SURVEY OF THE STEINHAUS TILING PROBLEM

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Abstract. We survey some results and problems arising from a classic problem of Steinhaus: Is there a subset S of $\mathbb{R}^2$ such that each isometric copy of $\mathbb{Z}^2$ (the lattice points in the plane) meets S in exactly one point.

1. Introduction

We survey in this paper some of the known results, methods, and open questions concerning *Steinhaus sets*. Although this is primarily a survey article, we include a few new results such as the existence of Steinhaus sets for certain other rectangular lattices, theorem 2.11. The notion of a Steinhaus set can be introduced at various levels of generality. Steinhaus asked in the late fifties if there could be a set $S \subseteq \mathbb{R}^2$ such that S meets every isometric copy of the integer lattice $\mathbb{Z} \times \mathbb{Z}$ in exactly one point. We are asking this question in the context of ZFC, that is, assuming choice, but the question remains interesting in the context of ZF as well. The question can be immediately extended to higher dimensions and other lattices as well. Thus, if $L \subseteq \mathbb{R}^m$ is a lattice and $n \geqslant m$, we can ask if there is a set $S \subseteq \mathbb{R}^n$ which meets every $\pi(L)$ is exactly one point, where $\pi\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isometry (and we view $\mathbb{R}^m \subseteq \mathbb{R}^n$ in the usual way). This will be the context for most of the discussion of this paper, and we will in fact generally have $m = n$. In this case, we refer to such a set S, if it exists, as a Steinhaus set for the lattice L. Let us comment that Steinhaus' question concerns the existence of a "simultaneous tiling" of the plane. A set E tiles the plane means the translates of E by the lattice points in Z^2 partitions the plane. So, S is a Steinhaus set means that each rotated copy of S tiles R^2.

We can generalize the basic notions considerably. Given $(X, \mathcal{A}, f)$, where X is a set, $\mathcal{A} \subseteq \mathcal{P}(X)$ is a family of subsets of X, and $f\colon \mathcal{A} \to \mathrm{CARD}$ is a function assigning a cardinality to each $A \in \mathcal{A}$, we may ask if there is a $S \subseteq X$ such that $|S \cap A| = f(A)$ for all $A \in \mathcal{A}$. If X is an uncountable Polish space, we may further ask if S could be Borel, have the Baire property, be measurable with respect to some Borel measure on X, etc. At this level of generality, several questions and results not normally thought of in connection with the Steinhaus problem can be viewed as instances. For example, the theorem of Mazurkiewicz that there is a set in the plane meeting every line in exactly two points is such an instance. We note that it is still open if such a set can be Borel; see [25] for this and related questions. Likewise, the Sierpinski type results on partitions of points and lines in $\mathbb{R}^n$ of [6], [7] can be viewed as instances. We can also use this formulation to

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suggest restrictions on the problem. For example, we might take $X = \mathbb{R}^2$ and $\mathcal{A} \subseteq \mathcal{L}(\mathbb{Z} \times \mathbb{Z})$, where $\mathcal{L}(A)$ denotes the collection of isometric copies $\pi(A)$ of A. $\mathcal{A}$ could be $\{\pi(\mathbb{Z} \times \mathbb{Z}) : \pi \in G\}$ where G is a subgroup of the isometry group, or perhaps all translations of $L_1, \dots, L_k$, where each L_i is a rotation of $\mathbb{Z} \times \mathbb{Z}$. This last possibility, it was observed by Kechris, has a connection with the theory of countable Borel equivalence relations. Many other generalizations are also possible. We note, though, that to be non-trivial, the question requires some sort of geometric condition on the collection $\mathcal{A}$. For example if $\mathcal{A}$ is the collection of all $A \subseteq X$ of size κ, where $\kappa \geqslant 2$ is a cardinal, then trivially there does not exist a corresponding Steinhaus set. It is precisely this interaction between the geometry of X (e.g., $\mathbb{R}^n$) and set theory which we believe makes the problems interesting.

Aside from a few excursions and minor variations, we shall in this paper stay mainly with the "traditional" formulation of the Steinhaus problem where $X = \mathbb{R}^n$, $\mathcal{A}$ is the collection of all isometric copies of a lattice $L \subseteq \mathbb{R}^n$, and f is the constant 1 function. In fact, we will spend a good deal of time on the case originally proposed by Steinhaus, that is, $X = \mathbb{R}^2$ and $\mathcal{A} = \mathcal{L}(\mathbb{Z} \times \mathbb{Z})$ is the collection of all isometric copies of $\mathbb{Z} \times \mathbb{Z}$. Here we have been able to show the existence of a Steinhaus set:

Theorem 1.1. *There is a set $S \subseteq \mathbb{R}^2$ such that $|S \cap L| = 1$ for any isometric copy $L \subseteq \mathbb{R}^2$ of $\mathbb{Z} \times \mathbb{Z}$.*

The arguments here already show interesting and surprising connections between the set theory of the continuum and the geometry of $\mathbb{R}^2$. Let us first discuss some of the history of results leading to theorem 1.1 and mention some of the many open problems remaining. We will discuss some of these problems in more detail later.

As we mentioned above, Steinhaus first raised the question of the existence of such a set in the late 50's. He also asked if there was any set $A \subseteq \mathbb{R}^2$ at all such that a corresponding Steinhaus set exists for the isometric copies of A (in our terminology, $X = \mathbb{R}^2$, $\mathcal{A} = \mathcal{L}(A)$). The trivial cases when $|A| = 1$ or $A = \mathbb{R}^2$ are dismissed. Shortly after this question was raised, Sierpiński [15] and independently later Erdős [5] showed that there was a set A for which a Steinhaus set exists. Komjáth [14] showed that a Steinhaus set $S \subseteq \mathbb{R}^2$ exists when $A = \mathbb{Z}$ (again, we view $\mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{R}^2$ so $\mathcal{L}(A)$ makes sense). Komjáth also showed that a Steinhaus set exists when $A = Q \times Q$. These results certainly pointed in the direction of theorem 1.1, but problems remained (there are still some interesting combinatorial questions related to these "obstructions" which we mention later). The full proof of theorem 1.1 can be found in [9] and [10]. The proof in [9] is shorter but proves only what is necessary to get theorem 1.1. In [10] a more detailed analysis is given which proves some more general results which give a better understanding of the problem. We will not give the detailed proofs here, but present special cases of the arguments which illustrate the main ideas.

It is currently unknown whether there is a Steinhaus set S in $\mathbb{R}^n$ for $n \geqslant 3$ (for the standard lattice $L = \mathbb{Z}^n$). One can also consider lower dimensional lattices, that is, $\mathcal{A} = \mathcal{L}(\mathbb{Z}^m)$ for $m < n$ in our notation. Even for $\mathbb{R}^3$ it is not obvious that either of the questions for $L = \mathbb{Z}^3$ and $L = \mathbb{Z}^2$ implies the other, and both are open. If we consider lattices L other than the standard ones, much remains open, even in $\mathbb{R}^2$. Of course, if a lattice L' differs from L by a uniform scaling, then a Steinhaus set for L can be scaled to get one for L'. We show in theorem 2.11 that Steinhaus sets exist for certain other rectangular lattices in $\mathbb{R}^2$, but for other lattices the problem remains open.

For a different family of questions, one can consider possible regularity properties of Steinhaus sets (we say "possible" since one can obtain results before the existence of the Steinhaus set is settled). For example, can Steinhaus sets $S \subseteq \mathbb{R}^n$ (for various lattices) have the Baire property or be Lebesgue measurable? Can they be Borel? Several people have investigated these questions over the years. It is relatively straightforward (by a category argument, for example) to see a Steinhaus set cannot be open or closed. More recently, the authors have shown it cannot have the Baire property (we give a proof in §3; in essence it is following an argument of Croft).

Theorem 1.2. *Let $L \subseteq \mathbb{R}^n$ be a lattice. Then there does not exists a Steinhaus set $S \subseteq \mathbb{R}^n$ for L having the Baire property. In particular, there does not exists a Borel measurable Steinhaus set for L.*

This result shows that the axiom of choice is necessary for the construction of a Steinhaus set even for the standard lattice in two dimensions. The situation for Lebesgue measurability is currently much less clear. Croft [2] and, independently J. Beck [1] showed that there is no bounded measurable Steinhaus set in the plane, and Koulountzakis obtained some further refinements [12]. For $n \geqslant 3$, Kolountzakis and Wolff [13] showed that there is no measurable Steinhaus set in R^n for the lattice $\mathbb{Z}^n$. Mauldin and Yingst have formulated an extension of some of these results to show that for an interesting array of lattices in R^3 there cannot be a measurable Steinhaus set. These results are discussed in §4. It is still open, however, if there can be a measurable Steinhaus set in the plane.

One can ask about other properties than descriptive complexity or regularity. For example, can there be a bounded Steinhaus set? This is unknown even for the standard lattice. We can consider topological properties of the set. Ciucu [18] showed that a Steinhaus set in $\mathbb{R}^n$ cannot contain an interior point. Must a Steinahus set be totally disconnected? This seems plausible, but it is unknown. Recently, however, Srivastava and Thangadurai [27] have noted that a Steinhaus set in the plane cannot be connected.

In the last section of this paper we collect together the problems which we pose in this article. We want to mention that we are not focusing here on the deeper analytic aspects these lattice tiling problems. Kolountzakis has discussed a number of these in his 1997 article [22] and in his 2001 lectures [23].

2. Existence of Steinhaus Sets in the Plane

We discuss now the proof of the existence of a Steinhaus set in $\mathbb{R}^2$, for the standard lattice $\mathbb{Z} \times \mathbb{Z}$. Although we do not give a complete proof here, we will consider some special cases of the key lemmas which suffice to illustrate the main ideas. In particular, we wish to highlight the interplay of methods from number theory, geometry, set theory, and mechanics which come into play. We let $\mathcal{L} = \mathcal{L}(\mathbb{Z}^2)$ denote the collection of all $L \subseteq \mathbb{R}^2$ which are isometric to $\mathbb{Z}^2$.

To get started, it is helpful to reformulate the problem a bit. By a *lattice distance* we mean a real number of the form $\sqrt{n^2 + m^2}$, where m, n are non-negative integers, not both 0. Let ρ denote the Euclidean metric in the plane. A moment's thought shows a Steinhaus set can be equivalently described as a set $S \subseteq \mathbb{R}^2$ such that

(1) $S \cap L \neq \varnothing$ for any isometric copy L of $\mathbb{Z} \times \mathbb{Z}$.
(2) $\forall x, y \in S$ $[\rho(x, y)$ is not a lattice distance$]$.

