Packing measure and dimension of random fractals.

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ABSTRACT. We consider random fractals generated by random recursive constructions. We prove that the box-counting and packing dimensions of these random fractals, K, equals α, their almost sure Hausdorff dimension. We show that some "almost deterministic" conditions known to ensure that the Hausdorff measure satisfies $0 < \mathcal{H}^\alpha(K) < \infty$ also imply that the packing measure satisfies $0 < \mathcal{P}^\alpha(K) < \infty$. When these conditions are not satisfied, it is known $0 = \mathcal{H}^\alpha(K)$. Correspondingly, we show that in this case $\mathcal{P}^\alpha(K) = \infty$, provided a random strong open set condition is satisfied. We also find gauge functions $\phi(t)$ so that the $\mathcal{P}^\phi$-packing measure is finite.

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1. Introduction

In this paper we consider a general type of random fractal and some dimension properties associated with it. The random fractals considered are generated by a random recursive construction first defined by Mauldin and Williams[18]. This consists of a probability space (Ω, Σ, P) and a family of random compact subsets of $\mathbf{R^d}$ indexed by an n-ary $(1 < n \leq \infty)$ tree: $\mathbf{J} = \{J_\sigma | \sigma \in \{1, 2, \ldots, n\}^* = \bigcup_{\nu=0}^{\infty} \{1, 2, \ldots, n\}^\nu\}$, where $J_\emptyset = J$, $J = \mathrm{Cl}(\mathrm{Int}(J))$, $J_{\sigma * i}$ is a proper subset of J_σ for all $\sigma \in \{1, 2, \ldots, n\}^*$ and $i \in \{1, 2, \ldots, n\}$ provided $J_\sigma \neq \emptyset$. Although for the general construction, n may be infinite, in this paper we assume n is finite. These random set constructions satisfy some additional properties: (i) the sets J_σ, if not empty, are geometrically similar to the fixed seed set J, (ii) setting $\mathrm{diam}(J_{\sigma * i}) = \mathrm{diam}(J_\sigma) T_{\sigma * i}$, then for every finite word $\sigma \in \{1, \ldots, n\}^*$, the random vectors $(T_{\sigma * 1}, \ldots, T_{\sigma * n})$ are independent and distributed as $(T_1, \ldots, T_n)$ and finally, (iii) the construction satisfies a random *open set condition*: if ω and τ are two words of the same length, then $\mathrm{Int}(\mathrm{J}_\omega) \cap \mathrm{Int}(\mathrm{J}_\tau) = \emptyset$. We study the random limit set or fractal $K(w) = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in \{1,2,\ldots,n\}^k} J_\sigma(w)$, $w \in \Omega$. We note that this setting allows random placement of the sets $J_{\sigma * i}$ within J_σ. Thus these constructions include as a special case the random self-similar sets defined independently by Mauldin and Williams[18] and by Graf[7] who first carefully studied them. These last constructions are obtained by choosing the similarity mappings according to some probability distribution and thus may be regarded as random iterated functions systems. We also recall that the only interesting case in all of these constructions occurs when there is a positive probability that a nontrivial limit set exist, *i.e.* when $E[\sum_{i=1}^{n} T_i^0] > 1$, (by convention, $0^0 = 0$); otherwise $K(w)$ is almost surely an empty set or a point. We will assume this condition holds throughout the paper.

We will be mainly concerned with the packing dimension and measure of these random fractals. Packing measure and dimension together with the classic Hausdorff measure and dimension and box counting or Minkowski dimensions are discussed by Falconer[2], Mattila[15], Taylor[20,21] and Taylor and Tricot[22]. But, for the convenience of the reader we recall the definitions and some basic facts about packing measure and dimension. Let $g : [0, +\infty) \to [0, +\infty)$ be a gauge function, or a non-decreasing function with $g(0) = 0$. The g-packing measures, $\mathcal{P}^g$, naturally arose in two different areas. They were defined by D. Sullivan[19] to analyze some problems in dynamics and independently by Taylor and Tricot[22]. In their paper not only are packing measures and dimensions defined, but the exact gauge function for transient Brownian trajectory is calculated. Let $A \subset \mathbf{R^d}$ and $\delta > 0$. We say that $\{(x_i, r_i)\}_{i=1}^n$ is a δ-packing of A, if $x_i \in A$, $\delta \geq 2r_i > 0$, and $r_i + r_j < \text{dist}(x_i, x_j)$ for $i,\ j = 1, \ldots, n,\ i \neq j$. Then the closed balls $B(x_i, r_i)$ are disjoint. We first define the prepacking measures $P^g_{0,\delta}$ and P^g_0 by

$$P^g_{0,\delta}(A) = \sup\left\{\sum_{i=1}^n g(2r_i) \mid \{(x_i, r_i)\}_{i=1}^n \quad \text{is a } \delta - \text{packing of A}\right\},$$

$$P^g_0(A) = \lim_{\delta \to 0} P^g_{0,\delta}(A).$$

Since P^g_0 is not countably subadditive, one needs a standard modification to get an outer measure out of it. The packing g-measure for $A \subset \mathbf{R^d}$ is defined by

$$\mathcal{P}^g(A) = \inf\left\{\sum_{i=1}^\infty P^g_0(A_i) \mid A \subset \bigcup_{i=1}^\infty A_i\right\}.$$

