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Quantization dimension for conformal iterated function systems

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Abstract. The quantization dimension function is determined for a certain class of probability measures generated by a finite conformal iterated function system satisfying the strong open set condition.

1. Introduction

The term *quantization* in this paper refers to the idea of estimating a given probability on $\mathbb{R}^d$ with a discrete probability. That is, a "quantized" version of the probability supported on a finite set. Hence, quantization in this context should not be confused with a similar term from quantum physics. For over fifty years engineers and mathematicians have been interested in the problem of efficiently quantizing a probability distribution. This problem arises in signal processing, data compression, cluster analysis, and pattern recognition, and it also has been studied in the context of economics, statistics, and numerical integration. Many useful theorems and algorithms have evolved over the years. (See [4] for a detailed survey of the history of quantization.) In general, these theorems have dealt almost exclusively with absolutely continuous distributions on $\mathbb{R}^d$. Two main goals have been (1) finding the exact configuration of a so-called "n-optimal set" which corresponds to the support of the quantized version of the distribution, and (2) estimating the rate at which some specified measure of the error (also called the distortion or noise, between the quantized distribution and the original distribution) goes to zero as n goes to infinity. This paper deals with the second problem in the case of singular distributions.

If the absolutely continuous part of the given probability does not vanish, then the asymptotics of the error behave nicely, and certain limits are known to exist. However, less is known in the strictly singular case. The goal of this paper is to generalize known results for certain "self-similar" probabilities to a class of (singular) probabilities supported on fractal sets generated by conformal iterated function systems in $\mathbb{R}^d$.

Following the work of Graf and Luschgy [2] and [3] we define the *quantization dimension* (or perhaps better, the *quantization dimension function*) as follows. Given a

Borel probability measure P on $\mathbb{R}^d$, a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the *n-th quantization error* of order r for P is defined by

$$e_{n,r} = \inf\{(\int d(x,\alpha)^r \mathrm{d}P(x))^{1/r} : \alpha \subset \mathbb{R}^d, \mathrm{card}(\alpha) \leq n\}$$

where $d(x, \alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^d$. We note that if $\int \|x\|^r \mathrm{d}P(x) < \infty$, then there is some $\alpha \subset \mathbb{R}^d$ for which the infimum is achieved [2]. As a side note we observe that this set α can then be used to give a best approximation of P by a discrete probability supported on a set of n points. Under suitable conditions this can be done by giving each point $a \in \alpha$ a mass corresponding to $P(A_a)$ where A_a is the set of points $x \in \mathbb{R}^d$ such that $d(x, \alpha) = d(x, a)$. Of course, the idea of "best approximation" is, in general, dependent on the choice of r.

Graf and Luschgy also define $e_{n,r}$ when $r = 0$ and $r = \infty$, but in this paper we only deal with the case $0 < r < +\infty$.

The *quantization dimension of order* r for P is defined to be

$$D_r = D_r(P) = \lim_{n\to\infty} \frac{\log n}{-\log e_{n,r}}$$

if the limit exists. If the limit does not exist then we define $\overline{D}_r$ as the limsup of the sequence and $\underline{D}_r$ as the liminf. One sees that the quantization dimension is actually a function $r \mapsto D_r$ which measures the asymptotic rate at which $e_{n,r}$ goes to zero. If D_r exists, then one can write

$$\log e_{n,r} \sim \log \left(\frac{1}{n}\right)^{1/D_r}.$$

The quantization dimension function is in some respects similar to the $f(\alpha)$ curve which gives the multifractal spectrum (see Falconer [1]). In fact, in Theorem 1 we indicate a relationship between the quantization dimension function and $\beta(q)$, the Legendre transform of $f(\alpha)$.

The use of the term "dimension" follows from the original work of Zador [12], in which he introduces the formula and compares it to box-counting dimension and Hausdorff dimension. In the engineering literature the case $r = 2$ is almost exclusively used, and interesting relationships to other types of dimension have been noted. Specifically, Pötzelberger [10] has shown that for distributions with bounded support

$$\underline{D}_2 \in [\dim_H(P), \underline{\dim}_B(P)] \text{ and } \overline{D}_2 \in [\dim_P(P), \overline{\dim}_B(P)]$$

where $\dim_H(P)$, $\dim_P(P)$, $\underline{\dim}_B(P)$, and $\overline{\dim}_B(P)$ denote the Hausdorff, packing, lower and upper box-counting dimension of P, respectively.

Another way of formulating the quantization error problem is by considering a random vector X from a probability space (Ω, μ) to $\mathbb{R}^d$ with probability distribution $P = \mu \circ X^{-1}$. We then want to approximate X with a random vector Y whose range consists of at most n points in $\mathbb{R}^d$. In this case, we minimize $\|X - Y\|_r$, where $\|X\|_r = (\int \|X\|^r d\mu)^{1/r}$ is the L^r-norm. It turns out that Y can be written in the form $f(X)$ where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a function with at most n points in its range.

