CH, $\mathbf{V} = \mathbf{L}$, DISINTEGRATION OF MEASURES, AND $\mathbf{\Pi}_1^1$ SETS

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Abstract. In 1950 Maharam asked whether every disintegration of a σ-finite measure into σ-finite measures is necessarily uniformly σ-finite. Over the years under special conditions on the disintegration, the answer was shown to be yes. We show here that the question is equivalent to the existence of a Borel uniformization of a certain set defined from the disintegration. Moreover, we show that the answer may depend on the axioms of set theory in the following sense. If CH, the continuum hypothesis holds, then the answer is no. Our proof of this leads to some interesting problems in infinitary combinatorics. Also, if Gödel's axiom of constructibility $\mathbf{V} = \mathbf{L}$ holds, then not only is the answer no, but, of equal interest is the construction of $\mathbf{\Pi}_1^1$ sets with very special properties.

1. Introduction and Background

Disintegration of a measure has long been a very useful tool in ergodic theory (see, for examples, [14] and [1]) and in the theory of conditional probabilities [15]. The origins of disintegration are hazy but the first rigorous definitions and results seem to be due to von Neumann [14]. We recall the formal definition of a disintegration considered in this paper.

Let $(Y, \mathcal{B}(Y))$ and $(X, \mathcal{B}(X))$ be uncountable Polish spaces each equipped with the σ-algebra of Borel sets, let $\phi : Y \to X$ be measurable, and let μ and ν be measures on $\mathcal{B}(Y)$ and $\mathcal{B}(X)$ respectively.

Definition 1.1. A **disintegration** of μ with respect to (ν, ϕ) is a family, $\{\mu_x : x \in X\}$, of measures on $(Y, \mathcal{B}(Y))$ satisfying:

(1) $\forall B \in \mathcal{B}(Y),\ x \mapsto \mu_x(B)$ is $\mathcal{B}(X)$-measurable
(2) $\forall x \in X,\ \mu_x(Y \setminus \phi^{-1}(x)) = 0$ and
(3) $\forall B \in \mathcal{B}(Y),\ \mu(B) = \int \mu_x(B) d\nu(x)$.

One could consider disintegrations in more general settings but we will consider only this setting or the setting where X and Y are standard Borel spaces, *i.e.*, measure spaces isomorphic to uncountable Polish spaces equipped with the σ-algebra of Borel sets.

Let us recall that if $\{\mu_x : x \in X\}$ is a disintegration of μ with respect to (ν, ϕ), then the image measure, $\mu \circ \phi^{-1}$, is absolutely continuous with respect to ν in the following sense. If $N \in \mathcal{B}(X)$ with $\nu(N) = 0$, then combining properties (2) and

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(3) we have

$$\begin{aligned}\mu \circ \phi^{-1}(N) &= \int \mu_x(\phi^{-1}N)d\nu(x)\\ &= \int_N \mu_x(\phi^{-1}N)d\nu(x)\\ &= 0.\end{aligned}$$

As is well known, the converse also holds in our setting.

Theorem 1.2. *Suppose $(Y, \mathcal{B}(Y))$ and $(X, \mathcal{B}(X))$ are standard Borel spaces, μ is a σ-finite measure on $\mathcal{B}(Y)$, ν is a σ-finite measure on $\mathcal{B}(X)$, and $\phi : Y \to X$ is a Borel measurable function. If $\mu \circ \phi^{-1} << \nu$ then there exists a σ-finite disintegration $\{\mu_x : x \in X\}$ of μ with respect to (ν, ϕ). Moreover this disintegration is unique in the sense that if $\{\hat{\mu}_x : x \in X\}$ is any σ-finite disintegration of μ with respect to (ν, ϕ), then there exists $N \subseteq X$ such that $\nu(N) = 0$ and $\forall x \notin N$ $\mu_x = \hat{\mu}_x$.*

In the late 1940's Rokhlin [16] and independently, Maharam [9] introduced canonical representations of disintegrations of a finite measure into finite measures. This situation naturally arises when one is considering a dynamical system with an invariant finite measure or when one obtains the conditional probability distribution induced by a given probability measure. Maharam also considered disintegrations of σ-finite measures. This situation arises when one has a dynamical system with a σ-finite invariant measure, but no finite invariant measure (see, for example, [3]). In her investigation of σ-finite disintegrations, Maharam found a basic problem which does not occur in the case of disintegrations of a finite measure. To explain this problem we make the following definitions.

Definition 1.3. If $\{\mu_x : x \in X\}$ is a disintegration of μ with respect to (ν, ϕ) such that $\forall x \in X$, μ_x is σ-finite, then we say that the disintegration is **σ-finite**. If $\{\mu_x : x \in X\}$ is a σ-finite disintegration of μ with respect to (ν, ϕ) we say that the disintegration is **uniformly σ-finite** provided there exists a sequence, (B_n), from $\mathcal{B}(Y)$ such that

(1) $\forall n \in \mathbb{N}\ \forall x \in X,\ \mu_x(B_n) < \infty$ and
(2) $\forall x \in X,\ \mu_x(Y \setminus \bigcup_n B_n) = 0$.

Problem 1.4. *Maharam* [9, 10]*: Let $\{\mu_x : x \in X\}$ be a σ-finite disintegration of μ with respect to (ν, ϕ). Is this disintegration uniformly σ-finite?*

The following theorem demonstrates in what manner a given disintegration is "almost" uniformly σ-finite.

Theorem 1.5. *Suppose $\{\mu_x : x \in X\}$ is a σ-finite disintegration of the σ-finite measure μ with respect to (ν, ϕ). Then there exists a sequence, (D_n), from $\mathcal{B}(Y)$ such that*

(1) $\forall x \in X$, $\mu_x(D_n) < \infty$
(2) for ν-a.e. $x \in X$, $\mu_x(Y \setminus \bigcup_n D_n) = 0$.

Proof. Define $F : \mathcal{B}(Y) \to \mathcal{B}(X)$ by

$$F(B) = \{x \in X : \mu_y(B) < \infty\}.$$

Note that $\forall B \in \mathcal{B}(Y)$, $F(B) = \bigcup_n \{x \in X : \mu_x(B) < n\}$. Thus F does map $\mathcal{B}(Y)$ into $\mathcal{B}(X)$.

Let (B_n) be a sequence from $\mathcal{B}(Y)$ such that $\forall n \in \mathbb{N}$, $\mu(B_n) < \infty$ and $Y = \bigcup_n B_n$. Note that for every n we have that $\mu(B_n) = \int \mu_x(B_n) d\nu(x) < \infty$. Thus $\mu_x(B_n) < \infty$ for ν-a.e. x and therefore $\nu(X \setminus F(B_n)) = 0$. Let $E = \bigcap_n F(B_n)$. Note that

$$\nu(X \setminus E) = \nu\left(X \setminus \bigcap_n F(B_n)\right) = \nu\left(\bigcup_n X \setminus F(B_n)\right) \leq \sum_n \nu(X \setminus F(B_n)) = 0,$$

and consequently

$$\mu(Y \setminus \phi^{-1}(E)) = \mu(\phi^{-1}(X \setminus E)) = \int_{X \setminus E} \mu_x(Y) d\nu(x) = 0.$$

For each $n \in \mathbb{N}$ define $D_n = \phi^{-1}(E) \cap B_n$. For every $x \in E$ we have $\mu_x(D_n) = \mu_x(\phi^{-1}(E) \cap B_n) \leq \mu_x(B_n) < \infty$ and for every $x \in X \setminus E$ we have $\mu_x(D_n) = \mu_x(\phi^{-1}(E \cap B_n)) \leq \mu_x(\phi^{-1}(E)) = 0$. Furthermore

$$\begin{aligned}\mu_x\left(Y \setminus \bigcup_n D_n\right) &= \mu_x\left(Y \setminus \left(\phi^{-1}(E) \cap \bigcup_n B_n\right)\right) \\ &= \mu_x\left(Y \setminus \phi^{-1}(E) \cup \left(Y \setminus \bigcup_n B_n\right)\right) \\ &\leq \mu_x\left(Y \setminus \phi^{-1}(E)\right) + \mu_x\left(Y \setminus \bigcup_n B_n\right) = 0 \text{ for } \nu\text{-a.e. } x.\end{aligned}$$

□

Corollary 1.6. *Suppose $\{\mu_x : x \in X\}$ is a σ-finite disintegration of the σ-finite measure μ with respect to (ν, ϕ). There exists a uniformly σ-finite disintegration $\{\hat{\mu}_x : x \in X\}$ of μ with respect to (ν, ϕ) such that $\mu_x = \hat{\mu}_x$ for ν-almost every $x \in X$.*

Proof. Let (D_n) be the sequence from $\mathcal{B}(Y)$ that is constructed in Theorem 1.5. Let $N \in \mathcal{B}(X)$ be such that $\nu(N) = 0$ and such that $\mu_x(Y \setminus \bigcup_n D_n) = 0$ for every $x \notin N$. Define $\hat{\mu}_x$ by

$$\hat{\mu}_x(B) = \begin{cases} \mu_x : x \notin N \\ 0 : x \in N \end{cases}$$

Clearly, $\{\hat{\mu}_x : x \in X\}$ has the required properties. □

Maharam's question is whether a given σ-finite disintegration must be altered in some fashion to be uniformly σ-finite or is it automatically already uniform. In view of Corollary 1.6, one might think that Maharam's question is not really about the interaction between measure theory and descriptive set theory. However, as we show in Theorem 5.3, the problem is equivalent to the existence of a Borel uniformization of a certain set defined measure theoretically.