Then $\mathcal{P}^g$ is a Borel regular outer measure. When $g(r) = r^s$, we denote $\mathcal{P}^g = \mathcal{P}^s$. In analogy with Hausdorff dimension, the packing dimension can be defined in terms of the packing measures:

$$\dim_P A = \inf\{s \mid \mathcal{P}^s(A) = 0\} = \sup\{s \mid \mathcal{P}^s(A) = \infty\}.$$

We note that the two stage definition of packing measure makes it somewhat more technical to handle than Hausdorff measure. In some sense there is no way around this. The complexity of packing measures has been analyzed by Mauldin and Mattila[16]. For example, it is shown there that the Hausdorff dimension function is a Borel class 2 mapping on $K(\mathbf{R^d})$, the space of compact sets, whereas the packing dimension function although measurable with respect to the σ algebra generated by analytic sets, is not Borel measurable.

Finally, we recall the upper Minkowski (or box-counting) dimension. For K, a bounded subset of R^d, and $\delta > 0$, let $N_\delta(K)$ be the smallest number of open balls of radius δ that are needed to cover K. The upper box-counting dimension, or Minkowski dimension, $\overline{\dim}_B K$, of K is defined by

$$\overline{\dim}_B K = \limsup_{\delta \to 0} \log N_\delta(K)/(-\log \delta) = \limsup_{j \to \infty} \log N_{2^{-j}}(K)/(j \log 2),$$

One very useful fact is that the packing dimension of a set can be calculated from the upper box-counting dimension:

$$\dim_P A = \inf\left\{ \sup_i \overline{\dim}_B K_i \mid A \subset \bigcup_{i=1}^{\infty} K_i,\ K_i \in K(\mathbf{R^d}) \right\}.$$

Mauldin and Williams determined the Hausdorff dimension of the random limit set $K(w)$ even when n is infinite as follows. Given $K(w) \neq \emptyset$, the Hausdorff dimension is almost surely α, where $\alpha = \inf\{\beta : E[\sum_{i=1}^{n} T_i^\beta] \leq 1\}$. Note that in case n is finite, we have $E[\sum_{i=1}^{n} T_i^\alpha] = 1$. Here the expectation E is taken with respect to P. Gatzouras[4] showed that Minkowski dimension of $K(w)$ coincides with its Hausdorff dimension. In Theorem 1 we present another short proof of this fact. Thus if n is finite, all four of the usual notions of dimension: the upper and lower box-counting dimensions, the Hausdorff and the packing dimensions, agree. Of course, if n is allowed to be infinite the box-counting dimension and packing dimension may be greater than the Hausdorff dimension even if the recursion is deterministic, see Mauldin and Urbanski[17], theorem 2.11.

The situation regarding the α-dimensional Hausdorff measure of these random fractals is fairly well understood. First, by a 0–1 law (see theorem 2 and the remark following it) the α-Hausdorff measure of $K(w)$ may be 0, $+\infty$, or positive and finite almost surely. Graf[7] found that in the case of random self-similar sets the α-Hausdorff measure of $K(w)$ is positive and finite provided the random similarity system is *almost deterministic*, specifically in case (i) $P(\sum_{i=1}^{n} T_i^\alpha = 1) = 1$ and (ii)$P(\min_{1\leq i\leq n} T_i \geq \delta | T_i \neq 0) = 1$ for some $\delta > 0$. Graf, Mauldin and Williams[8] extended this result to general random recursive constructions. The situation is similar for the α-dimensional packing measure. We show in Theorem 4 that under these same two conditions the α-packing measure is positive and finite. Moreover, Graf[7] showed that if $P(\sum_{i=1}^{n} T_i^\alpha = 1) < 1$, then the $\mathcal{H}^\alpha$-measure is 0 a.s. Correspondingly, we will prove in Theorem 5 that the α-packing measure in this situation is infinite in case a random strong open set condition is also satisfied.