This suggests the following definition.

Definition 2.1. A *partial Steinhaus set* (in the plane) is a set $S \subseteq \mathbb{R}^2$ satisfying (2) above.

In our proof of theorem 1.1 we will actually show the following stronger result.

Theorem 2.2. *There is a set $S \subseteq \mathbb{R}^2$ satisfying:*

(1) $S \cap L \neq \varnothing$ *for any isometric copy L of $\mathbb{Z} \times \mathbb{Z}$.*
(2) $\forall x \neq y \in S\ [\rho^2(x,y) \notin \mathbb{Z}]$.

When the problem is reformulated this way, it now makes sense to ask if there are partial Steinhaus sets meeting all the lattices in smaller subcollections of $\mathcal{L}$. For example, consider $\mathcal{L}_t \subseteq \mathcal{L}$ defined to be the set of all translations $\mathbb{Z}^2 + (x,y)$ of the standard lattice. Note that the original formulation of the problem is trivial when restricted to the lattices $\mathcal{L}_t$, namely, $[0,1) \times [0,1)$ meets every $L \in \mathcal{L}_t$ in exactly one point. However, the question of whether there is a partial Steinhaus set meeting all the lattices in $\mathcal{L}_t$ is still non-trivial. In fact, considering this restricted version of the problem motivates our approach.

We restrict even further. Let $\mathcal{L}_q \subseteq \mathcal{L}_t$ denote the lattices which are rational translations of $\mathbb{Z}^2$, that is, of the form $\mathbb{Z}^2 + (r,s)$ where $r,s \in \mathbb{Q}$. We then ask if there is a partial Steinhaus set $S \subseteq \mathbb{Q}^2$ which meets all of the lattice in $\mathcal{L}_q$. We can rephrase this question as follows. Do there exists functions $k,l\colon \mathbb{Q}^2 \to \mathbb{Z}$ such that $\{(r+k(x,y), s+l(x,y)\colon r,s \in \mathbb{Q}\}$ forms a partial Steinhaus set? This is a purely number-theoretic question, though an interesting one. The fact that this can be done is an important lemma which we refer to as "lemma A."

Lemma 2.3 (Lemma A)**.** *There are functions $k,l\colon \mathbb{Q}^2 \to \mathbb{Z}$ such that $\{(r+k(x,y), s+l(x,y)\colon r,s \in \mathbb{Q}\}$ forms a partial Steinhaus set.*

We note that lemma A as stated is equivalent to the seemingly stronger version that requires $\rho^2(x,y) \notin \mathbb{Z}$ whenever x,y are distinct points in $\{(r+k(x,y), s+l(x,y)\colon r,s \in \mathbb{Q}\}$. This is because of the fact from elementary number theory that an integer is the sum of two squares of rationals if and only if it is the sum of two squares of integers.

2.1. **Number Theory.** To investigate lemma A, it seems appropriate to restrict even further. Fix an integer $d > 1$. Let

$$R_d = \{\left(\frac{i}{d}, \frac{j}{d}\right) : 0 \leqslant i,j < d\}$$

be the rational points in the unit square which can be written with denominator d. We have the following approximation to lemma A.

Lemma 2.4. *Let $d > 1$ be an integer. Then there are functions $k,l\colon R_d \to \mathbb{Z}$ such that*

$(*)_d : \{(r+k(x,y), s+l(x,y)\colon (r,s) \in R_d\}$ *forms a partial Steinhaus set.*

Of course, lemma 2.4 is not sufficient to prove lemma A, as we must also guarantee that we may extend the functions k,l satisfying $(*)_d$ to functions k',l' satisfying $(*)_{d'}$ if $d|d'$. In [10] we actually show that the following general extension property is true.

Lemma 2.5. *Let $d|d'$ and assume $k, l\colon R_d \to \mathbb{Z}$ satisfy $(*)_d$. Then the k, l functions may be extended to functions k', $l'\colon R_{d'} \to \mathbb{Z}$ satisfying $(*)_{d'}$.*

Although lemma 2.5 is of some independent interest, it is significantly stronger than what is required for the proof of theorem 1.1. In [9] a simpler proof, suggested by one of the referees, of lemma 2.4 is given which also suffices to get lemma A, though it does not seem to give lemma 2.5. We will use these ideas in §2.4. We will not prove lemma 2.5 here in full generality. We will analyze the general case in this section, but then specialize to the case of d a prime power in §2.2. This will illustrate the main points, but it avoids some technical complications.

We first note that primes $p \equiv 3 \mod 4$, or $p = 2$ are "trivial" with respect to the proof of Lemma A. To see this, suppose z_1, $z_2 \in \mathbb{Q}^2$ and $\rho^2(z_1, z_2) \in \mathbb{Z}$. Let $z_1 - z_2 = (\frac{i}{d}, \frac{j}{e})$, with the fractions written in lowest terms. We must have $d = e$ as otherwise multiplying through by the square of the least common multiple m of d and e would give $i^2(\frac{m}{d})^2 + j^2(\frac{m}{e})^2 = am^2$ for some $a \in \mathbb{Z}$. Then some prime power would divide two of the three terms of this equation but not the other, a contradiction. We thus have $i^2 + j^2 = ad^2$ where $(i, d) = (j, d) = 1$. If d were divisible by a prime $p \equiv 3 \mod 4$, then we would have $i^2 + j^2 \equiv 0 \mod p$ where $i, j \not\equiv 0 \mod p$. This would give -1 being a square root mod p, which it is not as $p \equiv 3 \mod 4$. Easily we cannot have 2 dividing d either. Thus, d must be divisible by only "non-trivial" primes, those congruent to $1 \mod 4$. This says that if $d' = pd$ where p is a trivial prime, and we have k, l function defined on R_d satisfying $(*)_d$, then we may extend them to k', l' satisfying $(*)_{d'}$ simply by copying the k, l functions over to each coset of $G_d = \{(\frac{i}{d}, \frac{j}{d}) + \mathbb{Z}^2\}$ in $G_{d'} = \{(\frac{i}{d'}, \frac{j}{d'}) + \mathbb{Z}^2\}$. In particular, if d contains only trivial primes, then the k, l functions may be defined arbitrarily on R_d and satisfy $(*)_d$. We may assume henceforth that d is divisible by only non-trivial primes.

Let $(\frac{i_1}{d}, \frac{j_1}{d})$, $(\frac{i_2}{d}, \frac{j_2}{d})$ be distinct points in R_d. Let k_1, l_1, k_2, l_2 be integers, and let $z_1 = (\frac{i_1}{d} + k_1, \frac{j_1}{d} + l_1)$, $z_2 = (\frac{i_2}{d} + k_2, \frac{j_2}{d} + l_2)$. The statement that $\rho^2(z_1, z_2) \in \mathbb{Z}$ written out is:

$$(1) \qquad (i_1 - i_2)^2 + (j_1 - j_2)^2 + 2d[(i_1 - i_2)(k_1 - k_2) + (j_1 - j_2)(l_1 - l_2)] \in d^2\mathbb{Z}.$$

Let p^a be the exact power of the (non-trivial) prime p dividing d. If we assume that $\rho^2(z_1, z_2) \in \mathbb{Z}$, then we have $(i_1 - i_2)^2 + (j_1 - j_2)^2 \equiv 0 \mod p^a$. Let $(i_1 - i_2) = p^e u$, $(j_1 - j_2) = p^f v$, where $(u, p) = (v, p) = 1$. Assume for the moment that $\min\{e, f\} < a$. From equation (1) it follows readily that $e = f$ and $u^2 + v^2 \equiv 0 \mod p^{a-e}$. Recall that for primes $p \equiv 1 \mod 4$, there are for each a exactly two square roots of $-1 \mod p^a$, which we call λ_{p^a}, μ_{p^a} (of course, $\mu_{p^a} \equiv -\lambda_{p^a} \mod p^a$). Also, for $b < a$, $\lambda_{p_a} \mod p^b = \lambda_{p^b}$. Thus, $v \equiv \lambda_{p^a} u \mod p^{a-e}$. Multiplying by p^e gives $(j_1 - j_2) \equiv \lambda_{p_a}(i_1 - i_2) \mod p^a$ (renaming the roots perhaps). If $\min\{e, f\} \geqslant a$, then this equation also holds trivially.

Since this is true for each prime power p^a dividing d, it follows that for $\rho^2(z_1, z_2) \in \mathbb{Z}$ we must have $j_1 - j_2 \equiv \lambda(i_1 - i_2) \mod d$, where $\lambda^2 \equiv -1 \mod d$ (we call such a λ a d-root). Let $\tilde{\jmath} = j_1 - \lambda i_1 \mod d = j_2 - \lambda i_2 \mod d$. Let m_1, m_2 be the integers such that

$$(2) \qquad \begin{aligned} j_1 &= \tilde{\jmath} + \lambda i_1 - d m_1 \\ j_2 &= \tilde{\jmath} + \lambda i_2 - d m_2 \end{aligned}$$

Substituting into equation (1) and then dividing through by $2d$ (recall $(2,d)=1$; we write $\frac{1}{2}$ for the multiplicitive inverse of $2 \mod d$) we have

$$(3)\quad (i_1-i_2)^2\left(\frac{1}{2}\right)\left(\frac{1+\lambda^2}{d}\right)+(i_1-i_2)[(k_1-k_2)+\lambda(l_1-l_2-m_1+m_2)]\equiv 0 \mod d$$

Suppose $d=p_1^{a_1}\dots p_k^{a_k}$. Write $i_1-i_2=p_1^{b_1}\dots p_k^{b_k}u$ where $(d,u)=1$. Dividing through by i_1-i_2 this equation is equivalent to

$$(4)\quad (i_1-i_2)\left(\frac{1}{2}\right)\left(\frac{1+\lambda^2}{d}\right)+(k_1-k_2)+\lambda(l_1-l_2)-\lambda(m_1-m_2)\equiv 0 \mod p^{\eta(a_1-b_1)\dots\eta(a_k-b_k)}$$

where $\eta(x)=x$ if $x\geqslant 0$ and 0 otherwise. That is,

$$(5)\quad \begin{aligned}&(k_1+\lambda l_1)+i_1\left(\frac{1}{2}\right)\left(\frac{1+\lambda^2}{d}\right)-\lambda m_1\equiv\\ &(k_2+\lambda l_2)+i_2\left(\frac{1}{2}\right)\left(\frac{1+\lambda^2}{d}\right)-\lambda m_2 \mod p^{\eta(a_1-b_1)\dots\eta(a_k-b_k)}.\end{aligned}$$

Note that if we had chosen λ to be a square root of $-1 \bmod d^2$, then this would simplify to

$$(6)\quad k_1+\lambda l_1-\lambda m_1\equiv k_2+\lambda l_2-\lambda m_2 \mod p^{\eta(a_1-b_1)\dots\eta(a_k-b_k)}.$$

This suggests the following definition.

