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POSITIVELY REGULAR HOMEOMORPHISMS OF EUCLIDEAN SPACES*

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A homeomorphism of $\mathbf{R}^n$ onto itself is called *positively regular* (or EC^+) iff its family of non-negative iterates is pointwise equicontinuous. For EC^+ homeomorphisms of $\mathbf{R}^n$ such that some point of $\mathbf{R}^n$ has bounded positive semi-orbit, the *nucleus* M is defined, and the following theorems are proved.
Theorem 1. If such a homeomorphism $h:\mathbf{R}^n \twoheadrightarrow \mathbf{R}^n$ has compact nucleus M, then M is a fully invariant compact AR. Further, for $n \neq 4, 5$, $h:\mathbf{R}^n/M \twoheadrightarrow \mathbf{R}^n/M$ is conjugate to a contraction on $\mathbf{R}^n$.
Theorem 2. In $\mathbf{R}^n$, $n \neq 4, 5$, M is compact iff there exists a disk D such that $h(D) \subseteq \operatorname{Int} D$.
Theorem 3. In $\mathbf{R}^2$, either M is a disk and $h|M$ is a rotation, or $h|M$ is periodic. The relationship between M and the irregular set of h is also studied.

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positively regular homeomorphism	Euclidean space	contraction
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1. Introduction

A homeomorphism of $\mathbf{R}^n$ onto itself, whose family of non-negative iterates $\{h^n\}_{n\geq 0}$ forms a pointwise equicontinuous collection, is called EC^+ or *positively regular*. If the entire family of iterates has this property, it is called EC or *regular*.

In this paper, we study EC^+ homeomorphisms of Euclidean spaces for which there exists a point whose positive semi-orbit is bounded.

In Section 2, we define the *nucleus* of such homeomorphisms and we use semigroups to characterize such homeomorphisms provided they have compact nucleus M. Our main theorems of this section are Theorem 2.6 and Corollary 2.7. We are indebted to the referee for the present version of Section 2, which, by use of semigroups, considerably shortened our initial version of this paper, while generalizing our main Theorem (2.6). Theorems 2.6 and 2.7 say essentially that if the

* These results were presented at Auburn University and announced in [27].

nucleus is compact, it is an absolute retract, and if one looks at the map induced by h on $\mathbf{R}^2/M$, it is conjugate to a contraction on $\mathbf{R}^2$.

In Section 3, we study the nucleus M itself, obtaining particularly nice properties for M in case h is a homeomorphism of $\mathbf{R}^2$. We also relate the nucleus to the irregular set of a homeomorphism. A particularly nice by-product of this section is Theorem 3.5.

In Section 4, we study the action of h on the nucleus, in case h is a homeomorphism of $\mathbf{R}^2$. Our main theorems here are Theorem 4.11 and 4.12, which say that $h|M$ is a rotation if the nucleus M is a disk. Otherwise it is periodic.

Finally we raise some questions.

We note that in [13, 14, and 15], Kerekjarto, Homma and Kinoshita, and Husch have studied positively regular homeomorphisms of S^2, S^3, and S^n $(n \neq 4, 5)$, obtaining characterizations of the standard contraction $x \to \frac{1}{2}x$.

Notation. Let $h : \mathbf{R}^n \twoheadrightarrow \mathbf{R}^n$ be a homeomorphism. The *double arrow* indicates an *onto* function; for $X \subseteq \mathbf{R}^n$, if $h(X) \subseteq X$, we call X *invariant*, and if $h(X) = X$, we call X *fully invariant*. $O(x)$ is the *orbit* of x, $\{h^n(x) \mid n \in I\}$. $O^+(x)$ is the *positive orbit*, $\{h^n(x) \mid n \geq 0\}$. An *absolute retract* is an *AR*; a *continuum* is a compact, connected set. *Orientation preserving* and *reversing* homeomorphisms are denoted by *o.p.* and *o.r.*, respectively.

2. EC$^+$ homeomorphisms of $\mathbf{R}^n$

Let $h : \mathbf{R}^n \twoheadrightarrow \mathbf{R}^n$ be an EC$^+$ homeomorphism such that there exists an $x_0 \in \mathbf{R}^n$ such that $O^+(x_0)$ is bounded.

2.1. Lemma. *For all* $x \in \mathbf{R}^n$, $O^+(x)$ *is bounded.*

Proof. By equicontinuity, the set of $x \in \mathbf{R}^n$ such that $O^+(x)$ is bounded is both open and closed in $\mathbf{R}^n$.

Let $C(\mathbf{R}^n)$ be the space of continuous functions of $\mathbf{R}^n$ to $\mathbf{R}^n$ with compact open topology; let E be the closure of the set $\{h^i : i \geq 1\}$ in $C(\mathbf{R}^n)$. Note that E is a topological commutative semigroup under composition. By Ascoli's theorem, E is compact. By [28, Theorem 3.1.1] the cluster points of $\{h^i : i \geq 1\}$ form a group K and E contains a *unique idempotent* r, which is the unit of K; K is an ideal in E.