The problem of determining the quantization dimension function for a general probability is open. However, Graf and Luschgy have determined a formula for the quantization dimension function of a self-similar probability P defined for an iterated function system using a finite number of contracting similarities $S_1, \ldots, S_N$ on $\mathbb{R}^d$ satisfying the Open Set Condition and given a probability vector $(p_1, \ldots, p_N)$. The measure P satisfies

$$P = \sum_{i=1}^{N} p_i P \circ S_i^{-1}.$$

They show that $D_r := D_r(P)$ satisfies

$$\sum_{i=1}^{N} (p_i s_i^r)^{\frac{D_r}{r+D_r}} = 1 \tag{1}$$

where s_i is the contraction coefficient for the map S_i. In fact, they prove a stronger result. Namely, that the quantization dimension D_r also satisfies the following.

$$0 < \liminf_{n\to\infty} n e_{n,r}^{D_r} \leq \limsup_{n\to\infty} n e_{n,r}^{D_r} < +\infty. \tag{2}$$

In this paper we extend the results of Graf and Luschgy in the following ways. We consider conformal iterated function systems, and we obtain the quantization dimension function for probability measures supported on the limit set which are the Gibbs states or equilibrium measures for a Hölder potential. Our main theorem indicates the relationship between the quantization dimension function and the multifractal spectrum of the measure. Also, Lemma 6 shows that the right-hand side of (2) holds for (finite) conformal iterated function systems. However, it still remains open whether $\liminf_{n\to\infty} n e_{n,r}^{D_r}$ is strictly positive.

2. Definitions and background

In [2] Graf and Luschgy set

$$V_{n,r} = \inf\{\int d(x,\alpha)^r \mathrm{d}P(x) : \alpha \subset \mathbb{R}^d, \mathrm{card}(\alpha) \leq n\}$$

and point out that $e_{n,r} = V_{n,r}^{1/r}$. Also, in their proofs in [3] they make use of the value

$$u_{n,r} = \inf\{\int d(x,\alpha \cup U^c)^r \mathrm{d}P(x) : \alpha \subset \mathbb{R}^d, \mathrm{card}(\alpha) \leq n\}$$

where U is a set which comes from the Open Set Condition (see below for the definition) and U^c denotes the complement of U. We see that

$$u_{n,r}^{1/r} \leq V_{n,r}^{1/r} = e_{n,r}.$$

We will call sets $\alpha_n \subset \mathbb{R}^d$, for which the above infimums are achieved, *n-optimal sets* for $e_{n,r}$, $V_{n,r}$, or $u_{n,r}$, respectively. As stated above, Graf and Luschgy have shown that n-optimal sets exist when $\int \|x\|^r \mathrm{d}P(x) < \infty$. Since the probabilities associated with iterated function systems (as described below) are supported on compact sets we can make use of n-optimal sets in our proofs.

In the following we remind the reader of the basic setup for conformal iterated function systems. Our notation and assumptions are taken from [6]. Suppose $X \subset \mathbb{R}^d$ is a compact set such that $X = \overline{\mathrm{Int}(X)}$ (the closure of its interior). Let $I = \{1, 2, \ldots, N\}$ be an index set, $I^* = \bigcup_{n \geq 1} I^n$ be the set of all finite words, and I^∞ be the set of all infinite words. Suppose $\{\varphi_i : X \to X\}$ is a collection of conformal maps such that $\|\varphi_i'\| \leq s < 1$ for some s where $\|\varphi_i'\|$ denotes the supremum norm of the derivative. Let $|\varphi_i'(x)|$ be the norm of the derivative at $x \in \mathbb{R}^d$. For $\omega \in I^*$, let $|\omega|$ denote the length of ω. For $\omega = \omega_1\omega_2 \ldots \in I^\infty$, let $\sigma(\omega) = \omega_2\omega_3 \ldots$ be the "shift" map and $\omega|_n = \omega_1\omega_2 \cdots \omega_n$. For $\omega = \omega_1\omega_2 \cdots \omega_n \in I^n$, let $\varphi_\omega = \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n}$ and $\omega^- = \omega_1\omega_2 \cdots \omega_{n-1}$. Let J be the unique limit set which satisfies

$$J = \bigcup_{i \in I} \varphi_i(J).$$

An iterated function system satisfies the Open Set Condition if there exists a nonempty open set $U \subset X$ such that $\varphi_i(U) \subset U$ for every $i \in I$ and $\varphi_i(X) \cap \varphi_j(X) = \emptyset$ for every pair $i, j \in I$, $i \neq j$. Furthermore, the system satisfies the Strong Open Set Condition if U can be chosen such that $U \cap J \neq \emptyset$. A recent paper by Peres *et al* [8] shows that the Open Set Condition implies the Strong Open Set Condition in the case of a conformal iterated function system using a finite number of maps.

An iterated function system satisfying the Open Set Condition is said to be *conformal* if the following conditions are satisfied.

(i) $U = \mathrm{Int}_{\mathbb{R}^d}(X)$.

(ii) There exists an open connected set $X \subset V \subset \mathbb{R}^d$ such that all maps φ_i extend to conformal C^1-diffeomorphisms of V into V.

(iii) There exist $\gamma, \ell > 0$ such that for every $x \in \partial X \subset \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \gamma, \ell) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude ℓ.

(iv) Bounded Distortion Property. There exists $K \geq 1$ such that

$$|\varphi_\omega'(y)| \leq K|\varphi_\omega'(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$.