Definition 2.6. Let $d>1$ and $d=p_1^{a_1}\cdots p_k^{a_k}$ be its prime decomposition. We say a permutation $\pi=(\pi(0),\dots,\pi(d-1))$ of the set $(0,1,\dots,d-1)$ is a d-good permutation if whenever $0\leqslant i_1,i_2<d$ are distinct and $i_1-i_2=p_1^{b_1}\dots p_n^{b_n}v$ where $(v,d)=1$, then $\pi(i_1)\not\equiv\pi(i_2) \mod p_1^{\eta(a_1-b_1)}\cdots p_k^{\eta(a_k-b_k)}$.

To be precise, given the k, l functions on R_d, if we define for each $0\leqslant\tilde{\jmath}<d$ and each square root λ of $-1 \mod d$ the function

$$(7)\quad \pi_{\tilde{\jmath}}^{\lambda}(i)=(k+\lambda l)-\lambda m+\frac{1}{2}\left(\frac{1+\lambda^2}{d}\right)(i) \mod d,$$

then we have shown that if all of the $\pi_{\tilde{\jmath}}^{\lambda}$ are d-good permutations, then the k, l functions satisfy $(*)_d$.

The permutations $\pi_{\tilde{\jmath}}^{\lambda}$ must also satisfy a consistency condition. Suppose λ_1, λ_2 are two square roots of $-1 \mod d$, and $\lambda_1\equiv\lambda_2 \mod p^a$, where p^a is one of the prime powers of d. Say $\lambda_2=\lambda_1+\epsilon p^a$. Let $0\leqslant\tilde{\jmath}_1,\tilde{\jmath}_2<d$, and $0\leqslant i<d$ be such that $i(\lambda_1-\lambda_2)\equiv-(\tilde{\jmath}_1-\tilde{\jmath}_2) \mod d$. If we let $0\leqslant j_1,j_2<d$ and m_1, m_2 be defined by

$$\begin{aligned}j_1&=\tilde{\jmath}_1+\lambda_1 i-m_1 d\\ j_2&=\tilde{\jmath}_2+\lambda_2 i-m_2 d,\end{aligned}$$

then $j_1=j_2$, which we now denote by j. Thus,

$$\tilde{\jmath}_1-\tilde{\jmath}_2=-i(\lambda_1-\lambda_2)+d(m_1-m_2)=iep^a+d(m_1-m_2).$$

Let k, l be the values associated to the point $(\frac{i}{d},\frac{j}{d})$. From the definition of the $\pi_{\tilde{\jmath}}^{\lambda}$ we have:

$$k + \lambda_1 l \equiv \pi^{\lambda_1}_{\tilde{\jmath}_1}(i) + \lambda_1 m_1 - \frac{1}{2}\left(\frac{1+\lambda_1^2}{d}\right) i \mod p^a$$
$$k + \lambda_2 l \equiv \pi^{\lambda_2}_{\tilde{\jmath}_2}(i) + \lambda_2 m_2 - \frac{1}{2}\left(\frac{1+\lambda_2^2}{d}\right) i \mod p^a$$
$$\equiv \pi^{\lambda_2}_{\tilde{\jmath}_2}(i) + \lambda_1 m_1 - \frac{1}{2}\left(\frac{1+\lambda_1^2}{d}\right) i - \frac{\lambda_1(\tilde{\jmath}_1 - \tilde{\jmath}_2)}{d} \mod p^a$$

upon substituting the above values. Note that p^a divides $\tilde{\jmath}_1 - \tilde{\jmath}_2$, so the last equation makes sense. Thus, we have:

$$\pi^{\lambda_1}_{\tilde{\jmath}_1}(i) - \pi^{\lambda_2}_{\tilde{\jmath}_2}(i) \equiv -\frac{\lambda_1(\tilde{\jmath}_1 - \tilde{\jmath}_2)}{d} \mod p^a.$$

Thus, if the k, l functions on R_d satisfy $(*)_d$, the $\pi^{\lambda}_{\tilde{\jmath}}$ satisfy the following goodness and consistency conditions:

(d-goodness) For each $0 \leqslant \tilde{\jmath} < d$, and each d-root λ, $\pi^{\lambda}_{\tilde{\jmath}}$ is a d-good permutation.

(d-consistency) Suppose $0 \leqslant \tilde{\jmath}_1, \tilde{\jmath}_2 < d$ and λ_1, λ_2 are both d-roots. Suppose p^a is one of the prime factors $p_1^{a_1}, \ldots, p_n^{a_n}$ and $\lambda_1 \equiv \lambda_2 \mod p^a$. Then

$$\pi^{\lambda_1}_{\tilde{\jmath}_1}(i) - \pi^{\lambda_2}_{\tilde{\jmath}_2}(i) \equiv -\frac{\lambda(\tilde{\jmath}_1 - \tilde{\jmath}_2)}{d} \mod p^a \tag{8}$$

for any $0 \leqslant i < d$ such that

$$i(\lambda_1 - \lambda_2) \equiv -(\tilde{\jmath}_1 - \tilde{\jmath}_2) \mod d \tag{9}$$

(in equation 8, λ could be either λ_1 or λ_2; recall p^a must divide $(\tilde{\jmath}_1 - \tilde{\jmath}_2)$)

Conversely, if for each $0 \leqslant \tilde{\jmath} < d$ and d-root λ a d-good permutation $\pi^{\lambda}_{\tilde{\jmath}}$ is given and if these permutations together satisfy the d-consistency condition, then we may define k, l functions on R_d which satisfy $(*)_d$. Namely, for each prime power p^a of d, let λ be a d-root with $\lambda \equiv \lambda_{p^a}$. Given i, j, let Let $0 \leqslant \tilde{\jmath} < d$, m be such that $\tilde{\jmath} + \lambda i - dm = j$. We then use equation (7) to define $k + \lambda_{p^a} l \mod p^a$, where k, l are the to be determined values of the functions at the point $(\frac{i}{d}, \frac{j}{d})$. The d-consistency condition shows that this does not depend on the choice of λ. Similarly we determine the value of $k + \mu_{p^a} l \mod p^a$ for the other p^a root. This gives a non-singular 2×2 system mod p^a which we can then solve for k, l.

We summarize our analysis in the following theorem.

Theorem 2.7. *Let $d > 1$. If k, l functions are defined on R_d and satisfy $(*)_d$, and if for each $0 \leqslant \tilde{\jmath} < d$ and d-root λ we define $\pi^{\lambda}_{\tilde{\jmath}}$ by equation (7), then the $\pi^{\lambda}_{\tilde{\jmath}}$ satisfy the d-goodness and d-consistency conditions. Conversely, if the $\pi^{\lambda}_{\tilde{\jmath}}$ are given satisfying the d-goodness and d-consistency conditions, then we may define k, l functions via equation (7) which then satisfy $(*)_d$.*

2.2. **A Special Case.** We use the analysis of the previous section to prove lemma 2.5 in the case where d and d' are prime powers. The reader can consult [10] for the general case. Thus, we prove in this section the following special case of lemma 2.5.

Lemma 2.8. *Let $d = p^a$, $d' = p^{a+1}$, and assume k, l functions are given on R_d which satisfy $(*)_d$. Then we may extend these functions to functions k', l' on $R_{d'}$ satisfying $(*)_{d'}$.*

The simplification that arises in this case is that the d' consistency condition is now trivial. Let λ, μ be the two d'-roots (i.e., $\lambda^2 \equiv \mu^2 \equiv -1 \mod d'$), which we may also regard as d-roots. The given k, l functions which satisfy $(*)_d$ give by theorem 2.7 for each $0 \leqslant \tilde{\jmath} < p^a$ good permutations $\pi^\lambda_{\tilde{\jmath}}$, $\pi^\mu_{\tilde{\jmath}}$. We may in fact use equation (7) to define for each $0 \leqslant \tilde{\jmath} < p^a$, and hence $0 \leqslant p\tilde{\jmath} < p^{a+1}$, and each $0 \leqslant i < p^a$, and hence $0 \leqslant pi < p^{a+1}$, the values $\pi^\lambda_{p\tilde{\jmath}}(pi)$, $\pi^\mu_{p\tilde{\jmath}}(pi)$. Specifically,

$$\pi^\lambda_{p\tilde{\jmath}}(pi) = (k + \lambda l) - \lambda m + \frac{1}{2}\left(\frac{1+\lambda^2}{d'}\right)(pi) \mod d', \tag{10}$$

where k, l are the values associated to the point $(\frac{i}{d}, \frac{j}{d}) = (\frac{pi}{d'}, \frac{pj}{d'})$, and $pj = p\tilde{\jmath} + \lambda(pi) - md'$, which is equivalent to $j = \tilde{\jmath} + \lambda i - md$. The $\pi^\lambda_{p\tilde{\jmath}}$ and $\pi^\mu_{p\tilde{\jmath}}$ are partially d'-good, that is, they satisfy the d' goodness condition on their domains, the set of $0 \leqslant i < p^{a+1}$ which are divisible by p. In view of theorem 2.7, to prove lemma 2.8 it suffices to prove the following lemma.

Lemma 2.9. *Let π be defined on the $0 \leqslant i < p^{a+1}$ which are divisible by p, and assume that π is partially p^{a+1}-good. Then we may extend π to a p^{a+1}-good permutation.*

Proof. For $0 \leqslant i < p^{a+1}$, write $i = pi' + u$ where $0 \leqslant u < p$. Define $\pi(i) = \pi(pi') + p^a u$. It is easy to check that this defines a p^{a+1}-good permutation. □

It is perhaps worth considering a numerical example. Consider the case $d = 5$. We take $\lambda = 3$ and $\mu = 2$. Since any permutation of $\{0, \dots, p-1\}$ is a p-good permutation, we may take the π^λ_5, π^μ_5 to be arbitrary permutations. For example, we choose them both to be the identity permutation. Carrying out the algorithm results in the following values for the k, l functions (we write $k(i,j)$ for the value of the k function at the point $(\frac{i}{5}, \frac{j}{5})$).

