if S is uncountable, then since S is complete and separable, there is a subset A of S such that if M is a perfect subset of S, then $A\cdot M$ and $A'\cdot M$ are dense subsets of M. The characteristic function of A would not belong to $B(G)$, since it is not continuous on any perfect set and every second category set contains a perfect set.

The following theorem characterizes the σ-ideals R in a complete and separable metric space such that $B(G)$ should be $F(S)$.

THEOREM 1. *Suppose S is a complete and separable metric space and R is a σ-ideal of S. In order that $B(G)$ should be $F(S)$ it is necessary and sufficient that there be a scattered (or dispersed) subset M of S such that M' is in R. Moreover, if $F(S) = B(G)$, then $F(S) = B_1(G)$.*

Proof. Suppose $B(G) = F(S)$ and S is not scattered. Since S is complete, S contains a perfect set.

Let A be a subset of S such that if K is a perfect subset of S, then $K\cdot A$ and $K\cdot A'$ are dense subsets of S and let h be the characteristic function of A. If K is a perfect subset of S, then h_K is totally discontinuous; h belongs to $F(S) = B(G)$. It can be shown by transfinite induction that if f is in $B(G)$, then there is a countable subcollection G_f of G such that f belongs to the Baire system generated by G_f, $B(G_f)$. So, there is a countable subcollection G_h of G such that h belongs to $B(G_h)$. Let N be the subset of S to which x belongs if and only if some function in G_h is discontinuous it x. The set N is in the σ-ideal R.

Let $M = S - N$ and suppose M is not scattered. Let H be a subset of M which is dense in itself; $\bar{H}$ is perfect. If f is in G_h, then $f_{\bar{H}}$ is continuous at each point of H. Since H is a dense subset of $\bar{H}$, $f_{\bar{H}}$ is continuous at each point of H. Since H is a dense subset of $\bar{H}$, $f_{\bar{H}}$ is continuous except for a set of the first category with respect to $\bar{H}$. If f is in G_h, then $f_{\bar{H}}$ is in, T, the collection of all functions over $\bar{H}$ which are continuous except for a first category set with respect to $\bar{H}$. So, $h_{\bar{H}}$ belongs to $B(T)$. Therefore, by Kuratowski's theorem there is a subset E of $\bar{H}$ such that E is the first category with respect to $\bar{H}$ and $h_{\bar{H}-E}$ is continuous. Let L be a perfect subset of $\bar{H}-E$; h_L is continuous. This is a contradiction. So, no subset of M is dense in itself; M is scattered and $N = M'$ is in R.

Now, suppose M is a scattered subset of S. If S is M, then S is countable and it follows from Theorem 12 of [3] that $F(S) = B_1(G)$. So, suppose M is not S and $S - M$ is in the σ-ideal R.

The set M is an inner limiting set and since S is separable, M is countable; M: $t_1, t_2, t_3, \ldots$ Let $E_1, E_2, E_3, \ldots$ be a monotonic sequence of open sets whose intersection is M.

Let s_{11} be a spherical open set containing t_1 such that (1) $\bar{s}_{11}$ is a subset of E_1 and (2) no boundary point of s_{11} belongs to M. For each n, and for each p, $p \leqslant n+1$, let $s_{n+1,p}$ be a spherical open set containing t_p such that (1) $\overline{s_{n+1,p}}$ is a subset of E_{n+1}, (2) no boundary point of $s_{n+1,p}$ belongs to M, (3) if $1 \leqslant p \leqslant n$, $\overline{s_{n+1,p}}$ is a subset of s_{np} and (4) $\overline{s_{n+1,i}}$ and $\overline{s_{n+1,j}}$ are mutually exclusive if $i \neq j$.

For each n, let $D_n = \sum_{p=1}^{n} s_{np} + E_n \cdot \prod_{p=1}^{n} \overline{(s_{np})}'$; $D_1, D_2, D_3, \ldots$ is a sequence of open sets whose common point is M.

Suppose f belongs to $F(S)$. For each n, let

$$f(x) = \begin{cases} f(x), & \text{if } x \text{ is in } D_n', \\ 1, & \text{if } x \text{ is in } E_n \cdot \prod_{p=1}^{n} \overline{s_{np}}', \\ f(t_i), & \text{if } x \text{ is in } s_{ni},\ 1 \leqslant i \leqslant n. \end{cases}$$

For each n, f_n is continuous at each point of M. The sequence $f_1, f_2, f_3, \ldots$ is a sequence of functions in G which converge to f. So, $F(S) = B_1(G)$. This completes the proof of Theorem 1.

Theorem 1 easily yields a characterization of scattered subsets of complete and separable metric spaces.

