Some problems in set theory, analysis and geometry

by

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ABSTRACT. We discuss several problems raised by Erdős including the triangle of area one problem, a problem concerning sets meeting each line in two points and a problem from number theory involving series of translates of functions. We also discuss some joint work with Erdős on partitions of lines and planes. Finally, we discuss some recent work on a lattice problem of Steinhaus.

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In this note I will discuss several problems mentioned over the years by Erdős. As with so many of his problems, they are simply stated and beautiful and yet they get to the heart of some central issues. They also indicate the tremendous influence "Uncle Paul" had on my mathematical interests. I am most fortunate and grateful to have met him early in my career and that I was a fairly regular stop in his travels. Most of these problem are discussed in the book of Croft, Falconer and Guy [CFG]. For these, I am reproting either solutions or progress.

1. The triangle of area one problem.

Years ago Erdős posed the following problem, see e.g., [CFG] (G13),[Er79], [Er81] and [Er84].

PROBLEM 1.1 *Is there a positive finite constant c such that every measurable set in the plane with area larger than c contains the vertices of a triangle of area one?*

This problem remains unsolved. As a warm up for this problem Erdős would usually indicate an agrument for the fact that if A is a measurable subset of $\mathbf{R}^2$ with infinite planar Lebesgue measure, then A contains the vertices of a triangle of area one. This may be shown with an application of Steinhaus' theorem concerning the fact that the difference set of a measurable set of real numbers with positive measure contains an open interval about 0. In fact, if A has positive measure and A is unbounded, then A contains the vertices of such a triangle. Erdős conjectured the answer to his question is yes and conjectured the best constant:

Conjecture 1.2 *If A is a measurable subset of $\mathbf{R}^2$ with area $> c_0 = 4\pi/3\sqrt{3}$, then A contains the vertices of a triangle of area one. Of course, c_0 is the area of the disk such that the inscribed equilateral triangle has area one.*

There are several partial results and reductions. For a positive solution it suffices of course to show that there is a positive constant c such that a compact set with positive measure contains such a triangle. Also, I noted, by applying Besicovitch's covering theorem, it suffices to show that there is a positive constant c such that if one has a set with measure greater than c which is the finite union of pairwise disjoint balls all with the same radius, then this set contains the vertices of a triangle of area one. Weizsäcker and I tried this approach without success. It may be that using this approach one may succeed in obtaining a positive answer to Erdős' question, but not obtain the best constant.

I want to mention a result of Chris Freiling and myself which also lends some more evidence that Erdős conjecture is correct. I will state it as follows:

Theorem 1.3 *Let A be a set in the plane not containing the vertices of a triangle with area larger than one. Then the disk whose area is the same as the outer measure of A will also not contain the vertices of a triangle with area larger than one. Thus, the area of A is no more than $c_0 = 4\pi/3\sqrt{3}$.*

The proof starts with an arbitrary set A, not containing the vertices of such a triangle, and gradually transform A into a disk D. The area of D will be at least as large as the outer measure of A, and D also will not contain the vertices of such a triangle. To begin with, we might as well assume that A has positive outer measure, otherwise the claim is trivial. We may further assume A is bounded since otherwise, it will contain the vertices of triangles of arbitrarily large area.

Let us say a set is "large" if it contains three points such that the area of the triangle formed by these points is greater than one. A set is called "small" if it is not large.

Lemma 1.4 *If A is small, then so is $C(A)$, the closed convex hull of A.*

Proof. To prove the lemma, note that A is bounded. By way of contradiction suppose $C(A)$ is large. Since $C(A)$ is compact, there is a triangle T with area greater than one, whose vertices, v_1, v_2, v_3 are in $C(A)$ and such that the area of this triangle is maximal among all triangles with vertices in $C(A)$. Let l be the line containing v_1 and v_2. Let l' be the line parallel to l passing through v_3. Then no point of $C(A)$ lies on the side of l' opposite to l. Thus, moving v_3 along the line if necessary, we may take v_3 to be an extreme point of $C(A)$ and similarly for v_1 and v_2. Now, each of these vertices must be a limit point of A. From continuity of the area function, it follows that A contains the vertices of a triangle with area greater than one. ■

It follows that if A is a small set with outer measure μ, then there is a closed convex set which is also small with area at least μ. We recall the following well known fact from convex analysis.