We see from a result of Patzschke [9], Lemma 2.2, that there exists $\widetilde{K} \geq K$ such that

$$\widetilde{K}^{-1}\|\varphi_\omega'\|d(x, y) \leq d(\varphi_\omega(x), \varphi_\omega(y)) \leq \widetilde{K}\|\varphi_\omega'\|d(x, y)$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$.

As a specific example of such a system in $\mathbb{R}^2$ the reader is referred to Figure 1. In this example we let the initial set X be the disk centered at $(1/2, 0)$ with radius $1/2$.

Equating $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ we define three conformal maps φ_1, φ_2, and φ_3 on $\mathbb{C}$ by

$$\varphi_1(z) = \frac{1}{1+z}$$
$$\varphi_2(z) = \frac{1}{(1-i)+z}$$
$$\varphi_3(z) = \frac{1}{(1+i)+z}.$$

In the figure one can see "level 1" and "level 5" in the construction of the limit set J. These levels correspond to the $n = 1$ and $n = 5$ stages in the representation of J as an infinite intersection

$$J = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^n} \varphi_\omega(X).$$

The above example gives a fractal set whose elements are the points in $\mathbb{C}$ which have an infinite continued fraction representation using only the entries 1, $1-i$, and $1+i$.

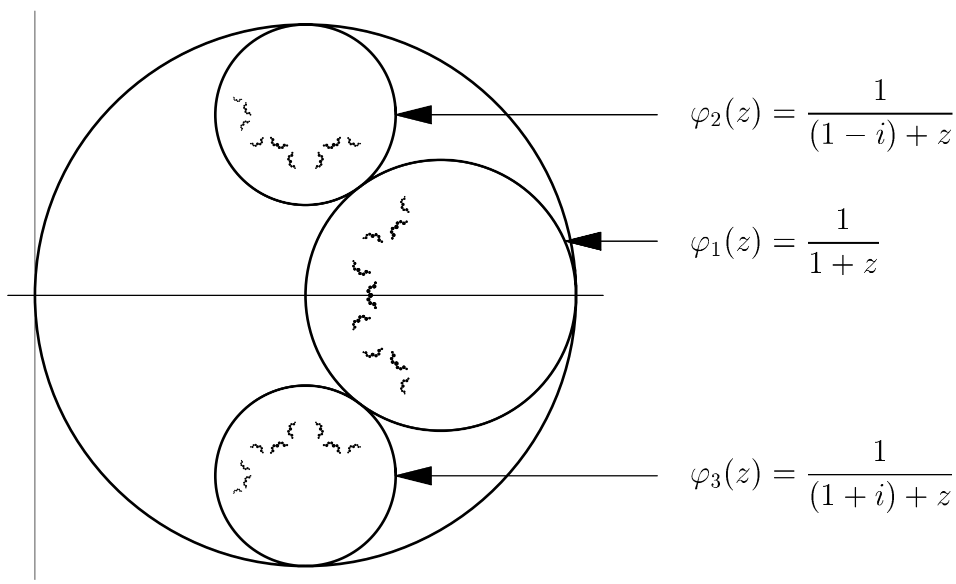

Figure 1. A conformal construction in $\mathbb{R}^2$. This figure shows the boundary of the original compact set X (the large circle), the boundaries of the images of X under the three conformal maps (the three smaller circles), and "level five" in the construction of the limit set J. Note that the conformal maps are given as maps from $\mathbb{C}$ to $\mathbb{C}$.

Now we return to our general setup and discuss the existence of certain probability measures on J. To this end let $F = \{f^{(i)} : X \to \mathbb{R}\}_{i \in I}$ be a strongly Hölder family of continuous functions (see [6]). That is, for some $\beta > 0$, there exists

$$V_\beta(F) = \sup_{n \geq 1}\{V_n(F)\} < \infty$$

where, for each $n \geq 1$,

$$V_n(F) = \sup_{\omega \in I^n} \sup_{x,y \in X} \{|f^{(\omega_1)}(\varphi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\varphi_{\sigma(\omega)}(y))|\} \mathrm{e}^{\beta(n-1)} \tag{3}$$

and also

$$\sum_{i \in I} \|\mathrm{e}^{f^{(i)}}\| < \infty. \tag{4}$$

For $n \geq 1$ and $\omega \in I^n$ denote $\sum_{j=1}^n f^{(\omega_j)} \circ \varphi_{\sigma^j(\omega)}$ by $S_\omega(F)$. We define the topological pressure of F by

$$P(F) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \| \exp(S_\omega(F))\|.$$

This is analogous to the free energy discussed by Tél [11]. By (4) we get that $P(F) < \infty$, and by subtracting $P(F)$ from each $f^{(i)}$ we can assume $P(F) = 0$.

From [6] (using equation (2.3), Lemma 2.4, and the proof of Proposition 2.5), we have the existence of a probability measure m on J (the F-conformal measure) such that for any continuous function $g : X \to \mathbb{R}$ and $n \geq 1$,

$$\int g \mathrm{d}m = \sum_{|\omega|=n} \int \exp(S_\omega(F)) \cdot (g \circ \varphi_\omega) \mathrm{d}m. \tag{5}$$

Furthermore, for $q, \beta \in \mathbb{R}$, let

$$G_{q,\beta} = \{g_{q,\beta}^{(i)} := \beta \log |\varphi_i'| + q f^{(i)}\}_{i \in I}.$$

We note that since our system is finite we have that for every $q \in \mathbb{R}$, $P(qF) < \infty$. Therefore, we have the following lemma (compare Lemma 7.2 in [6]).