In [5], it was noted that if each member of a disintegration, μ_x is locally finite, then the disintegration is uniformly σ-finite. Also, a canonical representation of uniformly σ-finite disintegrations was developed. We also point out that in [11] Maharam showed how spectral representations could be carried out for uniformly

σ-finite kernels. Whether these tools can be carried over the kernels that are not necessarily uniform remains open.

In section 2, we show that the continuum hypothesis implies the answer to Maharam's question is no. We note that after sending David Fremlin an earlier version of this work where we used $\mathbf{V} = \mathbf{L}$, but did not discuss the use of CH, he commented, [4], and may have independently proved, the answer is no assuming CH. Our argument leads to some interesting infinitary combinatorial questions.

In section 3, we begin a more detailed investigation of the relation between Maharam's problem and descriptive set theory. In particular, we assume the existence of a "special" coanalytic set, a coanalytic set with some specific properties in the product of the Baire space with itself. This assumption leads to a more descriptive σ-finite disintegration which is not uniformly σ-finite for $X = Y = \omega^\omega$. Of course, this result extends to any pair of uncountable Polish spaces.

In section 4, assuming Gödel's axiom of constructibility, $\mathbf{V} = \mathbf{L}$, we show that special coanalytic sets exist. As the existence of such sets is of perhaps equal interest as Maharam's problem, we present the construction of such a set in some detail from basic principles. Since our argument involves methods from logic and set theory that some readers may not be familiar with, we give specific references to Kunen's book where the necessary background may be found.

In section 5, we show that uniformly σ-finite kernels are jointly measurable. We don't know whether the converse holds. In this section we also show the equivalence of Marharam's problem with the existence of a certain Borel uniformization.

2. CH implies the answer is no

We show that the answer to Maharam's question is no assuming CH. The proof will involve the construction of a subset of the plane with some specific properties. We first show that such a construction is necessary and sufficient for a nonuniformly σ-finite disintegration into purely atomic measures (by a nonuniformly σ-finite disintegration we mean a σ-finite disintegration which is not uniformly σ-finite).

Theorem 2.1. *Let X and Y be Polish spaces, let $\phi : Y \to X$ be Borel measurable, and let $\{\mu_x : x \in X\}$ be a family of purely atomic measures each of which is supported on $\phi^{-1}(x)$. There exist measures μ on $\mathcal{B}(Y)$ and ν on $\mathcal{B}(X)$, such that $\{\mu_x : x \in X\}$ forms a nonuniformly σ-finite disintegration of μ with respect to (ν, ϕ) if and only if*

(1) *$\forall B \in \mathcal{B}(Y)$ the mapping $x \mapsto \mu_x(B)$ is $\mathcal{B}(X)$-measurable*
(2) *The set $W = \{(x, y) \in X \times Y : \mu_x(\{y\}) > 0\}$ is not the union of countably many graphs of Borel functions $f_n : X \to Y$.*

Proof. Suppose conditions 1 and 2 are satisfied. Fix $x_0 \in X$ and let ν be the Dirac measure concentrated at x_0. For each $B \in \mathcal{B}(Y)$ define $\mu(B) = \int \mu_x(B) d\nu(x)$.

By 1, the measures μ_x form a disintegration of μ with respect to (ν, ϕ) into σ-finite measures supported on the sections $W_x = \{y : \mu_x(\{y\}) > 0\} \subseteq \phi^{-1}(x)$. This disintegration is not uniformly σ-finite. If it were, then by theorem 5.3 which is proven later, the mapping $(x, y) \mapsto \mu_x(\{y\})$ would be measurable in $X \times Y$. Thus W would be a Borel set with countable sections and would be a countable union of Borel graphs, contradicting 2.

Now suppose $\{\mu_x : x \in X\}$ is a nonuniformly σ-finite disintegration of μ with respect to (ν, ϕ) into purely atomic measures. Let $W = \{(x, y) : \mu_x(\{y\}) > 0\}$. Condition 1 is satisfied by the definition of a disintegration.

Suppose W fails condition 2 and $f_n : X \to Y$ is a sequence of Borel functions such that $W = \bigcup_n \{(x, f_n(x)) : x \in X\}$. Since the sections W_x are disjoint, each f_n is one-to-one. Then $E_n = f_n(X)$ is a Borel subset of Y. For every x, $\mu_x(E_n) = \mu_x(\{f_n(x)\}) < \infty$ and $\mu_x(Y \setminus \bigcup_n E_n) = 0$ a contradiction.

□

Restating theorem 2.1 gives the following corollary.

Corollary 2.2. *A given disintegration into purely atomic measures is uniformly σ-finite if and only if the set $W = \{(x, y) : \mu_x(\{y\}) > 0\}$ of atoms is a countable union of Borel graphs.*

To aid the discussion we make the following definition.

Definition 2.3. Given Polish spaces $(X, \mathcal{B}(X))$, $(Y, \mathcal{B}(Y))$, a *measure kernel* is a map $x \mapsto \mu_x$ which assigns to $x \in X$ a σ-finite measure μ_x on Y and is such that for every $B \in \mathcal{B}(Y)$, the map $x \mapsto \mu_x(B)$ is Borel measurable.

Note that being a measure kernel is part of the definition of a disintegration, but here we do not necessarily have a function $\phi\colon Y \to X$ such that μ_x is supported on $\phi^{-1}(x)$. In particular, for $x_1 \neq x_2$, we do not necessarily have that μ_{x_1} and μ_{x_2} have disjoint supports.

The next fact show that from a wellordering of the reals of type ω_1 we get measure kernels which are not uniformly σ-finite (the definition of uniformly σ-finite immediately generalizes to measure kernels).

Fact 2.4. Suppose $\prec$ is a wellordering of the Polish space X of type ω_1. Define $W \subseteq X \times X$ by $W = \{(x, y)\colon y \prec x\}$. For each x, let μ_x be counting measure on the section W_x. Then $x \mapsto \mu_x$ is a measure kernel which is not uniformly σ-finite.

Proof. Each section W_x is countable as $\prec$ has length ω_1, so each μ_x is a σ-finite measure. To see this defines a measure kernel, fix a Borel $B \subseteq Y = X$, and fix $n \in \omega$. If $|B| \geq n+1$, let $b_{n+1} \in B$ be the $n+1^{\text{st}}$ element of B in the wellordering $\prec$. Then if $b_{n+1} \prec x$ we have that $|W_x \cap B| \geq n+1$, and so $\{x\colon \mu_x(B) \leq n\}$ is co-countable (hence Borel). If $|B| \leq n$, then $\{x\colon \mu_x(B) \leq n\}$ is all of X, hence Borel. Finally, this measure kernel cannot be uniformly σ-finite, for otherwise the relation $W(x, y)$, that is the relation $y \prec x$, would be Borel (being a countable union of Borel graphs). Thus the wellordering $\prec$ would be Borel, hence measurable, a contradiction to Fubini. □

It is tempting to think that a variation of the above argument might produce a nonuniformly σ-finite disintegration. Namely, fix Polish spaces X and $Y = X$, and let $\pi\colon X \times Y \to Z$ be a Borel bijection, for some Polish space Z. Suppose again that $\prec$ is a wellordering of X of type ω_1. Let μ_x be counting measure on $\{\pi(x, y)\colon y \prec x\}$. Then each μ_x is a σ-finite measure on Z, and if we let $\phi\colon Z \to X$ be defined by $\phi(z) = \pi_X \circ \pi^{-1}(x)$, then μ_x is supported on $\phi^{-1}(x)$. However, such a construction cannot give a measure kernel. To see this, suppose that the map $x \mapsto \mu_x$ as constructed was a measure kernel. Suppose $B \subseteq X \times Y$ is Borel, and let $B' = \pi(B)$, so B' is a Borel subset of Z (as π is a one-to-one Borel map). Define $R \subseteq X$ by

$$\begin{aligned} R(x) &\Leftrightarrow \exists y \prec x \; B(x,y) \\ &\Leftrightarrow \mu_x(B') > 0 \end{aligned}$$

Since $x \mapsto \mu_x$ is assumed to be a measure kernel, the second equivalence shows that R is Borel. In other words, this would give a wellordering of the Polish space X for which bounded quantification over Borel sets produces Borel sets (this means precisely a set defined as the set R above). However, this is impossible by the following fact. This fact is likely folklore, though we are unable to locate a reference.