Lemma 1.5. *If A is a bounded closed convex set in $\mathbf{R}^2$ with positive area and l is a line, then $S_l(A)$, the Steiner symmetrical of A about l, has the same area as A and also $S_l(A)$ is small, if A is small.*

By the preceding lemmas, we can assume that A is compact and convex and by translation, we can assume the center of mass of A is the origin. There is a sequence of lines $\{l_k\}_{k=1}^{\infty}$ passing through the origin such that the sequence of sets $\{A_n\}_{n=1}^{\infty}$ converges to a disk, D, with center the origin where $A_1 = A$ and for each positive integer $n > 1$, A_n is the Steiner symmetrical of A_{n-1} about the line l_{n-1}. Clearly, then the area of each A_n is μ and therefore the area of D is μ. Since each A_n is small, D is small. ■

Finally, in connection with this problem let me mention that Erdős discusses this problem further in [Er84]. There he gives some variations he and I considered.

2. Series of translates.

In [Er84], Erdős recalls a problem of W. Schmidt: Is there a set S on the line with infinite measure so that $\frac{x}{y}$ is never an integer if $x, y \in S$? J. Haight [H1] and E. Szemerédi [S] independently constructed such a set. He mentions a very nice problem of Haight [H2], also see [CFG], G16:

Problem 2.1 *Let E be a set of positive measure in $(0, \infty)$. Consider the set $E' = \cup_{r=1}^{\infty} rE$. Is it true that for almost all $x > 0$ there is an $M(x)$ so that for every $n > M(x), n \cdot x \in E'$?*

Erdős then proposes several problems connected with this sort of problem which remain unsolved as far as I know.

Buczolich and I solved Haight's problem [BM]. Actually, we were considering a more general problem which arose from several different sources. Let me state our result.

Theorem 2.2 *There is an open subset S of $(0, \infty)$ and two nonempty intervals $P, U \subset [\frac{1}{2}, 1)$ such that setting $f = \mathbb{1}_S$ for every $x \in U$ we have $\sum_{n=1}^{\infty} f(nx) = +\infty$ and for almost every $x \in P$ we have $\sum_{n=1}^{\infty} f(nx) < +\infty$.*

Our proof uses some elementary properties of the prime numbers and their distribution and some classical results from diophantine approximation. As Buczolich points out our theorem, as a sort of dual, gives a negative answer to Haight's problem as follows.

Choose N such that $P_N = \{x \in P : \sum_N^{\infty} f(nx) = 0\}$ has positive measure. Let $E = P_N$, define E' as above. Then $S \cap E' \cap [K, \infty) = \emptyset$ for a sufficiently large K. On the other hand, we also know that $\sum f(nx) = \infty$, for every $x \in U$. This means that for infinitely many n's we have $nx \in S$ and $nx \notin E'$. This gives a negative answer to Haight's question.

Shortly after Buczolich and I completed our paper, Buczolich, Kahane and I obtained more general results by different methods [BKM1, BKM2]. We changed our viewpoint from the multiplicative to the additive. Let me give the setting.

Given $f : \mathbf{R} \mapsto \mathbf{R}^+$ and Λ a discrete infinite subset of $\mathbf{R}^+$ we denote by $C(f, \Lambda)$ resp. $D(f, \Lambda)$ the set of x's where the series $\Sigma_\lambda f(x + \lambda)$ $(\lambda \in \Lambda)$ converges resp. diverges. The sets Λ break into two types. Type 1 consists of Λ such that the Lebesgue measure of either $C(f, \Lambda)$ or $D(f, \Lambda)$ vanishes no matter what measurable f is considered, and type 2 consists of all the other Λ. In this context the result of Buczolich and myself is that $\{\log n\}$ $(n = 2, \dots)$ is of type 2. Buczolich, Kahane and myself show that type 2 sets are generic and type 1 are rare with respect to the box topology. We also define a notion of Λ being asymptotically lacunary and show all such sets are type 2.