THEOREM 2. *Suppose M is a subset of a complete and separable metric space S. The set M is scattered if and only if every real function on S is the limit of a sequence of functions each continuous at each point of M.*

Theorem 2 follows from Theorem 1 by taking the σ-ideal R to be the class of all subsets of S which do not intersect M.

EXAMPLE 2. Suppose R is the class of all countable subsets of a metric space S. It follows from Theorem 3 of [1] that a function f on S is in $B_1(G)$ if and only if there is a function g in $B_1(C)$ such that the set $(f \neq g)$ is countable. C. Tucker [5] has shown that this is true in a more general setting. Certainly, $B_1(C)$ is a subset of $B_1(G)$. In general $B_1(C)$ is a proper subset of $B_1(G)$. For example, if S is $[0, 1]$ and f is 0 on the rational numbers and 1 on the irrational numbers, then f is the limit of a sequence of step functions. So, f is in $B_1(G)$ but f is not in $B_1(C)$.

However, if h is in $B_2(G)$, then by Theorem 3 of [4], there is a function g in $B_2(C)$ such that the set $(g \neq h)$ is countable. The function $h-g$ is 0 except for a countable set. It follows that $h-g$ is in $B_2(C)$. So, $h = (h-g)+g$ is in $B_2(C)$. Therefore, $B_2(C) = B_2(G)$.

So, if R is the σ-ideal of all countable subsets of S, then $B_1(C)$ may be a proper subset of $B_1(G)$, but if $\alpha > 1$, then $B_\alpha(G) = B_\alpha(C)$.

The final theorem and example are connected with measures.

In what follows, suppose S is a complete and separable metric space and μ is a complete, regular, σ-finite measure on S and $u(S) > 0$ and R is the σ-ideal of all sets of measure 0. If f is a function in $G = B_0(G)$, then it follows from Theorem 2a of [4] and the fact that μ is a topological measure that f is a measurable function. So, $B_0(G)$ is a collection of measurable functions. Hence, $B(G)$, the Baire system generated by G is a collection of measurable functions. If there is a scattered subset M of S such that $\mu(S-M) = 0$, then it follows from Theorem 1, that $F(S)$ is $B(G)$ and so $B(G)$ is the class of all measurable functions. Theorem 3 characterizes the measures μ, such that $B(G)$ is the collection of all measurable functions.

THEOREM 3. *Suppose μ is a complete, regular, σ-finite measure on the complete and separable metric space S and $\mu(S) > 0$ and R is the σ-ideal of all sets of measure 0. Each two of the following statements are equivalent:*

(1) *$B(G)$ is the collection of all measurable functions,*

(2) *there is a scattered subset M of S such that S is M or $\mu(S-M) = 0$.*

(3) *$B(G) = F(S)$, and*

(4) *$B(G) = B_1(G)$.*

Proof. Suppose $B(G)$ is the collection of all measurable functions and S is not scattered. Let S_1 be the subset of S to which p belongs if and only if $\mu(p) = 0$. There are not uncountably many points of S which do not belong to S_1.

Suppose $\mu(S_1) \neq 0$. Since μ is regular, there is a closed subset of S_1 having positive measure. Let S_2 be a perfect subset of S_1 having positive measure. If there is no point p of S_2 and open set D containing p such that $\mu(S_2 \cdot D) = 0$, then let $S_3 = S_2$. If there is such a point, let T be the set of all such points p and for each pcint p in T, let R_p be an open set containing p such that $\mu(R_p \cdot S_2) = 0$. Let E be a countable subcollection of the set of all R_p's covering T. The set E^*, the sum of the members of E, is an open set and $\mu(E^* \cdot S_2) = 0$. Let $S_3 = S_2 - E^* \cdot S_2$; S_3 is a closed subset of S_2. Suppose there is a point x of S_3 and an open set R_x containing x such that $\mu(R_x \cdot S_3) = 0$; $R_x + E^*$ is an open set containing x and $\mu(R_x + E^* \cdot S_2) = \mu(R_x \cdot S_3) = \mu(R_x \cdot S_3) + \mu(E^* \cdot S_2) = 0$. This is a contradiction. So, if R is an open set intersecting S_3, then $\mu(R \cdot S_3) > 0$. It follows that S_3 is perfect.

Let K be a countable dense subset of S_3; $\mu(K) = 0$. Let A be an inner limiting set containing K such that $\mu(A) = 0$; $S_3 \cdot A$ is an inner limiting set with respect to S_3, $\mu(S_3 \cdot A) = 0$ and $S_3 - S_3 \cdot A$ is of the first category with respect to S_3.