$k(0,0) = 0,\quad l(0,0) = 0,\quad k(0,1) = 0,\quad l(0,1) = 0$
$k(0,2) = 0,\quad l(0,2) = 0,\quad k(0,3) = 0,\quad l(0,3) = 0$
$k(0,4) = 0,\quad l(0,4) = 0,\quad k(1,0) = 4,\quad l(1,0) = 3$
$k(1,1) = 4,\quad l(1,1) = 3,\quad k(1,2) = 3,\quad l(1,2) = 0$
$k(1,3) = 4,\quad l(1,3) = 2,\quad k(1,4) = 4,\quad l(1,4) = 2$
$k(2,0) = 2,\quad l(2,0) = 3,\quad k(2,1) = 3,\quad l(2,1) = 0$
$k(2,2) = 3,\quad l(2,2) = 0,\quad k(2,3) = 3,\quad l(2,3) = 0$
$k(2,4) = 2,\quad l(2,4) = 2,\quad k(3,0) = 2,\quad l(3,0) = 3$
$k(3,1) = 1,\quad l(3,1) = 0,\quad k(3,2) = 1,\quad l(3,2) = 0$
$k(3,3) = 1,\quad l(3,3) = 0,\quad k(3,4) = 2,\quad l(3,4) = 2$
$k(4,0) = 0,\quad l(4,0) = 3,\quad k(4,1) = 0,\quad l(4,1) = 3$
$k(4,2) = 1,\quad l(4,2) = 0,\quad k(4,3) = 0,\quad l(4,3) = 2$
$k(4,4) = 0,\quad l(4,4) = 2$

Let $d' = p^2 = 25$, and we extend these k, l functions from R_5 to R_{25}. First we extend the p roots $\lambda = 3$, $\mu = 2$ to p^2 roots $\lambda' = 18$, $\mu' = 7$. Using equation (7) (and our new roots for the λ in that equation), the existing k, l functions on R_5 define for each $0 \leqslant \tilde{\jmath} < 25$ which is divisible by 5 partial 25-good permutations $\pi_{\tilde{\jmath}}^{\lambda'}$, $\pi_{\tilde{\jmath}}^{\mu'}$ (they are defined on the $i < 25$ which are divisible by 5). For example, the original identity permutation $\pi_1^{\lambda} = (0,1,2,3,4)$ lifts to $\pi_5^{\lambda'} = (0,6,17,18,14)$. Our extension algorithm extends this to the following 25-good permutation:

$$\pi_5^{\lambda'} = (0,5,10,15,20,6,11,16,21,1,17,22,2,7,12,18,23,3,8,13,14,19,24,4,9).$$

For $\tilde{\jmath}$ not divisible by 5, we are free to choose any 25-good permutation for the $\pi_{\tilde{\jmath}}^{\lambda'}$, $\pi_{\tilde{\jmath}}^{\mu'}$. We take the following:

$$(0,5,10,15,20,1,6,11,16,21,2,7,12,17,22,3,8,13,18,23,4,9,14,19,24).$$

Equation (7) then gives for each $0 \leqslant i,j < 25$ a 2×2 system mod 25 which we solve for the corresponding values of the k, l functions. For example, for $i = 5$ we get the following values for $\langle k(i,j) \colon 0 \leqslant j < 25 \rangle$ and $\langle l(i,j) \colon 0 \leqslant j < 25 \rangle$ (where $k(i,j)$ denotes the value of the k function at the point $(\frac{i}{25}, \frac{j}{25})$):

$$\langle 4,19,19,19,19,4,19,19,19,19,3,3,3,3,3,4,19,19,19,19,4,19,19,19,19, \rangle$$

$$\langle 3,13,13,13,13,3,13,13,13,13,0,0,0,0,0,2,12,12,12,12,2,12,12,12,12 \rangle.$$

2.3. **A Finite Obstruction.** We can use the analysis of §2.1 to show that "finite obstructions" to the Steinhaus problem exist. By this we mean a finite set $F \subseteq \mathbb{R}^2$ which is a partial Steinhaus set, but which cannot be enlarged to a Steinhaus set. In fact, we can get F to consist of points with rational coordinates none of which lies on the integer lattice, but for which every point on the integer lattice is at a lattice distance from one of the points of F.

Take two non-trivial primes, say $p = 5$ and $q = 13$. Let $d = pq = 65$. Let λ be a d root, say $\lambda = 8$. We construct a partial 65-good permutation which cannot be extended to a 65-good permutation. For $1 \leqslant i < 13$ let $\pi(5i) = i$. For $1 \leqslant i < 5$, let $\pi(13i) = 15 + i$. Define also $\pi(1) = 0$. Clearly π satisfies the goodness condition on its domain. However, π cannot be extended to a 65-good permutation as the first clauses in the definition of π would force $\pi(0) \equiv 0 \mod 13$ and $\pi(0) \equiv 0 \mod 5$, hence $\pi(0) = 0$, which violates $\pi(1) = 0$. Using this partial π and equation (7) we determine $k + \lambda l$ for 17 points in R_{65} of the form $(\frac{i}{65}, \frac{\lambda i \mod 65}{65})$ for $i \in \mathrm{dom}(\pi)$. We arbitrarily assign $l = 0$ for these points, and thereby determine the k values for these points. This results in the following partial Steinhaus set:

$(\frac{404}{13}, \frac{8}{13})$ $(\frac{67}{13}, \frac{3}{13})$ $(\frac{471}{13}, \frac{11}{13})$ $(\frac{134}{13}, \frac{6}{13})$
$(\frac{642}{13}, \frac{1}{13})$ $(\frac{201}{13}, \frac{9}{13})$ $(\frac{709}{13}, \frac{4}{13})$ $(\frac{268}{13}, \frac{12}{13})$
$(\frac{776}{13}, \frac{7}{13})$ $(\frac{439}{13}, \frac{2}{13})$ $(\frac{843}{13}, \frac{10}{13})$ $(\frac{506}{13}, \frac{5}{13})$
$(\frac{251}{5}, \frac{3}{5})$ $(\frac{142}{5}, \frac{1}{5})$ $(\frac{318}{5}, \frac{4}{5})$ $(\frac{209}{5}, \frac{2}{5})$
$(\frac{2081}{65}, \frac{8}{65})$

By construction, there are no k, l values which can be assigned to the the point $(0,0)$ so that the resulting extension of π is still good. Thus, this partial Steinhaus set cannot be extended to a partial Steinhaus set which meets the integer lattice (that is, every point of $\mathbb{Z} \times \mathbb{Z}$ has a squared integer distance from one of these 17 points, which by a fact mentioned earlier implies that the distance is a lattice distance).

On the other hand, it is not difficult to see that for any prime power $d = p^n$, any partial d-good permutation may be extended to a d-good permutation. For suppose $i \notin \mathrm{dom}(\pi)$. By cyclically shifting we may assume $i = 0$. The values $\pi(p^{n-1}i)$, $1 \leqslant i < p$, (where defined) omit at least one value mod p, say a_p. If we consider the p values mod p^2 which are congruent to a_p mod p, then at least one of them, say a_{p^2}, is not taken on as a value $\pi(p^{n-2}i)$, $1 \leqslant i < p^2$, as otherwise these p values occur as $\pi(p^{n-2}i)$ where $p \nmid i$, and hence for some $1 \leqslant i \neq j < p^2$ we have $i \equiv j \mod p$ and $\pi(p^{n-2}i) \equiv \pi(p^{n-2}j) \mod p$. This contradicts the partial goodness of π. Continuing, we define a value a_{p^n} such that if we set $\pi(0) = a_{p^n}$, then this extension still satisfies goodness.

It follows that if k, l functions are defined on a subset $A \subseteq R_d$ (for $d = p^n$) and satisfy $(*)_d$ for points in A, then we may extend these functions to all of R_d satisfying $(*)_d$. This is because the consistency condition for prime powers is trivial and we checked above that we may extend partial d permutations and satisfy goodness. The procedure for constructing a Steinhaus set sketched in the next section (and given in detail in [10]) may then be applied starting from these k, l functions on R_d. In fact, if d is divisible by only one prime power p^a for which $p \equiv 1 \mod 4$, then this extension result is still valid, as we may apply the argument to each coset of R_{p^a} in R_d (viewed as Abelian subgroups of $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$) separately. Summarizing, we have the following, where $\mathbb{Q}_d$ denotes the rationals that can be written with denominator d.

Theorem 2.10. *Let $d > 1$ be an integer divisible by at most one prime p such that $p \equiv 1 \mod 4$. Then any partial Steinhaus set $S \subseteq \mathbb{Q}_d \times \mathbb{Q}_d$ can be extended to a Steinhaus set.*

In particular, $d = 65$ is the smallest integer such that a finite obstruction can be found in $\mathbb{Q}_d \times \mathbb{Q}_d$.

2.4. **Another Approach.** One of the referees of [10] suggested another approach to the proof of lemma 2.3 which has some advantages (though it does not seem to yield lemma 2.5). In particular, it avoids the need to solve a 2×2 system and thereby the consistency conditions. Suppose for example that $d = p$ is prime. The idea is to write each point in R_d as a linear combination of the form $\frac{a(1,\lambda)}{p} + \frac{b(1,\mu)}{p}$ mod $\mathbb{Z} \times \mathbb{Z}$, where $0 \leqslant a, b < p$. We assume that λ, μ satisfy $\lambda^2 = \mu^2 = -1 \mod p^2$ (which eliminates the term containing $\frac{1+\lambda^2}{p}$ in equation (7)). For x of this form (that is, $x = \frac{a(1,\lambda)}{p} + \frac{b(1,\mu)}{p}$) we can take $k = 0$, $l = a + b$. To see this works, note that if $x_1 = \frac{a_1(1,\lambda)}{p} + \frac{b_1(1,\mu)}{p}$ and $x_2 = \frac{a_2(1,\lambda)}{p} + \frac{b_2(1,\mu)}{p}$ and so $x_1 - x_2 = \frac{u(1,\lambda)}{p} + \frac{v(1,\mu)}{p}$ with $-p < u, v < p$, then for $\rho^2(z_1, z_2) \in \mathbb{Z}$ we must have $u = 0$ or $v = 0$ (since for a point $(\frac{a}{p}, \frac{b}{p})$ to have the square of its a norm an integer, we must have $b \equiv \lambda a$ or $b \equiv \mu a \mod p$). If say $v = 0$ (hence $u \neq 0$) and if we let $z_1 = x_1 + (k(x_1), l(x_1))$ and likewise for z_2, then $\rho^2(z_1, z_2) = \left(\frac{u}{p}\right)^2 + \left(\frac{u\lambda}{p} + u\right)^2 \equiv \frac{2u^2\lambda}{p} \mod \mathbb{Z}$ which is not an integer.