For many of the cases where the α-Hausdorff measure is 0, the exact Hausdorff dimension function has been determined. Specifically, Graf, Mauldin and Williams[8] have found a gauge function g, so that $0 < \mathcal{H}^g(K(w)) < \infty$ a.s. provided $K(w) \neq \emptyset$. The corresponding situation for the exact packing dimension function is more complicated and largely unsolved. There are only two types of constructions for which the solution has been determined. They are the almost deterministic case mentioned above and those constructions such that the limit set is the image of a subordinator, for example, the zero set in Brownian bridge. Feng and Sha[3] showed that in this case there is no exact packing function in the following sense. If $\phi(t) = t^\alpha L(t)$ where L satisfies a doubling condition: there is some $c > 0$ such that $L(2t) \le cL(t)$ for small t, then either $\mathcal{P}^\phi(K) = 0$ a.s. or else $\mathcal{P}^\phi(K) = \infty$ a.s., depending on the convergence of some integral. Also Fristedt and Taylor[4] have found criteria when the image of a general subordinator has the exact packing dimesion and when it does not. Other than these cases, the exact packing measure function problem remains open. Let us comment that Liu[13] claimed to have the exact packing measure gauge function for a Galton-Watson tree in case the number of offspring is at least 2. However, as we shall show in section 5, there is a mistake in the proof that this measure is positive a.s. Along these lines Xiao[23] proved that there is no exact packing dimension for a branching process in case the number of offspring has a geometric distribution. The exact packing dimension of many other stochastic processes has been investigated, for example, Fristedt and Taylor[4], Gatzouras and Lalley[6], X. Hu[9,10], and Y. Hu[11]. In Theorem 6 we find an upper estimate for the exact packing dimension function for general random recursive constructions.

2. Preliminaries

We will use the following notation:

$B(x,r)$ is an open ball with center x and radius r, $D = \{1,\dots,n\}^{\mathbf{N}}$, for a finite sequence $\sigma \in \{1,\dots,n\}^*$, its length is $|\sigma|$ and $\sigma|_k$ denotes the first k elements of that sequence, for $\eta \in D$, $f_w(\eta) = \bigcap\limits_{k=1}^{\infty} J_{\eta|_k}(w)$, $l_\sigma(w) = \operatorname{diam}(J_\sigma(w))$, for $\Gamma \subset \{1,2,\dots,n\}^*$, $S_\Gamma^\alpha = \sum\limits_{\sigma\in\Gamma} l_\sigma^\alpha$.

Let $\mathcal{E}_k$ be the σ-algebra generated by the maps $w \to l_\sigma(w)$ where $|\sigma| \le k$. A basic fact is that the sequence $(S^\alpha_{\{1,\dots,n\}^k}, \mathcal{E}_k)$ forms an L^p-bounded martingale for all $p \ge 1$. We denote $X = \lim\limits_{k\to\infty} S^\alpha_{\{1,\dots,n\}^k}$. It is known (see Graf[7] , Graf, Mauldin and Williams[8] , Mauldin and Williams[18]) that $E[X] = \operatorname{diam}(J)^\alpha$. Similarly, we define $X_\sigma = \lim\limits_{k\to\infty} \sum\limits_{\tau\in\{1,\dots,n\}^k} \prod\limits_{i=1}^{|\tau|} T^\alpha_{\sigma*\tau|_i}$. It is also known that X_σ is distributed as $X/\operatorname{diam}(J)^\alpha$, $E[X_\sigma] = 1$, for $\sigma,\tau \in \{1,\dots,n\}^*$ such that $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$, X_σ and X_τ are independent, where $\sigma \prec \tau$ signifies that τ begins with σ.

We call $\Gamma \subset \{1,\ldots,n\}^*$ an antichain if for all $\tau,\sigma \in \Gamma$ $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$. An antichain Γ is maximal, if for all $\eta \in D$ there exists a unique $k \in \mathbf{N}$ such that $\eta|_k \in \Gamma$ (we denote $\eta|_k$ by $\eta|_\Gamma$), in other words, a maximal antichain is a cut. Especially useful for us will be antichains of the form $\Gamma_r(w) = \{\sigma \in \{1,2,\ldots,n\}^* \mid l_{\sigma|_{|\sigma|-1}} > r,\ l_\sigma \le r\}$. For $\tau \in \Gamma_r$ let $\Gamma_{r,\tau} = \{\sigma \in \Gamma_r \mid \mathrm{dist}(J_\sigma, J_\tau) < r\}$. For $\eta \in D$ let $\mathcal{G}_r(\eta,w) = \Gamma_{r,\eta|_{\Gamma_r}}$. These last sets reflect geometric clustering.

Graf, Mauldin and Williams[8] have demonstrated that with each construction one can associate 3 measures, denoted ν_w (the construction measure), μ_w and Q as follows. First, ν_w is determined by setting for a compact set $A \subset \mathbf{R^d}$

$$\nu_w(A) = \lim_{k\to\infty} \sum_{\substack{\sigma\in\{1,\ldots,n\}^k \\ J_\sigma \cap A \neq \emptyset}} l_\sigma^\alpha(w) X_\sigma(w).$$