Fact 2.5. Let $\prec$ be a wellordering of an uncountable Polish space X. Then there is a Borel set $B \subseteq X \times X$ such that the relation R defined by $R(x) \Leftrightarrow \exists y \prec x \; B(x,y)$ is not Borel. In fact, there is a single Borel set B such that for every wellordering $\prec$ of X, the corresponding set $R_\prec$ is not Borel.

Proof. Let $\sim$ be a Borel equivalence relation on X which is not smooth, that is, such that there is no Borel transversal for the equivalence relation (i.e., a Borel set meeting each equivalence class in exactly one point). For example, we could take the Vitali equivalence relation on $\mathbb{R}$ (so $x \sim y$ iff $x - y \in \mathbb{Q}$). Given the wellordering $\prec$, let $S_\prec$ be the corresponding transversal for $\sim$, namely

$$S_\prec(x) \Leftrightarrow \forall y \; (y \sim x \wedge y \neq x \rightarrow x \prec y).$$

So, $x \notin S_\prec$ iff $\exists y \prec x \; (x \sim y)$ and so $X - S_\prec$ (which cannot be Borel), is defined by a bounded quantification over the Borel set $\sim$. □

Part of the difficulty in dealing with Maharam's problem and related questions is the fact that the set of σ-finite measures on a Polish space X does not admit a reasonable Borel structure. We make this precise in the following theorem. Recall that if X is a Polish space, then the set of Borel probability measures $\mathcal{M}(X)$ is a standard Borel space in such a way that for every Borel $B \subseteq X$ the map $\mu \mapsto \mu(B)$ is a Borel function on $\mathcal{M}(X)$ (recall that a standard Borel space is a set with a σ-algebra $\mathcal{B}$ which is the collection of Borel subsets of X for some Polish topology on X).

Theorem 2.6. *Let X be an uncountable Polish space and $\mathcal{M}^+_\sigma(X)$ the set of positive σ-finite Borel measures on X. Then there does not exists a σ-algebra Σ on $\mathcal{M}^+_\sigma(X)$ such that $(\mathcal{M}^+_\sigma(X), \Sigma)$ is a standard Borel space and the map $(\mu, B) \mapsto \mu(B)$ from $\mathcal{M}^+_\sigma(X) \times \mathcal{B}(X)$ to $[0, +\infty]$ (with the standard one-point compactification topology) is Borel on the codes for Borel sets (that is, there are $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$ relations S, $R \subseteq \mathcal{M}^+_\sigma(X) \times \mathcal{B}(X) \times [0, +\infty]$ such that for $\mu \in \mathcal{M}^+_\sigma(X)$, y a codes of a Borel set $B_y \in \mathcal{B}(X)$, and $r \in [0, +\infty]$, we have $S(\mu, y, r) \leftrightarrow R(\mu, y, z) \leftrightarrow \mu(B_y) = r$).*

Proof. Let X, $\mathcal{M}^+_\sigma(X)$ be as in the statement, and suppose toward a contradiction that Σ a standard Borel structure on M_X as in the statement. Let $\{U_n\}_{n\in\omega}$ be a base for the Polish topology on X (where we assume $U_n \neq \emptyset$ for all n). Without loss of generality we may assume $X = 2^\omega$ (as all standard Borel spaces are Borel isomorphic). When referring to $\mathcal{M}^+_\sigma(X)$, "Borel" will refer to this Borel structure. Let $A = \{\mu \in \mathcal{M}^+_\sigma(X) \colon \mu \text{ is atomic}\}$. Then A is a $\boldsymbol{\Sigma}^1_1$ set in $\mathcal{M}^+_\sigma(X)$ as we have

$$\begin{aligned} \mu \in A &\leftrightarrow \exists y \in \omega^\omega \; [\mu \text{ is supported on } \{y_n\}_{n\in\omega}] \\ &\leftrightarrow \exists y \in \omega^\omega \; [\mu(X - \{y_n\}_{n\in\omega}) = 0] \end{aligned}$$

Since there is a recursive function mapping $y \in \omega^\omega$ to a Borel (even G_δ) code for the set $X - \{y_n\}_{n\in\omega}$, our hypothesis on Σ gives that $A \in \mathbf{\Sigma}^1_1$. In fact, $A \subseteq \mathcal{M}^+_\sigma(X)$ is a Borel set since we can compute $\mathcal{M}^+_\sigma(X) - A$ to be $\mathbf{\Sigma}^1_1$ as follows:

$$\begin{aligned}\mu \in \mathcal{M}^+_\sigma(X) - A \leftrightarrow \exists P \subseteq X\ [&P \text{ is non-empty, perfect } \wedge \mu(P) > 0 \wedge \mu(P) < \infty \\ &\wedge \forall n\ (P \cap U_n \neq \emptyset \rightarrow \mu(P \cap U_n) > 0) \\ &\wedge \forall n\ (P \cap U_n \neq \emptyset \rightarrow \forall k\ \exists m\ \forall p\ (U_p \subseteq U_n \wedge \operatorname{diam}(U_p) < \frac{1}{m} \\ &\quad \rightarrow \mu(P \cap U_p) < \frac{1}{k}))]\end{aligned}$$

Consider the relation $R \subseteq \mathcal{M}^+_\sigma(X) \times X$ defined by $R(\mu, x) \leftrightarrow (\mu \in A) \wedge \mu(\{x\}) > 0$. By our hypothesis, the second conjunct is Borel, and so the relation R is Borel. Note that each section or R is countable, and if $\mu \in A$ then the section R_μ is just the support of the atomic measure μ. So, there are countably many Borel functions $f_n\colon A \to X$ such that for all $\mu \in A$ we have that $R_\mu = \{f_n(\mu)\}_{n\in\omega}$.

Let $A' \subseteq A$ be defined by $\mu \in A' \leftrightarrow (\mu \in A) \wedge \forall n(\mu(f_n(\mu)) = 1$. So, A' is also a Borel set in $\mathcal{M}^+_\sigma(X)$, so it too is a standard Borel space. Note that A' is just the set of atomic measures on X which give each atom a measure of 1. Clearly A' is in bijection with the set of countable subsets of X (identifying a countable set with counting measure on it). This is a contradiction as A' cannot be a standard Borel space.

To see this, consider the equivalence relation E_c on ω^ω giving equality on countable sets, more precisely, $x\, E_c\, y \leftrightarrow \{x_n\}_{n\in\omega} = \{y_n\}_{n\in\omega}$. E_c is a Borel equivalence relation on ω^ω and it is well-known that E_c is not smooth, that is, has no Borel selector (in fact, any countable Borel equivalence relation Borel embeds into E_c). Consider the relation $C \subseteq A' \times \omega^\omega$ defined by $C(\mu, z) \leftrightarrow \{z_n\}_{n\in\omega} = \{f_n(\mu)\}_{n\in\omega}$. Clearly C is Borel. By Jankov-von Neumann uniformization, there is a function $g\colon A' \to \omega^\omega$ which uniformizes C, and such that g is measurable with respect to the σ-algebra generated by the $\mathbf{\Sigma}^1_1$ sets (and so g is universally measurable). Let E_0 be the Vitali equivalence relation on 2^ω, that is, $a\, E_0\, b \leftrightarrow \exists k\ \forall l \geq k\ (a(l) = b(l))$. E_0 is a non-smooth countable Borel equivalence relation, and so by Harrington-Kechris-Louveau it Borel (even continuously) embeds into E_c, say by the Borel function π. That is, $a\, E_0\, b$ iff $\pi(a)\, E_c\, \pi(b)$. Let $H = [\operatorname{ran}(\pi)]_{E_c}$ be the saturation of $\operatorname{ran}(\pi)$ under the equivalence relation E_c. Note that $\operatorname{ran}(\pi)$ is Borel (being a Borel, one-to-one image of a Borel set), and thus so is its saturation H under E_c, as E_c is generated by a Polish group action (a theorem of Kuratowski and Ryll-Nardzewski).