As an example of Type 1 sets we show

Theorem 2.3 *Let* (n_k) *be an increasing sequence of positive integers and let* $\Lambda = \cup_{k \in N} \Lambda_k$ *where* $\Lambda_k = 2^{-k} N \cap [n_k, n_{k+1})$. *Then* Λ *is of type* 1.

On the other hand, it is possible to modifiy the type 1 sets given in theorem 2.3 so that they become type 2 sets. This is accomplished by making them become asymptotically dense at a faster rate:

Theorem 2.4 *Let* (n_k) *be a given increasing sequence of positive integers. There is an increasing sequence of integers* $(m(k))$ *such that the set* $\Lambda = \cup_{k \in N} \Lambda_k$ *with* $\Lambda_k = 2^{-m(k)} N \cap [n_k, n_{k+1})$ *is of type* 2.

Problem 2.5 *What is the situation with theorems 2.3 and 2.4 when the integer 2 in those theorems is replaced by some number* $a > 1$*?*

There are several other interesting unsolved problems posed in [BKM]. One of them is the following:

QUESTION 2.6 *Is it true that* Λ *is of type 2 if and only if there is a* $\{0, 1\}$ *valued measurable function* f *such that both* $C(f, \Lambda)$ *and* $D(f, \Lambda)$ *have positive Lebesgue measure?*

3. Sets meeting each line in two points

In 1914, Mazurkiewicz [Maz] showed that there is a "two-point" subset M of $\mathbf{R}^2$, *i.e.*, M meets each line in exactly 2 points. One can easily modify Mazurkiewicz's argument to show that for each positive integer n, $n \geq 2$, there is an "n-point" subset M of $\mathbf{R}^2$; a set M which meets each line in exactly n points. More refined generalizations of this result were given by Bagemihl and Erdős [BE]. The axiom of choice played a central role in these constructions. There is one indication that perhaps the axiom of choice is not needed. Consider the set M which is the union of all circles with center the origin and radius a positive integer. This F_σ set meets every line in a countably infinite set. Thus, the question naturally arises as to how effective a construction of an n-point set can be.

Problem 3.1 *Can a set which meets every line in exactly 2 (or more generally n) points be a Borel set?*

This question has been known for many years. I first heard the problem from Erdős, who said it had been around since he was a "baby." (It is mentioned by Sierpinski, [S4], p. 450.) Many people have studied this problem from various viewpoints. Larman showed that if there is such a Borel set, then it must be somewhat complex [Lar]. He showed that a 2 point set cannot be an F_σ set. This naturally leads to a restricted version of the main problem which I posed in [Mau90]. (This article contains further references and discussion.)

Problem 3.2 *Can a set which meets every line in exactly 2 (or generally n) points be a G_δ set?*

It seems likely to me that the solution to this restricted question would entail an answer to the general question. Recently, J. J. Dijkstra and J. van Mill in a preprint showed that that if a 2-point set is a G_δ set, then it must be nowhere dense in the plane.