Second, μ_w, a measure on D, is determined from each set $A(\sigma) = \{\eta \in D \mid \sigma \prec \eta\}$, a clopen subset of D, by

$$\mu_w(A(\sigma)) = l_\sigma^\alpha(w) X_\sigma(w)$$

and μ_w is extended to a Borel measure on D. Finally, Q is a measure on the product space $D \times \Omega$: for a Borel set B, let $B_w = \{\eta \in D \mid (\eta, w) \in B\}$. Then

$$Q(B) = \frac{\int \mu_w(B_w)\,dP(w)}{\mathrm{diam}(J)^\alpha}.$$

Expectations with respect to measures P and Q are connected in the following way (Graf, Mauldin and Williams[8]): if Γ is a map from Ω into the countable set of all maximal antichains in $\mathcal{P}(\{1,\ldots,n\}^*)$ such that for each maximal antichain Δ, $\Gamma^{-1}(\Delta)$ is in the σ-algebra generated by $\{J_\sigma | \sigma \preceq \Delta\}$ and $Y : D \times \Omega \to \mathbf{R}$ is a random variable such that $Y(\eta,w) = Y(\eta',w)$ provided $\eta|_{\Gamma(w)} = \eta'|_{\Gamma(w)}$, then

$$E_Q[Y] = \frac{E\left[\sum_{\sigma\in\Gamma} l_\sigma^\alpha X_\sigma Y(\sigma,\cdot)\right]}{\mathrm{diam}(J)^\alpha}.$$

In particular, for all $p > 0$ and $\sigma \in \{1,\ldots,n\}^*$, $E_Q[X_\sigma^p] = E[X_\emptyset^{p+1}] < \infty$. Without loss of generality, we assume that $\mathrm{diam}(J) = 1$.

3. Results

Theorem 1. $\dim_P K(w) = \overline{\dim}_B K(w) = \dim_H K(w) = \alpha$ a.s.

Proof: Let $\beta > \alpha$ and let $p = E[\sum_i^n T_i^\beta] < 1$. Since $\mathrm{diam}(J) = 1$, for each m, we have $\sum_{|\sigma|=m} l_\sigma^\beta = p^m$. Suppose that there exists a set A such that $P(A) > 0$ and for all $w \in A$, $\mathcal{P}_0^\beta(K(w)) > c > 0$. Therefore for every $\gamma > 0$ we can find $\{B_j(x_j, r_j)\}$ – a γ-packing of $K(w)$ such that $\sum_j |B_j|^\beta > c$. From this we construct random antichains $\Gamma_\gamma = \{\sigma | x_j \in J_\sigma, l_\sigma < r_j, l_{\sigma|_{|\sigma|-1}} \geq r_j\}$, where $\sigma = \sigma(x)$ is a code for $x = \cap_n J(\sigma|_n)$. Then $\sum_{\sigma \in \Gamma_\gamma} l^\beta_{\sigma|_{|\sigma|-1}} > c/2^\beta$. Let $|\Gamma| = \min_{\sigma\in\Gamma} |\sigma|$. Since for all $w \in A$ $\lim_{\gamma\to 0} |\Gamma_\gamma(w)| = \infty$, we can find a set $B \subset A$ with positive measure on which the divergence is uniform.

Then for all $w \in B$, $c \leq 2^\beta \limsup_{m\to\infty}\{\sum_{\sigma\in\Gamma} l^\beta_{\sigma|_{|\sigma|-1}} \mid |\Gamma| \geq m\} \leq 2^\beta \limsup_{m\to\infty} \sum_{k\geq m} \sum_{|\sigma|=k} l^\beta_{\sigma|_{|\sigma|-1}}$. Let $R_m = \sum_{k\geq m}\sum_{|\sigma|=k} l^\beta_{\sigma|_{|\sigma|-1}}$. Then $E[R_m] = \sum_{k\geq m} np^{k-1}$, $\lim_{m\to\infty} E[R_m] = 0$ and since R_m is non-increasing, we obtain $\lim_{m\to\infty} R_m = 0$ a.s. which is a contradiction. Hence $\overline{\dim}_B K(w) = \inf\{\beta | \mathcal{P}_0^\beta(K(w)) = 0\} \leq \alpha$ and the result follows from the general fact that $\alpha = \dim_H K(w) \leq \dim_P K(w) \leq \overline{\dim}_B K(w)$. ■

Next we prove a $0-1$ law for packing measures which can be applied in other situations as well. Here we assume there are only finitely many offspring a.s., although their number may be unbounded.

Theorem 2. Suppose $P(\mathrm{card}(\{i \mid T_i \neq 0\}) < \infty) = 1$. Then $P(\mathcal{P}^\alpha(K(w)) = 0 | K(w) \neq \emptyset) = 0$ or 1.

Proof: Let $C = \{w | K(w) = \emptyset\}$, $A = \{w | \mathcal{P}^\alpha(K(w)) = 0\}$, $B_i = \{w |$ exactly i among $T_1, T_2, \ldots \neq 0\}$, $y = P(A)$, $p_i = P(B_i)$. So, $P(C) < 1$ and $P(C) \leq y$. Since the sets B_i partition the probability space, $P(A) = \sum_{i=0}^\infty P(A|B_i)P(B_i)$. Let $K_i(w) = K(w) \cap J_i(w)$. Then $K(w) = \bigcup_{i=1}^\infty K_i(w)$, $P(A|B_i) = P(\mathcal{P}^\alpha(K(w)) = 0 |$ exactly i among $J_1, J_2, \ldots \neq \emptyset) = y^i$. Thus $y = \sum_{i=0}^\infty p_i y^i$. Consider the function $f\colon [0,1] \to \mathbf{R}$ defined by the formula $f(x) = p_0 + x(p_1 - 1) + \sum_{i=2}^\infty x^i p_i$. So, $f(y) = 0$ and the same considerations show that $f(P(C)) = 0$. Since $\sum_{i=0}^\infty p_i = 1$, $f(1) = 0$. Not all of $p_i, i \geq 2$ can equal 0 because $P(C) < 1$. Therefore f'' is positive on the interval and the only root of f on $[0, 1)$ is $P(C)$.