Let $D \subseteq H \times 2^\omega$ be defined by $D(y, a) \leftrightarrow (y \in H \wedge \pi(a)\, E_c\, y)$. Let $h\colon H \to 2^\omega$ be Borel and uniformize D (as H is Borel with countable sections). We now define a selector $S \subseteq 2^\omega$ for E_0 by:

$$a \in S \leftrightarrow \exists \mu \in \mathcal{M}^+_\sigma(X)\ \exists y \in \omega^\omega\ [g(\mu) = y \wedge h(y) = a].$$

Note that we also have:

$$a \notin S \leftrightarrow \exists \mu \in \mathcal{M}^+_\sigma(X)\ \exists y \in \omega^\omega\ [g(\mu) = y \wedge h(y)\, E_0\, a \wedge h(y) \neq a].$$

These computations show that S is $\mathbf{\Delta}^1_2$, in fact they show that S is absolutely $\mathbf{\Delta}^1_2$ (that is, there are $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_2$ formulas φ, ψ defining S such that $\mathrm{ZF} \vdash \forall x\ (\varphi(x) \leftrightarrow \psi(x))$). If we let φ and $\neg\psi$ be the statements from the above expressions, then it is not hard to check that it is a theorem of ZF that $\forall x\ (\varphi(x) \leftrightarrow \psi(x))$. The main point is that the definitions of the functions g and h arise from the uniformizations of certain Borel sets, and it is a theorem of ZF that these definitions produce uniformizations of these set. It is now a theorem of Solovay that S is (universally) measurable, which is a contradiction as it is a selector for the Vitali equivalence relation. □

Remark 2.7. It follows easily from Theorem 2.6 that there cannot be a countable separating family for the space of σ-finite measures, that is, there cannot exists a sequence of Borel set $B_n \subseteq X$ such that every σ-finite measure μ on X is determined by its values $\mu(B_n)$ on the family (this would give a standard Borel structure on $M^+_\sigma(X)$, and one easily sees that the map $(\mu, B) \mapsto \mu(B)$ would be Borel in the codes). However, this corollary is easy to see directly. Namely, given a sequence of Borel set B_n it is easy to construct directly a σ-finite $\mu \neq 0$ such that $\mu(B_n) = 0$ or $+\infty$ for all n, and thus μ and 2μ agree on the B_n.

The following problem asks if we can weaken the hypotheses of Theorem 2.6.

Problem 2.8. *Let X be an uncountable Polish space. Does there exists a standard Borel structure Σ on the set $\mathcal{M}^+_\sigma(X)$ of positive σ-finite measures on X satisfying:*
(1) For every Borel set $B \subseteq X$ the map $\mu \mapsto \mu(B)$ is Borel?
Does there exists one satisfying:
(2) The map $(\mu, K) \mapsto \mu(K)$ on $\mathcal{M}^+_\sigma(X) \times \mathcal{K}(X)$ is Borel ($\mathcal{K}(X)$ denotes the standard Borel space of compact subsets of X).

We note that if $\mathcal{M}(X)$ is the standard Borel space of probability measures on a Polish space X, then the map $(\mu, B) \mapsto \mu(B)$ from $\mathcal{M}(X) \times \mathcal{B}(X)$ to $[0, 1]$ as in the statement of Theorem 2.6 is Borel in the codes. Thus, Theorem 2.6 exhibits an essential difference between the finite and the σ-finite measures on an uncountable Polish space.

Despite Fact 2.5, it is still true that under ZFC + CH there is a nonuniformly σ-finite disintegration. To see this, we introduce a combinatorial principle $P(\kappa)$ for κ an uncountable cardinal.

Definition 2.9. $P(\kappa)$ is the statement that for every sequence $\{B_\alpha\}_{\alpha<\kappa}$ of sets $B_\alpha \subseteq \kappa$, and every family $\{f_{\alpha,n} \colon \alpha < \kappa,\ n \in \omega\}$ of functions $f_{\alpha,n} \colon \kappa \to \kappa$, there is a sequence $\{S_\alpha\}_{\alpha<\kappa} \subseteq \mathcal{P}_{\omega_1}(\kappa)$ of countable subsets of κ satisfying:
(1) $\forall \alpha < \kappa\ \exists \beta < \kappa\ S_\beta \neq \{f_{\alpha,n}(\beta)\}_{n\in\omega}$.
(2) $\forall \alpha < \kappa\ \forall n \in \omega\ [\{\beta < \kappa \colon |S_\beta \cap B_\alpha| = n\}$ is countable or co-countable in κ].

Theorem 2.10. $P(2^\omega)$ *implies there is a purely atomic σ-finite disintegration which is not uniformly σ-finite.*

Proof. Take $\{B_\alpha\}_{\alpha<2^\omega}$ to consist of all Borel sets and take $\{f_{\alpha,n} \colon \alpha < 2^\omega,\ n \in \omega\}$ to be the family of all sequences of Borel measurable functions. Then, by theorem 2.1, taking μ_α to be counting measure on S_α, we have such a disintegration. □

We are interested in the strength of $P(\kappa)$.

Theorem 2.11 (ZF)**.** $P(\omega_1)$ *holds. In particular, assuming CH we have* $P(2^\omega)$.

Proof. Let the B_α and $f_{\alpha,n}$ be as in the hypothesis of $P(\omega_1)$. We define the countable sets S_β, $\beta < \omega_1$, as follows. Assume $S_{\beta'}$ has been defined for all $\beta' < \beta$. We let S_β be such that

(i) $\min(S_\beta) > \sup_{\beta'<\beta} \sup(S_{\beta'})$.
(ii) for all $\beta' < \beta$, if $B_{\beta'}$ is uncountable then $|S_\beta \cap B_{\beta'}| = \omega$.
(iii) $S_\beta \nsubseteq \{f_{\beta,n}(\beta) : n \in \omega\}$.

Since there are only countably many β' less than β, we can get a countable S_β which meets the second requirement above, and adding an extra point will meet the third requirement. It is now easy to verify the statements of $P(\omega_1)$. Property 1 of definition 2.9 follows from (iii) above (using $\beta = \alpha$). To see property 2, fix B_α and $n \in \omega$. If B_α is countable then by (i) above we have that for large enough β that $S_\beta \cap B_\alpha = \emptyset$, which gives 2 in definition 2.9. If B_α is uncountable, then for $\beta > \alpha$ we have $B_\alpha \cap S_\beta$ is infinite. This again gives 2. □

We show that it is consistent that $P(2^\omega)$ fails.

Theorem 2.12. *Assume $2^\omega = 2^{\omega_1} = \omega_2$. Then $P(2^\omega)$ fails.*

Proof. Let κ denote $2^\omega = \omega_2$. We define the sets B_α and functions $f_{\alpha,n}$ witnessing the failure of $P(\kappa)$. Consider the collection of all ω sequences $(f_0, f_1, \dots)$ of functions $f : \kappa \to \kappa$ which are eventually constant. Under our hypothesis there are only κ many such ω sequences of functions, so we may fix the $f_{\alpha,n}$ so that every such sequence occurs as $(f_{\alpha,0}, f_{\alpha,1}, \dots)$ for some $\alpha < \kappa$. For α a successor ordinal let $B_\alpha = \{\alpha - 1\}$. From our hypothesis we may let $\{D_\alpha\}$, for $\alpha < \kappa$ a limit ordinal, enumerate all subsets $D \subseteq \kappa$ of order type ω_1. Let B_α, for α a limit ordinal, be given by $B_\alpha = D_\alpha \cup (\sup(D_\alpha), \kappa)$.

