Fubini-type theorems for general measure constructions

by

K. J. Falconer

**Department of Mathematics,
University of St Andrews,
St Andrews, Fife KY16 9SS, Scotland**

and

R. Daniel Mauldin[1]

**Department of Mathematics,
University of North Texas,
Denton, Texas 76203, USA**

Abstract. We use methods from descriptive set theory to derive Fubini-like results for the very general Method I and Method II (outer) measure constructions. Such constructions, which often lead to non-σ-finite measures, include Carathéodory and Hausdorff-type measures. We encounter several questions of independent interest, such as the measurability of measures of sections of sets, the decomposition of sets into subsets with good sectional properties, and the analyticity of certain operators over sets. We indicate applications to Hausdorff and generalised Hausdorff measures and to packing dimensions.

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1. Method I and Method II measures

Throughout this paper (X,d) and (T,ρ) will be Polish spaces, that is, complete separable metric spaces. If $B \subset T \times X$, and $t \in T$, then by B_t, the *t-section* or *fiber* of B, we mean the set $\{x \in X : (t,x) \in B\}$. It is sometimes convenient to regard B_t as the set $\{t\} \times B_t$; it should be clear from the context which interpretation is intended.

Let τ be a nonnegative set function defined on the subsets of X, such that $\tau(\emptyset) = 0$, and $\tau(E) = \infty$ if $\overline{E}$ is not compact. Let ν be a complete Borel probability measure on T. This means that Σ, the σ-algebra of subsets on which ν is defined, includes the Borel subsets of T and if $N \subset M$ and $\nu(M) = 0$, then $N \in \Sigma$. Since ν is complete, every analytic subset of T is ν-measurable and consequently every set in $\mathcal{B}A(T)$, the σ-algebra generated by the analytic subsets of T, is ν-measurable.

We first consider the Method I measure induced by τ and then the Method II measure, see Rogers [Ro] for a general treatment of such measures. Method I measures will be denoted by an asterisk, *. Thus τ^*, the usual Method I measure induced by τ, is defined by setting

$$\tau^*(E) = \inf_{E \subset \cup E_i} \sum \tau(E_i),$$

for $E \subset X$, where, as always, $\{E_i\}$ is a countable cover. (Often Method I measures are defined in terms of coverings by sets E_i from a restricted class of sets $\mathcal{C}$. However, the same definition of τ^* may be achieved by redefining $\tau(E)$ for all $E \notin \mathcal{C}$, for example by setting $\tau(E) = \infty$ for $E \notin \mathcal{C}$, and this allows the convenience of having $\tau(E)$ defined for all E.)

We define the set function μ on $T \times X$ by setting

$$\mu(B) = \int_T^* \tau(B_t)\,d\nu(t), \tag{1}$$

where $\int^*$ denotes the upper integral. Let μ^* be the outer measure on $T \times X$ constructed from μ by Method I. Thus, for $B \subset T \times X$,

$$\mu^*(B) = \inf_{B \subset \cup B_i} \sum \mu(B_i). \tag{2}$$

A major aim of this paper is to establish conditions that enable $\mu^*(B)$ to be expressed as an integral of the sectional measures $\tau^*(B_t)$ with respect to ν, that is to obtain identities such as

$$\mu^*(B) = \int_T \tau^*(B_t)\,d\nu(t).$$

for certain sets B. There a basic inequality relating these set functions:

Lemma 1. Let $B \subset T \times X$. Then

$$\int_T^* \tau^*(B_t)\,d\nu(t) \le \mu^*(B). \tag{3}$$

Proof.

$$
\begin{aligned}
\int^* \tau^*(B_t)d\nu(t) &= \int^* \left(\inf_{B_t\subset\ \cup E_i^t} \sum \tau(E_i^t)\right) d\nu(t) \\
&= \inf_{B\subset\cup B_i} \int^* \sum \tau((B_i)_t)d\nu(t) \le \inf_{B\subset\cup B_i} \sum_i \int^* \tau((B_i)_t)d\nu(t) \\
&= \inf_{B\subset\cup B_i} \sum \mu(B_i) = \mu^*(B),
\end{aligned}
$$

where we take the cover of B defined by $B_i = \cup_{t\in T}\{t\} \times E_i^t$. ∎

We seek conditions for equality in (3). We note that for a given set $B \subset T \times X$ inequality (3) becomes an equality provided for each $\epsilon > 0$, there is a sequence of sets $\{B_i\}_{i=1}^\infty$ such that (i) $B \subset \cup B_i$, (ii) for each i, $t \mapsto \tau((B_i)_t)$ is ν-measurable, and (iii) for ν a.e. t, $\sum \tau((B_i)_t) \le \tau^*(B_t) + \epsilon$. These are very general conditions and it is desirable to have some more easily-checked conditions on τ that lead to equality in (3) for a reasonably large class of sets B. Thus we list below various verifable conditions on τ. We denote the space of compact subsets of X endowed with the topology inherited from the Hausdorff metric by $K(X)$.

(C1) τ is monotone,

(C2) $\tau(E) = \tau(\overline{E})$,

(C3) for each closed set F, $\tau(F) = \sup\{\tau(K) : K \in K(X) \text{ and } K \subset F\}$,

(C4) $K \mapsto \tau(K)$ is a Borel measurable map on the space $K(X)$,

(C5) For each compact set K, $\tau(K) = \inf\{\tau(V) : V \text{ is open, } K \subset V\}$,

(C6) $\tau(E) = +\infty$, if $\overline{E}$ is not compact.