Thus $P(A) = 1$ or $P(A) = P(C)$, and

$$P(A|\overline{C}) = \frac{P(A\cap\overline{C})}{P(\overline{C})} = \frac{P(A) - P(A\cap C)}{P(\overline{C})} = \frac{P(A) - P(C)}{P(\overline{C})} = 0 \text{ or } 1. \blacksquare$$

Remarks. Similarly one can prove that $P(\mathcal{P}^\alpha(K(w)) < \infty | K(w) \neq \emptyset) = 0$ or 1. The proof of Lemma 1 remains valid for any property (or negation thereof) that holds for the whole construction $K(w)$ if and only if it holds independently for every non-empty first-level offspring. We also can replace the measure $\mathcal{P}^\alpha$ with $\mathcal{P}^g$ where g is a gauge function or with prepacking measure $\mathcal{P}_0^\alpha$, if the number of offspring is finite.

The next theorem states that under a commonly occurring clustering growth rate, the packing measure is almost surely positive. This growth rate condition was studied extensively by Graf, Mauldin and Williams[8] in connection with calculating the exact Hausdorff gauge function.

Theorem 3. If there exist $C > 0$ and $b \in (0,1)$ such that for every $r > 0$ and all $k \in \mathbf{N}$ $Q(\text{card}(\mathcal{G}_r(\eta, w)) = k) \leq Cb^k$, then $P(\mathcal{P}^\alpha(K(w)) > 0 | K(w) \neq \emptyset) = 1$.

Proof: Let $F = \{w | K(w) \neq \emptyset\}$. By Lemma 1 it is enough to prove that $P(\mathcal{P}^\alpha(K(w)) > 0 | F) > 0$. Therefore it is enough to find E such that $P(E \cap F) > 0$ and for all $w \in E \cap F$ there exists $K'(w) \subset K(w)$ such that $P^\alpha(K'(w)) > 0$. For any $x \in K(w)$ we can find an $\eta(x) \in \{1, \ldots, n\}^{\mathbf{N}}$, so that $\{x\} = f_w(\eta)$. Notice that

$$\varliminf_{r\to 0} \frac{\nu_w(B(f_w(\eta), r) \cap K(w))}{r^\alpha} \leq \varliminf_{r\to 0} \frac{\sum\limits_{\substack{\sigma\in\Gamma_r \\ J_\sigma \cap B(f_w(\eta), r) \neq \emptyset}} l_\sigma^\alpha X_\sigma}{r^\alpha} \leq \varliminf_{r\to 0} \frac{\sum\limits_{\sigma\in\mathcal{G}_r(\eta,w)} l_\sigma^\alpha X_\sigma}{r^\alpha} \leq$$

$$\leq \varliminf_{r\to 0} \sum_{\sigma\in\mathcal{G}_r(\eta,w)} X_\sigma.$$

We will show the expected value of this last limit is finite. This in turn allows us to apply the density theorem for packing measures (see Mattila[15], Theorem 6.11). We estimate

$$E_Q\left[\varliminf_{r\to 0} \sum_{\sigma\in\mathcal{G}_r(\eta,w)} X_\sigma\right] \leq \varliminf_{r\to 0} E_Q\left[\sum_{\sigma\in\mathcal{G}_r(\eta,w)} X_\sigma\right] = \varliminf_{r\to 0} \sum_{\Delta} \int_{\{\mathcal{G}_r=\Delta\}} \sum_{\sigma\in\Delta} X_\sigma dQ \leq$$

$$\leq \varliminf_{r\to 0} \sum_{\Delta} \sum_{\sigma\in\Delta} \int_{D\times\Omega} X_\sigma 1\!\!1_{\{\mathcal{G}_r=\Delta\}} dQ.$$

For $p > 1$ and $q > 1$ with $1/p + 1/q = 1$, we have

$$\sum_{\Delta} \sum_{\sigma\in\Delta} \int_{D\times\Omega} X_\sigma 1\!\!1_{\{\mathcal{G}_r=\Delta\}} dQ \leq$$

$$\leq \sum_{\Delta} \sum_{\sigma\in\Delta} E_Q[X_\sigma^p]^{1/p} E_Q\left[1\!\!1^q_{\{\mathcal{G}_r=\Delta\}}\right]^{1/q} = E[X_\emptyset^{p+1}]^{1/p} \sum_{\Delta} Q(\mathcal{G}_r = \Delta)^{1/q} \mathrm{card}(\Delta) \leq$$

$$\leq E[X_\emptyset^{p+1}]^{1/p} \sum_{k=1}^{\infty} k^2 Q(\mathrm{card}(\mathcal{G}_r) = k)^{1/q} \leq E[X_\emptyset^{p+1}]^{1/p} \sum_{k=1}^{\infty} k^2 C^{1/q} b^{k/q} < M < \infty,$$

for some $M \in \mathbf{R}$.