Lemma 1. *For each $q \in \mathbb{R}$ there exists a unique $\beta(q) \in \mathbb{R}$ such that $P(G_{q,\beta(q)}) = 0$.*

The function $\beta(q)$ is sometimes denoted $T(q)$ and called the temperature function. A more general discussion of this function can be found in Halsey *et al* [5], where our $\beta(q)$ function would correspond to $-\tau(q)$ using their notation. We note from [6] that in our conformal setup $\beta(q)$ is continuous, strictly decreasing, convex, $\beta(0) = \dim_H(J)$ (the Hausdorff dimension of J), and $\beta(1) = 0$. Also, note that in the (finite) self-similar case $\beta(q)$ satisfies the relation

$$\sum_{i=1}^N p_i^q s_i^{\beta(q)} = 1. \tag{6}$$

In addition, we have the existence of a probability measure m_q on J such that for any continuous function $g : X \to \mathbb{R}$ and $n \geq 1$,

$$\begin{aligned} \int g \mathrm{d}m_q &= \int \sum_{|\omega|=n} \exp(S_\omega(G_{q,\beta(q)})) \cdot (g \circ \varphi_\omega) \mathrm{d}m_q \\ &= \sum_{|\omega|=n} \int |\varphi_\omega'|^{\beta(q)} (\exp(S_\omega(F)))^q \cdot (g \circ \varphi_\omega) \mathrm{d}m_q. \end{aligned} \tag{7}$$

Note that equations (5) and (7) are true if the summations are taken over any finite, maximal antichain. We call $\Gamma \subset I^*$ a finite, maximal antichain if Γ is a finite set of words such that every $\omega \in I^*$ is an extension of some word in Γ, but no word of Γ is an extension of another word in Γ. Of course, this requires that I is finite. We will make this assumption in the remainder of this paper. By $|\Gamma|$ we denote the cardinality of Γ.

3. Main result

The relationship of the quantization dimension function and the temperature function $\beta(q)$ of certain probability measures m where the temperature function is the Legendre transform of the $f(\alpha)$ curve (the definition of $f(\alpha)$ and the Legendre transform are given in [1]) is given by the following theorem. For a graphical description see Figure 2.

Theorem 1. *Let $\varphi_1, \ldots, \varphi_N$ be a conformal iterated function system satisfying the strong open set condition and $F = \{f^{(1)}, \ldots, f^{(N)}\}$ a strongly Hölder family. Let m be the F-conformal measure generated by this system. That is, for any continuous function $g : X \to \mathbb{R}$ and $n \geq 1$,*

$$\int g\mathrm{d}m = \sum_{|\omega|=n} \int \exp(S_\omega(F)) \cdot (g \circ \varphi_\omega)\mathrm{d}m.$$

Let $\beta(q)$ be the temperature function of the system. For each $r \in (0, +\infty)$ choose q_r such that $\beta(q_r) = rq_r$. Then the quantization dimension (of order r) of the probability measure m is given by

$$D_r = \frac{\beta(q_r)}{1 - q_r}.$$

Our motivation for this theorem comes from the self-similar case. By comparing equations (1) and (6) we see that in the self-similar case we have simultaneously $\beta(q) = rq$ and $D_r = \frac{rq}{1-q}$. Solving equation (1) for D_r corresponds in our more general setting to finding a value κ_r such that

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} (\|\varphi'_\omega\|^r \| \exp(S_\omega(F))\|)^{\frac{\kappa_r}{r+\kappa_r}} = 0. \tag{8}$$

Hence, we wish to show that for finite conformal iterated function systems we still get $D_r = \kappa_r$. Then by comparing the solution κ_r in (8) to the solution $\beta(q)$ which gives $P(G_{q,\beta(q)}) = 0$ we arrive at the statement of the theorem.

To show $D_r = \kappa_r$ we prove a series of lemmas. The following closely parallels the proofs of Graf and Luschgy for the self-similar case. In this more general case we make extensive use of the Bounded Distortion Property (using K and $\widetilde{K}$ in order to get our inequalities) and the following lemma.

Lemma 2. *There exists $C \geq 1$ such that for any $x, y \in X$ and $\omega \in I^*$,*

$$\frac{\exp(S_\omega(F)(x))}{\exp(S_\omega(F)(y))} \leq C.$$

In particular, for any $x \in X$ and $\omega \in I^*$, $\exp(S_\omega(F)(x)) \geq C^{-1}\| \exp(S_\omega(F))\|$.