Suppose $\{S_\beta\}_{\beta<\kappa}$ satisfied 1 and 2. We first claim that for any $\alpha, \beta < \kappa$ there is a $\gamma > \beta$ such that $S_\gamma \nsubseteq \alpha$. To see this, suppose α, β were to the contrary. For every $\alpha' < \alpha$ we have that for large enough γ_1, γ_2 that $\alpha' \in S_{\gamma_1} \leftrightarrow \alpha' \in S_{\gamma_2}$. For otherwise $B_{\alpha'+1} = \{\alpha'\}$ would violate 2. But this then gives that for all large enough γ that $S_\gamma = S_\gamma \cap \alpha$ is the same. Let $f_n : \kappa \to \kappa$ be such that $S_\beta = \{f_n(\beta)\}_{n\in\omega}$ for all $\beta < \kappa$. We may assume that the f_n are eventually constant, since the S_β are eventually constant. So, there is an $\alpha_0 < \kappa$ such that $f_n(\beta) = f_{\alpha_0,n}(\beta)$ for all $n \in \omega$ and $\beta < \kappa$. This α_0 then violates 1. This proves the claim. We next claim that there is an $\alpha_0 < \kappa$ such that for all $\alpha, \beta < \kappa$ there is a $\gamma > \beta$ such that $\min(S_\gamma - \alpha_0) > \alpha$. Suppose this claim fails. We construct inductively an increasing sequence α_η, for $\eta < \omega_1$, such that for all $\eta < \omega_1$ and all large enough γ we have $\alpha_\eta \in S_\gamma$. This will contradict the fact that all the S_γ are countable. Suppose α_η is defined for $\eta < \eta'$. Let $\alpha = \sup\{\alpha_\eta : \eta < \eta'\}$. By the assumed failure of the claim, there is an $\alpha' > \alpha$ such that for κ many $\gamma < \kappa$ we have $\min(S_\gamma - \alpha) < \alpha'$. We may then fix $\alpha_{\eta'} \in (\alpha, \alpha')$ such that for κ many γ we have $\alpha_{\eta'} \in S_\gamma$. As in the proof of the first claim above, 2 implies that for all large enough γ that $\alpha_{\eta'} \in S_\gamma$. Thus, we may continue to construct the α_η for all $\eta < \omega_1$, a contradiction. This proves the second claim. Fix $\bar\alpha$ as in the second claim. From the second claim, we can get an increasing ω_1 sequence $\{\gamma_\eta\}_{\eta<\omega_1}$ such that $\inf(S_{\gamma_\eta} - \bar\alpha) > \sup_{\eta'<\eta}(\sup S_{\gamma'_\eta})$ for all $\eta < \omega_1$. Let $\alpha_\eta \in S_{\gamma_\eta} - \bar\alpha$ for all $\eta < \omega_1$. Let $D = \{\gamma_\eta : \eta \text{ is even }\}$. Let δ be a limit ordinal such that $B_\delta = D \cup (\sup(D), \kappa)$. Then $A = \{\beta < \kappa : |S_\beta \cap B_\delta| = 0\}$ and $\kappa - A$ both meet $\{\gamma_\eta : \eta < \omega_1\}$ in a set of size ω_1, contradicting 2. □

Problem 2.13. *Is it consistent that CH fails and $P(2^\omega)$ holds?*

Problem 2.14. *Is it consistent that every σ-finite disintegration be uniformly σ-finite?*

3. Construction of a nonuniformly σ-finite disintegration assuming the existence of a special $\mathbf{\Pi}^1_1$ set

In this section, let both X and Y be the Baire space. So, $X = Y = \omega^\omega$ where ω has the discrete topology and X and Y have the product topology. Let P be a closed subset of $X \times Y$ such that $\forall x \in X$, P_x is nonempty and perfect and if $x \neq x'$, $P_x \cap P_{x'} = \emptyset$. We say G is a special coanalytic set for P provided $G \subseteq P$ is a $\mathbf{\Pi}^1_1$ set with the following properties:

(1) $\forall x \in X \ |G_x| = \omega_0$,
(2) G is not the union of countably many $\mathbf{\Pi}^1_1$ graphs over X,
(3) for every $n \in \omega$ and for every $B \in \mathcal{B}(Y)$, $\{x \in X : |B \cap G_x| = n\} \in \mathcal{B}(X)$.
(4) there is a nonempty Borel set (or even perfect) $H \subseteq X$ such that $G \cap (H \times Y)$ is the union of countably many pairwise disjoint Borel graphs over H.

We note that this last condition is not necessary to construct a nonuniformly σ-finite disintegration. One may simply take the set H to be a singleton and construct a disintegration with respect to a Dirac measure ν. However, we include this last condition so that the measure ν may be chosen to be non-atomic.

Theorem 3.1. *Let $X = Y = \omega^\omega$. Let $P = \{((x_i), (y_i)) \in \omega^\omega \times \omega^\omega : \forall i \in \omega[y_{2i} = x_i]\}$.*

If G is a special coanalytic set for P, then there exists a σ-finite measure μ on Y, a σ-finite measure ν on X, a Borel measurable map $\phi : Y \mapsto X$, and a σ-finite disintegration $\{\mu_x : x \in X\}$ of μ with respect to (ν, ϕ) which is not uniformly σ-finite.

Proof. Let $\pi_i : \omega^\omega \times \omega^\omega \to \omega^\omega$ be the projection map onto the ith coordinate. Note P is closed, $\pi_1(P) = \omega^\omega = \pi_2(P)$, and if $x, x' \in \omega^\omega$ with $x \neq x'$ then $P_x \cap P_{x'} = \emptyset$. Note the sections P_x are disjoint and perfect. Define the function $\phi : Y \to X$ by $\phi(y) = x \iff y \in P_x$. The function ϕ is Borel measurable since its graph is a Borel set. Next define a σ-finite transition kernel $\{\mu_x : x \in X\}$. For each $x \in X$ and $B \in \mathcal{B}(Y)$ define $\mu_x(B) = |B \cap G_x|$, *i.e.*, counting measure on the fibers of G. Since each fiber G_x is countably infinite, μ_x is σ-finite for all x in X. Also since the fibers are pairwise disjoint, $\mu_x(Y \setminus \phi^{-1}(x)) = 0$. If $B \in \mathcal{B}(Y)$ then $\{x : \mu_x(B) \geq n\} = \{x : |B \cap G_x| \geq n\}$ which is a Borel subset of X since G is special. Thus for every $B \in \mathcal{B}(Y)$ the function $x \to \mu_x(B)$ is $\mathcal{B}(X)$-measurable and $\{\mu_x : x \in X\}$ is a transition kernel.

Since G is special, there is a Borel set $H \subseteq X$ and Borel functions $f_n : X \to Y$ with pairwise disjoint graphs such that for every $x \in H$ $G_x = \bigcup_n \{f_n(x)\}$. Note that since the sections of G are pairwise disjoint, each f_n is 1-to-1 over H. Let ν be a probability measure on $\mathcal{B}(X)$ such that $\nu(H) = 1$.

Define a measure μ on the Borel subsets of Y by

$$\mu(B) = \int \mu_x(B) d\nu(x).$$

We first show that μ is σ-finite. Let $B_n = f_n(H)$ and note that $\forall x \in H$, $G_x \subseteq \bigcup_n B_n$. Each B_n is Borel since each f_n is 1-to-1 over H, and $\forall x \in H$,

$\mu_x(B_n) = |B_n \cap G_x| = 1$. Furthermore

$$\begin{aligned}\mu\left(Y \setminus \bigcup_n B_n\right) &= \int \mu_x\left(Y \setminus \bigcup_n B_n\right) d\nu(x)\\ &= \int_{X\setminus H} \left|\left(Y \setminus \bigcup_n B_n\right) \cap G_x\right| d\nu(x) + \int_H \left|\left(Y \setminus \bigcup_n B_n\right) \cap G_x\right| d\nu(x)\\ &= \int_H \left|\left(Y \setminus \bigcup_n B_n\right) \cap G_x\right| d\nu(x) = 0.\end{aligned}$$

The measure μ is thus a σ-finite measure on Y and the family $\{\mu_x : x \in X\}$ is a disintegration of μ with respect to (ν, ϕ) into σ-finite measures. However, this disintegration cannot be uniformly σ-finite. If it were, there would exist countably many Borel sets $E_n \subseteq Y$ such that $\forall x \in X$, $\mu_x(E_n) < \infty$ and $\mu_x(Y \setminus \cup_n E_n) = 0$. Thus for each $x \in X$, $|G_x \cap E_n| < \infty$ and $G \subseteq \bigcup_n X \times E_n$. For each n, $G \cap (X \times E_n)$ is $\mathbf{\Pi}^1_1$ with finite sections and is thus a countable union of $\mathbf{\Pi}^1_1$ graphs (see [6]) implying that $G = \bigcup_n G \cap E_n$ is a countable union of $\mathbf{\Pi}^1_1$ graphs, a contradiction. □

This argument shows that in fact there does not exist countably many $E_n \in \mathcal{B}(X \times Y)$ satisfying $\forall x\ \mu_x(E_{nx}) < \infty$ and $\mu_x(Y \setminus \bigcup_n E_{nx}) = 0$.