The proof for the general case of lemma 2.3 using this approach is given in [9]. Here we use this method to answer a question posed in [10]: does there exist a Steinhaus set for the 2×1 lattice (that is, the lattice with basis vectors $(2, 0)$, $(0, 1)$)? We show more generally the following.

Theorem 2.11. *Let r, $s \geqslant 1$ be integers with r, s divisible only by trivial primes (i.e., $p = 2$ or $p \equiv 3 \mod 4$). Then there is a Steinhaus set for the $r \times s$ lattice.*

Actually the construction we outline in the next section works for these lattices as well as the standard lattice, provided we can establish the analog of lemma 2.3. Thus, we sketch here the proof of the following lemma. Let $R^{r,s}$ denote the points with rational coordinates in the $r \times s$ rectangle. By an r, s lattice distance we mean a real of the form $\sqrt{a^2r^2 + b^2s^2}$ where $a, b \in \mathbb{Z}$.

Lemma 2.12. *Let r, s be as in theorem 2.11. Then there are k, l functions $k, l\colon R^{r,s} \to \mathbb{Z}$ such that if $x_1 \neq x_2 \in R^{r,s}$ and $z_1 = x_1 + (k(x_1), l(x_1))$ and likewise for x_2, then $\rho(z_1, z_2)$ is not an r, s lattice distance.*

Proof. We sketch the proof for the reader familiar with the corresponding argument for the standard lattice in [9]; we will concentrate on the differences. Since everything is invariant under a uniform scaling, we may assume that $(r, s) = 1$. We assume that $r \geqslant s$, and hence $r > 1$ (if $r = s = 1$, the argument of [9] applies, which is a slight simplification of the following argument). For $d \geqslant 1$ an integer, write $R_d^{r,s}$ for the points of the form $(\frac{i}{d}, \frac{j}{d})$ where $0 \leqslant i < rd$ and $0 \leqslant j < sd$.

Lemma 2.13. *There are functions $f, g\colon R_1^{r,s} \to \mathbb{Z}$ such $f(a, b) \equiv 0 \mod r$, $g(a, b) \equiv 0 \mod s$ for any $(a, b) \in R_1^{r,s}$, and for any distinct points x_1, x_2 in $R_1^{r,s}$ if $z_1 = x_1 + (f(x_1), g(x_1))$, $z_2 = x_2 + (f(x_2), g(x_2))$, then $\rho(z_1, z_2)$ is not an r, s lattice distance. In fact, if c, d are integers divisible by rs, then $||(z_1 - z_2) + (c, d)||$ is not an r, s lattice distance.*

Proof. Recall we are assuming $r \geqslant s$. First assume $s \geqslant 3$. Let γ be such that $\gamma^2 + 1$ is a non-square mod r (starting from a square, keep adding 1 until the first non-square is found), and $\delta^2 + 1$ a non-square mod s. Let $f(a, b) \equiv 0 \mod r$, and $f(a, b) \equiv -a + \delta b \mod s$. Let $g(a, b) \equiv 0 \mod s$ and $g(a, b) \equiv -b + \gamma a \mod r$. Let $x_1 = (a_1, b_1)$, $x_2 = (a_2, b_2)$, and z_1, z_2 be as in the lemma. Then $\rho^2(z_1, z_2) \equiv (a_1 - a_2)^2(1 + \gamma^2) \mod r$. If $a_1 \neq a_2$, then this is a non-square mod r. This suffices since the square of a lattice distance is of the form $e^2r^2 + f^2s^2$ and hence is a square mod r. If $a_1 = a_2$, then $b_1 \neq b_2$. Also, $\rho^2(z_1, z_2) \equiv (b_1 - b_2)^2(1 + \delta^2) \mod s$, and so is a non-square mod s, which suffices. Clearly if rs divides c and d, then $||(z_1 - z_2) + (c, d)||^2 \equiv ||z_1 - z_2||^2 \mod rs$ and the last statement follows. If $s = 1$ and $r \geqslant 3$, this argument also applies since in this case we must have $a_1 \neq a_2$ (in this case we can take $f = 0$ and choose g as above).

If $s = 2$ (so $r \geqslant 3$), we let δ be such that $\delta^2 + 1$ is a non-square mod 4 (e.g., $\delta = 1$) and take $f \equiv 0 \mod r$, $f \equiv -a + \delta b \mod 4$, $g \equiv -b + \gamma a \mod r$, and $g \equiv 0 \mod 4$. Since the square of a lattice distance is a square mod 4, this suffices. The last statement again easily follows.

Finally, if $s = 1$ and $r = 2$ we let $\gamma^2 + 1$ be a non-square mod 4 and take $f = 0$ and $g \equiv -a + \gamma b \mod 4$. □

Following [9], let $P_1 < P_2 < \ldots$ enumerate all of the prime powers, say $P_i = p_i^{e_i}$, in such a way that if $P_i | P_j$ then $i \leqslant j$. Let I denote the trivial primes and J the non-trivial primes. Fix two sequences of integers A_i, B_i for $i \in J$ satisfying:

(1) $(A_i, P_i) = 1$ and if $j < i$, $j \in J$, and $(P_j, P_i) = 1$ then $P_j | A_i$.
(2) $rs | A_i, B_i$
(3) For all $j < i$ with $j \in J$, $P_j | B_i$ but $P_i \nmid B_i$.

For each i fix integers λ_i, μ_i with $\lambda_i^2 \equiv \mu_i^2 \mod P_i^2$. Also, name the roots so that if $P_j | P_i$ then $\lambda_i \equiv \lambda_j \mod P_j$.

Let d_n be the least common multiple of $P_1, \ldots, P_n$. At stage n we define the k, l functions at all points $(x, y) \in R^{r,s}_{d_n} - R^{r,s}_{d_{n-1}}$. Consider such an (x, y). An easy argument shows that we may write (x, y) uniquely in the form

$$(x,y) = (a,b) + \sum_{\substack{i \leqslant n \\ i \in I}} \left(\frac{a_i}{P_i}, \frac{b_i}{P_i} \right) + \sum_{\substack{i \leqslant n \\ i \in J}} A_i \frac{a_i(1,\lambda_i) + b_i(1,\mu_i)}{P_i} \mod (r,s) \tag{11}$$

where $0 \leqslant a_i, b_i < p_i$ are integers, and $0 \leqslant a < r$, $0 \leqslant b < s$. Also, at least one of a_n, b_n is non-zero. Let D_n be the least common multiple of the P_i, $i \leqslant n$, for which p_i is a non-trivial prime.

Let $f, g \colon R^{r,s}_1 \to \mathbb{Z}$ be as in lemma 2.13.

We define the k, l values for the point given by the right-hand side of equation (11) (which is equivalent to $(x, y) \mod (r, s)$) by

$$(k,l) = (f(a,b), g(a,b)) + (0, \sum_{\substack{i \leqslant n \\ i \in J}} (a_i + b_i) B_i)$$

This completes the definition of the k, l functions. To see this works, consider two points $(x, y) \in R^{r,s}_{d_n} - R^{r,s}_{d_{n-1}}$ and $(x', y') \in R^{r,s}_{d_m} - R^{r,s}_{d_{m-1}}$. We may assume $n \geqslant m$. We may assume (x, y) and (x', y') are written as in the right-hand side of equation (11). Say the coefficients for (x, y) are a, b, a_i, b_i and for (x', y'), a', b', a'_i, b'_i. We may extend the sum for (x', y') by adding zeros so that the last term for both sums involves P_n. Subtracting we get

$$(x-x', y-y') = (a-a', b-b') + \sum_{\substack{i \leqslant n \\ i \in I}} \left(\frac{u_i}{P_i}, \frac{v_i}{P_i} \right) + \sum_{\substack{i \leqslant n \\ i \in J}} A_i \frac{u_i(1,\lambda_i) + v_i(1,\mu_i)}{P_i} \mod (r,s)$$

where $-p_i < u_i, v_i < p_i$. Let $z = (x + k(x, y), y + l(x, y))$ and $z' = (x' + k(x', y'), y' + l(x', y'))$ be the translated points. We must show that $\rho(z, z')$ is not an r, s lattice distance. We may assume that for all $i \in I$, $u_i = v_i = 0$ as otherwise $\rho^2(z, z') \notin \mathbb{Z}$ regardless of the k, l values. Let $(\bar{a}, \bar{b}) = (a, b) + (f(a, b), g(a, b))$ and $(\bar{a}', \bar{b}') = (a', b') + (f(a', b'), g(a', b'))$ Thus,

$$z - z' = (\bar{a} - \bar{a}', \bar{b} - \bar{b}') + \sum_{\substack{i \leqslant n \\ i \in J}} A_i \frac{u_i(1,\lambda_i) + v_i(1,\mu_i)}{P_i} + (0, \sum_{\substack{i \leqslant n \\ i \in J}} (u_i + v_i) B_i). \tag{12}$$

First assume that $(a, b) \neq (a', b')$. Let $(w_1, w_2) = (\bar{a} - \bar{a}', \bar{b} - \bar{b}')$. Thus, $w_1^2 + w_2^2$ is either a non-square mod r or a non-square mod s (if r or s equals 2, then a non-square mod 4). The sum of the remaining terms in equation (12) is of the form $(\frac{e}{D_n}, \frac{f}{D_n})$ where rs divides e and f. It follows that $(w_1 + \frac{e}{D_n})^2 + (w_2 + \frac{f}{D_n})^2$ is either a non-integer or else a non-square mod r or a non-square mod s. (if it is an integer, then it is congruent mod rs to $w_1^2 + w_2^2$). In either case, $||z - z'||$ is not an r, s lattice distance.

Assume next that $(a,b) = (a',b')$. Thus,

$$z - z' = \sum_{\substack{i \leqslant n \\ i \in J}} A_i \frac{u_i(1,\lambda_i) + v_i(1,\mu_i)}{P_i} + (0, \sum_{\substack{i \leqslant n \\ i \in J}} (u_i + v_i) B_i). \tag{13}$$

We claim in this case that $||z - z'||^2$ is not an integer, which suffices. Actually, the argument is now identical to that given in [9]. For the sake of completeness we give a sketch.