2.2. Definition. $M = \text{image } r$. We will call M the *nucleus of* $\mathbf{R}^n$ *under* h, or more simply, the *nucleus of* h.

2.3. Lemma. *If* $r' \in K$, *then* $\text{image } r' = M$.

Proof. Let $s \in K$ such that $sr' = r's = r$. Then

$$r'(R^n) = rr'(R^n) \subseteq r(R^n)$$

and

$$r(R^n) = r's(R^n) \subseteq r'(R^n).$$

2.4. Lemma. *If* $z \in \limsup\{h^i(x): i \geq 1\}$, *then* $z \in M$.

Proof. Suppose $z = \lim_{i \to +\infty} h^{j_i}(x)$ for some subsequence $\{j_i\}$ of the positive integers. Since E is compact, $\{h^{j_i}\}$ has a cluster point $s \in K$; note that $s(x) = z$. By Lemma 2.3, $z \in M$.

2.5. Lemma. *If* $z \in \limsup\{h^{-i}(x): i \geq 1\}$, *then* $x \in M$.

Proof. Suppose $z = \lim_{i \to +\infty} h^{-j_i}(x)$ for some subsequence $\{j_i\}$ of the positive integers. Let s be a cluster point of $\{h^{j_i}\}$. By equicontinuity, $s(z) = x$ [for, let $\varepsilon > 0$; choose δ so that if $d(z, w) < \delta$, then $d(h^i(z), h^i(w)) < \frac{1}{2}\varepsilon$ for all $i \geq 0$. Choose j_i so that $d(z, h^{j_i}(x)) < \delta$ and $d(s(z), h^{j_i}(z)) < \frac{1}{2}\varepsilon$. Then $d(s(z), x) \leq d(s(z), h^{j_i}(z)) + d(h^{j_i}(z), h^{j_i}h^{-j_i}(x)) < \varepsilon$]. By Lemma 2.3, $x \in M$.

2.6. Theorem. *Let* $h : \mathbb{R}^n \to \mathbb{R}^n$ *be an o.p.* EC^+ *homeomorphism such that there exists* $x_0 \in R^n$ *such that* $O^+(x_0)$ *is bounded. Suppose M is compact. Then M is a compact AR which does not separate* $\mathbb{R}^n$, *and* $h(M) = M$. *If* $n \neq 4, 5$, *then the induced map* $\tilde{h} : \mathbb{R}^n/M \to \mathbb{R}^n/M$ *is conjugate to the contraction* $x \to \frac{1}{2}x$ *on* $\mathbb{R}^n$.

Proof. Since $r^2 = r$, r is a retraction of $\mathbb{R}^n$ onto M. Let B be an n-cell in $\mathbb{R}^n$ such that $M \subseteq \operatorname{int} B$. Let V be a neighborhood of M in $\mathbb{R}^n$. Since r is a cluster point of $\{h^i : i \geq 1\}$, there exists $i \geq 1$ such that $h^i(B) \subseteq V$ [Could also use Lemma 2.4 and compactness of B and equicontinuity to show this]. Hence M is cellular in $\mathbb{R}^n$ and $\mathbb{R}^n/M$ is homeomorphic to $\mathbb{R}^n$.

We show that $h(M) = M$. Since K is an ideal in E, $rh \in K$. Let $s \in K$ such that $(rh)s = r$. Then

$$h(M) = hr(\mathbb{R}^n) = rh(\mathbb{R}^n) \subseteq r(\mathbb{R}^n) = M$$

and

$$M = r(\mathbb{R}^n) = rh\, sr(\mathbb{R}^n) = hr\, sr(\mathbb{R}^n) \subseteq hr(\mathbb{R}^n) = h(M).$$

Hence h induces a homeomorphism $\tilde{h} : \mathbb{R}^n/M \to \mathbb{R}^n/M$. If we identify $[M] = 0$, then it follows from Lemmas 2.4 and 2.5 that if $x \in \mathbb{R}^n/M$, $x \neq 0$, then

$$\lim_{i \to +\infty} \tilde{h}^i(x) = 0 \quad \text{and} \quad \lim_{i \to -\infty} \tilde{h}^i(x) = +\infty.$$

Hence, if $n \neq 4, 5$, $\tilde{h}$ is top. equivalent to the contraction [13, 14, 15].