4. Construction of a "special" $\mathbf{\Pi}^1_1$ set assuming $\mathbf{V} = \mathbf{L}$

In this section we consider the Polish spaces $X = Y = \omega^\omega$ and we prove the existence of a "special" $\mathbf{\Pi}^1_1$ set assuming $\mathbf{V} = \mathbf{L}$. In order to do this we first put in place the formal logical structures which will be needed. We let ZF_N denote a finite fragment of ZF that is large enough such that Π^1_1 and Σ^1_1 formulas are absolute for transitive models of ZF_N.

It will be necessary to code models by elements of ω^ω. We now make this coding specific.

For each n let ϕ_n be the n-th formula in the Gödel numbering of the formulas in the language $\mathcal{L}^\in$ (see [7] Def 1.4 pp 155). Given $x \in \{0,1\}^\omega \subseteq \omega^\omega$, we will define the theory Th_x by $\phi_n \in Th_x$ if and only if $x(n) = 1$. Let $\phi_{<L}$ be a formula defining the canonical well-ordering of $\mathbf{L}$ and let $M \in \omega$ be the integer such that $\phi_M =$ "$\phi_{<L}$ is a well ordering of the universe."

Let $C \subseteq \omega^\omega$ be the collection of codes of theories, *i.e.*, $x \in C$ iff:

(1) $x \in \{0,1\}^\omega$
(2) Th_x is a consistent and complete theory of $\mathrm{ZF}_N + (\mathbf{V} = \mathbf{L})$
(3) $x(M) = 1$.

Note that C is a $\mathbf{\Delta}^1_1$ set.

Given a formula $\phi_n(w, x_1, \dots, x_k)$ with free variables $w, x_1, \dots, x_k$ define the **Skolem term** for ϕ_n to be the corresponding formula $\tau_n(z, x_1, \dots, x_k)$ where $\tau_n(z, x_1, \dots, x_n)$ is

$$\begin{aligned}&(\exists w\, \phi_n(w, x_1, \dots, x_k) \wedge z \text{ is the } <_L \text{ least such } w) \vee\\ &(\neg\exists w\, \phi_n(w, x_1, \dots, x_k) \wedge z = 0)\,.\end{aligned}$$

For each $x \in \{0,1\}^\omega$ if S is a collection of Skolem terms, define an equivalence relation, $\equiv_x$, on S by

$$\tau_n \equiv_x \tau_m \iff Th_x \vdash \tau_n = \tau_m.$$

For $x \in C$, define M_x to be the set of equivalence classes of all Skolem terms arising from formulas $\phi(w)$ such that $Th_x \vdash \exists w[\phi(w)]$. We note the Skolem hull of $\emptyset$ inside of M_x is all of M_x. In other words, M_x is the smallest model of the theory Th_x. Define the relation E_x on $M_x \times M_x$ by

$$[\tau_i]E_x[\tau_j] \iff Th_x \vdash \tau_i \in \tau_j.$$

Recall that a structure M with binary relation E is well-founded if every nonempty subset of M contains an E-minimal element (see [7] Ch. 3). For each $x \in C$, note that M_x does not necessarily code a well-founded structure. However, if M_x is well-founded, then there exists a countable ordinal α such that $M_x \cong L_\alpha$ (see [7] Thm. 3.9(b) p. 172). The following proposition shows that codings of well-founded models are unique.

Proposition 4.1. *Suppose $x, x' \in C$ and there is an ordinal α such that $M_x \cong L_\alpha \cong M_{x'}$. Then $x = x'$.*

Proof. Let T be the theory of L_α. Since $M_x \cong L_\alpha$ and $M_{x'} \cong L_\alpha$, both x and x' code T. Then for every n, $x(n) = 1 \iff \phi_n \in T \iff x'(n) = 1$. Thus $x = x'$. □

We next show that if an element of ω^ω is constructed at an ordinal α then there exists a code $x \in C$ for a structure (M_x, E_x) that is isomorphic to L_α.

Proposition 4.2. *If $\omega^\omega \cap L_{\alpha+1} \setminus L_\alpha \neq \emptyset$ then $\exists x \in C$ such that $M_x \cong L_\alpha$.*

Proof. Let T be the theory of L_α and let $x \in C$ such that $Th_x = T$. Then (M_x, E_x) is an elementary submodel of $(L_\alpha, \in)$ (see [7] Lemma 7.3 p.136). Since L_α is well-founded, M_x is well-founded. Then $\in$ is well-founded on the transitive collapse $TC(M_x)$ (see [7] Thm. 5.14 p. 106) and thus $(M_x, E_x) \cong (TC(M_x), \in) \cong (L_\beta, \in)$ for some $\beta \leq \alpha$. So $w \in L_{\beta+1}$ and thus $\beta = \alpha$. □

Theorem 4.3. *Assume $\mathbf{V} = \mathbf{L}$. Let $X = Y = \omega^\omega$. Let P be a closed subset of $X \times Y$ such that $\forall x \in X$, P_x is nonempty and perfect and if $x \neq x'$, $P_x \cap P_{x'} = \emptyset$. Then there exists a $\mathbf{\Pi}^1_1$ set $G \subseteq P$ with the following properties:*

1. *$\forall x \in X, |G_x| = \omega_0$*
2. *For every $n \in \omega$ and for every $\mathbf{\Delta}^1_1$ set $B \subseteq Y$, $\{x \in X : |B \cap G_x| \geq n\}$ is $\mathbf{\Delta}^1_1$*
3. *G is not the union of countably many $\mathbf{\Pi}^1_1$ graphs over X.*
4. *There is a nonempty $\mathbf{\Delta}^1_1$ (or even perfect) set $H \subseteq X$ such that $G \cap (H \times Y)$ is the union of countably many pairwise disjoint $\mathbf{\Delta}^1_1$ graphs over H.*

Proof. Fix a pair of recursive bijections, $x \mapsto (x^n)_{n=0}^\infty$ from ω^ω onto $(\omega^\omega)^\omega$ and $x \mapsto (x^0, x^1)$ from ω^ω onto $\omega^\omega \times \omega^\omega$. Denote the inverse of the second bijection by $(y, z) \mapsto \langle y, z \rangle$. Call an ordinal β **good** if $L_\beta \models \mathrm{ZF}_N + (\mathbf{V} = \mathbf{L})$.

Let $p \in \omega^\omega$ be a code for P. In this regard, when we say "z codes the Borel set B" we mean a coding such that the statement "w is in the set coded by z" is absolute to all transitive models of ZF_N (for example, we could have z code a wellfounded tree on ω which gives an inductive construction of B from the basic open sets).

For each $n \in \omega$ let $f_n : X \to Y$ be a $\mathbf{\Delta}^1_1$ function such that $\forall x \in X$ and for $n \neq m$ $f_n(x) \neq f_m(x)$ and such that $\forall x \in X$ $\forall n \in \omega$ $f_n(x) \in P_x$.

For a given $w \in \omega^\omega$ and an $x \in C$ coding an ω-model M_x (*i.e.* ω is in the well-founded part of M_x), we will make frequent use of the shorthand "$w \in M_x$" to mean (for convenience, we identify here ω^ω with $\mathcal{P}(\omega)$)

$$\exists \tau \in \operatorname{dom}(M_x)\ [(M_x \models \text{“}\tau \subseteq \omega\text{”}) \wedge TC(\tau) = w].$$

Define $U \subseteq C$ by $x \in U$ if and only if there exists an ordinal $\alpha(x) \geq \omega_0$ such that $M_x \cong L_{\alpha(x)}$ and $p \in L_{\alpha(x)}$. Define $V \subseteq C$ by $x \in V$ iff M_x is an ω-model, and "$p \in M_x$". Note that $U \subseteq V$, V is $\mathbf{\Delta}^1_1$, and that the elements of U code well-founded structures.