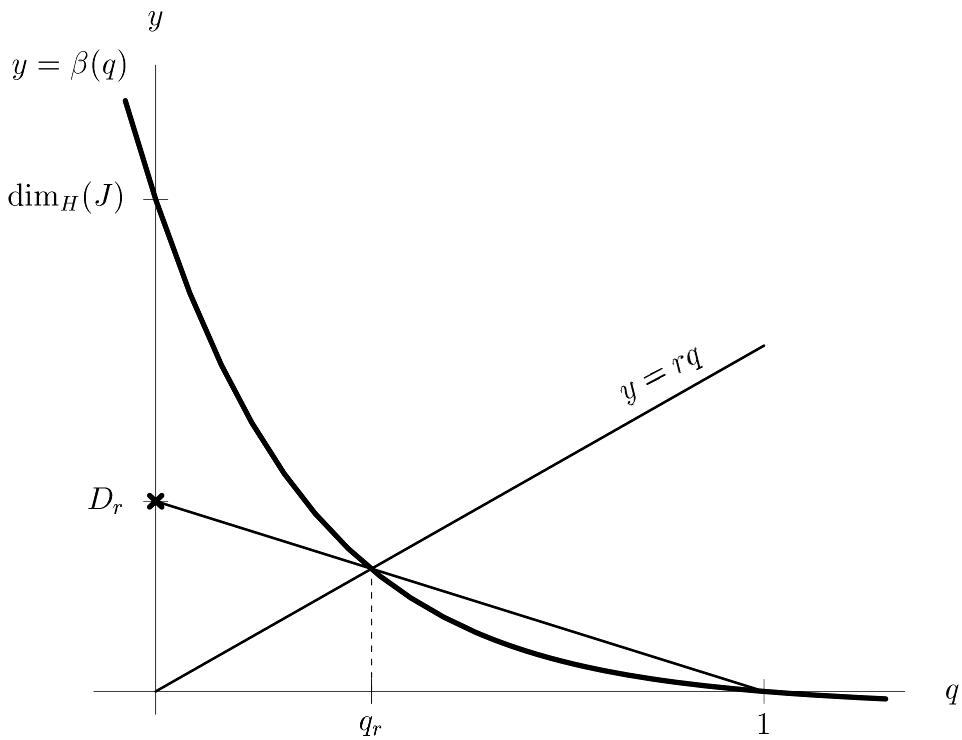

Figure 2. To determine D_r first find the point of intersection of $y = \beta(q)$ and the line $y = rq$. Then D_r is the y-intercept of the line through this point and the point $(1, 0)$.

Proof. We have for any $x, y \in X$ and $\omega \in I^*$, letting $n = |\omega|$,

$$\begin{aligned}|S_\omega(F)(x) - S_\omega(F)(y)| &= \left|\sum_{j=1}^{n}(f^{(\omega_j)}(\varphi_{\sigma^j\omega}(x)) - f^{(\omega_j)}(\varphi_{\sigma^j\omega}(y)))\right| \\ &\leq \sum_{j=1}^{n} V_\beta(F)\mathrm{e}^{-\beta(n-j)} \\ &\leq V_\beta(F)\frac{1}{1-\mathrm{e}^{-\beta}}.\end{aligned}$$

Let $C = \exp(V_\beta(F)\frac{1}{1-\mathrm{e}^{-\beta}})$. The lemma follows. □

Lemma 3. *Let $\Gamma \subseteq I^*$ be a finite, maximal antichain. Then there exists $n_0 = n_0(\Gamma)$ such that for every $n \geq n_0$, there exists a set of positive integers $\{n_\omega = n_\omega(n)\}_{\omega\in\Gamma}$ such that $\sum_{\omega\in\Gamma} n_\omega \leq n$ and*

$$u_{n,r} \geq \frac{1}{C\widetilde{K}^r}\sum_{\omega\in\Gamma}\|\varphi'_\omega\|^r\|\exp(S_\omega(F))\|u_{n_\omega,r}.$$

Proof. Let U be the open set from the strong open set condition. Then there exists $\tau \in I^*$ such that $\varphi_\tau(X) \subset U$. Let $\varepsilon = d(\varphi_\tau(X), U^c)$ and $s_{\min} = \min_{\omega\in\Gamma}\{\|\varphi'_\omega\|\}$. We get

$$d(\varphi_\omega(\varphi_\tau(X)), \varphi_\omega(U)^c) \geq \frac{1}{\widetilde{K}}\|\varphi'_\omega\|d(\varphi_\tau(X), U^c) \geq \frac{1}{\widetilde{K}}s_{\min}\varepsilon$$

which implies for any $x \in \varphi_\omega(\varphi_\tau(X))$,

$$d(x, U^c) \geq d(x, \varphi_\omega(U)^c) \geq \frac{1}{\widetilde{K}} s_{\min}\varepsilon.$$

For each n let α_n be an n-optimal set for $u_{n,r}$ and let $\delta_n = \max\{d(x, \alpha_n \cup U^c) : x \in J\}$. Since $\delta_n \to 0$ as $n \to \infty$ we can choose n_0 such that $\delta_n < \frac{1}{\widetilde{K}} s_{\min}\varepsilon$ for any $n \geq n_0$.

Suppose $n \geq n_0$ and $x \in \varphi_\omega(\varphi_\tau(J))$. There exists $a \in \alpha_n \cup U^c$ such that

$$d(x, \alpha_n \cup U^c) = d(x, a) \leq \delta_n < \frac{1}{\widetilde{K}} s_{\min}\varepsilon$$

which implies $a \in \varphi_\omega(U)$. Therefore, letting $\alpha_{n,\omega} = \alpha_n \cap \varphi_\omega(U)$, we get $n_\omega := |\alpha_{n,\omega}| \geq 1$ and $\sum_{\omega\in\Gamma} n_\omega \leq n$.