For some $i \in J$ we have that $u_i \neq 0$ or $v_i \neq 0$. If we write $z - z' = (\frac{e}{d}, \frac{f}{d'})$ where $(d,e) = (d',f) = 1$, then we have $\rho^2(z,z') \notin \mathbb{Z}$ unless $d = d'$ and $f \equiv \lambda e \mod d$ for some d root λ. Note in this case that d is the least common multiple of the P_i for $i \in J$ such that at least one of u_i, v_i is non-zero, but no higher power of p_i has this property. We may assume that $\lambda^2 \equiv -1 \mod d^2$. Thus, $\lambda^2 \equiv -1 \mod P_i^2$ for all of the P_i occuring in the sum of equation (13) (where at least one of the u_i, v_i is non-zero). By renaming if necessary we may assume $\lambda \equiv \lambda_i \mod P_i^2$. From $\frac{f}{d} - \lambda\frac{e}{d} \in \mathbb{Z}$ and $\lambda \equiv \lambda_i \mod P_i^2$, we get from equation (13) that all of the v_i are 0. So,

$$z - z' = \sum_{\substack{i \leqslant n \\ i \in J}} A_i \frac{u_i(1,\lambda_i)}{P_i} + (0, \sum_{\substack{i \leqslant n \\ i \in J}} u_i B_i).$$

Let $i_0 \leqslant n$ be least such that $u_{i_0} \neq 0$. If we replace each λ_i by $\lambda + (\lambda_i - \lambda)$, recall $\lambda \equiv \lambda_i \mod P_i^2$, and use the definitions of A_i, B_i we have that $z - z'$ can be written as

$$z - z' = \left(\frac{e}{d}, \frac{\lambda e}{d} + B_{i_0} u_{i_0} + X \right)$$

where $(e,d) = 1$ and P_{i_0} divides d and X. Recall $P_{i_0} \nmid B_{i_0}$. Since $d^2 | (1 + \lambda^2)$ it follows easily that $\rho^2(z,z')$ is not an integer.

2.5. **The Construction.** We discuss now the construction of the Steinhaus set. We do not give the complete details (which can be found in [10]), but try to motivate the main ideas. We will use of course the number theoretic lemma A but the need for a geometric lemma, which we call lemma B, also arises.

Throughout, by a lattice L we mean an isometric copy of the standard lattice $\mathbb{Z} \times \mathbb{Z}$, although the reader can check that all our arguments here remain valid for isometric copies of a fixed rectangular $r \times s$ lattice, where r and s are rational. By a rational translation of L we mean a lattice of the form $L_{a,b} = L + a\vec{u} + b\vec{v}$ where $a, b \in \mathbb{Q}$ and $\vec{u}$, $\vec{v}$ are the basis vectors of L. In other words, we are referring to a translation which is rational with respect to the coordinate system of L. Similarly, by a rational rotation we mean a transformation which in the coordinate system of L is given by a rational rotation matrix.

Definition 2.14. We define $L \sim L'$ if L' can be obtained from L by successive rational translations and rotations.

It is easy to check this is an equivalence relation and $L \sim L'$ iff all the points of L' are rational in the coordinate system of L. Also, if two distinct points of L' are rational with respect to L, then $L \sim L'$.

Lemma A says that given any L we may get a partial Steinhaus set which meets all of the rational translations of L. The next lemma says that such a partial

Steinhaus set automatically meets all lattices $L' \sim L$. The proof, which is not difficult, is given in [10].

Lemma 2.15. *Suppose S is a partial Steinhaus set, L is a lattice, and $S \cap L_{r,s} \neq \emptyset$ for all $r, s \in \mathbb{Q}$. Then for any $L' \sim L$, $S \cap L' \neq \emptyset$.*

We remark that lemma 2.15 is not true for arbitrary fields. For example, using the method of lemma A we can construct a partial Steinhaus set which meets all translations of $\mathbb{Z} \times \mathbb{Z}$ by elements of $\mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2})$, but which misses a lattice obtained from $\mathbb{Z} \times \mathbb{Z}$ by a rotation over this field.

In view of this, a natural attempt to build a Steinhaus set would be to enumerate the equivalence classes $\{\mathcal{L}_\alpha\}_{\alpha<2^\omega}$ of lattices, and then successively build partial Steinhaus sets $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_\alpha \subseteq$ such that at step α, $S_\alpha \cap L \neq \emptyset$ for all $L \in \mathcal{L}_\alpha$. At limit stages we would take unions and there would be no problem. Suppose that S_α is defined, and we attempt to extend to $S_{\alpha+1}$. Let $L \in \mathcal{L}_{\alpha+1}$. Although S_α is a partial Steinhaus set by assumption, it may be that every point on L lies at a lattice distance from some point of S_α, in which case the extension is impossible. This presents our second "obstruction" which we overcome with the "hull" method. To investigate this, suppose $z_1 \in L$, $c_1 \in S_\alpha$, and $\rho^2(c_1, z_1) \in \mathbb{Z}$. Let us assume that c_1 does not have rational coordinates with respect to L (we comment on the general case below). The following lemma is easily verified.

Lemma 2.16. *Let L be a lattice and suppose z does not have rational coordinates with respect to L. Then there is a line $l = l(z, L)$ such that if $w \in L$ and $\rho^2(z, w) \in \mathbb{Q}$, then $w \in l$.*

Thus, the point c_1 can only rule out a line $l_1 = l(c_1, L)$ of points on L. Choose $z_2 \in L - l_1$. Suppose there is a $c_2 \in S_\alpha$ with $\rho^2(c_2, z_2) \in \mathbb{Z}$. Suppose again that c_2 does not have rational coordinates with respect to L. Let $l_2 = l(c_2, L)$, and let $z_3 \in L - (l_1 \cup l_2)$. Finally, suppose there is a $c_3 \in S_\alpha$ with $\rho^2(c_3, z_3) \in \mathbb{Z}$. Let $r_1 = \rho(c_1, z_1)$ and likewise for r_2, r_3. Let C_1 be the circle with center c_1 and radius r_1, and likewise for C_2, C_3. So the three circles are definable from the points c_1, c_2, c_3 of S_α. Clearly the congruence class of the triangle $\triangle z_1 z_2 z_3$ is definable. We would like to assert that for any triangle T there are only finitely many z_1, z_2, z_3 with $\triangle z_1 z_2 z_3 \cong T$ with $z_1 \in C_1, \dots, z_3 \in C_3$. If so, then the z_i will be definable from the c_i. This would then be a contradiction if we assume the S_α are sufficiently closed and L is not definable from the points of S_α. (note that at most one point of L can lie in S_α as L is definable from any two of its points). There is, however, an obvious exception to the above assertion. Namely, the case where $r_1 = r_2 = r_3$ and $\triangle z_1 z_2 z_3 \cong \triangle c_1 c_2 c_3$. This exceptional case does not arise in the argument though, as in this case we would have $\rho(c_1, c_2) = \rho(z_1, z_2)$ is a lattice distance, contradicting S_α being a partial Steinhaus set. The following geometric lemma says that this is the only exceptional case to our assertion.

Lemma 2.17 (Lemma B)**.** *Let c_1, c_2, c_3 be three distinct points in the plane, $r_1, r_2, r_3 > 0$, and C_1, C_2, C_3 the corresponding circles. Let T be a triangle. Then there are only finitely many (z_1, z_2, z_3) such that $z_i \in C_i$ and $\triangle z_1 z_2 z_3 \cong T$ except in the exceptional case described above.*

Granting this lemma, we now briefly outline the actual construction (the details may be found in [9] or [10]). To simplify matters we assume CH (in the general case we use an iteration of the hull method). Let $M_0 \subseteq M_1 \subseteq M_\alpha \subseteq$ be an increasing, continuous sequence of hulls, that is, substructures of some large V_κ which are sufficiently closed. We construct the partial Steinhaus sets S_α so that S_α meets all lattices in M_α. Limit stages again are trivial. For $\alpha < \omega_1$ a successor, let $\mathcal{L}_n$ enumerate the equivalence classes in $M_\alpha - M_{\alpha-1}$, and let L_n be a representative for $\mathcal{L}_n$. To keep the construction going, we need to assume also the following inductive hypothesis:

$(*)$: for any $\beta < \alpha$ and any $x, y \in S_\beta$, if $\rho^2(x, y) \in \mathbb{Q}$, then for some lattice $L \in M_\beta$ we have that x, y are both rational with respect to L.

We then diagonalize the construction of the k, l functions of lemma A for all of the L_n, that is, at each step we extend the k, l functions from the rational points of some L_n with denominators d_m to those of denominator d_{m+1} (where $d_1|d_2|d_3 \dots$, and every integer divides some d_m). Lemma B gives us finitely many lines that we must avoid at each step. From $(*)$ we have there is at most one point in $S_{\alpha-1}$ which has rational coordinates with respect to L_n. If this point exists, we make sure that in defining the k, l functions for L_n that this point is thrown in at the first step of the Lemma A construction for L_n. There is enough freedom in the lemma A construction (from the A_i and B_i) so that we can meet these demands at each step. This completes the outline of the proof, granting lemma B.

2.6. **Geometry.** We finish our overview of the construction with some comments on the geometry, that is, on lemma B. This lemma is really a result in the branch of engineering mathematics known as the theory of mechanical linkages. A (planar) mechanical linkage can be thought of as a collection of rigid rods which are joined by hinges which allow rotation. Lemma B corresponds to the classical case of a four-bar linkage (sometimes described in the literature as a three bar linkage). In the terminology of lemma B, let $C_1 = C(c_1, r_1)$, and $C_2 = C(c_2, r_2)$ be circles with $c_1 \neq c_2$ and $r_1, r_2 > 0$. Consider the mechanism consisting of four bars linked to from a quadrilateral with one side (which we view as immovable) of length $\rho(c_1, c_2)$, two adjacent sides of lengths r_1 and r_2, and the remaining side of length $\rho(z_1, z_2)$. This is our four-bar linkage. Consider a rigid triangle congruent to $\triangle z_1 z_2 z_3$ attached to the linkage so that the $z_1 z_2$ edges are identified. As the linkage moves, the point z_3 traces out a path in the plane. This is referred to as a *coupler curve* for the linkage.

Thus, lemma B is the statement that the coupler curve of a four-bar linkage has, except in the exceptional case noted, only a finite intersection with any circle (equivalently, does not have a circular component).