Hence, for all $\delta > 0$ there exist $A_\delta \subset D \times \Omega$ such that $Q(A_\delta) > 1 - \delta$ and $M_\delta \in \mathbf{R}$ such that for all $(\eta, w) \in A_\delta$ $\varliminf_{r\to 0} \nu_w(B(f_w(\eta), r) \cap K(w))/r^\alpha \leq M_\delta$. Since $Q(A_\delta) = \int \mu_w(A_{\delta w}) dP(w)$, we can find a set E_δ such that $P(E_\delta) > 0$, for all $w \in E_\delta$ $\mu_w(A_{\delta w}) > 1 - \delta$ and $\nu_w(K(w)) > 0$. Let $K_\delta(w) = f_w(A_{\delta w})$. Then $K_\delta(w) \subset K(w)$ and as $\delta \searrow 0$, $\nu_w(K_\delta(w))$ goes up to $\nu_w(K(w))$. Thus there is some δ' such that $\nu_w(K_{\delta'}(w)) > 0$. We let $K'(w) = K_{\delta'}(w)$ and $E = E_{\delta'}$. ■

We recall from the monograph of Graf, Mauldin and Williams[8] that a subset J of $\mathbf{R}^m$ is said to satisfy the neighborhood boundedness property provided that there exists an $n_0 \in \mathbf{N}$ such that for every $\varepsilon > \mathrm{diam}(J)$, if $J_1, \ldots, J_k$ are non-overlapping sets which are all similar to J with $\mathrm{diam}(J_i) \geq \varepsilon > \mathrm{dist}(J, J_i); i = 1, \ldots, k$, then $k \leq n_0$. As shown in Graf, Mauldin and Williams[8] , there are several different easily verifiable and commonly occurring conditions on the seed set J under which this condition is satisfied.

Corollary 1. Suppose J satisfies the neighborhood boundedness property and there exist $\kappa > 0$, such that $E[1/\min T_i^\kappa | T_i > 0] < \infty$. Then $P(\mathcal{P}^\alpha(K(w)) > 0 | K(w) \neq \emptyset) = 1$.

Proof: Graf, Mauldin and Williams[8] have shown in lemmas 4.4 and 4.6 that this condition implies that the clustering growth rate condition of theorem 3 is satisfied. ■

Remark. The corollary holds for many known examples, e.g. the zero set of the Brownian bridge, Mandelbrot percolation process, etc.

Next, we turn to the *almost deterministic* setting, i.e., the sum of the random reduction ratios is almost surely 1 and the δ condition holds: if the reduction ratio is nonzero it is greater than δ.

Theorem 4. If $P(T_1^\alpha + \cdots + T_n^\alpha = 1) = 1$ and there is some $\delta > 0$ so that $P\big(T_i \geq \delta | T_i \neq 0\big) = 1$ for all $1 \leq i \leq n$, then $\mathcal{P}^\alpha(K(w)) \leq (2/\delta)^\alpha < \infty$ a.s.

Proof: By definition $\mathcal{P}_0^\alpha(K(w)) = \limsup_{\delta \to 0} \{\sum_i |B_i|^\alpha \mid B(x_i, r_i)$ is a δ-packing of $K(w)\}$. For a δ-packing $B(x_i, r_i)$, consider the set $\Gamma_0(w) = \{\sigma \in \{1, \ldots, n\}^* \mid x_i \in J_\sigma,\ l_\sigma < r_i,\ l_{\sigma|_{|\sigma|-1}} \geq r_i\}$ and extend it to a maximal antichain $\Gamma(w)$. Then $\sum_i |B_i|^\alpha \leq \sum_{\sigma \in \Gamma(w)} (2l_\sigma)^\alpha / \delta^\alpha$. It is known (see Graf[7] , theorem 6.11) that for a maximal antichain Δ, $P\Big(\sum_{\sigma \in \Delta} l_\sigma^\alpha = 1\Big) = 1$. Therefore $P\Big(\sum_{\sigma \in \Gamma(w)} l_\sigma^\alpha = 1\Big) = \sum_{\Delta \subset \{1,\ldots,n\}^*} P\Big(\sum_{\sigma \in \Delta} l_\sigma^\alpha = 1 \text{ and } \Gamma = \Delta\Big) = \sum_{\Delta \subset \{1,\ldots,n\}^*} P(\Gamma = \Delta) = 1$. Hence, $\sum_{\sigma \in \Gamma(w)} (2l_\sigma)^\alpha / \delta^\alpha = (2/\delta)^\alpha$ a.s. ■

As mentioned before it is known that under the hypotheses of theorem 3, $\mathcal{H}^\alpha(K(w)) > 0$. Since α-packing measure dominates α-Hausdorff measure, we have:

Corollary 2. Suppose that $P(\sum_{i=1}^n T_i^\alpha = 1) = 1$ and for some $\delta > 0$ $P(T_i \geq \delta | T_i \neq 0) = 1$. Then $0 < \mathcal{H}^\alpha(K(w)) \leq \mathcal{P}^\alpha(K(w)) < \infty$ a.s. Moreover, $\mathcal{P}^\alpha(K(w))$ and $\nu_w(K(w))$ are absolutely continuous with respect to each other given that $K(w) \neq \emptyset$.

