SOME SELECTION THEOREMS AND PROBLEMS

R. Daniel Mauldin
Department of Mathematics
North Texas State University
Denton, Texas 76203/USA

Let I be the closed unit interval, [0,1]. Let B be a Borel subset of I×I such that for each $x, B_x = \{y : (x,y) \in B\} \neq \phi$. Using the axiom of choice, we find that the Borel set B contains a uniformization (= the graph of some function f mapping I onto I). The question was raised concerning how nice or describable the function f is in the famous letters exchanged among Baire, Borel, Hadamard and Lebesgue [1]. Novikov gave the first example of a Borel subset of I×I which does not possess a Borel uniformization [2]. Kondo proved that every such Borel set B possesses a uniformization which is coanalytic [3]. Yankov [4] and von Neumann [5] proved that B contains the graph of a function f which is measurable with respect to the σ-algebra generated by the analytic subsets of I. In fact, they proved this result assuming only that B is an analytic set. Whether every Borel set B possesses a uniformization which is the difference of two coanalytic sets seems to be an unsolved problem.

Various extensions of these uniformization problems have been considered not only for their intrinsic interest but for their applications. One such problem was discussed by A.H. Stone at the 1975 Oberwolfach conference on Measure Theory [6]. The problem is essentially that of filling up a set with pairwise disjoint uniformizations so that the uniformizations are indexed in some reasonable manner. For simplicity let me formulate the problem as follows.

PARAMETRIZATION PROBLEM.
Let B be a subset of I×I so that for each x, B_x is uncountable. A parametrization of B is a map g from I×I onto B so that for each x, $g(x,\cdot)$ maps I onto $\{x\} \times B_x$. Given a description of B how describable can a parametrization of B be?

Stone was particularly interested in the case where B is a Borel set and he inquired about the existence of a universally measurable parametrization. Wesley proved by forcing techniques that the answer is yes. Cenzer and I proved the following theorem [7].

THEOREM 1.

Let B be an analytic subset of I×I so that for each x, B_x is uncountable.

Then there is a parametrization g of B such that both g and g^{-1} are measurable with respect to $S(I\times I)$.

Here $S(I\times I)$ denotes the family of C-sets of Selivanovskii, the smallest family of subsets of $I\times I$ containing the open sets and closed under operation (A). $S(I\times I)$ is a very nice family of universally measurable sets.

In the course of proving this theorem, we showed that such an analytic set B possesses $2^{\aleph_0}$ pairwise disjoint uniformizations which are the graphs of functions which are measurable with respect to the σ-algebra generated by the analytic subsets of I. Whether such an analytic set can be filled up by pairwise disjoint uniformizations each of which is the graph of function measurable with respect to this σ-algebra is an unsolved problem. Whether the function g chosen so that both g and g^{-1} are measurable with respect to the σ-algebra generated by the analytic subsets of $I\times I$ is also unsolved. Necessary and sufficient conditions for a Borel parametrization are given in the following theorem [8].

THEOREM 2.

Let B be a Borel subset of $I\times I$ such that for each x, B_x is uncountable. The following are equivalent.

1. B has a Borel parametrization;
2. there is an atomless conditional probability distribution μ so that for each x, $\mu(x,B_x) > 0$;
3. B contains a Borel set M such that for each x, M_x is a nonempty perfect subset of I.

In connection with this I showed that there is a closed uncountable subset B of $I\times I$ such that for each x, B_x is uncountable and yet B does not have a Borel parametrization.

In view of this last result, let me pose the following problem which seems to be unsolved.

PROBLEM.

Let B be a Borel subset of $I\times I$ so that for each x, B_x is a closed uncountable set. Does B have $2^{\aleph_0}$ pairwise disjoint Borel uniformizations?

That such a Borel set B has a Borel uniformization was proven by Novikov [9]. That B possesses $\aleph_1$ Borel uniformizations was shown by Larman [10] We can obtain Larman's result from the following selection theorem.

THEOREM 3.

There are Borel measurable maps, $f_\alpha, \alpha < \omega_1$ from 2^I into I such that for each closed set $K, f_\alpha(K) \in K$ and if K is uncountable, then $f_\alpha(K) \neq f_\beta(K)$, if $\alpha \neq \beta$.

Before proving this theorem, let us indicate how this gives $\aleph_1$ pairwise disjoint Borel measurable uniformizations. One simply takes the map ϕ from I into 2^I defined by $\phi(x) = B_x$. It is well known that this is a Borel measurable map. For $\alpha < \omega_1$, set $g_\alpha(x) = f_\alpha(\phi(x))$. The graphs of the functions g_α are pairwise disjoint Borel uniformizations of B.

PROOF OF THEOREM 3.

Let τ be a Borel isomorphism of I onto 2^I. For each $\alpha < \omega_1$, let D_α be the αth derived set map of 2^I onto 2^I. D_1 is a Borel measurable map of class 2. Thus, for each $\alpha < \omega_1$, D_α is a Borel measurable map. (The exact class of these maps is an unsolved problem [11]). Let $P = \{F \in 2^I : F \text{ is perfect}\}$. Let $Z_\alpha = D_\alpha^{-1}(\phi)$ and let $h: 2^I \to B$ by $h(E) = \{x \in E : x \text{ is an isolated point of } E\}$.