The best known examples of Method I measures are the pre-Hausdorff measures. Fixing $X = \mathbb{R}^n$ and $s, \delta > 0$, we define, for $E \subset X$, $\tau(E) = |E|^s$ if $|E| \le \delta$ and $\tau(E) = \infty$ if $|E| > \delta$, where $|E|$ is the diameter of E. Thus only sets E with $|E| \le \delta$ provide useful covering sets. (Later on we will consider Method II where we let $\delta \to 0$ to give Hausdorff measures.) This example may be generalised, by taking an outer measure λ on X and $s, q > 0$, and setting, for $E \subset X$, $\tau(E) = |E|^s \lambda(E)^q$ if $|E| \le \delta$ and $\tau(E) = \infty$ if $|E| > \delta$. Then τ satisfies (C1) and (C3) if λ is regular and (C4) and (C5) if λ is outer regular. We note that if condition (C5) holds, then (C4) holds. If fact, if condition (C5) holds, then not only is $K \mapsto \tau(K)$ upper semi-continuous, but also $K \mapsto \tau^*(K)$ is upper semi-continuous.

Theorem 2. Suppose τ satisfies conditions (C2) and (C4). Let A be an analytic subset of $T \times X$ such that $\overline{A_t}$ is compact for each $t \in T$. Then the map $t \mapsto \tau(A_t)$ is ν-measurable. Indeed, this map is measurable with respect to the σ-algebra $\mathcal{BA}(T)$ of subsets of T generated by the analytic subsets of T.

Proof. Let G be the sectionwise closure of A. Thus $(t, x) \in G$ if and only if there is some sequence $\{x_n\}_{n=1}^\infty$ with $\{x_n\}$ converging to x and $(t, x_n) \in A$ for all n. Since

$G = \pi_{T\times X}(\{(t,x,x_1,x_2,x_3,...) \in T \times X \times X^{\mathbb{N}} : \forall n (t,x_n) \in A \text{ and } x_n \to x\})$ where π denotes projection, the set G is the projection of an analytic set and so is analytic. We check that the map $g : T \mapsto K(X)$ given by $g(t) = G_t$ is $\mathcal{BA}(T)-$measurable. Fix a nonempty open subset U of X. Let $I(U) = \{K \in K(X) : K \cap U \neq \emptyset\}$ and let $C(U) = \{K \in K(X) : K \subset U\}$. Then $g^{-1}(I(U)) = \{t : G_t \cap U \neq \emptyset\} = \pi_T((T \times U) \cap G)$ is an analytic set and $g^{-1}(C(U)) = T \setminus \{t : G_t \cap X \setminus U \neq \emptyset\}$ is a coanalytic set. Since the sets of the form $I(U)$ and $C(U)$ form a subbasis for the topology of $K(X)$, g is a $\mathcal{BA}(T)$-measurable function. Finally, $t \mapsto \tau(A_t) = \tau(G_t)$ is $\mathcal{BA}(T)$-measurable, since it is the composition of g with a Borel measurable map. ■

Theorem 3. Suppose X is locally compact and τ satisfies conditions (C1), (C2), (C3) and (C4). Let B be an analytic subset of $T \times X$, then the map $t \mapsto \tau(B_t)$ is $\mathcal{BA}(T)$-measurable.

Proof. Let G be the sectionwise closure of B. Let $\{U_n\}$ be an ascending sequence of open subsets of X such that $\overline{U_n}$ is compact for each n, and $\cup U_n = X$. For each n, let $f_n(t) = \tau(G_t \cap \overline{U_n})$ for $t \in T$. By Theorem 2, f_n is $\mathcal{BA}(T)$-measurable for each n. Note that by property (C1) and the local compactness of X, for each t, $f(t) := \lim_{n\to\infty} f_n(t) = \sup\{\tau(K) : K \in K(X) \text{ and } K \subset G_t\}$. By properties (C2) and (C3), $f(t) = \tau(G_t) = \tau(B_t)$. Since f is $\mathcal{BA}(T)$-measurable, the proof is finished. ■

We use the following theorem of Saint Raymond [Ra] in several places. Let T and X be complete separable metric spaces and let B be a Borel subset of $T \times X$ such that for each $t \in T$, the t-section of B, B_t, is σ-compact. Then $\pi_T(B)$ is a Borel set, and there exist Borel sets $B_n \subset T \times X$ such that $B = \cup_n B_n$, and $(B_n)_t$ is compact for each t.

Theorem 4. Let X be locally compact. Let B be a Borel subset of $T \times X$ such that each t-section of B is σ-compact. Let

$$F = F(B) = \{(t,(K_n)) \in T \times K(X)^{\mathbb{N}} : \cup \text{int} K_n \supset B_t\}.$$

Then F is a Borel set.

Proof. Notice $T \times K(X)^{\mathbb{N}} \setminus F = \pi_{T\times K(X)^{\mathbb{N}}}(H)$, where $H = \{(t,(K_n),x) \in T \times K(X)^{\mathbb{N}} \times X : (t,x) \in B \text{ and } \forall n,\ x \notin \text{int} K_n\})$. Thus, H is a Borel subset of $T \times K(X)^{\mathbb{N}} \times X$ Also, for each $(t,(K_n))$, the section $H_{(t,(K_n))} = B_t \setminus \cup \text{int} K_n$ is σ-compact. So, by Saint Raymond's theorem, F is a Borel set. ■

We recall that a map $f : D \mapsto K(X)$, where D is a Borel subset of T is Borel measurable if and only if the graph of f, $\text{Gr}(f) = \{(t,x) : x \in f(t)\}$, is a Borel set in $T \times X$. This fact also follows easily from Saint Raymond's theorem.

Theorem 5. Let X be locally compact and let τ satisfy conditions (C1)-(C6). Let B be a Borel subset of $T \times X$ such that each t-section of B is σ-compact. Then the map $t \mapsto \tau^*(B_t)$ is $\mathcal{BA}(T)$-measurable. Moreover, for each $\epsilon > 0$, there are Borel sets $B_i \subset T \times X$, $i = 1,2,3,...$, with compact sections, and a Borel set $N \subset \pi_T(B)$ with $\nu(N) = 0$, such that if $t \in T \setminus N$, then $B_t \subset \cup_i (B_i)_t$ and $\Sigma_i \tau((B_i)_t) \leq \tau^*(B_t) + \epsilon$.