Let me mention that it is also known that if M is analytic and M is an n point set, then M is a Borel set. This follows from the fact that every analytic subset A of $\mathbf{R}^2$ such that each vertical fiber A_x has cardinality $\leq n$ lies in a Borel set B such that each vertical fiber has cardinality $\leq n$. Also, Arnie Miller has shown that if one assumes Gödel's axiom of constructibility, $V = L$, then there is a 2 point set which is a coanalytic set [Mil]. It is also known that a two point set must have topological dimension zero [Kul]. I have discussed this in problem 1069 in [Mau90]. In [Mau98] I proved the following geometric measure theoretic result about this problem:

Theorem 3.3 *Let n be an integer, $n \geq 2$. Let M be a subset of $\mathbf{R}^2$ which meets every line in exactly n points. Then M is not the union of countably many rectifiable 1-sets.*

I also related this problem to another old unsolved problem in geometric measure theory (see [Mat., p. 258]) whether the following proposition is true:

(P2) every purely unrectifiable or irregular compact 1-set in $\mathbf{R}^2$ must meet some line in at least three points or,

more generally

(Pn) every purely unrectifiable or irregular compact 1-set in $\mathbf{R}^2$ must meet some line in at least $n+1$ points.

Theorem 3.4 *Suppose $n \geq 2$ and proposition (Pn) is true. Then there is no analytic subset of $\mathbf{R}^2$ which meets every straight line in exactly n points.*

Another variant of this problem is

Problem 3.5 *Can a compact partial two-point set (meaning a set which meets each line in no more than 2 points) with linear Hausdorff measure zero always be extended to a two point set?*

Clearly, some condition must be added here as a circle cannot be extended. Some fairly definitive results about this problem have been obtaind by Dijkstra, Kunen and van Mill [DKM]:

Theorem 3.6 *There exist compact partial 2-point sets with linear Hausdorff measure zero (even with Hausdorff dimension zero) that are not contained in a 2-point set. Also, if a partial 2-point set has the property that the linear Hausdorff measure of its square is zero then it is extendible to a 2-point set (provided that Martin's Axiom holds).*

4. Partitions of lines and planes.

In 1951, Sierpiński [S2] showed that the continuum hypothesis is equivalent to the following: for the partition of the lines in $\mathbf{R}^3$ parallel to one of the coordinate axes into the disjoint sets, L_1, L_2, and L_3, where L_i consists of all lines parallel to the ith axis, there is a partition of $\mathbf{R}^3$ into disjoint sets, S_1, S_2, and S_3 such that any line in L_i meets at most finitely many points in S_i. He also showed that the corresponding statement for $\mathbf{R}^4$, using L_1, L_2, L_3 and L_4 and four sets S_1, S_2, S_3 and S_4 is equivalent to $2^\omega \leq \omega_2$. Also, the corresponding statement for $\mathbf{R}^2$, using sets of lines L_1 and L_2 and sets S_1 and S_2 is false. He obtained analogous results by replacing "finite"by "countable". Thus, CH is equivalent to the assertion that $\mathbf{R}^2$ can be divided into two disjoint sets S_1 and S_2 with each line in L_i meeting S_i in a countable set [S1]. He showed that the countable version for R^3 with three sets is equivalent to $2^\omega \leq \omega_2$. These theorems were generalized by Kuratowski [Ku] and Sikorski [Si]. Erdős [Er53] raised the issue of whether these results could be further strengthened by considering partitions of all lines rather than just those lines parallel to some coordinate axis. There has been quite a bit of research on these possibilites. Simms [Sm] gave an extensive historical survey and we also indicate a number of further results in [EJM1]. In [EJM1], we develop a general framework for studying arbitrary partitions of all lines (or planes, or more general objects) and not necessarily special partitions or families of lines. The central issues are the number of sets of lines in the partition, the allowed size of the intersection of a line in a given set with the corresponding set in the decomposition of the space, and the value of the continuum. In order to state one of the main results in [EJM1] let us establish some conventions. If t is a positive integer, then $\mathrm{card}(A) = |A| \leq \omega_{-t}$ means A is finite. If $\theta = \bar{\theta} + s$, where $\bar{\theta} > 0$ is a limit ordinal and s is an integer, and t is an integer t, $t > s$, then $|A| \leq \omega_{\theta - t}$ means $|A| < \omega_\theta$.