Now, we show that if there is enough randomness in the reduction ratios and a random *strong open set condition* holds then the α-packing measure is infinite.

Definition. The construction satisfies a (random) *strong open set condition* means there are ρ_0, $p_0 > 0$ such that $P(\exists x \in K(w) \cap J_\sigma \text{ and } \operatorname{dist}(x, \partial J_\sigma) \geq \rho_0 l_\sigma \mid J_\sigma \neq \emptyset) \geq p_0$.

Theorem 5. If $P(T_1^\alpha + \cdots + T_n^\alpha = 1) < 1$ and the construction satisfies the random strong open set condition, then $\mathrm{P}(\mathcal{P}^\alpha(K(w)) = \infty | K(w) \neq \emptyset) = 1$.

Proof: First, we deal with the α-prepacking measure. Note that $X = \sum_{i=1}^n T_i^\alpha X_i$ which implies that $\operatorname{ess\,inf} X = 0$ and there exist $\epsilon, \kappa > 0$ such that for all sufficiently large k, $P(S^\alpha_{\{1,\ldots,n\}^k} < 1 - \kappa) > \epsilon$.

According to the strong open set condition there exists $\rho > 0$, perhaps a smaller $\epsilon > 0$ and Z_ϵ such that $P(Z_\epsilon) > 1 - \epsilon/2$ and for all $w \in Z_\epsilon$ there is $\tau \in \{1, \ldots, n\}^*$ such that $\operatorname{dist}(J_\tau, \partial J) > \rho$. Therefore we can find $k_0 \in \mathbf{N}$ such that for all $k \geq k_0$, $P(\{\sum_{|\sigma|=k} T_\sigma^\alpha < 1 - \kappa\} \cap \{\exists\, \tau, |\tau| = k \text{ with } \operatorname{dist}(J_\tau, \partial J) > \rho\}) > 0$ and that event (which will be denoted by A) is in the σ-algebra $\mathcal{E}_k$ generated by the maps $w \to l_\sigma(w)$ where $|\sigma| \le k$.

Now let $h_m = \sup\{S^\alpha_\Gamma | \Gamma$ is an antichain, $\Gamma \ne \{\emptyset\}$, $\forall \sigma \in \Gamma\ |\sigma| \le m$, and $\exists x \in J_\sigma \cap K(w)\colon \mathrm{dist}(x, \partial J_\sigma) \ge \rho l_\sigma\}$. Obviously, $h_{m+1} \ge h_m$. Let $h = \lim_{m\to\infty} h_m$. Then it is easy to see that

$$h_{m+1} = \sum_{i=1}^{n} T_i^\alpha \max(1\!\!1_{\{\exists\, x \in K(w)\cap J_i:\ \mathrm{dist}(x,\partial J_i) > \rho l_i\}}, h^i_m).$$

Therefore

$$h = \sum_{i=1}^{n} T_i^\alpha \max(1\!\!1_{\{\exists\, x \in K(w)\cap J_i:\ \mathrm{dist}(x,\partial J_i) > \rho l_i\}}, h^i),$$

where h^σ is like h but the supremum is taken over antichains whose elements properly extend σ. Taking the expected value of both sides we obtain that $h \ge 1$ a.s. on the set $B = \{w \mid \exists\, x \in K(w) \cap J\colon\ \mathrm{dist}(x, \partial J) \ge \rho\}$ and therefore $h = \sum_{|\sigma|=k} l^\alpha_\sigma h^\sigma$ a.s. Let $\zeta = \operatorname{ess\,inf}_B h \ge 1$. Suppose that $\zeta \in \mathbf{R}$. Then for all $k \ge k_0$ we have $\sum_{|\sigma|=k} l^\alpha_\sigma h^\sigma \ge \zeta$ a.s. Let $C_\sigma = \{w | h^\sigma < \zeta + \kappa\}$. These events are independent of each other and of the σ-algebra $\mathcal{E}_k$. Hence, $P\big(\bigcap_{|\sigma|=k} C_\sigma \cap A\big) > 0$ and with positive probability we get $(\zeta + \kappa)(1 - \kappa) \ge \zeta$ which is a contradiction. This shows that $P(\mathcal{P}^\alpha_0(K(w)) = \infty | K(w) \ne \emptyset) > 0$, and hence equals 1 by the 0–1 law.