Mechanical linkages, and in particular the four-bar linkage, have been studied extensively and there is a considerable literature on the subject (c.f. [11]). It appears to us that lemma B was implicitly known before, but we were unable to find a rigorous explicit statement of it. However in [8] the authors use algebraic geometry to analyze the four-bar linkage and obtain results which imply lemma B (private communication). This is explained in more detail in [10]. Finally, the current authors give two elementary proofs of lemma B in [10]. □

3. Steinhaus sets for Z^2, the Baire property and measurability

In this section, we will prove the following theorem.

Theorem 3.1. *Suppose S is a Steinhaus set for Z^2. Then S does not have the Baire property. In particular, no Steinhaus set can be a Borel set or an analytic or coanalytic set.*

The argument we present for this is a category version of an argument given by Croft in [2] to show that no Steinhaus set in R^2 could be an essentially bounded measurable set.

The proof is based on the fact that the gaps in the lattice distances converge to 0. For the sake of completeness we indicate an elementary argument for this fact. Let n be a positive integer and consider the lattice distances $g(n,i) = ||(n,i)|| = \sqrt{n^2+i^2}$, for $0, \leqslant i \leqslant \lceil\sqrt{2n+1}\rceil = b_n$. So, $g(n,0) = n, g(n,b_n) \geqslant n+1$ and $g(n,i+1) - g(n,i) \leqslant \frac{2b_n+1}{2n}$. Thus, for n large the gaps between these distances is small which certainly means the gaps between consecutive lattice distances must converge to 0.

From this point on let $d_1 = 1, d_2 = \sqrt{2}, d_3 = 2, ...$ enumerate the lattice distances in increasing order.

Our first observation is that S must be essentially bounded in the sense of category.

Lemma 3.2. *Suppose S has the Baire property. Then there is some $R > 0$ such that $\{x \in S : ||x|| \geqslant R\}$ is meager.*

Proof. Since $R^2 = \bigcup_{z\in Z^2}(S+z)$, S cannot be meager. So, there is a ball such that the part of S in the ball is comeager in the ball. Since a translate of a Steinhaus set is a Steinhaus set, we may assume that S is comeager in $B(0,\epsilon)$. Note that if $||x|| = d_n$, then $S \cap B(x,\epsilon)$ is meager, since otherwise by translating this ball to the by $-x$ we would find two points u and v of S such that $v = u+x$, contradicting the fact that no two points of S are at a lattice distance apart. From this we see that the part of S in the annulus $A(d_n-\epsilon, d_n+\epsilon)$ is meager. If n is sufficiently large any two consecutive annuli overlap and the lemma follows. □

Let G be the largest open set in which S is comeager. The set G is bounded and let $N = \overline{G}\backslash G$. Then N is a bounded closed nowhere dense set and $N \neq \varnothing$. The set N is the category essential boundary of S. It consists of all points x such neither S nor the complement of S is meager in any neighborhood of x.

Lemma 3.3. *There is some isometric copy of Z^2 which meets N in exactly one point.*

Proof. By way of contradiction, let us suppose that every lattice $I(Z^2)$, where I is an isometry of R^2 either misses N entirely or else contains at least two points of N. Let $w_0 \in N$ and let $L_0 = Z^2 + w_0$. For each θ, there must be a point $w \neq w_0$ which is in $T_\theta(L_0)$, where T_θ is the rotation or angle θ about w_0. Let $F_n = \{\theta \colon \exists w \in N \cap T_\theta(L_0)$ and $||w - w_0|| = d_n\}$. Since the sets F_n are closed there is some n_0 such that F_{n_0} contains a closed arc C_0 on the unit circle. Let $u_1, ..., u_l$ be the points of L_0 at distance d_n from w_0 and let $H_i = \{\theta \in C_0 : T_\theta(u_i) \in N\}$. One of the sets H_i contains a subarc of C_0. Therefore, there is a closed arc $\Gamma_0 = [\alpha_0 - \epsilon_0, \alpha_0 + \epsilon_0]$ such that every point of the form $w_\theta = w_0 + d_{n_0}e^{i\theta}$ is in N for all $\theta \in \Gamma_0$. Now let us repeat the argument just given starting with a point w_θ. For d a lattice distance, Let $H_{d,n}$ be the set of $\theta_0 \in \Gamma_0$ such that there is an arc C_1 with length $\geqslant \frac{1}{n}$ such that $w_0 + d_{n_0}e^{i\theta_0} + de^{i\theta} \in N$ for all $\theta \in C_1$. By the Baire category theorem, there are two arcs γ_0 and γ_1 and a lattice distance d_{n_1} such that $\gamma_0 \subset \Gamma_0$ and for every $\theta \in \gamma_0$ and $\phi \in \gamma_1$, $w_{\theta,\phi} = w_0 + d_{n_0}e^{i\theta} + d_{n_1}e^{i\phi} \in N$.

Now, consider the map h on $\gamma_0 \times \gamma_1$ defined by $h(\theta,\phi) = w_{\theta,\phi}$. At most points the derivative of h is nonsingular and therefore the image of h, which is a subset of N, contains a nonempty open set. This contradiction establishes the lemma. □

Proof of Theorem. Let us suppose that a Steinhaus set S has the Baire property. By the preceding lemmas we can suppose that the one and only point of $N \cap Z^2$ is the origin and the the part of S at distance greater than D is meager.

Consider a point $(u,v) \in Z^2$ with $(u,v) \neq (0,0)$. Either $(u,v) \in G$ or $(u,v) \in R^2\backslash\overline{G}$. If (u,v) were in G, then there would be some $0 < \delta$ such that S is comeager in $B((u,v),\delta)$. But since S is not meager in $B((0,0),\delta)$ there would be two points of S at distance $||(u,v)||$ apart. Therefore, there is some $0 < \epsilon$ such that if $p \in Z^2$ and $0 < ||p|| \leqslant D$, then the part of S in $B(p,\epsilon)$ is meager. So in fact this is true for all $0 \neq p \in Z^2$. Translate each of these balls to be centered at the origin. Since the translated parts of S are meager, there is some point x with $||x|| < \epsilon$ such that $x \notin S$ and for all $p \in Z^2$, $x + p \notin S$. Thus, $(Z^2 + x) \cap S = \varnothing$. This contradiction completes the proof of the theorem. □

The arguments we have just given are modifications of ones given by Croft [2]. Croft proved the measure theoretic versions of the lemmas used in this section and proved the following theorem which was also proved independently by Beck [1].

Theorem 3.4. *Let S be a Steinhaus set for Z^2. Then S cannot be Lebesgue measurable and essentially bounded, i.e., there exists some $R > 0$ such that $\lambda(\{x \in S\colon ||x|| > R\}) = 0$.*

Beck's proof of this theorem has an entirely different viewpoint. Beck uses a Fourier transform approach. Kolountzakis in [21] and Kolountzakis and Wolff in [13] have much more detailed results about possible measurable Steinhaus sets for Z^2. Their approach is also via the Fourier transform. Kolountzakis proved the following theorem.

Theorem 3.5. *Suppose S is a measurable Steinhaus set for Z^2. Then $\int_S |x|^\alpha dx = \infty$ for all $\alpha > 10/3$. In particular, S cannot be essentially bounded.*

Kolountzakis and Wolff have made a connection to the famous circle problem.

Theorem 3.6. *Assume a bound of the form*

$$n(r) = \pi r^2 + O(r^\beta),$$

where $n(r) = card(Z^2 \cap B(0,r))$. Then

$$\int_S |x|^\alpha dx = \infty,$$

for all $\alpha > \beta/(1-\beta)$.

At present the best result concerning β is that of Huxley [19] who showed that $\beta > 46/73$ may be used in estimating $n(r)$. Thus, if S is a measurable Steinhaus set, then $\int_S |x|^\alpha dx = \infty$, for all $\alpha > 46/27$. We will discuss some more aspects of the Fourier transform approach in the next section.

The following basic problem remains open.

Problem. Can there be a Lebesgue measurable Steinhaus set for Z^2?

We remark that although it remains open whether a Steinhaus set for $\mathbb{Z}^2$ must be non-measurable, we can show the existence of non-measurable Steinhaus sets. We give a sketch for the reader familiar with the construction. It is enough to produce a Steinhaus set S such that the difference set $S - S$ does not contain a neighborhood of the origin, and for this it suffices to arrange that $\rho^2(x, y) \notin \{\frac{1}{2^n} : n \geqslant 1\}$ for all $x, y \in S$. From (∗) of the construction of §2.5 we have that if $x, y \in S$ and $\rho^2(x, y) = \frac{1}{2^n}$, then x and y are both rational with respect to some lattice L considered at some step β of the construction. In the diagonalization of lemma A to the countably many lattices $L_1, L_2, \ldots$ considered at stage β, we can easily arrange that as we extend the partial k, l functions on some L_m (say from those points with denominator d_n to d_{n+1}) we have that no two of the corresponding translated points x, y have $\rho^2(x, y) \leqslant 1$. This suffices. In fact, we produce a Steinhaus set S such that if $x, y \in S$ and $\rho^2 \in \mathbb{Q}$, then $\rho(x, y) \geqslant 1$.

Finally, we want to mention a result of M. Ciucu about Steinhaus sets of Z^2 [18].

Theorem 3.7. *If S is a Steinhaus set for Z^2, then S has empty interior.*

Ciucu's approach is geometric. By following his proof one can obtain another proof of the fact that no Steinhaus set can have the Baire property.

4. Higher Dimensional Lattices

In this section, we consider lattices L in $\mathbb{R}^d$, for $d \geqslant 2$. This means there is some invertible linear transformation $A \in GL(d, \mathbb{R})$ such that $L = A\mathbb{Z}^d$. Sometimes we denote L by L_A. We say S is a Steinhaus set for the lattice L provided $|S \cap T(L)| = 1$, for all isometries T of R^d. Thus, S is a Steinhaus set for the lattice L provided $S \cap T(L) \neq \varnothing$ for all isometries T of R^d and no two distinct points of S have the same distance as the distance between two points of L.

The basic unsolved problem is the following:

EXISTENCE PROBLEM. Fix $d \geqslant 3$ and a lattice L in R^d. Is there a Steinhaus set for L?

Let us note that the arguments given in the preceding section actually can be carried out for any lattice in R^d, $d > 1$. We only need to know that if L is a lattice in R^d then the gaps in the lattice distances converge to 0. An argument similar to the one we indicated for Z^2 can be carried out for any planar lattice and since a lattice in R^d, $d > 2$ contains a planar lattice, this is true for any lattice. So, we have the following theorem.