Proof. Since the theorem is trivially true if $\nu(\pi_T(B)) = 0$, we may assume the projection $\pi_T(B)$ has positive measure. It follows from assumption (C4) that the map $f : K(X)^{\mathbb{N}} \mapsto \mathbb{R}$

defined by $f((K_n)) = \Sigma\tau(K_n)$ is Borel measurable. It also follows from (C6) and the local compactness of X that for each $t \in T, g(t) = \tau^*(B_t) = \inf\{f((K_n)) : (t,(K_n)) \in F(B)\}$, where $F = F(B)$ is defined in Theorem 4. If q is a positive rational or ∞, $F \cap (f \leq q)$ is a Borel set, where $(f \leq q) = \{(t,(K_n)) : \sum \tau(K_n) \leq q\}$. Therefore, by the Jankov-von Neumann Theorem [K], there is a function $s_q : D_q \to K(X)^{\mathbb{N}}$ where $D_q \equiv \pi_T(F \cap (f \leq q))$ is analytic, such that s_q is a $\mathcal{BA}(T)$ measurable selector for $F \cap (f \leq q)$. Let s_{qi} be the i-th coordinate function of s_q. Noting that $g(t) = \inf\{q : t \in D_q \text{ and } q \text{ is rational}\}$, it follows that g is $\mathcal{BA}(T)$-measurable. Next, fix $\epsilon > 0$ and enumerate the rationals as $\{q_n\}$. For each p, let $A_p = \{t : q_p$ is the first rational with $s_{q_p}(t) \leq g(t) + \epsilon\}$. We may find a set $N \subset \pi_T(B)$ such that $\nu(N) = 0$ and $s_{q_p}|(A_p \setminus N)$ is Borel measurable for each p. Since for each p and i the map $s_{q_p i}$ is a Borel measurable map of $A_p \setminus N$ into $K(X)$, the graph $B_{pi} = \{(t,x) : x \in s_{q_p i}(t)\}$ is a Borel set. For each i, setting $B_i = \cup_{p=1}^{\infty} B_{pi}$ gives sets with the required properties. ■

Remarks

1. It is possible to strengthen Theorem 5 if τ^* satisfies the increasing sets lemma, that is if for any increasing sequence A_n of subsets of X we have

$$\tau^*(\bigcup_1^{\infty} A_n) = \lim_{n\to\infty} \tau^*(A_n).$$

If (C1)-(C6) and the increasing sets lemma holds, then if B is a Borel set in $T \times X$ such that each section B_t is σ-compact, we may conclude that the map $t \mapsto \tau^*(B_t)$ is Borel measurable. To see this, we first apply Saint-Raymond's theorem to deduce that $B = \cup B_n$ for Borel sets B_n where, for each t, $(B_n)_t$ is an increasing sequence of compact sets converging to B_t. Then for each n the map $g_n(t) = B_{nt}$ is a Borel measurable map of T into $K(X)$. Since τ^* is upper semi-continuous on $K(X)$, the composition $f_n = \tau^* \circ g_n$ is Borel measurable. By the increasing sets lemma, $t \mapsto \tau^*(B_t)$ is Borel measurable as the limit of the f_n.

We note that if τ satisfies (i) $\tau(\emptyset) = 0$, (ii) τ is monotone, (iii) if $\text{diam}(A) > 0$ then $\tau(A) > 0$, and (iv) τ is continous in the Hausdorff pseudo metric, and if X is compact, then τ satisfies (C1)-(C6) and τ^* satisfies the increasing sets lemma by a theorem of Sion and Sjerve [SS]. Thus if (i)-(iv) are satisfied, the map $t \mapsto \tau^*(B_t)$ is Borel measurable.

Sion and Sjerve also show that if X is σ-compact and τ^* satisfies the increasing sets lemma for compact sets then τ^* satisfies the increasing sets lemma.

We note the following example. Let $X = \{0, 1, 1/2, 1/3, ...\}$. Define τ by $\tau(A) = 2$, if $0 \in \overline{A}$; $\tau(A) = 1$ if $0 \notin \overline{A}$ and $A \neq \emptyset$ and $\tau(\emptyset) = 0$. Then τ satisfies (C1)-(C6) and (i)-(iii), but τ is not continuous in the Hausdorff pseudo metric.

2. These theorems or variants doubtless hold under more general conditions on X or by relaxing some of the conditions on τ. We do not know whether the map $t \mapsto \tau^*(B_t)$ is always Borel measurable when B has σ-compact sections or whether the set N in Theorem 5 can be eliminated.

What is the situation if we do not require τ to satisfy condition (C5)? Some further structure is needed. Consider the example with $\tau(E) = 0$, if E is finite and $\tau(E) = \infty$, otherwise; then τ satisfies (C1)-(C4) and (C6). However, since $\tau^*(E) = 0$ if K is countable and is ∞ otherwise, $K \mapsto \tau^*(K)$ is not Borel measurable.

We need a condition such as (C6) for Theorem 5 to hold. For example, let $X = \mathbb{N}$, the positive integers, and let $\tau(E) = \text{card}(E)/(1 + \text{card}(E))$. Then τ satisfies (C1)-(C5), but if (K_n) are compact sets covering an infinite set E, then $\tau^*(E) = 1$ and $\sum \tau(K_n) = \infty$.

We now consider the relationship between the Method I measures of a set and those of its sections.