Define the set $G' \subseteq X \times Y$ by $(x,y) \in G' \iff$

$$\begin{aligned}&[x \notin V \wedge \exists n(y = f_n(x))] \vee [x \in V \wedge (x,y) \in P \wedge [\text{“}y \in M_x\text{”} \vee \\ &\quad \exists \text{ a well-founded extension } M \text{ of } M_x\ \exists \alpha', \alpha < \omega_1 \\ &\qquad (L_{\alpha'} \cong M_x \subseteq M \cong L_\alpha \wedge y \in L_\alpha \wedge \\ &\qquad [\forall \alpha' \leq \gamma < \alpha\ (\neg(\gamma \text{ is good and a limit of good ordinals}) \vee \\ &\qquad\quad \exists \phi \in \Sigma^1_2\ \exists \tau > \gamma\ (L_\gamma \models \neg\phi \wedge L_\tau \models \phi))])]].\end{aligned}$$

To clarify, if $x \in V$ and M_x is ill-founded then G'_x consists of all reals in M_x. If $x \in V$ and M_x is well-founded then we continue adding reals to the section G'_x until the truth of Σ^1_2 statements stabilize to be true.

Note that G' is Σ^1_2 and let $\Omega'(x,y)$ be the above Σ^1_2 formula defining G'.

We first show that the sections of G' are countable. Clearly G'_x is countable for every $x \notin V$. Since each model M_x is countable, G'_x is countable for every $x \in V \setminus U$. Finally suppose $x \in U$. Let M be a well-founded extension of M_x as in the definition above for G'. Let α be the ordinal such that $M \cong L_\alpha$. Let β be the least good ordinal less than ω_1 such that L_β is a Σ_2 elementary substructure of $\mathbf{L}$. Then for every $\beta' > \beta$ and every Σ^1_2 formula ϕ we have

$$L_\beta \models \phi \iff L_{\beta'} \models \phi \iff \mathbf{L} \models \phi.$$

We clearly have that β is good and a limit of good ordinals, and by the definition of G' we must have $\beta \geq \alpha$. Thus $G'_x \subseteq L_\beta$ and is therefore countable.

Let G be a Π^1_1-uniformization of G', *i.e.* a subset of $\omega^\omega \times \omega^\omega$ such that for every $x \in \omega^\omega$

$$G'(x,y) \iff \exists z\, G(x, \langle y,z\rangle) \iff \exists! z\, G(x, \langle y,z\rangle).$$

Let Ω be a Π^1_1 formula defining G. We assume that ZF_N was chosen large enough such that the following is a theorem of ZF_N.

$$\forall x\, \forall y [\Omega'(x,y) \iff \exists z\, \Omega(x, \langle y,z\rangle) \iff \exists! z\, \Omega(x, \langle y,z\rangle)].$$

Note that since the sections of G' are countable so too are the sections of G. Note also that if $H = X \setminus V$ then property (4) holds for G. Next we proceed to show that the Borel condition in property (2) holds for G.

Fix a $\mathbf{\Delta}^1_1$ set $B \subseteq Y$, fix an $n \in \omega$, let $K_n = \{x \in X : |B \cap G_x| \geq n\}$, let $b \in \omega^\omega$ be a code for B, and since we are assuming $\mathbf{V} = \mathbf{L}$ let τ be the level of L at which b is constructed. Then τ is well-defined and $\tau < \omega_1$. Partition V into the following $\mathbf{\Delta}^1_1$ sets: $E = \{x \in V : \text{“}b \notin M_x\text{”}\}$ and $D = \{x \in V : \text{“}b \in M_x\text{”}\}$.

Define the formula

$$\psi(x) = \exists \text{ distinct } a_1, \ldots, a_n\ [\text{“}a_1, \ldots, a_n \in M_x\text{”} \wedge (a_1, \ldots, a_n \in B)]$$

Clearly $\psi(x)$ is a Σ^1_1 statement about x.

By the definition of G, ψ correctly defines K_n on $V \setminus U$. For $x \in U \cap D$, "$b \in M_x$" and since Σ^1_1 statements are absolute between transitive models of ZF_N, ψ correctly defines K_n on $U \cap D$. Since $\tau < \omega_1$ and distinct $x \in U$ determine distinct well-founded L_α, there can be only countably many $x \in U$ which code L_α with $\alpha < \tau$. If $x \in U \cap E$ then $M_x \cong L_\alpha$ where $\alpha < \tau$. Thus $U \cap E$ is countable. Therefore the formula ψ correctly defines K_n on V except for the countable set $U \cap E$.

To see that $(K_n \cap V) \setminus (U \cap E)$ is Δ^1_1, note that the formula ψ is equivalent to the Σ^1_1 formula

$$\exists i_1, \dots, \exists i_n \in \omega, \exists a_1, \dots, \exists a_n \in \omega^\omega \ [M_x \models \text{“} i_1, \dots, i_n \in \omega^\omega \text{”} \wedge \\ TC(i_1) = a_1, \dots, TC(i_n) = a_n \wedge a_1, \dots, a_n \in B]$$

which is equivalent to the Π^1_1 formula

$$\exists i_1, \dots, \exists i_n \in \omega, \forall a_1, \dots, \forall a_n \in \omega^\omega \ [M_x \models \text{“} i_1, \dots, i_n \in \omega^\omega \text{”} \wedge \\ (TC(i_1) = a_1, \dots, TC(i_n) = a_n) \Rightarrow a_1 \in B, \dots, a_n \in B].$$

Thus ψ defines a Δ^1_1 set which gives K_n on $V \setminus (U \cap E)$, and since $(U \cap E)$ is countable, $K_n \cap V$ is Δ^1_1.

For $x \in X \setminus V$ each section $G_x = \bigcup_{n=1}^\infty f_n(x)$. Thus for each $x \in X \setminus V$ we have

$$|B \cap G_x| \geq n \iff \exists \text{ distinct } k_1, \dots, k_n [f_{k_1}(x) \in B, \dots, f_{k_n}(x) \in B] \\ \iff x \in \bigcup_{(k_1, \dots, k_n)} f^{-1}_{k_1}(B) \cap \dots \cap f^{-1}_{k_n}(B).$$

Therefore $K_n \cap X \setminus V$ is $\mathbf{\Delta}^1_1$.

Finally we show that property (3) holds for G. Proceeding by contradiction suppose that G could be written as a countable union of $\mathbf{\Pi}^1_1$ graphs G_m. Choose a sequence (x^m) from ω^ω and formulas $\psi_m(x, y)$ so that ψ_m are $\mathbf{\Pi}^1_1(x^m)$ formulas defining the G_m. Let $x' \in \omega^\omega$ be such that x' codes the sequence $(x^m)^\infty_{m=0}$ and choose $x \in U$ and α such that $M_x \cong L_\alpha$ and $x' \in L_\alpha$. Next let $\beta \geq \alpha$ be the least ordinal such that (β is good and a limit of good ordinals) $\wedge\, \forall \phi \in \Sigma^1_2\, (L_\beta \models \neg\phi \Rightarrow \forall \tau > \beta\, L_\tau \models \neg\phi)$.

From the definition of G' we have that $\omega^\omega \cap L_\beta \subseteq G'_x$. Furthermore if $y \in L_\beta$ then for some good ordinal $\delta < \beta$, $y \in L_\delta$. Since β was chosen to be minimal, we have that $\forall \gamma < \delta\, [\neg(\gamma \text{ is good and a limit of good ordinals }) \vee \exists \phi \in \Sigma^1_2\ (L_\delta \models \neg\phi \wedge \exists \tau > \gamma\, (L_\tau \models \phi))]$. In fact we may replace "$\exists \tau > \gamma$" in the previous statement with "$\exists \tau > \gamma, \tau < \beta$". Thus δ witnesses that $L_\beta \models \Omega'(x, y)$.

Since β was chosen so that Σ^1_2 statements are stabilized at β, we have that $L_\beta \models$ "$\{y : \exists m\, \psi_m(x^m, y)\}$ is countable". However, $L_\beta \models$ "ω^ω is uncountable". Thus we may let $y, z \in L_\beta$ such that

$$L_\beta \models \Omega(x, \langle y, z\rangle) \text{ and} \\ L_\beta \models \forall m\ \neg\psi_m(x^m, \langle y, z\rangle).$$

Then by absoluteness $\mathbf{L} \models \forall m\, \neg\psi(x^m, \langle y, z\rangle)$. Thus $\forall m\, (x, \langle y, z\rangle) \notin G_m$. However this contradicts the fact that $\mathbf{L} \models \Omega(x, \langle y, z\rangle)$ by absoluteness and therefore $(x, \langle y, z\rangle) \in G$.