2.7. Corollary. *If h is o.p. on* $\mathbb{R}^2$ *such that the positive semi-orbit of some point is bounded, then h is* EC^+ *with compact nucleus M if and only if there exists a fully* *invariant, locally connected, non-separating continuum K such that*

(1) *$h|K$ is EC,*

(2) *h is EC^+ at all points of* Bd K, *and*

(3) *$\tilde{h}:\mathbf{R}^2/K \twoheadrightarrow \mathbf{R}^2/K$ ($\cong \mathbf{R}^2$) is conjugate to a contraction. Moreover, in this case, K is M.*

Proof. M is fully invariant by 2.6, and locally connected, non-separating, since $M=$ image r, and r is a retraction. Further, any locally connected, non-separating continuum X in R^2 satisfies $\mathbf{R}^2/X\cong\mathbf{R}^2$.

The converse is clear.

Remark. Condition 2 is necessary because of the following example:

$$\text{Let}\quad h(r,\theta)=(r,\theta) \qquad \text{if } 0\leqslant r\leqslant 1,$$
$$=\left(\frac{r+1}{2},\theta+\left(1-\frac{1}{r}\right)\pi\right) \qquad \text{if } r>1.$$

Then h is the identity on M, the unit disk, and $\tilde{h}$ is conjugate to a contraction. But h is not EC^+ at any point of the unit circle.

3. The nucleus of h in $\mathbf{R}^2$

Let $h:\mathbf{R}^2\twoheadrightarrow\mathbf{R}^2$ be an EC^+ homeomorphism such that $O^+(x_0)$ is bounded for some point $x_0\in\mathbf{R}^2$. Using equicontinuity and ε-sequential growths as in [5] and [6], one can show that $\mathbf{R}^2$ can be filled up with an increasing sequence of disks $D_1\subseteq \text{Int } D_2\subseteq D_2\subseteq \text{Int } D_3\subseteq D_3\subseteq\cdots$, such that $h(D_i)\subseteq D_i$.

In this section, we will show that the nucleus can be defined in terms of these disks (3.2) and that M is compact iff there exists a disk D such that $h(D)\subseteq \text{Int } D$ (Theorem 3.4). We have not been able to prove this latter result for $\mathbf{R}^n$, $n=4,5$. We also make some observations about the relationship between the nucleus and the irregular set of h.

3.1. Lemma. *Let X be a compactum in $\mathbf{R}^2$ such that $h(X)\subseteq X$. Then $\bigcap_{i=1}^{\infty} h^i(X)$ is $M\cap X$. In particular, $\bigcap_{i=1}^{\infty} h^i(D_n)=M\cap D_n$, where M is the nucleus of $\mathbf{R}^2$ under h, and D_n is as above.*

Proof. Let $x\in\bigcap_{i\geq 1} h^i(X)$; then $x=h^i(x_i)$ for some $x_i\in X$. Choose a subsequence $\{x_{i_j}\}$ so that $\lim x_{i_j}=z$. Let s be a cluster point of $\{h^{i_j}\}$, thus $\lim_{j\to+\infty} h^{i_j}(z)=s(z)$. It follows from EC^+ that $s(z)=x$ (cf. proof of lemma 2.5). Hence $\bigcap_{i=1}^{\infty} h^i(X)\subseteq M\cap X$.

Let $x\in M\cap X$. Consider $E_x=\{f|X: f\in E\}$; since X is compact, E_x is metrizable. Thus $r|X=\lim_{k\to+\infty}(h|X)^{i_k}$ for some subsequence $\{i_k\}$. Hence $x=r(x)=\lim_{k\to+\infty} h^{i_k}(x)$. It is easily checked that $x\in\bigcap_{i=1}^{\infty} h^i(X)$.

3.2. Theorem. *Let $M_n=\bigcap_{i=1}^{\infty} h^i(D_n)$ and let $N=\bigcup_{n=1}^{\infty} M_n$. Then $N=M$; that is, N is the nucleus of $\mathbf{R}^2$ under h. (Here D_n is as in the introduction to this section.)*

Proof. $N=\bigcup_{n=1}^{\infty} M_n=\bigcup_{n=1}^{\infty}(M\cap D_n)=M\cap\bigcup_{n=1}^{\infty} D_n=M.$

3.3. Theorem. (1) *$h|M$ is EC on M.*

(2) *$M=\{x\mid O(x)$ is bounded$\}$.*

Proof. (1) It is well known that an EC^+ homeomorphism on a compact space is EC. See, for example, [12, p. 124].

(2) Clear.

3.4. Theorem. *M is compact iff there exists a disk D such that $h(D)\subseteq \operatorname{Int} D$. (This argument works for $\mathbf{R}^n$, $n\neq 4, 5$.)*

Proof. ($\Rightarrow$) (We thank the referee for this shorter proof of ($\Rightarrow$).) Let K be the unit disk in $\mathbf{R}^2/M$; then by Theorem 2.6, $\tilde{h}(K)\subseteq \operatorname{Int} K$. Let D be the preimage of K in $\mathbf{R}^2$; then $h(D)\subseteq \operatorname{Int} D$.