Theorem 6. Let X be locally compact and let τ satisfy conditions (C1)-(C6). Let $B \subset T \times X$ be a Borel set with each t-section B_t σ-compact. Then we have equality in (3):

$$\mu^*(B) = \int_T \tau^*(B_t) d\nu(t). \tag{4}$$

Proof. Let $\epsilon > 0$. Let B_i be the Borel sets and N the ν-null set given by Theorem 5. By Theorem 2, $t \mapsto \tau((B_i)_t)$ is ν-measurable, so by definition

$$\mu^*(B) \leq \sum_i \mu(B_i) = \sum_i \int \tau((B_i)_t) d\nu(t) \leq \int \tau^*(B_t) d\nu(t) + \epsilon,$$

where we have interchanged summation and integration and used the final conclusion of Theorem 5. Taking ϵ arbitrarily small and combining with Lemma 1 gives the result. ■

Equation (4) specialises to the following product fromula for Borel rectangles $U \times E$:

$$\mu^*(U \times E) = \nu(U)\tau^*(E).$$

Thus, the outer measure μ^* on $T \times X$, defined in terms of μ using Method I, may be regarded as a product measure of ν and τ^*. However, for τ^* non-σ-finite (as occurs in many applications) a product measure is generally far from uniquely defined by the product formula on rectangles. It also follows from (4) that a Borel set $B \subset T \times X$ is μ^*-measurable if and only if $B_t \subset X$ is τ^*-measurable for ν-almost all t, see Rogers [Ro, Chapter 1.2].

The following refinement of Theorem 6, which restricts covering sets to Borel rectangles, is required for the Method II results which follow.

Theorem 7. Let X be locally compact and let τ satisfy conditions (C1)-(C6). Let $B \subset T \times X$ be a Borel set such that each t-section B_t is σ-compact. Then

$$\mu^*(B) = \inf \left\{ \sum_i \mu(B_i) : B \subset \cup_i B_i \text{ and } B_i \text{ are Borel rectangles} \right\}. \tag{5}$$

Proof. Let $\epsilon > 0$. By Theorem 5, there are a Borel set N with $\nu(N) = 0$ and a sequence of Borel sets G_i, such that for $t \in T \setminus N$, each $(G_i)_t$ is compact, $B_t \subset \cup_i (G_i)_t$ and $\sum_i \tau((G_i)_t) \leq \tau^*(B_t) + \epsilon$. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of sets forming a base for the topology of X such that $\overline{U_n}$ is compact for each n, and with this sequence closed under finite unions. For each i, n, let $T_{in} = \{t \in T \setminus N : n$ is the first integer such that $U_n \supset (G_i)_t$ and $\tau(\overline{U_n}) < \tau((G_i)_t) + \epsilon\}$. Then $R_{in} = T_{in} \times \overline{U_n}$ is a Borel rectangle. By Theorem 6,

$$\begin{aligned}
\mu^*(B) = \int \tau^*(B_t) d\nu(t) &\geq \int (\sum_i \tau((G_i)_t) - \epsilon) d\nu(t) \\
&= \sum_i \sum_n \int_{T_{in}} \tau((G_i)_t) d\nu(t) - \epsilon \\
&\geq \sum_i \sum_n \int_{T_{in}} (\tau(\overline{U_n}) - \epsilon) d\nu(t) - \epsilon \\
&= \sum_i \sum_n \mu(R_{in}) + \mu(N \times X) - 2\epsilon,
\end{aligned}$$

as $\nu(N) = 0$. Since $\epsilon > 0$ is arbitrary, and $B \subset \cup_i \cup_n R_{in} \cup (N \times X)$ is a cover of B by rectangles,

$$\mu^*(B) \geq \inf \left\{ \sum_i \mu(B_i) : B \subset \cup_i B_i \text{ and } B_i \text{ are Borel rectangles} \right\},$$

with the opposite inequality immediate from (2). ∎

We now introduce Method II constructions which by their definition depend on the metric structure of the sets. For these constructions we make the additional assumption that d is a metric on X with the property that for some $\delta_0 > 0$, if $|E| < \delta_0$, then $\overline{E}$ is compact. We work with the metric $d_0 = \max\{d, \rho\}$ on $T \times X$, and write $|\cdot|$ for the diameter of a set in any of the metric spaces.

For $\delta > 0$, define for $E \subset X$

$$\tau_\delta(E) = \tau(E) \text{ if } |E| \leq \delta \text{ and } \tau_\delta(E) = \infty \text{ if} |E| > \delta.$$

This is equivalent to seeking covers by sets of diameters at most δ. As before, we set

$$\tau_\delta^*(E) = \inf_{E \subset \cup E_i} \sum \tau_\delta(E_i) = \inf_{E \subset \cup E_i, |E_i| \leq \delta} \sum \tau_\delta(E_i).$$

The Method II measure on X constructed from the set function τ is then defined by

$$\tau^{**}(E) = \lim_{\delta \to 0} \tau_\delta^*(E).$$

This is a metric outer measure on X and thus all the Borel sets and analytic sets are measurable, see Rogers [Ro]. Proceeding as before, we set, for $B \subset T \times X$,

$$\mu_\delta(B) = \int_T \tau_\delta(B_t) d\nu(t) \tag{6}$$

and

$$\mu_\delta^*(B) = \inf_{B \subset \cup B_i} \sum \mu_\delta(B_i). \tag{7}$$

The set function μ_δ^* may be presented in several different ways for certain Borel sets.