We now get

$$\begin{aligned}
u_{n,r} &= \int d(x, \alpha_n \cup U^c)^r \mathrm{d}m(x) \\
&= \sum_{\omega\in\Gamma} \int \exp(S_\omega(F)) d(\varphi_\omega(x), \alpha_n \cup U^c)^r \mathrm{d}m(x) \\
&\geq \sum_{\omega\in\Gamma} \int \exp(S_\omega(F)) d(\varphi_\omega(x), \alpha_n \cup \varphi_\omega(U)^c)^r \mathrm{d}m(x) \\
&\geq \sum_{\omega\in\Gamma} \frac{\|\exp(S_\omega(F))\|}{C} \int d(\varphi_\omega(x), \alpha_{n,\omega} \cup \varphi_\omega(U)^c)^r \mathrm{d}m(x) \\
&\geq \sum_{\omega\in\Gamma} \frac{\|\exp(S_\omega(F))\|}{C} \frac{1}{\widetilde{K}^r} \|\varphi'_\omega\|^r \int d(x, \varphi_\omega^{-1}(\alpha_{n,\omega}) \cup U^c)^r \mathrm{d}m(x) \\
&\geq \sum_{\omega\in\Gamma} \frac{\|\exp(S_\omega(F))\|}{C} \frac{1}{\widetilde{K}^r} \|\varphi'_\omega\|^r u_{n_\omega,r}.
\end{aligned}$$

The second to last inequality in the above display is verified by the following argument. For any $x \in J$ there exists $y \in \alpha_{n,\omega} \cup \varphi_\omega(U)^c$ such that $d(\varphi_\omega(x), \alpha_{n,\omega} \cup \varphi_\omega(U)^c) = d(\varphi_\omega(x), y)$. The claim is that $z := \varphi_\omega^{-1}(y) \in \varphi_\omega^{-1}(\alpha_{n,\omega}) \cup U^c$. But this is obviously true, for if $y \in \alpha_{n,\omega}$ then $z = \varphi_\omega^{-1}(y) \in \varphi_\omega^{-1}(\alpha_{n,\omega})$, and if $y \in \varphi_\omega(U)^c$ then $z \in U^c$, otherwise we would have $z = \varphi_\omega^{-1}(y) \in U$, which would imply $y = \varphi_\omega(\varphi_\omega^{-1}(y)) \in \varphi_\omega(U)$, a contradiction. Therefore,

$$\begin{aligned}
d(\varphi_\omega(x), \alpha_{n,\omega} \cup \varphi_\omega(U)^c) &= d(\varphi_\omega(x), \varphi_\omega(z)) \\
&\geq \frac{1}{\widetilde{K}} \|\varphi'_\omega\| d(x, z) \\
&\geq \frac{1}{\widetilde{K}} \|\varphi'_\omega\| d(x, \varphi_\omega^{-1}(\alpha_{n,\omega}) \cup U^c).
\end{aligned}$$

Hence, the lemma is proved. □

Lemma 4. *Let* $0 < r < +\infty$ *and* $0 < t < \kappa_r$. *Then* $\liminf_{n\to\infty} n^{\frac{r}{t}} u_{n,r} > 0$.

Proof. Since $t < \kappa_r$ we see from the uniqueness of κ_r and (8) that

$$\sum_{|\omega|=m} (\|\varphi'_\omega\|^r \|\exp(S_\omega(F))\|)^{\frac{t}{r+t}} \to \infty \text{ as } m \to \infty.$$

Choose m such that the above sum is greater than $C\widetilde{K}^r$, and let $\Gamma = \{\omega \in I^* : |\omega| = m\}$. Then Γ is a finite, maximal antichain. By the previous lemma we have n_0 and for $n \geq n_0$ the numbers $\{n_\omega(n)\}_{\omega\in\Gamma}$ which satisfy the conclusion of that lemma. Set $c = \min\{n^{\frac{r}{t}} u_{n,r} : n \leq n_0\}$. Clearly each $u_{n,r} > 0$, and hence $c > 0$. Suppose $n \geq n_0$ and $k^{\frac{r}{t}} u_{k,r} \geq c$ for all $k < n$. Using the previous lemma we get

$$\begin{aligned} n^{\frac{r}{t}} u_{n,r} &\geq n^{\frac{r}{t}} \frac{1}{C\widetilde{K}^r} \sum_{\omega\in\Gamma} \|\varphi'_\omega\|^r \|\exp(S_\omega(F))\| n_\omega(n)^{-\frac{r}{t}} n_\omega(n)^{\frac{r}{t}} u_{n_\omega(n),r} \\ &\geq c \frac{1}{C\widetilde{K}^r} \sum_{\omega\in\Gamma} \|\varphi'_\omega\|^r \|\exp(S_\omega(F))\| \left(\frac{n_\omega(n)}{n}\right)^{-\frac{r}{t}}. \end{aligned}$$

Using Hölder's inequality (with exponents less than 1) we get

$$n^{\frac{r}{t}} u_{n,r} \geq c \frac{1}{C\widetilde{K}^r} \left(\sum_{\omega\in\Gamma} (\|\varphi'_\omega\|^r \|\exp(S_\omega(F))\|)^{\frac{t}{r+t}} \right)^{1+\frac{r}{t}} \left(\sum_{\omega\in\Gamma} \left(\frac{n_\omega(n)}{n}\right)^{(-\frac{r}{t})(-\frac{t}{r})} \right)^{-\frac{r}{t}}.$$

By our choice of Γ, which depended only on t and not on n, and the fact that $\sum n_\omega(n) \leq n$, we see that

$$n^{\frac{r}{t}} u_{n,r} \geq c.$$

By induction, we have $\liminf_{n\to\infty} n^{\frac{r}{t}} u_{n,r} \geq c > 0$. □

The following problem remains open: Is $\liminf_{n\to\infty} n^{\kappa_r/r} u_{n,r} > 0$?