THEOREM 4.1. *Let θ be an ordinal, $\theta = \bar{\theta} + s \geq 1$, where $\bar{\theta}$ is 0 or a limit ordinal, and let $s \in \omega$. The following statements are equivalent:*

(1.1) $2^\omega \leq \omega_\theta$.

(1.2) For each $n \geq 2$ and for each partition of L, the set of all lines in $\mathbf{R}^n$ into $p \geq 2$ disjoint sets, $L = L_1 \cup L_2 \cup ... \cup L_p$, there is a partition of $\mathbf{R}^n$ into p disjoint sets, $\mathbf{R}^n = S_1 \cup S_2 \cup ... \cup S_p$, such that each line in L_i meets S_i in a set of size $\leq \omega_{\theta-p+1}$.

(1.3) For some $n \geq 2$, some p, with $s + 2 \geq p \geq 2$ and some non-parallel lines

$l_1, ..., l_p$ in $\mathbf{R}^n$, if we let L_i be the set of all lines in $\mathbf{R}^n$ parallel to l_i, then there is a partition $\mathbf{R}^n = S_1 \cup ... \cup S_p$ such that every line in L_i meets S_i in a set of size $\leq \omega_{\theta-p+1}$.

I note that the fact that (1.3) implies (1.1) was proven earlier by Roy Davies [D1] and that some of the key ideas of our arguments go back to combinatorial arguments of Erdős and Hajnal [EH]. Several corollaries are given in [EJM1]. For example, Sierpiński's theorem and the answer to question a) in [Er53]:

COROLLARY 4.2 *The following are equivalent:*

(i) CH, the continuum hypothesis holds: $2^\omega = \omega_1$,

(ii) if the lines in $\mathbf{R}^3$ are colored with three colors L_i $(i = 1, 2, 3)$, then there exists a coloring of $\mathbf{R}^3$ with the same three colors such that each line meets only finitely many points with its color.

Proof. Take $\theta = 1$, $n = 3$ and $p = 3$ in theorem 4.1. Then each line in L_i meets S_i in a set of size at most $\omega_{\theta-p+1} = \omega_{-1}$, which by our convention means finite.

A second corollary notes that the condition that we are in $\mathbf{R}^3$ in corollary 4.2 is not necessary. This also answers question *b*) in [Er53]. ■

COROLLARY 4.3 *The following are equivalent:*

(i) $2^\omega = \omega_1$,

(ii) If the lines in $\mathbf{R}^2$ are colored with three colors, then there exists a coloring of $\mathbf{R}^2$ with the same colors such that each line contains only finitely many points with its color.

Proof. Take $\theta = 1$, $n = 2$ and $p = 3$ in theorem 4.1. ■

We also gave some theorems concerning partitions or colorings of other objects in $\mathbf{R}^n$, for example hyperplanes, and corresponding colorings of the points in $\mathbf{R}^n$ so that one had prescribed bounds on how many points an object contained with its color.

We also considered infinite colorings of lines and points in [EJM1]. One corollary of our theorems is the following theorem of Davies [D2]:

Theorem 4.4 *Assume ZFC. For every infinite partition $L = \bigcup_{i\in\omega} L_i$ of L, the set of all lines in $\mathbb{R}^n$, there is a partition, $\mathbb{R}^n = \bigcup_{i\in\omega} S_i$, of the points in $\mathbb{R}^n$ such that $\forall l \in L_i$ $(|l \cap S_i|$ is finite). Furthermore, if $2^\omega \leq \omega_m$, then "finite" may be replaced by $m + 1$.*

Davies had suggested that the converse of theorem 4.4 might hold. That is, does the partition property with size $m+1$ intersection imply $2^\omega \le \omega_m$, or any bound on 2^ω? In [EJM2] we showed that this is not the case by proving the following theorem.