Lemma 8. Let X be locally compact and let τ satisfy conditions (C1)-(C6). Let $B \subset T \times X$ be a Borel set with each t-section B_t σ-compact. For each $\delta > 0$,

$$\mu_\delta^*(B) = \inf\left\{\sum \mu_\delta(B_i) : B \subset \cup_i B_i \text{ and } B_i \text{ are Borel rectangles}\right\} \tag{8}$$

$$= \inf\left\{\sum \mu_\delta(B_i) : B \subset \cup_i B_i \text{ and } |B_i| \leq \delta\right\} \tag{9}$$

$$= \inf\left\{\sum \mu_\delta(B_i) : B \subset \cup_i B_i,\ |B_i| \leq \delta \text{ and } B_i \text{ are Borel rectangles}\right\}. \tag{10}$$

Proof. The equality in (8) is just Theorem 7. Certainly, the right-hand side of (8) is no greater than expression (10), and if (8) is infinite then these two expressions are equal. So suppose (8) is finite and $B_i = T_i \times E_i$ is a family of Borel rectangles covering B with $\mu_\delta(B_i) < \infty$. We may decompose $T_i = \cup_{j=1}^\infty T_{ij}$, where the T_{ij} are disjoint Borel subsets of T with $|T_{ij}| \leq \delta$. Then $B_i = \cup_{j=1}^\infty T_{ij} \times E_i$. Since B_i is a rectangle,

$$\mu_\delta(B_i) = \sum_{j=1}^\infty \tau_\delta(E_i)\nu(T_{ij}) = \sum_{j=1}^\infty \mu_\delta(T_{ij} \times E_i). \tag{11}$$

If $\tau_\delta(E_i) = \infty$, we must have $\nu(T_{ij}) = 0$ for all j, so $\mu_\delta(T_{ij} \times E_i) = \mu_\delta(B_i) = 0$ for all j. Otherwise, $\tau_\delta(E_i) < \infty$, so $|E_i| \leq \delta$ and $|T_{ij} \times E_i| \leq \delta$ for all j. It follows, using (11) that the sum in (8) is unchanged if we replace each set $B_i = T_i \times E_i$ by the countable union $\cup_j T_{ij} \times E_i$ of sets of diameter at most δ. Thus expressions (8) and (10) are equal. Finally, expression (9) lies between $\mu_\delta^*(B)$ as defined by (7) and (10). ■

We now relate the Method II measure on the sections X_t obtained from τ to the Method II measure on $T \times X$ obtained from μ. Thus we set

$$\mu^{**}(B) = \lim_{\delta \to 0} \mu_\delta^*(B); \tag{12}$$

this is the Method II measure on $T \times X$ obtained from μ by virtue of (9).

Theorem 9. Let X be locally compact and let τ satisfy conditions (C1)-(C6). Let $B \subset T \times X$ be a Borel set with each t-section B_t σ-compact. Then

$$\mu^{**}(B) = \int_T \tau^{**}(B_t)\, d\nu(t). \tag{13}$$

Proof. For each $\delta > 0$, applying Theorem 6 to τ_δ gives

$$\mu_\delta^*(B) = \int_T \tau_\delta^*(B_t)\, d\nu(t).$$

Letting $\delta \to 0$ we have $\tau^*_\delta(B_t) \to \tau^{**}(B_t)$ for all t, and $\mu^*_\delta(B) \to \mu^{**}(B)$. Identity (13) follows by the monotone convergence theorem. ■

Since μ^{**} is a Method II measure, it is a metric measure, and all Borel and analytic subsets of $T \times X$ are μ^{**} measurable. We again have a product formula for Borel rectangles,

$$\mu^{**}(U \times E) = \nu(U)\tau^{**}(E),$$

so μ^{**} is a product of ν and τ^{**}, though once again extensions to X may be far from unique given that τ^{**} is likely to be non-σ-finite.

Example I Our principle example is Hausdorff measure. For $s \geq 0$, setting $\tau(E) = |E|^s$, where, as usual, $|\cdot|$ denotes diameter, we get that τ^{**} is the usual s-dimensional Hausdorff measure, $\mathcal{H}^s$ on X, see Rogers [Ro]. Thus, by Theorem 9, if B is a Borel set with σ-compact t-sections,

$$\mu^{**}(B) = \int_T \mathcal{H}^s(B_t) d\nu(t),$$

where μ^{**} is the Method II measure constructed from the set function $\mu(B) = \int |B_t|^s d\nu(t)$. It follows from Lemma 8 that we may use Borel rectangles $R_i = U_i \times E_i$ in covers for finding μ^*_δ and μ^{**}, so

$$\begin{aligned}\mu^{**}(B) &= \liminf_{\delta \to 0} \sum_{B \subset \cup R_i, |R_i| \leq \delta} \mu(R_i) \\ &= \liminf_{\delta \to 0} \sum_{B \subset \cup R_i, |R_i| \leq \delta} \nu(U_i)|E_i|^s.\end{aligned}$$

Now let ν be the restriction of m-dimensional Hausdorff measure $\mathcal{H}^m$ to a compact set $T \subset R^n$ with $0 < \mathcal{H}^m(T) < \infty$; we lose little by assuming that $\mathcal{H}^m(T) = 1$. By a standard result on upper densities, see [M], we have that $\limsup_{r \to 0} \nu(B(x,r))(2r)^{-m} \leq 1$ for ν-almost all x. Thus, if $\epsilon > 0$, we may take an increasing sequence of Borel sets $T_i \to T_0$, where $\nu(T_0) = 0$, and $\delta_i \to 0$, such that $\nu(U) \leq (1+\epsilon)2^m|U|^m$ if $|U| \leq \delta_i$ and $U \cap T_i \neq \emptyset$. Then

$$\begin{aligned}\mu^{**}(B \cap (T_i \times X)) &\leq \liminf_{\delta \to 0} \sum_{B \subset \cup R_i, |R_i| \leq \delta} (1+\epsilon)2^m|U_i|^m|E_i|^s \\ &\leq \liminf_{\delta \to 0} \sum_{B \subset \cup R_i, |R_i| \leq \delta} (1+\epsilon)2^m|R_i|^{m+s} \\ &= (1+\epsilon)2^m\mathcal{H}^{m+s}(B)\end{aligned}$$

for all i. Using Theorem 9 and taking the limit as $i \to \infty$, $\mu^{**}(B) \leq (1+\epsilon)2^m\mathcal{H}^{m+s}(B)$ so, since ϵ may be taken arbitrarily small,

$$\mu^{**}(B) = \int_T \mathcal{H}^s(B_t) d\mathcal{H}^m(t) \leq 2^m\mathcal{H}^{m+s}(B).$$

The right-hand inequality is well-known, see [M]. Here we have given an alternative derivation of a somewhat stronger fact, that $B \mapsto \int_T \mathcal{H}^s(B_t) d\mathcal{H}^m(t)$ is itself a Method II measure on $T \times E$ constructed from the set function $\mu(B) = \int |B_t|^s d\mathcal{H}^m(t)$.