Theorem 4.1. *Let L be a lattice in $\mathbb{R}^d$, for $d \geqslant 2$. There is no Steinhaus set for the lattice L which has the Baire property. In particular, no Steinhaus set for a lattice in R^d, $d > 1$ can be a Borel set.*

Although the existence problem remains completely open for all lattices in R^d, $d > 2$, the question of whether there is a measurable Steinhaus set has been solved in the negative for some lattices in R^d with $d > 2$. The method is based on the use of Fourier transforms. This approach was started by Beck [1]. Kolountzakis extended and simplfied this approach [20]. It was deeply studied by Kolountzakis and Wolff [13]. They proved the following theorem.

Theorem 4.2. *There is no Lebesgue measurable Steinhaus set for the lattices Z^d for $d > 2$.*

We will sketch the Fourier transform technique used for proving theorem 4.2 and a slight generalization which yields some other lattices for which we know there is no measurable Steinhaus set. Since we will discussing measurable sets, it makes sense to talk about "almost sure" Steinhaus sets, a notion introduced in [13].

Definition 4.3. A set S is said to have the *almost sure Steinhaus property on the lattice* L_A provided that under almost every rotation T, and almost every point x, $|(TS+x) \cap (A\mathbb{Z}^d)| = 1$.

Observe that this property may be described as follows:

$$\sum_{n \in A\mathbb{Z}^d} \mathbf{1}_{TS}(x-n) = 1, \quad \text{a.e. } x \in \mathbb{R}^d, \text{ a.e. rotation T.} \tag{14}$$

We can directly compute, $\mu(S)$, the Lebesgue measure of such a set S. In particular, if a measurable set S has this property for almost all x for just some fixed isometry T, then we may integrate both sides over the fundamental domain $D = A([0,1)^d)$ to obtain:

$$\begin{aligned} |detA| &= \int_D 1 dx = \int_D \sum_{n \in A\mathbb{Z}^d} \mathbf{1}_{TS}(x-n)dx = \sum_{n \in A\mathbb{Z}^d} \int_D \mathbf{1}_{TS}(x-n)dx \\ &= \sum_{n \in A\mathbb{Z}^d} \int_{n+D} \mathbf{1}_{TS}(x)dx = \int_{R^d} \mathbf{1}_{TS}(x)dx = \mu(T(S)) = \mu(S). \end{aligned} \tag{15}$$

So, if S is an almost sure measurable Steinhaus set for the lattice $A\mathbb{Z}^d$, then the Lebesgue measure of S is $|detA|$. More importantly, there is a characterization of almost sure measurable Steinhaus sets by applying basic Fourier transform methods; see Chapter VII of [26].

To explain this, let $L_A^* = A^{-T}\mathbb{Z}^d$ be the dual lattice to L_A.

From elementary harmonic analysis, we have that the following lemma.

Lemma 4.4. *Let f be an L^1 function. Then*

$$\sum_{\lambda \in L_A} f(x-\lambda) = C, \quad a.e.\ x \tag{16}$$

if and only if its Fourier transform satisfies:

$$\hat{f}(\lambda) = 0, \quad \forall \lambda : \lambda \in L_A^* \backslash \{0\}. \tag{17}$$

Moreover, if (16) holds, then by integrating both sides of (16) over D, the fundamental domain or parallelepiped spanned by the columns of A, we find that $C = \int f(x)dx/|det(A)|$.

Proof. Again, let $D = A([0,1)^d)$. Consider

$$
\begin{aligned}
\hat{f}(y) &= \int_{\mathbb{R}^d} f(x)e^{-2\pi i(x\cdot y)}dx = \sum_{\lambda\in AZ^d}\int_{\lambda+D} f(x)e^{-2\pi i(x\cdot y)}dx \\
&= \sum_{\lambda\in AZ^d}\int_D f(\lambda+v)e^{-2\pi i(\lambda+v)\cdot y}dv \\
&= \sum_{\lambda\in AZ^d}|det(A)|\int_{[0,1)^d} f(\lambda+Au)e^{-2\pi i((\lambda+Au)\cdot y)}du.
\end{aligned}
\tag{18}
$$

Denote the columns of A by a_i and the columns of A^{-T} as a_i^*. Consider $y \in L_A^*\backslash 0$. Then $y = A^{-T}c = c_1a_1^* + ... + c_da_d^*$, for some $c = (c_1, ..., c_n) \in \mathbb{Z}^d$, $c \neq 0$. For $u \in [0,1)^d$, we have $Au = u_1a_1 + ... + u_da_d$. Now, substituting into (18), and using $\lambda \cdot y \in \mathbb{Z}$, we get:

$$\hat{f}(y) = |\det(A)|\int_{[0,1)^d}\sum_{\lambda\in A\mathbb{Z}^d} f(\lambda+Au)e^{-2\pi i(u\cdot c)}du.$$

The function $g(u) = \sum_{\lambda\in A\mathbb{Z}^d} f(\lambda+Au)$ is in $L^1([0,1)^d)$ and is periodic: $g(u+z) = g(u)$ for all $u \in R^d, z \in Z^d$ Thus, $\hat{f}(y) = |det(A)|\hat{g}(A^Tc)$, where $\hat{g}(z) = \int_{[0,1)^d} g(u)e^{-2\pi i(u\cdot z)}du$.

Now, all the Fourier coefficients other than the constant term of a periodic function are 0 if and only if the function is constant a. e. on the unit cube. The proof of the equivalence is complete.

Note that integrating both sides of (16) over D gives $C = \frac{\int f(x)dx}{\det(A)}$ □

Thus, we can characterize an almost Steinhaus set S for a lattice L in terms of the properties of its Foruier transform.

Corollary 4.5. *A measurable set S has the almost sure Steinhaus property for the lattice L_A if and only if the Lebesgue measure of S, $\mu(S) = |det(A)|$, and the Fourier transform $\widehat{\mathbf{1}_S}$ vanishes on all points x, such that $\|x\| = \|\lambda\|$ for some $\lambda \in L_A^*$, $\lambda \neq 0$.*

We are now in a position to give sufficient conditions under which there is no measurable set with the almost sure Steinhaus property for the lattice L_B. Let us set some notation. Given a matrix M, let $\mathcal{D}(M) = \{\|Mx\|^2 \colon x \in \mathbb{Z}^d\}$, the set of possible square distances between points of the lattice $M\mathbb{Z}^d$. If A and B are matrices such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, we say B norm dominates A, and write $B \succ A$ or $A \prec B$. If $B \succ A$ and we have that $det(A)/det(B)$ is irrational, we say B strongly norm dominates A, and write $B \succ_s A$. If $B \succ A$ and we have $det(A)/det(B)$ not an integer, we say B weakly norm dominates A, and write $B \succ_w A$. Finally, if $B \succ A$ and $det(A)/det(B) \in \mathbb{Z}$, we say B trivially norm dominates A, and write $B \succ_t A$. With this terminology in place, we have the following theorem.

Theorem 4.6. *Let $B \in GL(d,\mathbb{R})$ and suppose there exists a matrix $A \in GL(d,\mathbb{R})$, where $B^{-T} \succ_w A^{-T}$. Then there is no measurable set with the almost sure Steinhaus property on the lattice L_B.*

Proof. Suppose by way of contradiction, that there is a measurable set S with the almost sure Steinhaus property on L_B. By corollary 4.5, $\int \mathbf{1_S}(x)dx = |\det(B)|$ and $\widehat{\mathbf{1}_S}$ vanishes on all nonzero points with norm square in $\mathcal{D}(B^{-T})$. So, $\widehat{\mathbf{1}_S}$ vanishes on all nonzero points with norm square in $\mathcal{D}(A^{-T})$. In particular, $\widehat{\mathbf{1}_S}$ vanishes on $L_A^*\backslash\{0\}$. By lemma 4.4, $\sum_{\lambda\in\Lambda_A} f(x-\lambda) = \frac{\int \mathbf{1_S}(x)dx}{|detA|} = \frac{|det(B)|}{|det(A)|} = \frac{|det(A^{-T})|}{|det(B^{-T})|}$ for almost all x. However, the left side must be an integer,whereas we have supposed that the right side is not. □

As an immediate corollary let us prove the following theorem.

Theorem 4.7. *There is no measurable Steinhaus set for the lattices Z^d for $d > 2$.*

Proof. It is perhaps easiest to see this when $d = 4$. Let B be the identity matrix and let A^{-T} be

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}.$$

We have $\mathcal{D}(A^{-T}) \subseteq \mathcal{D}(B)$, since every integer is the sum of 4 squares. Also, Since $det(A^{-T})/det(B^{-T}) = \sqrt{2}$, $B^{-T} \succ_w A^{-T}$. By theorem 4.6, there is not even an almost sure measurable Steinhaus set for Z^4. This method clearly works for all Z^d with $d > 3$. The case when $d = 3$ was dealt with by Kolountzakis and Papadimitrakis [24]. They showed

$$B^{-T} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \succ_w \begin{bmatrix} \sqrt{2} & & \\ & \sqrt{11} & \\ & & \sqrt{6} \end{bmatrix} = A^{-T}.$$

So, there can be no measurable Steinhaus set for the lattice Z^3. □

It is useful to note the Theorem leads to the following two part strategy: if we can find a matrix C such that $C \succ_t B$ and $B \succ_s A$, then $C \succ_s A$ and of course, $C \succ_w A$. In [16] it is shown that this strategy can be applied to a number of other lattices. The method uses some special quadratic forms and the method of descent. On the other hand, there seem to be some severe restrictions to this approach. For example, Kolountzakis and Papadimitrakis [24] have shown that in case $d = 2$ and with B being the identity matrix, there is no such A. So this strategy cannot be applied in the plane. It seems likely that this method cannot be applied to any planar lattice, but there is no proof of this to the authors' knowledge.

In [16] some further limitations of this strategy are shown. It is shown that there is a class of diagonal matrices B such that if $A \succ B$, then $A \succ_t B$.

5. Problems

Here we gather some of the problems that remain unsolved.

1. Is there a Steinhaus set S for the lattice Z^n in R^n for $n \geqslant 3$? Is there a Steinhaus set for the rectangular lattices in R^2? More generally, for which lattices L in R^n, $n \geqslant 2$, is there a Steinhaus set?

2. Can a Steinhaus set S for Z^2 be Lebesgue measurable? Is there any lattice with dimension greater than 1 with a Lebesgue measurable Steinhaus set?

3. Can a Steinhaus set for Z^2 or, for that matter any lattice, be bounded? Must it always be totally disconnected?

4. Is there a measurable partial Steinhaus set meeting all translates of Z^2? What if we allow all translates in two different directions?

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