Lemma 5. *Let $\Gamma \subseteq I^*$ be a finite, maximal antichain, $n \geq |\Gamma|$ and $0 < r < +\infty$. Then*

$$V_{n,r} \leq \widetilde{K}^r \min \left\{ \sum_{\omega\in\Gamma} \|\varphi'_\omega\|^r \|\exp(S_\omega(F))\| V_{n_\omega,r} : n_\omega \geq 1, \sum_{\omega\in\Gamma} n_\omega \leq n \right\}.$$

Proof. Suppose $n_\omega \geq 1$ for each $\omega \in \Gamma$ and $\sum n_\omega \leq n$. For each $\omega \in \Gamma$ let α_ω be an n_ω-optimal set for $V_{n_\omega,r}$. Since $|\bigcup_{\omega\in\Gamma} \varphi_\omega(\alpha_\omega)| \leq n$, we get

$$\begin{aligned} V_{n,r} &\leq \int d(x, \bigcup \varphi_\omega(\alpha_\omega))^r \mathrm{d}m(x) \\ &= \sum_{\omega\in\Gamma} \int \exp(S_\omega(F)) d(\varphi_\omega(x), \bigcup \varphi_\omega(\alpha_\omega))^r \mathrm{d}m(x) \\ &\leq \sum_{\omega\in\Gamma} \|\exp(S_\omega(F))\| \int d(\varphi_\omega(x), \varphi_\omega(\alpha_\omega))^r \mathrm{d}m(x) \\ &\leq \sum_{\omega\in\Gamma} \|\exp(S_\omega(F))\| \widetilde{K}^r \|\varphi'_\omega\|^r \int d(x, \alpha_\omega)^r \mathrm{d}m(x) \\ &= \sum_{\omega\in\Gamma} \|\exp(S_\omega(F))\| \widetilde{K}^r \|\varphi'_\omega\|^r V_{n_\omega,r} \end{aligned}$$

which implies the lemma. □

Lemma 6. *Suppose κ_r satisfies (8). Then*

$$\limsup_{n\to\infty} n e_{n,r}^{\kappa_r} < \infty.$$

Proof. Write $q = \frac{\kappa_r}{r+\kappa_r}$ and note that $\beta(q) = rq$. For $\omega \in I^*$ let $q_\omega = m_q(\varphi_\omega(J))$. Recall that m_q is the probability measure on J which satisfies equation (7). Let $\varepsilon_0 = \min\{q_1, \dots, q_N\}$. Fix m. Choose n such that $C^q K^{rq}(\frac{m}{n}) < \varepsilon_0^2$. Let $\varepsilon = \varepsilon_0^{-1}(\frac{m}{n})C^q K^{rq}$. Note that $0 < \varepsilon < 1$. Let $\Gamma(\varepsilon) = \{\omega \in I^* : q_\omega < \varepsilon \le q_{\omega^-}\}$. Since I is finite, Γ is a finite, maximal antichain. Now

$$\begin{aligned}
q_\omega = m_q(\varphi_\omega(J)) &= \int (\exp(S_\omega(F))^q |\varphi_\omega'|^{rq} \mathrm{d}m_q \\
&= \int (\exp(S_{\omega^-}(F)(\varphi_{\omega_{|\omega|}})))^q (\exp(S_{\omega_{|\omega|}}(F)))^q |\varphi_{\omega^-}'|^{rq} |\varphi_{\omega_{|\omega|}}'|^{rq} \mathrm{d}m_q \\
&\ge \frac{1}{C^q K^{rq}} \| \exp(S_{\omega^-}(F))\|^q \|\varphi_{\omega^-}'\|^{rq} \int (\exp(S_{\omega_{|\omega|}}(F)))^q |\varphi_{\omega_{|\omega|}}'|^{rq} \mathrm{d}m_q \\
&\ge \frac{1}{C^q K^{rq}} \left(\int (\exp(S_{\omega^-}(F)))^q |\varphi_{\omega^-}'|^{rq} \mathrm{d}m_q \right) q_{\omega_{|\omega|}} \\
&= \frac{1}{C^q K^{rq}} (q_{\omega^-})(q_{\omega_{|\omega|}})
\end{aligned}$$

which gives us

$$1 = \sum_{\omega \in \Gamma(\varepsilon)} q_\omega \ge \frac{1}{C^q K^{rq}} \sum_{\omega \in \Gamma(\varepsilon)} (q_{\omega^-})(q_{\omega_{|\omega|}}) \ge \frac{1}{C^q K^{rq}} \varepsilon \varepsilon_0 |\Gamma(\varepsilon)|.$$