Theorem 4.5 *Assume ZFC + MA. Then for any partition* $L = \bigcup_{i\in\omega} L_i$ *of the lines in* $\mathbb{R}^n$ $(n \ge 2)$*, there is a partition* $\mathbb{R}^n = \bigcup_{i\in\omega} S_i$ *of the points in* $\mathbb{R}^n$ *such that* $\forall l \in L_i(|l \cap S_i| \le 3)$.

However, our result left open the questions of whether MA is necessary and what is the situation for $n = 2$. Both of these questions were answered by the conclusive work of Schmerl [Sch]:

Theorem 4.6: *Let* $L = L_0 \cup L_1 \cup L_2 \cup \cdots$ *be a countable partition of the set of lines of* $\mathbb{R}^n$*. Then there is a countable partition* $\mathbb{R}^n = S_0 \cup S_1 \cup S_2 \cup \cdots$ *of the set of points such that whenever* $\ell \in L_i$*, then* $|\ell \cap S_i| \le 2$.

5. Steinhaus' lattice problem.

Sometime in the 1950's, Steinhaus posed the following problem. Do there exist two sets A and B in the plane such that for every set congruent to A has exactly one point in common with B? The trivial case where one of the sets is the plane is ruled out. Sierpinski [S3] showed the answer is yes and later Erdős rediscovered this result [Er85]. Komjáth showed that such a set exists if $B = Z$, the set of all integers [K]. Steinhaus also asked about a specific case:

Problem 5.1 *Does there exist a set* $E \subset \mathbf{R}^2$ *which meets every isometric copy of* Z^2 *in exactly one point?*

The first reference in the literature to this problem that I know of is Sierpinski's 1958 paper [S3]. See also [CFG], E10,G9. Now, the situation regarding this problem seems somewhat unusual. J. Beck showed that there is no bounded measurable subset of the plane which is a "Steinhaus" set [B]. Also, Kolountzakis and Wolff have shown that for the analogous problem for $\mathbf{R}^d$ with $d > 2$, there is no such set E which is measurable [KW]. However, their methods do not quite work in the plane. So, whether there can be a measurable Steinhaus set in the plane remains open although there are numerous partial results as indicated in [KW]. On the other hand, Steve Jackson and I have shown that the answer to Steinhaus' planar problem is yes [JM]. However, our methods do not directly generalize to the case when $d > 2$. So, the higher dimensional version of Steinhaus' problem remains open.

I want to make a few comments about the proof Jackson and I gave. In fact, we proved a slightly stronger theorem:

Theorem 5.2 *There is a set $S \subseteq \mathbf{R}^2$ satisfying:*

(1) For every isometric copy L of Z^2 we have $S \cap L \neq \emptyset$,

and

(2) for all distinct $z_1, z_2 \in S$, $(\rho(z_1, z_2))^2 \notin Z$.

Of course, for S to be a Steinhaus set we only need condition (1) to hold and (2') for all distinct $z_1, z_2 \in S$, the distance between z_1 and z_2 is not the distance between two lattice points.

Note that viewed this way, the Steinhaus problem has a natural interpretation for smaller sets of lattices. Namely, given an arbitrary set $\mathcal{L}$ of lattices (each of which is an isometric copy of Z^2), we may ask whether there is a set S satisfying (2) of theorem 5.2 and such that $S \cap L \neq \emptyset$ for all $L \in \mathcal{L}$. Indeed, establishing this restricted version of the problem for the case where $\mathcal{L}$ is the (countable) family of rational translations of Z^2 is a central step towards proving our theorem.