Let A_1 be a subset of A such that if H is a perfect subset of A, then $H \cdot A_1$ is a dense subset of H and $H \cdot (A - A_1)$ is a dense subset of H and let h be the characteristic function of A_1. If L is a perfect subset of A_1, then h_L is not continuous. Since the measure μ is complete, the function h is a measurable function; h belongs to $B(G)$. It follows from Theorem 3 of [4] that there is a function g in $B(C)$ and an inner limiting set L such that $\mu(S-L) = 0$ and $g_L = h_L$.

Let $S - L = \sum_{p=1}^{\infty} F_p$, where each F_p is closed. For each p, if F_p intersects S_3, then $\mu(F_p \cdot S_3) = 0$ and so $F_p \cdot S_3$ is a closed nowhere dense subset of S_3. So, $(S-L)S_3$ is of the first category in S_3. Let K_1 be a perfect subset of $S_3 - [(S-L) \cdot S_3 + S_3 - S_3 \cdot A]$; K_1 is a perfect subset of A and $g_{K_1} = h_{K_1}$. The function g_{K_1} belongs to $B\big(C(K_1)\big)$ and it follows from Example 1 that there is a subset K_2 of K_1 of the first category in K_1 such that $g_{K_1-K_2}$ is continuous. Let K_3 be a perfect subset of $K_1 - K_2$; K_3 is a perfect subset of A and $h_{K_3} = g_{K_3}$ is continuous. This is a contradiction. So, $\mu(S_1) = 0$.

Let $M = S - S_1$; M is countable and every function on S is a measurable function. Also, a function belongs to G if and only if f is continuous at each point of M. It follows from Theorem 1 that M is scattered. So, statement 1 implies statement 2.

Certainly, statement 2 implies statement 1 and it follows from Theorem 2 that statement 2 and 3 are equivalent and that statement 3 implies statement 4.

Suppose $B(G) = B_1(G)$ and S is not scattered. Let S_1 be the set of all points p such that $\mu(p) = 0$. Suppose $\mu(S_1) > 0$. Let S_3 be a perfect subset of S_1 such that if D is an open set intersecting S_3, then $\mu(D \cdot S)_3 > 0$.

Let A be an inner limiting set such that $S_3 - S_3 \cdot A$ is of the first category in S_3 and $\mu(A) = 0$. Let h be the characteristic function of A. h is in $B_2(C)$. So, h is in $B(G)$ and h is in $B_1(G)$.

By Theorem 3 of [4], there is an inner limiting set A_1 and a function g_1 in $B_1(C)$ such that $\mu(S - A_1) = 0$ and $g_{1A_1} = h_{A_1}$.

The set A_1 intersects S_3, since S_3 has positive measure; $A_3 \cdot S_3$ is an inner limiting set with respect to S_3 and $\mu(S_3 - A_1 \cdot S_3) = 0$. $S_3 - A_1 \cdot S_3 = \sum_{p=1}^{\infty} F_p$, where for each p, F_p is closed and $\mu(F_p) = 0$. Since S_3 has positive measure at each of its points, for each p, F_p is nowhere dense in S_3. So, $S_3 - A_1 \cdot S_3$ is of the first category with respect to S_3.

The function g_{1S_3} is in $B_1\big(C(S_3)\big)$. So, the set $(g_{1S_3} > 1/2)$ is the sum of countably many closed sets. Also, since g_1 and h agree almost everywhere, $\mu(g_{1S_3} > 1/2) = 0$. So $(g_{1S_3} > 1/2)$ is of the first category in S_3.

So, we have $S_3 \cdot A = A \cdot (g_{1S_3} > 1/2) + A \cdot (S_3 - A_1 \cdot S_3)$. Since each set on the right hand side is of the first category in S_3, $S_3 \cdot A$ is of the first category in S_3. This is a contradiction. So, $\mu(S_1) = 0$.

Let $M = S - S_1$, $\mu(S - S_1) = 0$. Suppose M contains a subset M_1 which is dense in itself. $\bar{M}_1$ is perfect and if D is an open set intersecting $\bar{M}_1$, then $\mu(D \cdot \bar{M}_1) > 0$. Let A be an inner limiting set such that $\mu(A) = 0$ and $\bar{M}_1 = A$ is of the first category with respect to $\bar{M}_1$. Using an argument similar to the one given above, we would have a contradiction. So, M is scattered. So, statement 2 follows from statement 4. This completes the argument for Theorem 3.

In the last part of the argument for Theorem 3, it was shown that if the second statement were not true, then there would be a function in $B_2(C)$ which would not be in $B_1(G)$. This generalizes a result of L. Kantorovitch. In [1], he gave an example of a function in $B_2(C)$ which is not in $B_1(G)$, where S is the interval $[0, 1]$ and μ is Lebesgue measure.

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Reçu par la Rédaction le 12. 2. 1970