This implies $|\Gamma(\varepsilon)| \le C^q K^{rq} (\varepsilon\varepsilon_0)^{-1} = \frac{n}{m}$, which implies $n \ge m|\Gamma(\varepsilon)|$. Using Lemma 5 we get

$$\begin{aligned}
V_{n,r} &\le \widetilde{K}^r \sum_{\omega \in \Gamma(\varepsilon)} \|\varphi_\omega'\|^r \| \exp(S_\omega(F))\| V_{m,r} \\
&= \widetilde{K}^r \sum_{\omega \in \Gamma(\varepsilon)} (\|\varphi_\omega'\|^r \| \exp(S_\omega(F))\|)^{\frac{\kappa_r}{r+\kappa_r}} (\|\varphi_\omega'\|^r \| \exp(S_\omega(F))\|)^{\frac{r}{r+\kappa_r}} V_{m,r} \\
&\le \widetilde{K}^r C^q K^{rq} V_{m,r} \max_{\omega \in \Gamma(\varepsilon)} \{ (\|\varphi_\omega'\|^r \| \exp(S_\omega(F))\|)^{\frac{r}{r+\kappa_r}} \} \\
&\le \widetilde{K}^r C^q K^{rq} V_{m,r} \left(C^{\frac{2r}{r+\kappa_r}} K^{\frac{2r^2}{r+\kappa_r}} \varepsilon_0^{-\frac{r}{\kappa_r}} m^{\frac{r}{\kappa_r}} \right) n^{-\frac{r}{\kappa_r}}.
\end{aligned}$$

The second to last inequality is true because

$$\begin{aligned}
1 = \int 1 \mathrm{d}m_q &= \sum_{\omega \in \Gamma(\varepsilon)} \int (\exp(S_\omega(F)))^q |\varphi_\omega'|^{rq} \mathrm{d}m_q \\
&\ge \sum_{\omega \in \Gamma(\varepsilon)} \frac{\| \exp(S_\omega(F))\|^q \|\varphi_\omega'\|^{rq}}{C^q K^{rq}}.
\end{aligned}$$

The last inequality in the previous display is true by the following. For any $\omega \in \Gamma(\varepsilon)$,

$$\begin{aligned}
(\|\varphi_\omega'\|^r \| \exp(S_\omega(F))\|)^{\frac{\kappa_r}{r+\kappa_r}} &= \|\varphi_\omega'\|^{rq} \| \exp(S_\omega(F))\|^q \\
&\le C^q K^{rq} \int (\exp(S_\omega(F)))^q |\varphi_\omega'|^{rq} \mathrm{d}m_q \\
&= C^q K^{rq} q_\omega < C^q K^{rq} \varepsilon
\end{aligned}$$

which implies

$$(\|\varphi_\omega'\|^r \| \exp(S_\omega(F))\|)^{\frac{r}{r+\kappa_r}} \le (C^q)^{\frac{r}{\kappa_r}} (K^{rq})^{\frac{r}{\kappa_r}} \varepsilon^{\frac{r}{\kappa_r}}$$

$$= C^{\frac{r}{r+\kappa_r}} K^{\frac{r^2}{r+\kappa_r}} (\varepsilon_0^{-1} (\frac{m}{n}) C^q K^{rq})^{\frac{r}{\kappa_r}}$$
$$= \left(C^{\frac{2r}{r+\kappa_r}} K^{\frac{2r^2}{r+\kappa_r}} \varepsilon_0^{-\frac{r}{\kappa_r}} m^{\frac{r}{\kappa_r}} \right) n^{-\frac{r}{\kappa_r}}.$$

Since m was fixed and n was chosen so that $C^q K^{rq}(\frac{m}{n}) < \varepsilon_0^2$, we now have that there exists a number $L < +\infty$ such that for all but finitely many n

$$V_{n,r} \leq L n^{-\frac{r}{\kappa_r}},$$

which implies

$$\limsup_{n\to\infty} n e_{n,r}^{\kappa_r} \leq L^{\frac{\kappa_r}{r}} < \infty$$

as desired. □

Proof of Theorem 1. Lemma 4 tells us that $\liminf_{n\to\infty} n e_{n,r}^t > 0$ whenever $0 < t < \kappa_r$, which implies $D_r > t$ for every $t < \kappa_r$. Hence, $D_r \geq \kappa_r$. Lemma 6 tells us that $D_r \leq \kappa_r$. Therefore, the theorem is proved. □

Remark. It is interesting to note that the formula for the quantization dimension function corresponds with the Hentschel-Procaccia generalized dimension given by

$$\hat{D}_q(\mu) = \lim_{r\to 0} \frac{1}{q-1} \frac{\log \int (\mu(B(x,r)))^{q-1} \mathrm{d}\mu(x)}{\log r}$$

where $q \neq 1$ (see [7] Equation (3.14)). Patzschke notes in [9] that for a self-conformal measure P we have

$$\hat{D}_q(P) = \frac{\beta(q)}{1-q}. \tag{9}$$

The difference between this result and our result is that Equation (9) holds for any $q \neq 1$, whereas the same formula gives the quantization dimension only when q is chosen such that $\beta(q) = rq$.

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