($\Leftarrow$) We will show that $\mathbf{R}^2=D\cup h^{-1}(D)\cup h^{-2}(D)\cup\ldots$. Suppose that $K=D\cup\bigcup_{k=1}^{\infty} h^{-k}(D)\neq\mathbf{R}^2$. Let p be a boundary point of K. We show that h is not EC^+ at p. Let $N_D=\bigcap_{i=1}^{\infty} h^i(D)$ be called the *nucleus of D*. The nucleus N_D of D is a subset of $\operatorname{Int} D$. Therefore $d(N_D, \operatorname{Bd} K)=\varepsilon>0$. Thus $d(h^n(p), N_D)\geq\varepsilon$, for all n. But arbitrarily close to p there are points whose nth iterates get arbitrarily close to N_D. Therefore small neighborhoods of p have large images. This contradicts the fact that h is EC^+ at p. It follows that $K=\mathbf{R}^2$.

Thus $N_D=M$, so M is compact.

3.5. Irregular sets. We note that in [16], and in other papers, Husch studies positively regular homeomorphisms and their irregular sets. The irregular set, $\operatorname{Irr}(h)$, is the set of points at which the full family $\{h^n\}_{n\in I}$ is *not* regular. He defines the set $K(x)=\bigcap_{i\geq 0} O_i(x)$, where $O_i(x)$ is the orbit closure of x under h^i, $i\geq 0$. From Theorem 2.7 of the present paper, it is easy to see that the nucleus of h is just $\bigcup_{x\in\mathbf{R}^2} K(x)$, if h is a homeomorphism of $\mathbf{R}^2$.

It follows from our results in this paper that $K=^{\text{def}}\bigcup_{p\in\mathbf{R}^2} K(x)$ is the set $\operatorname{Irr}(h)$ provided $\operatorname{Irr}(h)$ does not separate $\mathbf{R}^2$, where the metric on $\mathbf{R}^2$ is that of the one-point compactification – that is, the metric inherited from S^2.

We also note that it is not unusual that $\operatorname{Irr}(h)$ separates the plane, as the following simple example from differential equations shows. Consider the differential equation

$$(*)\qquad \begin{cases} \dfrac{dr}{dt}=\begin{cases} 0 & \text{if } r^2\leq 4\cos^2\theta,\\[2ex] \tfrac12(2|\cos\theta|-r) & \text{if } r^2>4\cos^2\theta,\end{cases}\\[4ex] \dfrac{d\theta}{dt}=0,\\[3ex] r(0)=r_0 \quad\text{and}\quad \theta(0)=\theta_0.\end{cases}$$

Here we are considering the plane with polar coordinates and $r \geq 0$. This could be written $(r'(t), \theta'(t)) = A(r(t), \theta(t))$, where A is jointly continuous in r and θ. Now, the critical values of this equation consist of those points (r_0, θ_0) such that $r_0^2 \leq 4\cos^2\theta_0$; i.e., the solution $(r(t), \theta(t))$ of (*) at these points is $\theta(t) = \theta_0$, $r(t) = r_0$, for all t. Thus, the critical values of (*) consist of the two disks of radius 1 and centers (1, 0) and $(1, \pi)$ (in polar coordinates).

If $r_0^2 > 4\cos^2\theta_0$, then the solution of (*) is:

$$r(t) = (r_0 - 2|\cos\theta_0|)\,e^{-\pi/2} + 2|\cos\theta_0|; \qquad \theta(t) = \theta_0.$$

Now consider the time one homeomorphism, h, induced by (*). For each (r_0, θ_0) in E^2, let $h(r_0, \theta_0) = (r(1), \theta(1))$, where $(r(t), \theta(t))$ is the corresponding solution of (*). It can be easily checked that h is positively regular under the usual Euclidean metric or under the metric inherited from the usual metric on S^2. The nucleus, M, of h consists of the critical values of the differential equation (*). Under the Euclidean metric, the irregular set of h is the closure of the complement of the nucleus. Under the metric inherited from S^2, the irregular set of h is the boundary of the nucleus of h.

3.5.1. Theorem. *Let* Irr(h) *denote the set of points at which h is not regular. Then* Irr$(h) =$ Bd M, *if the metric inherited from* $\mathbf{R}^2$ *is used.*

3.5.2. Theorem. *The class of sets which can be irregular sets of h in* $\mathbf{R}^2$ *(where the metric on* $\mathbf{R}^2$ *is inherited from* S^2*) is precisely the class of all locally connected continua I such that* $I = \mathrm{Bd}(I \cup \mathrm{Int}\, I)$.

This follows from Corollary 4.3 below.

4. The action of h on M

In this section we first show that any locally connected continuum which doesn't separate E^2, can be the nucleus of some EC$^+$ homeomorphism h of E^2 onto itself, with $h|M =$ identity.