For each ordinal α, $0 \leq \alpha < \omega_1$, let B_α be the graph of $D_\alpha o \tau$ in I×I, let

$$T_\alpha = (D_\alpha o \tau)^{-1}(P),$$

let

$$K_\alpha = T_\alpha - \bigcup\{T_\gamma : \gamma < \alpha\};$$

and let

$$M_\alpha = B_\alpha \cap (K_\alpha \times I).$$

Notice that for each α, M_α is a Borel subset of I×I and if $x \in \pi_1(M_\alpha) = K_\alpha$ then $(M_\alpha)_x$ is a perfect set. According to Theorem 2 each M_α has continuumly many pairwise disjoint Borel uniformizations. So, let $\{V_\beta^\alpha\}_{\beta<\omega_1}$ be a family of $\aleph_1$ pairwise disjoint Borel uniformizations of M_α.

For each $0 \leq \alpha < \omega_1$, let

$$E^\alpha = \bigcup_{\gamma \leq \alpha} V_\alpha^\gamma \cup (\tau^{-1}(Z_\alpha) \times I) \cap B \cup \mathscr{G}r((h o D_\alpha o \tau) | I - T_\alpha)$$

We note the following properties of these sets: (1) E^α is a Borel subset of I×I, (2) $E^\alpha \subset B$, (3) for each x in I, E_x^α is a nonempty countable set and finally (4) if $0 \leq \alpha < \beta < \omega_1$ and Y is an uncountable compact subset of I, then

$$E^\alpha{}_{\tau^{-1}(Y)} \cap E^\beta{}_{\tau^{-1}(Y)} = \phi$$

For each α, let g_α be a Borel map of I into I which uniformizes E^α. Let $f^\alpha(X) = g_\alpha(\phi^{-1}(X))$ for each $X \in 2^I$. Then $\{f_\alpha : \alpha < \omega_1\}$ is a family of Borel measurable selectors for 2^I such that if $\alpha \neq \beta$ and X is an uncountable closed subset of I, $f_\alpha(X) \neq f_\beta(X)$. Q.E.D.

The preceding proof is a variation on an argument of Larman[10]. I was simply unable to obtain Theorem 3 as a corollary of the results in [10]

There is one closely related problem which seems to bear stating.

PROBLEM.

Let B be a Borel subset of $I \times I$ so that for each x, B_x is uncountable and is the union of countably many compact sets. Does B have $2^{\aleph_0}$ pairwise disjoint Borel uniformization?

Again it was shown by Larman that B has $\aleph_1$ Borel uniformizations provided each B_x is also a G_δ set [10]. It may appear that this problem can be reduced to the previous problem by employing the beautiful result of Saint-Raymond [12] which states that B is the union of countably many Borel sets B_n so that for each n and each x, B_{nx} is compact. If one could insure that for some n each B_{nx} is uncountable then this last problem would be reduced to the previous problem. However, we give the following example.

EXAMPLE.

There is an F_σ subset B of $I \times I$ so that for each x, B_x is uncountable and yet B does not contain a Borel set each section of which an is an uncountable compact set.

Let $\{A_n\}_{n=1}^{\infty}$ be an increasing sequence of analytic subsets of I so that $\bigcup A_n = I$ such that there does not exist a sequence of Borel sets D_n so that for each n, $D_n \subset A_n$ and $\bigcup D_n = I$. The existence of such sets was shown by Leo Harrington. I understand that this is a classical result although I do not know any reference.

For each n, let M_n be a closed subset of $I \times [1/2n+1, 1/2_n]$ so that $(M_n)_x$ is uncountable if and only if x is in A_n. Let $B = \bigcup M_n$. For each n, let $T_n = \bigcup \{M_p : p \leq m\}$. Suppose B contains a Borel set K so that each K_x is compact and uncountable. Notice that for each x there is some n so that $K_x \subset T_{nx}$. For each n, let $D_n = \{x : K_x \subset T_{nx}\}$. Then each set D_n is a Borel set, $D_n \subset A_n$ and $\bigcup D_n = I$. This contradiction establishes that the F_σ set B has the required properties. □

Let me mention that one can easily obtain the following improvement of Larman's theorem [10,p.244] by combining the results of Saint-Raymond with those of Larman.

THEOREM 4.

Let B be a Borel subset of $I \times I$ so that for each x, B_x is uncountable and is the union of countably many compact sets. Then B has $\aleph_1$ pairwise disjoint Borel uniformizations.

PROOF.

First, express $B = \bigcup\{B_n : n \geq 1\}$ where each B_n is a Borel set and for every x, B_{nx} is compact. For each n, let g_n map $I\times[1/2n+1\ 1/2n]$ onto $I\times I$ by $g_n(x,y) = (x, t_n(y))$, where $t_n(y)$ is a linear map of $[1/2n+1, 1/2n]$ onto $[0,1]$.

Set $B_0 = \phi$ and for each positive integer n, $M_n = B_n - B_{n-1}$. Let $K_n = g_n^{-1}(M_n)$. Let $K = \bigcup\{K_n : n \geq 1\}$. Clearly, K is a Borel subset of $I\times I$ and for each x, K_x is both a K_σ and a G_δ set. Also, if $(x,y) \in K$, there is a unique n so that $(x,y) \in K_n$. So, define $g(x,y) = g_n(x,y) \in K$. Then g is a Borel isomorphism of K onto B so that for each x, $g(x,\cdot)$ maps $\{x\}\times K_x$ onto $\{x\}\times B_x$. According to Larman's theorem, there exists $\aleph_1$ pairwise disjoint Borel uniformizations of K. The images of these uniformizations of K are pairwise disjoint Borel uniformizations of M.Q.E.D.

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