Example II Let λ be a given probability measure on $\mathbb{R}^n$ which we assume satisfies (C1)-(C5), and let $s, q \geq 0$. In connection with multifractal measures, several authors, for example Olsen [Ol], have considered measures of Hausdorff type which are Method II measures constructed from the set functions such as $\tau(E) = |E|^s \lambda(E)^q$ for $E \subset \mathbb{R}^n$. (For certain purposes, this τ may be modified so that $\tau(E) = \infty$ unless $\overline{E}$ is a ball.) This leads to Borel measures $\mathcal{H}_\lambda^{s,q}$ given by

$$\mathcal{H}_\lambda^{s,q}(E) = \tau^{**}(E) = \liminf_{\delta \to 0} \sum_{E \subset \cup E_i} |E_i|^s \lambda(E_i)^q.$$

Just as in Example I, we get a formula for the integral of sections of a Borel set B with σ-compact sections as a Method II measure. Thus

$$\mu^{**}(B) = \int_T \mathcal{H}_\lambda^{s,q}(B_t) d\nu(t),$$

where μ^{**} is the Method II measure constructed from the set function $\mu(B) = \int^* |B_t|^s \lambda(B_t) d\nu(t)$. Such formulae may be applied to problems on sections of multifractal measures.

2. Analytic operators and packing dimensions

In this section we use properties of analytic operators to obtain some stronger results relating to packing dimensions of sections. For our purposes, it is enough to use the definition of packing dimension via upper box-counting dimension. For K a compact subset of some seperable metric space Y we set $N_r(K)$ for the least number of open balls of radius r that are needed to cover K. The *upper box-counting dimension* $\overline{\dim}_B K$ of K is defined by

$$\overline{\dim}_B K = \limsup_{r \to 0} \log N_r(K) / -\log r. \tag{14}$$

We define the *packing dimension* $\dim_P B$ of $B \subset Y$ by

$$\dim_P B = \inf \left\{ \sup_i \overline{\dim}_B K_i : B \subset \cup_i K_i \text{ with } K_i \text{ compact} \right\}. \tag{15}$$

For further properties of these dimensions, and the equivalent definition of packing dimension via packing measure, see [F,M].

We recall that the Borel operators over a Polish space X are generated in much the same way as the Borel sets [CM]. Thus, a function Δ mapping $P(X)$, the power set of X, into itself is said to be a *Borel operator* provided it is in the smallest family $\mathcal{F}$ of operators containing the following operators:

(a) $\Delta(K) = B$, B is a fixed Borel subset of X,

(b) $\Delta(K) = f^{-1}(K)$, where f is a fixed Borel map from X into X,

(c) $\Delta(K) = X \setminus K$,

and such that the family is closed under the operations of composition and countable unions:

$$(d)\quad \Delta(K) = \Delta_1(\Delta_2(K)), \quad \Delta_1, \Delta_2 \in \mathcal{F}$$
$$(e)\quad \Delta(K) = \cup_{n=0}^{\infty}\Delta_n(K), \quad \Delta_n \in \mathcal{F}.$$

An operator $\Theta : P(X) \to P(X)$ is said to be *analytic* if and only if there is a Polish space Y and a Borel operator $\Delta : P(X \times Y) \to P(X \times Y)$ such that $\Theta(M) = \pi_X(\Delta(M \times Y))$ for each $M \subset X$.

For each $d \geq 0$, let $\Gamma = \Gamma^{(d)} : P(X) \to P(X)$ be the operator defined by

$$x \in \Gamma(M) \iff \forall \epsilon > 0\, [\, \overline{\dim}_B(M \cap B(x, \epsilon)) \geq d\,].$$

Theorem 10. The operator Γ is analytic, that is Σ_1^1.

Proof. For each n, defining $\Gamma_n : P(X) \to P(X)$ by

$$x \in \Gamma_n(M) \iff \overline{\dim}_B(M \cap B(x, 1/n)) \geq d,$$

we have

$$\Gamma(M) = \bigcap_{n=1}^{\infty} \Gamma_n(M).$$

Since the intersection of a sequence of analytic operators is analytic, it suffices to show that each operator Γ_n is analytic. To this end, we consider the Polish space $Y = X^{\mathbb{N}}$.

Let $D = \{(y_p) \in X^{\mathbb{N}} : \overline{\dim}_B\{y_p : p \in \mathbb{N}\} \geq d\}$. We note that D is a Borel subset of Y. There are several ways to prove this. For example, one can easily check that $I = \{(y_p) \in Y : \{y_p : p \in \mathbb{N}\}$ is not conditionally compact$\}$ is a Borel set and the map $\phi : Y \setminus I \mapsto K(X)$, defined by $\phi((y_p)) = \overline{\{y_p : p \in \mathbb{N}\}}$ is Borel measurable. In [MM] it is shown that the map $K \mapsto \overline{\dim}_B(K)$ is Borel measurable, and composing these maps gives that D is a Borel set.