We then study the action of an arbitrary EC$^+$ homeomorphism, h, on its nucleus, and show that if M is bounded, then either

(1) $h|M$ is periodic, or

(2) M is a disk and $h|M$ is an irrational rotation.

These results are proved using prime end theory. See [7, 8, 17, 18, 25] for a discussion of prime ends and for the necessary definitions. See also [29, Section 2].

Our use of prime end theory was motivated by the work of Mason [18]. See also [19].

By a *C-map* from the interior of the unit disk B onto a simply connected region, we mean a 1–1 continuous function such that

(1) inverses of crosscuts are crosscuts, and

(2) the endpoints of inverses of crosscuts are dense on Bd B.

Remark. The following theorem establishes the converse of the Corollary to Theorem 2.21 of [17, p. 68]. We have recently become aware that a version of this theorem appears in *Univalent Functions* by Chr. Pommerenke, Vandenhoeck and Ruprecht in Gottingen, 1975, Theorem 9.8 page 279. We proved this theorem independently as a lemma for Theorem 4.2, where we use it to show that φ can be extended to the closed disk B. We refer to this on p. 340 of our announcement in [27].

4.1. Theorem. Let U be a simply connected domain in S^2 such that the boundary, K of U is locally connected. Then every prime end of U is of the first kind.

Proof. Let P be a prime end of U and $\{C_i\}_{i=1}^{\infty}$ a chain of crosscuts of U defining P. Thus, $U-C_i=U_i\cup(U-U_i)$ and $U_{i+1}\cup C_{i+1}\subseteq U_i$. Let $\lim_{i\to\infty} C_i = x$.

Assume P is not of the first kind. This means there is a point $y\in\bigcap_{i=1}^{\infty}\bar{U}_i=I(P)$, the impression of P, and $x\neq y$.

Since K is locally arcwise connected [11], it is possible to obtain a neighborhood V of x such that $y\notin\bar{V}$ and $\bar{V}\cap\bar{C}_1=\emptyset$ and a neighborhood W of x such that $W\subseteq V$ and every point of $K\cap W$ can be joined to x by an arc lying in $V\cap K$.

Now pick n so that $\bar{C}_n\subset W$. Since the endpoints of C_n are points of $K\cap W$, it follows that there is an arc B_n lying in $V\cap K$ which joins the endpoints of C_n.

Let $J=C_n\cup B_n$. Notice that $\operatorname{Int} J\subset V$ and $J\cap U_n=\emptyset$. Thus, $U_n\subseteq\operatorname{Int} J$ or $U_n\subseteq\operatorname{Ext} J$. Since $y\in\bar{U}_n$ and $y\in\operatorname{Ext} J$, $U_n\subseteq\operatorname{Ext} J$.

Let a be a point of C_n. Let b and c be points of C_n on opposite sides of a. Let α be an arc lying in U_n except for its endpoints connecting b to c and let β be an arc lying in $U-(U_n\cup C_n)$ except for its endpoints connecting b to c. Let $T=\alpha\cup\beta$ and let $D=\operatorname{Int} T$. The arc, τ, from b to c on C_n cuts D into two connected open sets $D=S\cup R$, with $S\subseteq U_n$ and $R\subseteq U-(U_n\cup C_n)$, $\operatorname{Bd} S=\alpha\cup\tau$, $\operatorname{Bd} R=\beta\cup\tau$.

Since $S\subseteq U_n\subseteq\operatorname{Ext} J$ and J locally separates $\operatorname{Int} J$ from $\operatorname{Ext} J$, it follows that $R\subseteq\operatorname{Int} J$.

Let γ be an arc connecting a to a point of C_1 such that $\gamma-\{a\}\subseteq U-(U_n\cup C_n)$. Notice that $\gamma\cap R\neq\emptyset$. Thus, there is a subarc γ_1 of γ connecting a point of the $\operatorname{Int} J$ to a point of $C_1\subseteq\operatorname{Ext} J$. But, $\gamma_1\cap J=\emptyset$. This contradiction establishes the theorem.

4.2. Theorem. *Let M be a locally connected, non-separating continuum in E^2. Then there exists a sequence of disks $\{D_i\}_{i=-\infty}^{\infty}$ with boundaries $\{C_i\}_{i=-\infty}^{\infty}$ and a homeomorphism $h:E^2\twoheadrightarrow E^2$ such that*

(1) $D_{i-1}\subseteq\operatorname{Int} D_i$,

(2) $\bigcup D_i=E^2$,

(3) $\bigcap_{i=-\infty}^{\infty} D_i=M$,

(4) $h(D_{i+1})=D_i$, *and*

(5) $h\,|\,M=$ identity.