In proving theorem 5.2, it is natural to procede inductively. That is, we build the desired set S in (transfinitely many) stages. At limit stages, we take unions, and at successor stages we enlarge S_α to $S_{\alpha+1}$ so as to meet a new lattice, while at the same time keeping property (2). Note that (2) is then trivially satisfied at limit stages. If we can meet every lattice L along the way, then the final set $S = \bigcup_\alpha S_\alpha$ will be as desired. While this is the general plan for the proof, there are several steps that must be taken to ensure its success. For example, we do not simply enumerate the lattices $\mathcal{L}$ in type 2^ω. Rather, it turns out to be important that we use the "hull construction" which has played a significant role in several other theorems of this general character in our work with Erdős, [EJM1], [EJM2]. The idea, described abstractly, is to consider a continuous elementary chain $\{M_\alpha\}_{\alpha<2^\omega}$ of substructures (say of some large V_κ) with each M_α of size $< 2^\omega$, but $\mathbf{R} \subseteq \bigcup_{\alpha<2^\omega} M_\alpha$. Let $\mathcal{L}_\alpha$ denote the isometric copies of Z^2 which are in M_α. At successor steps, we now enlarge S_α to $S_{\alpha+1}$ which meets all lattices $L \in \mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha$, while, of course, keeping property (2). The point is that while this gives us more to do at each successor step, it also provides us with a powerful inductive assumption, namely, the closure of $\mathcal{L}_\alpha$ under various operations.

REFERENCES

[BE] F. Bagemihl and P. Erdős, Intersections of prescribed power, type, or measure, Fund. Math. 41(1957), 57-67.

[B] J. Beck, On a lattice-point problem of H. Steinhaus, Studia Sci. Math. Hung. 24 (1989), 261-268.

[BM] Z. Buczolich and R.D. Mauldin, On the convergence of $\sum f(nx)$ for measurable functions, Mathematika, to appear.

[BK2] Z. Buczolich and R. D. Mauldin, On series of translates of positive functions, Indag. Math., to appear.

[BKM1] Z. Buczolich, J.-P. Kahane, and R. D. Mauldin, Sur les series de translates de fonctions positives, C. R. Acad. Sci. Paris, t. 329, Serie I, 261-264.

[BKM2] Z. Buczolich, J.-P. Kahane, and R. D. Mauldin, On series of translates of positive functions, Acta Mathematica Hungarica, to appear.

[CFG] H. T. Croft, K. J. Falconer and R. K. Guy, Unsolved problems in geometry, Problem Books in Mathematics. Unsolved Problems in Intuitive mathematics, II. Springer-Verlag, New York, 1994

[D1] R. O. Davies, On a Problem of Erdős Concerning Decompositions of the Plane, Proc. of the Cambridge Philosophical Society, vol. 59(1963), $p.33-36$.

[D2] R. O. Davies, On a denumerable partition problem of Erdős, Proc. Cambridge Philos. Soc. 59 (1963), 501–502.

[DKM] J. J. Dijkstra, K. Kunen, and J. van Mill, Hausdorff measures and two point sets, Fund. Math. 157 (1998), 43-60

[DM] Jan J. Dijkstra, Jan van Mill, Two point set extensions - a counterexample, Proc. Amer. Math. Soc., **125** no. 8 (1997) 2501–2502.

[EGH] P. Erdős, F. Galvin, and A. Hajnal, On Set-systems Having Large Chromatic Number and not Containing Prescribed Subsystems, Infinite and Finite Sets (Colloq., Keszthely; dedicated to P. Erdős on his 60^{th} Birthday), vol. 1(1973), $p.425-513$, Colloq. Math. Soc. János Bolyai, vol. 10(1975), North-Holland, Amsterdam.

[EH] P. Erdős and A. Hajnal, On Chromatic Number of Graphs and Set-systems, Acta. Math. Acad. Sci. Hungar., 17(1966), $p.61 - 99$.

[Er53] P. Erdős, Some Remarks on Set Theory. IV, Mich. Math. J. 2(1953-54), $p.169 - 173$.

[Er79] P. Erdős, Set-theoretic, measure-theoretic, combinatorial, and number - theoretic praoblems concerning point sets in Euclidean space, Real Analysis Exchange 4(1978-79), 113-138.