Next, define the operator Δ over $X \times Y$ by:

$$\Delta(A) = (X \times D) \cap \bigcap_{k=1}^{\infty} \big(B_k \cap f_k^{-1}(A)\big),$$

where $B_k = \{(x, (y_p)) : d(x, y_k) < 1/n\}$ is a Borel subset of $X \times Y$ for each k, and $f : X \times Y \mapsto X \times Y$ given by $f_k(x, (y_p)) = (y_k, (y_p))$ is a Borel measurable map. Thus $\Delta : P(X \times Y) \to P(X \times Y)$ is a Borel operator, see [CM, p. 58]. Since

$$x \in \Gamma_n(M) \iff \exists (y_p) \in Y\ [\,(x, (y_p)) \in \Delta(M \times Y)\,],$$

or

$$\Gamma_n(M) = \pi_X(\Delta(M \times Y)),$$

the operator Γ_n is an analytic, that is Σ_1^1, operator [CM, p. 58]. Therefore, $\Gamma = \bigcap \Gamma_n$ is an analytic operator. ■

Let $\Gamma_0(M) = M \cap \Gamma(M)$, so $x \in \Gamma_0(M)$ if and only if $x \in M$ and, for every neighborhood U of x, $\overline{\dim}_B(U \cap M) \geq d$. As the intersection of two analytic operators, Γ_0 is an analytic, or Σ^1_1, operator and therefore the dual operator Ψ defined by

$$\Psi(M) = X \backslash \Gamma_0(X \backslash M)$$

is a coanalytic, or Π^1_1, operator. The operator Ψ is also monotone, that is $M \subset \Psi(M)$, since

$$\Psi(M) = M \cup \{x \in X \backslash M : \exists \text{ an open set } U\,[x \in U \text{ and } \overline{\dim}_B(U \cap (X \setminus M)) < d]\}.$$

Thus Ψ adds to M all points x of $X \setminus M$ at which $X \setminus M$ is small in the sense that there is some neighborhood U of x such that $U \cap (X \setminus M)$ has upper box counting dimension less than d. An important feature of this operator is that Ψ adds to M a relatively open subset of $X \setminus M$.

We next consider the effect of iterating the operator Ψ. By transfinite recursion, we set $\Psi^0(E) = E$, and $\Psi^{\alpha+1}(E) = \Psi(\Psi^\alpha(E))$ for each ordinal α, and $\Psi^\lambda(E) = \cup_{\gamma<\lambda}\Psi^\gamma(E)$ if λ is a limit ordinal. We note some properties of the operator Ψ including a simple boundedness principle or stabilization property.

Lemma 11. For each $M \subset X$, there is an ordinal $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal, such that $\Psi^\alpha(M) = \Psi^{\alpha+1}(M)$. If $X \setminus M$ is compact, then for each ordinal α, the set $X \setminus \Psi^\alpha(M)$ is compact, and if in addition $\dim_P(X \setminus M) < d$, then there is a countable ordinal α such that $\Psi^\alpha(M) = X$.

Proof. Let $(U_n)_{n\in N}$ be a base for the topology of X. Suppose that for each countable ordinal α, $\Psi^\alpha(M)$ is a proper subset of $\Psi^{\alpha+1}(M)$. For each such α choose $n(\alpha)$ such that $U_{n(\alpha)} \cap X \setminus \Psi^\alpha(M) \subset \Psi^{\alpha+1}(M)$. Thus we may choose two countable ordinals $\alpha < \beta$ such that $n = n(\alpha) = n(\beta)$, and $x \in U_n \cap (X \setminus \Psi^\beta(M))$, so that $x \in \Psi^{\beta+1}(M) \setminus \Psi^\beta(M)$. On the other hand, $x \in \Psi^{\alpha+1}(M) \subset \Psi^\beta(M)$. This contradiction establishes the first part of the lemma.

For the second part, suppose $X \setminus M$ is compact. Since Ψ adds to M a relatively open subset of $X \setminus M$, the set $X \setminus \Psi(M)$ is compact. It follows by transfinite induction that $X \setminus \Psi^\alpha(M)$ is compact for each ordinal α. Finally, suppose in addition that $\dim_P(X \setminus M) < d$, that $\Psi^\alpha(M) = \Psi^{\alpha+1}(M)$ and that $Z = X \setminus \Psi^\alpha(M) \neq \emptyset$. Since $\dim_P(Z) < d$, there is a cover of Z by compact sets K_n such that for each n, $\overline{\dim}_B(K_n) < d$. By Baire's category theorem, for some n, the set K_n has nonempty interior U with respect to Z. Then $\emptyset \neq U \subset \Psi^{\alpha+1}(M) \setminus \Psi^\alpha(M)$. This last contradiction completes the proof of the lemma. ∎

We need a parametrized version of the operator Ψ. Let us define the operator Φ over $P(T \times X)$ by:

$$\Phi(M) = \bigcup_{t\in T} \{t\} \times \Psi(M_t).$$

We note some of the basic properties of this operator.

Lemma 12. The operator Φ is monotone and coanalytic. Let M be a Borel subset of $T \times X$ such that for each $t \in T, X \setminus M_t$ is compact. Then $\Phi(M)$ is a Borel set and for each $t, X \setminus (\Phi(M))_t$ is compact. Moreover, for each countable ordinal $\alpha, \Phi^\alpha(M)$ is a Borel set, and for each $t, X \setminus (\Phi^\alpha(M))_t$ is compact.