Proof. We use prime end theory by approaching M from the exterior. Let $U=S^2-M$. See [25]. Then U is a simply connected region and there exists a C-map

$\varphi:\operatorname{Int} B \twoheadrightarrow U$, where B is the unit disk in E^2. Further Bd U is locally connected [26, Theorem VI.2.2], and therefore each prime end of U is of the first kind by Theorem 6.1. Thus φ can be extended to a continuous function $\bar{\varphi}:B \twoheadrightarrow U$. See [17, Theorem 2.21].

Let $\{B_i\}_{i=-\infty}^{\infty}$ be a sequence of circular disks centered at the origin such that

(1) $\bigcup_{i=-\infty}^{\infty} B_i = \operatorname{Int} B$,

(2) $\bigcap B_i = \{0\}$,

(3) $B_i \subseteq \operatorname{Int} B_{i+1}$,

(4) $\lim_{i\to\infty} B_i =$ unit circle.

Let $g:B \twoheadrightarrow B$ be a homeomorphism such that $g(0)=0$, $g|\operatorname{Bd} B =$ identity, and $g(B_i)=B_{i+1}$ carrying radial segments onto themselves. Then $\varphi g \varphi^{-1}$ can be extended to a homeomorphism $\overline{\varphi g \varphi^{-1}}:\bar{U} \twoheadrightarrow \bar{U}$ by making it the identity on Bd $M =$ Bd $\bar{U}$. Let $D_i=\varphi(B_i)$ and let $h=\overline{\varphi g \varphi^{-1}}|S^2-\{0\}$, where "0" is the point $\bigcap_{i=-\infty}^{\infty} D_i$. The sequence $\{D_i\}$ and homeomorphism h satisfy the theorem.

4.3. Corollary of Proof. Any locally connected, non-separating continuum M can be the nucleus of some EC^+ homeomorphism of E^2 onto itself, in such a way that $h|M$ is the identity.

4.4. Notation for remainder of this section. We assume that h is an EC^+ homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$ whose nucleus M is a continuum. In Section 3, it is shown that M is locally connected. Thus by [26, Theorem VI 2.2], Bd M is also a locally connected continuum.

We again use prime end theory by approaching M from the exterior. Let B be the unit disk in E^2, and let $\varphi:\operatorname{Int} B \twoheadrightarrow U = S^2 - M$ be a "C-map". By Theorem 4.1, each prime end of U is of the first kind, so by [17, Theorem 2.21], φ can be extended to a continuous function $\bar{\varphi}:B \twoheadrightarrow \bar{U}$. Now we think of h as defined on S^2 by $h(\infty)=\infty$, (note that $\infty \in U$), and let $\Psi=\varphi^{-1}h\varphi$ and $\Psi:\operatorname{Int} B \twoheadrightarrow \operatorname{Int} B$. Then, see [18], Ψ can be extended to a homeomorphism $\tilde{\Psi}:B \twoheadrightarrow B$, since h is a homeomorphism of $\bar{U}$ onto itself. Since $\varphi\Psi = h\varphi$, $\bar{\varphi}\Psi = h\bar{\varphi}$. Thus, the following lemma holds.

4.5. Lemma. *If p is a fixed or periodic point of $\tilde{\Psi}$, then $\bar{\varphi}(p)$ is a fixed or periodic point of h, respectively.*

4.6. Lemma. *If $\tilde{\Psi}|\operatorname{Bd} B$ has a periodic point, then $\tilde{\Psi}|\operatorname{Bd} B$ is periodic.*

Proof. Suppose $\tilde{\Psi}$ is not of finite order. Let p be a point of Bd B, of order k. Now $\tilde{\Psi}^k=(\Psi^k)^{\sim}$, since $[(\varphi^{-1}h\varphi)^k]^{\sim}=(\varphi^{-1}h^k\varphi)$ on Int B, and the extension to the boundary is unique.

Let $g=h^k$, and note that g is EC on M, and $(\varphi^{-1}g\varphi)^{\sim}$ has a fixed point, p, on Bd B. By Lemma 4.5 $g(\bar{\varphi}(p))=\bar{\varphi}(p)$. Let $[p_0, q_0]$ be a maximal interval (possibly $p_0=q_0$) on Bd B such that $(\varphi^{-1}g\varphi)^{\sim}$ has no fixed points on the open interval (p_0, q_0). Then if

$x_0 \in (p_0, q_0)$, then $x_n \to p_0$ (say), where $x_n = (\varphi^{-1} g \varphi)^{\sim}(x_{n-1})$. Clearly $\bar{\varphi}(x_n) \to \bar{\varphi}(p_0)$.