[Er81] P. Erdős, My Scottish Book "Problems", The Scottish Book, Edited by R. Daniel Mauldin, Birkhäuser Boston, 1981

[Er84] P. Erdős, Some combinatorial, geometric and set theoretic problems in measure theory, Measure Theory Oberwolfach 1983, Lecture Notes in Mathematics 1089, Springer-Verlag, 1984.

[Er85] P. Erdős, Problems and results in combinatorial geometry, Discrete Geometry and Convexity, 44 (1985), 1-10.

[EJM1] P. Erdős, S. Jackson and R. D. Mauldin, On partitions of lines and space, Fund. Math. 145 (1994), 101-119.

[EJM2] P. Erdős, S. Jackson and R. D. Mauldin, On infinite partitions of lines and space, Fund. Math. 152 (1997), 75-95.

[Fal] K.J.Falconer, The Geometry of Fractals, Cambridge University Press, 1990.

[H1] J.A. Haight, A linear set of infinite measure with no two points having integral ratio, Mathematika 17(1970), 133-138.

[H2] J.A. Haight, A set of infinite measure whose ratio set does not contain a given sequence, Mathematika 22(1975), 195-201.

[K] P. Komjáth, A lattice point problem of Steinhaus, Quart. J. Math. Oxford 43 (1992), 235-241.

[L] C. G. Lekkerkerker, Lattice points in unbounded point sets, I., Indag. Math., 20 (1958) 197-205.

[JM] S. Jackson and R. D. Mauldin, On a lattice problem of H. Steinhaus, preprint.

[KW] M. N. Kolountzakis and T. Wolff, On the Steinhaus tiling problem, Mathematika, to appear.

[Kul] J. Kulesza, A two point set must be zero dimensional, Proc. Amer. Math. Soc. 116(1992), 551-553.

[Ku] C. Kuratowski, Sur une Caracterisation des Alephs, Fund. Math., 38(1951), 14-17.

[Lar] D. G. Larman, A problem of incidence, J. London Math. Soc. 43(1968), 407-409.

[Mat] P. Mattila, Geometry of sets and measures in Euclidean spaces, Cambridge University Press, 1995.

[Mau90] R. D. Mauldin, Problems in topology arising from analysis, in Open Problems in Topology, J.van Mill and G.M. Reed, editors, North-Holland, Amsterdam, 1990.

[Mau98] R. D. Mauldin, On sets which meet each line in exactly two points, Bull. London Math. Soc. 30 (1998), no. 4, 397–403.

[Maz] S. Mazurkiewicz, Sur un ensemble plan qui a avec chaque droite deux et seulement deux points communs, C. R. Varsovie 7(1914), 382-384.

[Mil] A. Miller, Infinite Combinatorics and definability, Ann. Pure Appl. Logic 41(1989), 179-203.

[Sch] J. H. Schmerl, Countable partitions of sets of points and lines, Fund. Math. 160 (1999), 183-196.

[S1] W. Sierpiński, Hypothèse du Continu, Warsaw, 1934.

[S2] W. Sierpiński, Sur Quelques Propositions Concernant la Puissance du Continu, Fund. Math., 38(1951), $p.1 - 13$.

[S3] W. Sierpiński, Sur un probleme de H. Steinhaus concernant les ensembles de points sur le plan, Fund. Math. 46 (1958), 191-194.

[S4] W. Sierpinński, Cardinal and ordinal numbers, PWN, 1958.

[Si] R. Sikorski, A Charactérization of Alephs, Fund. Math. 38(1951), 18-22.

[Sm] J. Simms, Is 2^ω Weakly Inaccessible?, to appear.

[S] E. Szemerédi, On a problem of W. Schmidt, Studia Sci. Math. Hungar. 6(1971), 287-288.

[W] H. v. Weizsäcker, Zum Konvergenzverhalten der Reihe $\Sigma_{n=1}^{\infty} f(nt)$ für λ-messbare Funktionen $f : \mathbf{R}^+ \mapsto \mathbf{R}^+$, Diplomarbeit, Universität München, 1970.