Proof. Clearly, Φ is monotone and it is shown in [CM] that an operator on a product space which is the sectionwise application of a coanalytic operator is coanalytic. Let (U_n) be a basis for the topology of X. We note

$$(t,x) \in \Phi(M) \iff (t,x) \in M \text{ or } \exists U_n[x \in U_n \cap X \setminus M_t \text{ and } \overline{\dim}_B(U_n \cap X \setminus M_t) < d].$$

For each n, set $S_n = ((T \times X) \setminus M) \cap (T \times U_n)$. Then S_n is a Borel subset of $T \times X$ and each t-section of S_n is σ-compact. Thus, $D_n = \pi_T(S_n)$ is a Borel set. So, $G_n = (D_n \times X) \cap T \times X \setminus M$ is a Borel set and each t-section of G_n is compact. Therefore, the map $\phi_n : D_n \mapsto K(X)$ defined by $\phi(t) = (G_n)_t$ is a Borel measurable map. Since the map $K \mapsto \overline{K \cap U_n}$ is Borel measurable and the map $K \mapsto \overline{\dim}_B(K)$ is Borel measurable [MM], the set $E_n = \{t \in D_n : \overline{\dim}_B(G_n)_t \cap U_n = \overline{\dim}_B((X \setminus M_t) \cap U_n) = \overline{\dim}_B(\overline{X \setminus M_t \cap U_n}) < d\}$ is a Borel set. Since

$$\Phi(M) = M \cup \cup_n (E_n \times U_n) \cap (T \times X \setminus M),$$

it follows that $\Phi(M)$ is a Borel set. This finishes the proof of the middle part of the lemma. The last part follows by transfinite induction using the middle conclusion of the lemma. ■

We now deduce that reasonable Borel sets $B \subset T \times X$ have a countable decomposition into subsets, such that the packing dimension of the sections of B are determined by the upper box-counting dimensions of the sections of the subsets.

Theorem 13. Let T and X be Polish spaces and let B be a Borel subset of $T \times X$ such that for all $t \in T$, the t-section B_t is σ-compact with $\dim_P(B_t) < d$. Then there is a sequence of Borel sets $\{E_n\}_{n=1}^\infty$ such that $B = \bigcup_{n \in N} E_n$, and for all $t \in T$ and $n \in \mathbb{N}$ the section $(E_n)_t$ is compact with $\overline{\dim}_B(E_n)_t < d$.

Proof. By Saint Raymond's theorem the Borel set B can be expressed as a countable union of Borel sets each of which have all t-sections compact. Thus it suffices to prove the theorem under the assumption that each t-section of B is compact. For each ordinal α, let $B_\alpha = \Phi^\alpha((T \times X) \setminus B) \subset T \times X$. For each $t \in T$, Lemma 11 implies that there is some countable ordinal $\alpha(t)$ such that $B_{\alpha(t)} = X$. In the terminology of [CM] this means $T \times X$ is the closure of the operator Φ on the Borel set $T \times X \setminus B$ which is defined to be $\cup_\alpha \Phi^\alpha((T \times X) \setminus B)$. By the boundedness principle for monotone coanalytic operators, [CM, Theorem 1.6(e)], there is a countable ordinal α such that

$$T \times X = B_\alpha,$$

so

$$B = \bigcup_{\gamma < \alpha} B_{\gamma+1} \setminus B_\gamma.$$

By Lemma 12, for each γ, the set $B_{\gamma+1}\backslash B_\gamma$ is Borel, and for each t, the set $(B_{\gamma+1}\backslash B_\gamma)_t$ is σ-compact. Also, if K is compact and $K \subset (B_{\gamma+1}\backslash B_\gamma)_t$, then $\overline{\dim}_B K < d$. Applying Saint Raymond's theorem (see Section 1), we can express each set $B_{\gamma+1}\backslash B_\gamma$ as a countable union of Borel sets each with every t-section compact. The theorem now follows. ■

We now apply Theorem 13 to give an alternative derivation of a formula for the essential supremum of the packing dimension of sections of sets, originally presented in [FJ]. For this illustration we take $T = \mathbb{R}^m$ and $X = \mathbb{R}^n$ with ν as m-dimensional Lebesgue measure, although the results extend to other homogeneous metric spaces.

We express our results in terms of a generalised packing dimension defined analogously to the usual packing dimension, see [FJ]. For K a compact subset of $T \times X$ we set

$$N_r^*(K) = \inf\left\{\sum_i \nu(\pi_T(K \cap U_i)) : K \subset \cup_i U_i \text{ with } |U_i| \leq r\right\}.$$

The *generalised upper box-counting dimension* $\overline{\dim}_B^* K$ of K is defined by

$$\overline{\dim}_B^* K = \limsup_{r\to 0} \log N_r^*(K)/-\log r.$$

Analogously to the usual dimensions, we define the *generalised packing dimension* $\dim_P^* B$ of $B \subset T \times X$ by

$$\dim_P^* B = \inf\left\{\sup_i \overline{\dim}_B^* K_i : B \subset \cup_i K_i \text{ with } K_i \text{ compact }\right\}.$$

For further properties and a measure approach to these dimensions see [FJ].

As in [FJ, Proposition 3.5], a straightforward integration argument establishes that for all $B \subset T \times X$ we have

$$\dim_P B_t \leq \dim_P^* B \tag{16}$$

for ν-almost all t. Another integration argument gives that for B bounded and analytic,

$$\dim_P^* B \leq \text{esssup}_t \overline{\dim}_B(B_t). \tag{17}$$

A much more technical argument is used in [FJ, Proposition 9] to obtain the natural and useful identity

$$\dim_P^* B = \text{esssup}_t \dim_P(B_t) \tag{18}$$

which gives an expression for the packing dimension of a typical section of a compact set B.

Equation (18) may alternatively be obtained as a simple corollary of Theorem 13. Let B be a compact subset of $T \times X$, and let $d > \text{esssup}_t \dim_P(B_t)$ so $\dim_P(B_t) < d$ for almost all t. Theorem 13 applied to a subset of T of full measure (noting that $\dim_P(B_t)$ is measurable) gives a Borel decomposition $B = \cup B_n$ with $\overline{\dim}_B B_{nt} < d$ for almost all t, for all n. By (17), $\dim_P^* B_n \leq d$ for all n, so $\dim_P^* B \leq \sup_n \dim_P^* B_n \leq d$. This gives $\dim_P^* B \leq \text{esssup}_t \dim_P(B_t)$ and the opposite inequality is immediate from (16).

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