We observe that $\varphi\psi = h\varphi$ which implies $\bar{\varphi}\tilde{\psi} = h\bar{\varphi}$. Thus $\bar{\varphi}\psi^k = h^k\bar{\varphi}$ which implies $\bar{\varphi}(\psi^k) = h^k\bar{\varphi} = g\bar{\varphi}$. It follows that $\bar{\varphi}(x_{n+1}) = g(\bar{\varphi}(x_n))$. But this implies that g is not EC on M. This is a contradiction, and it follows that $\tilde{\Psi}|\text{Bd } B$ is periodic.

4.7. Lemma. *If $\tilde{\Psi}\,|\text{Bd } B$ is not periodic, then each point on* Bd B *has a dense orbit.*

Proof. Suppose there exists an $x_0 \in \text{Bd } B$ such that $O(x_0)$ is not dense. Since $\bar{O}(x_0)$ is fully invariant, it follows that $C(\bar{O}(x_0))$ is also fully invariant. Now $C(\bar{O}(x_0)) = \cup\{V_i\}$ is a countable union of open intervals. Suppose there are n, i such that $\tilde{\Psi}^n(V_i) = V_i$. Then there exists $y \in \bar{V}_i$ such that $\tilde{\Psi}^n(y) = y$, and $\tilde{\Psi}\,|\text{Bd } B$ has a periodic point. But by Lemma 4.6, this means that $\tilde{\Psi}\,|\text{Bd } B$ is periodic, and this is a contradiction.

Thus $\{V_i\}$ is a countably infinite collection of open intervals, and for any $x \in \cup V_i$, $O(x)$ meets infinitely many members of the collection. Further, the diameters of the V_i's have limit 0, and $\bar{\varphi}$ is continuous, so diam $\bar{\varphi}(V_i)$ have limit 0. Let $x \in \cup V_i$, and let $\{W_i\}$ be the subcollection of $\{V_i\}$ whose elements contain images of x. Let $d_i =$ diam $\bar{\varphi}(W_i)$. Then diam $W_i \to 0$ and diam $\bar{\varphi}(W_i) \to 0$.

Now let $\varepsilon = \frac{1}{2}\max\{d_i\}$. There exists $\delta > 0$ such that $h^n(\delta\text{-set})$ has diameter less than ε for all integers n, since h is EC on M. Also there exists $\gamma > 0$ such that $\bar{\varphi}(\gamma\text{-set})$ has diameter $<\delta$, by the uniform continuity of $\bar{\varphi}$. Let $\{d_i\}$ be named so that $d_1 = \max\{d_i\}$. There exists W_i such that diam $W_i < \gamma$, and there exists n such that $\tilde{\Psi}^n(W_i) = W_1$. Thus $h^n(\bar{\varphi}(W_i)) = \bar{\varphi}(W_1)$. But $\bar{\varphi}(W_i)$ has diameter $<\delta$ and $\bar{\varphi}(W_1)$ has diam $> \varepsilon$. This contradicts the fact that h is EC on M.

It follows that each point of Bd B must have a dense orbit.

4.8. Lemma. *If M has interior points and each interior point of M is a periodic point, then $\tilde{\Psi}\,|\text{Bd } B$ is periodic.*

Proof. Since M is a locally connected continuum, the components $\{W_i\}$ of Int M have diameters with limit 0, by [26, IV 4.2]. Thus since h is EC on M, we see that for each W_i, there exists n_i such that $h^{n_i}(W_i) = W_i$. Now W_i is a manifold, and h is pointwise periodic on W_i, so h^{n_i} is also pointwise periodic on W_i. Thus by [22], h^{n_i} is periodic on W_i and on $\bar{W}_i$. Thus $\bar{\varphi}^{-1}(\text{Bd } W_i)$ is a union of non-degenerate continua on Bd B, and is fully invariant in Bd B. It follows that any orbit of the complement of this set on Bd B is not dense. Thus by Lemma 4.7, $\tilde{\Psi}\,|\text{Bd } B$ is periodic.

4.9. Lemma. *If M has no interior points, then $\tilde{\Psi}\,|\text{Bd } B$ is periodic.*

Proof. M is a locally connected, non-separating continuum with no interior points. Therefore M is a dendrite. Thus there exists an $x \in M$ such that $h(x) = x$. Then $\bar{\varphi}^{-1}(x)$ is a closed, fully invariant subset of Bd B, whose complement is non-empty and fully invariant. Thus $\tilde{\Psi}\,|\text{Bd } B$ has a point whose orbit is not dense. Therefore, by Lemma 4.7, $\tilde{\Psi}\,|\text{Bd } B$ is periodic.

4.10. Lemma. *If M has interior points, and each interior point of M is periodic, then $h|\mathrm{Bd}\, M$ is periodic.*

Proof. By Lemma 4.8, $\tilde{\Psi}|\mathrm{Bd}\, B$ is periodic, say of period k. Let E be any prime end of S^2-M, and let $e\in \mathrm{Bd}\, B$ be the corresponding point. By Theorem 4.1, $I(E)$ is a singleton $\{x\}$, so that $\bar{\varphi}(e)=x$. Now, as in the proof of Lemma 4.6, $\bar{\varphi}\tilde{\Psi}^k=h^k\bar{\varphi}$, so $\bar{\varphi}\tilde{\Psi}^k(e)=\bar{\varphi}(e)=x=h^k\bar{\varphi}(e)=h^k(x)$. Thus $h^k(x)=x$.