□

This naturally leads us to ask:

Problem 4.4. *Can one show in ZFC that special $\mathbf{\Pi}^1_1$ sets exist?*

5. Uniformly σ-finite implies joint measurability, and an equivalence to Maharam's problem

Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be Polish spaces, let $\phi\colon Y \to X$ be $\mathcal{B}$-measurable and let μ and ν be measures on $\mathcal{B}(Y)$ and $\mathcal{B}(X)$. Let $x \mapsto \mu_x$ be a *measure kernel*, that is, each μ_x is a measure on the Borel subsets of Y and such that for each Borel set E in Y, the map $x \mapsto \mu_x(E)$ is Borel measurable (this is part of the definition of a disintegration). Let $\mathcal{K}(Y)$ be the space of compact subsets of Y equipped with the Vietoris topology or equivalently the topology generated by the Hausdorff metric.

Lemma 5.1. *If for every x, $\mu_x(Y) < \infty$, then the map $F\colon X \times \mathcal{K}(Y) \mapsto \mathbb{R}$, given by $F(x, K) = \mu_x(K)$, is Borel measurable.*

Proof. Fix a basis for the topology of Y, say $\{V_n\}_{n=1}^{\infty}$. Enumerate sets of the form $\{K\colon K \subseteq V_{i_1} \cup \ldots \cup V_{i_j}\}$, say $\{U_n\}_{n=1}^{\infty}$. U_n is an open set in $\mathcal{K}(Y)$, and let $\tilde{U}_n = V_{i_1} \cup \ldots \cup V_{i_j}$ be the corresponding open set in Y. Fix a number c. For each n, let $D_n = \{x\colon \mu_x(\tilde{U}_n) < c\}$. We have $\{(x, K)\colon \mu_x(K) < c\} = \bigcup_n (D_n \times U_n)$. □

Lemma 5.2. *If for every x, $\mu_x(X) < \infty$, then for each $\epsilon > 0$, there is a Borel measurable map $x \mapsto K \in \mathcal{K}(Y)$ such that for every x, $\mu_x(Y \setminus K_x) < \epsilon$.*

Proof. This lemma follows from Theorem 2.2 of [12]. □

For the statement of the next theorem we introduce the following notations. Let $\mathcal{H} \subseteq X \times \mathcal{K}(Y)^\omega$ be the set:

$$\mathcal{H} = \{(x, \{K_n\}_{n\in\omega})\colon \forall n\ \mu_x(K_n) < +\infty \wedge \mu_x(Y - \bigcup_n K_n) = 0\}.$$

Let $\mathcal{M}$ denote the function $(\mu, B) \mapsto \mu(B)$ defined on $M^+_\sigma(Y) \times \mathcal{K}(Y)$.

Theorem 5.3. *Suppose $\{\mu_x : x \in X\}$ is a σ-finite disintegration of μ with respect to (ν, ϕ). Consider the following statements.*

1. *$\{\mu_x : x \in X\}$ is uniformly σ-finite.*
2. *There is a Borel uniformization of $\mathcal{H}$, that is, there is a sequence of Borel mappings $x \mapsto K_n(x)$ from X into $\mathcal{K}(Y)$ satisfying*
   - $\forall x \forall n\ \mu_x(K_n(x)) < \infty$
   - $\forall x\ \mu_x(Y \setminus \bigcup_n K_n(x)) = 0$.
3. *$\mathcal{M}$ is Borel measurable and $\mathcal{H}$ is Borel.*
4. *$\mathcal{M}$ is Borel measurable.*

Then statements (1) and (2) are equivalent and each of them implies statement (3), and (3) implies (4). Moreover, if each measure μ_x is purely atomic, then statements (1),(2), (3), and (4) are equivalent.

Proof. (1) ⇒ (2)
Fix $\{B_n\}$ witnessing the kernel $x \mapsto \mu_x$ is uniformly σ-finite. We may and do assume that for each $n, B_n \subseteq B_{n+1}$. For each n, let $\mu_{nx}(E) = \mu_x(E \cap B_n)$. then by Lemma 5.2, we obtain Borel measurable maps $x \mapsto K_{nmx} \in \mathcal{K}(Y)$ such that for every x, $\mu_{nx}(Y \setminus \bigcup_m K_{nmx}) = 0$. The implication follows.

(1) ⇒ (4)
Continuing with the preceding argument, we see that for each n, the map $F_n(x, K) = \mu_x(B_n \cap K)$ is Borel measurable and $F_n(x, K)$ converges up to $F(x, K)$.

(2) $\Rightarrow$ (1)
For each n let G_n be the 'epigraph' of the mapping $x \mapsto K_n(x)$. By 'epigraph' we mean

$$G_n = \{(x, y) : y \in K_n(x)\}.$$

Note that a function $f\colon X \to \mathcal{K}(Y)$ is Borel iff the epigraph, $\{(x, y)\colon y \in f(x)\}$ is Borel in $X \times Y$.

Let $B_n = \pi_Y(G_n \cap Graph(\phi))$. This projection is 1-to-1 therefore B_n is Borel. Observe that

$$\begin{aligned}\mu_x(B_n) &= \mu_x(K_n(x) \cap \phi^{-1}(x)) \\ &= \mu_x(K_n(x)) < \infty \quad \text{and} \\ \mu_x(Y \setminus \bigcup_n B_n) &= \mu_x\left(Y \setminus \bigcup_n (K_n(x) \cap \phi^{-1}(x))\right) \\ &= \mu_x\left(Y \setminus \bigcup_n K_n(x)\right) = 0.\end{aligned}$$

(2) $\Rightarrow$ (3)
We just need to show (2) imples that $\mathcal{H}$ is Borel. Let the Borel functions $x \mapsto K_n(x)$ be as in (2). We have $(x, \{T_n\}) \notin \mathcal{H}$ iff $\exists n\ (\mu_x(T_n) = +\infty)$ or $\exists m\ \mu_x(K_m(x) - \bigcup_n T_n) > 0$. The first disjunct clearly defines a Borel set (given (4), which we have by the above (2) $\Rightarrow$ (4)). The second disjunct is equivalent to (since each $\mu_x(K_m) < \infty$)

$$\exists m\ \exists p > 0\ \forall k\ (\mu_x(\bigcup_{n=1}^{k} T_n \cap K_m(x)) \leq \mu_x(K_m(x)) - \frac{1}{p}.$$

This is Borel by (4) and the fact that for each n, the map $(K_1, \ldots, K_n) \mapsto K_1 \cap \cdots \cap K_n$ from $\mathcal{K}(X)^n$ to $\mathcal{K}(X)$ is Borel (see, for example, page 180 of [8]).

Finally, let us assume that for every x, the measure μ_x is purely atomic and statement (4) holds. Let $W = \{(x, K)\colon \mu_x(K) > 0 \text{ and } card(K) = 1\}$. Then W is a Borel subset of $X \times \mathcal{K}(Y)$ with countable sections. Therefore, there are Borel functions $x \mapsto \mathcal{K}(Y)$ whose graphs fill up W. This means statement (2) holds.

Remark 5.4. Thus, if (1) and (4) are equivalent we have that $\mathcal{H}$ must be Borel. It is not clear, however, if (3) implies (1). We thank the referee for mentioning (3) to us.

□

Problem 5.5. *Is it true that a disintegration is uniformly σ-finite if and only if the map $(x, K) \mapsto \mu_x(K)$ is jointly measurable?*

We would like to mention the following problem concerning the mixture operator defined by a measure transition kernel.

Problem 5.6. *Suppose we are given a measure kernel $x \mapsto \mu_x$ (defined at the beginning of this section). Consider the mixture operator T defined by*

$$T(\lambda)(E) := \int_X \mu_x(E) d\lambda(x).$$

Suppose this operator has the property that it maps σ-finite (signed) measures on X to σ-finite (signed) measures on Y and the operator T is lattice preserving, i.e., T takes mutually singular measures to mutually singular measures. Is there a universally measurable map $\phi : Y \mapsto X$ such that for each x, $\mu_x(Y \setminus \phi^{-1}(x)) = 0$?

We mention that it was shown in [13] that the answer is yes assuming Martin's axiom or even weaker that a medial limit exists provided for each x, μ_x is a probability measure.

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