Since this holds for each prime end, and each prime end is accessible, $h|\mathrm{Bd}\, M$ is of period $\leq k$ on a dense subset of $\mathrm{Bd}\, M$. It follows that $h|\mathrm{Bd}\, M$ is of period $\leq k$.

4.11. Theorem. *If M has interior points, then either*

(1) *M is a disk and $h|M$ is an irrational rotation, or*

(2) *$h|M$ is periodic.*

Proof. (1) If there exists a non-periodic interior point of M, then M is a disk and $h|M$ is an irrational rotation, by Theorem 5.1 of [12].

(2) If each interior point of M is a periodic point, then by Lemma 4.11 $h|\mathrm{Bd}\, M$ is periodic. Thus there exists an integer n such that $h^n|\mathrm{Bd}\, M$ is the identity. Therefore, if W is any component of $\mathrm{Int}\, M$, then $h^n(\bar{W})=\bar{W}$, $h^n|\mathrm{Bd}\, \bar{W}$ is the identity, and $h^n|W$ is pointwise periodic. But W is homeomorphic to R^2 by Theorem 15 of [21], and therefore by [22], $h^n|W$ must be periodic. Now suppose h^n is not the identity on W; say it is of period $k>1$. Let $g:\mathbf{R}^2\twoheadrightarrow\mathbf{R}^2$ be defined by

$$g(x)=\begin{cases} h^n(x) & x\in\bar{W},\\ x & x\in C(\bar{W}).\end{cases}$$

Then g is periodic of period k on E^2. We will show that this is impossible.

Let $\varepsilon>0$, $\varphi:R^2\twoheadrightarrow R^2$ be a homeomorphism carrying $\bar{W}$ into the ε-neighborhood of the origin. Then $\varphi g\varphi^{-1}$ is a homeomorphism of R^2 onto itself, which is periodic and ε-close to the identity. But this contradicts Newman's theorem [23], which says that there does not exist arbitrarily small periodic homeomorphisms of E^2 onto itself.

If follows that $h^n|M$ is the identity, and $h|M$ is periodic.

4.12. Theorem. *If M has no interior points, then $h|M$ is periodic.*

Proof. By Lemma 4.9, $\tilde{\Psi}|\mathrm{Bd}\, B$ is periodic. Thus, as in the proof of 4.11, $h|M$ is periodic.

4.13. Remark. We observe that if W is a component of $\mathrm{Int}\, M$, then $\mathrm{Bd}\, \bar{W}$ is a simple closed curve by [26, Theorem VI. 2.3] and therefore $\bar{W}$ is a disk.

4.14. Theorem. *Let K be a locally connected, non-separating continuum in $\mathbf{R}^2$, and let $h:K\twoheadrightarrow K$ be an EC homeomorphism. Then*

(1) *If K is a disk, h is a rotation,*

(2) *If K is not a disk and h is extendable to a homeomorphism* $H: \mathbf{R}^2 \twoheadrightarrow \mathbf{R}^2$, *then h is periodic, and*

(3) *If K is not a disk and h is not extendable, then h is not necessarily periodic.*

Proof. (1) This is proved in [12].

(2) We note that our arguments of 4.5–4.13 of this section, only require that $h = H|M$ be EC. It is not necessary that the extension to R^2 be EC^+. Thus (2) follows.

(3) If h is not extendable to the plane, it is not necessarily periodic. For example, let K be a dendrite which is the union of a countable sequence of arcs $\{A_i\}_{i=1}^{\infty}$ whose diameters tend to 0, the arcs A_i all meeting at the only branch point of K. Let $B_1 = A_1$; $B_2 = A_2 \cup A_3$; ... ; B_n = union of first n arcs of $\{A_i\}$ not in $\bigcup_{j=1}^{n-1} B_j$. Let $h|B_n$ be of period n, interchanging cyclicly, the n arcs in B_n. Then h is pointwise periodic, but not periodic. Yet h is EC on K. Note that h cannot be extended to a homeomorphism of R^2 onto itself.

5. Questions

(1) If M is not compact, and h is o.p., can h be imbedded in a flow?

(2) Let $h: \mathbf{R}^n \to \mathbf{R}^n$, $n = 4, 5$, be an EC^+ homeomorphism such that some point has a bounded positive semi-orbit. Does there exist a disk D such that $h(D) \subseteq D$? such that $h(D) \subseteq \text{Int}\, D$?

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