Now let $\{E_i\}_{i=1}^\infty$ be an arbitrary cover of $K(w)$ by closed sets such that $E_i \cap K(w) \ne \emptyset$. Since $K(w) \subset \mathbf{R}^\mathbf{d}$ is compact, one of the E_i's must have a non-empty interior, and therefore there exists σ such that $J_\sigma \cap K(w) \subset K(w) \cap E_i$. As it has already been proved, $\mathcal{P}^\alpha_0(K(w) \cap J_\sigma) = \infty$ a.s. on $K(w) \ne \emptyset$. The result now follows from the definition of the packing measure. ■

Question: Does theorem 5 still hold if we only assume the random open set condition?

Now we improve the estimates obtained by considering packing measures with respect to a gauge function $\phi(t) = t^\alpha g(t)$. Let $D = \{1, \dots, n\}^\mathbf{N}$ and for $k \in \mathbf{N}$ define random variables T_k, l_k on $D \times \Omega$ by $T_k(\eta, w) = T_{\eta|_k}(w)$ and $l_k(\eta, w) = l_{\eta|_k}(w)$ respectively. Fix $c > E_Q[|\log T_1|]$ and let $N = e^c$.

Lemma 1. Let $\{Y_i\}_{i\ge1}$ be a sequence of i.i.d. random variables with expectation 0 and finite fourth moment and let $k \in \mathbf{N}$. Then for all $m \in \mathbf{R}$ there exists an $\tilde{M} > 0$ such that $\sum_{j=k+1}^{\infty} Q\big(\sum_{i=1}^{j} Y_i < (k+1-j)c + m\big) < \tilde{M} < \infty$.

The proof of Lemma 1 is straightforward in the spirit of the strong law of large numbers.

In making upper estimates of the Hausdorff measure with respect to a gauge function Graf, Mauldin and Williams[8] used antichains consisting of the cells on the same level in the construction. Here to get upper estimates of the packing measure with respect to a gauge function, we use antichains consisting of the cells of comparable size. Lemma 2 gives an estimate on the number of such cells. One can replace the number 2 in the lemma with any number greater than or equal to 1.

Lemma 2. There is an $M > 0$ such that for all $k \in \mathbf{N}$, $E\big[\operatorname{card}\{\sigma | N^{-k-1} < 2l_\sigma \leq N^{-k}\}\big] \leq MN^{k\alpha}$.

Proof: Fix k and set $A = N^{-k\alpha}E\big[\operatorname{card}\{\sigma | N^{-k-1} < 2l_\sigma \leq N^{-k}\}\big] = N^{-k\alpha}\sum_{j=1}^{\infty}\sum_{|\sigma|=j} E\Big[1\!\!1_{\{N^{-k-1}<2l_\sigma\leq N^{-k}\}}\Big]$. Since X_σ and $1\!\!1_{\{N^{-k-1}<2l_\sigma\leq N^{-k}\}}$ are independent and $E[X_\sigma] = 1$, we have

$$A = \sum_{j=1}^{\infty} N^{-k\alpha}E\Bigg[\sum_{|\sigma|=j} l_\sigma^\alpha X_\sigma l_\sigma^{-\alpha} 1\!\!1_{\{N^{-k-1}<2l_\sigma\leq N^{-k}\}}\Bigg] =$$

$$= \sum_{j=1}^{\infty} N^{-k\alpha}E_Q\Big[l_j^{-\alpha}1\!\!1_{\{N^{-k-1}<2l_j\leq N^{-k}\}}\Big] \leq \sum_{j=1}^{\infty}(2N)^\alpha E_Q\Big[1\!\!1_{\{N^{-k-1}<2l_j\leq N^{-k}\}}\Big] =$$

$$= \sum_{j=1}^{\infty}(2N)^\alpha Q(kc \leq |\log l_j| - \log 2 < (k+1)c) \leq (2N)^\alpha \sum_{j=1}^{k} Q\big(|\log l_j| \geq kc\big) +$$

$$+(2N)^\alpha \sum_{j=k+1}^{\infty} Q\Bigg(\sum_{i=1}^{j} |\log T_i| < (k+1)c + \log 2\Bigg) \leq (2N)^\alpha(kb_1c_2^k + \tilde{M}),$$

where the first sum is estimated using inequality (3.30) from the article of Graf, Mauldin and Williams[8] and the second sum is estimated by Lemma 1 with $Y_i = |\log T_i| - c$ and $m = \log 2$. The numbers $b_1 > 0$ and $c_2 \in (0,1)$ depend only on the construction. The result follows. ■

In the proofs and statements of the next two lemmas, we suppose a number M has been fixed such that lemma 2 holds.

Lemma 3. Let $\lambda > 0$ and set $M_k(w) = \operatorname{card}\{\sigma | N^{-k-1} < 2l_{\sigma|_{|\sigma|-1}} \leq N^{-k}, l_\sigma^\alpha X_\sigma < \lambda\phi(2l_{\sigma|_{|\sigma|-1}})\}$. Then $E[M_k] \leq nMN^{k\alpha}P\big(X < n(2N)^\alpha \lambda g(N^{-k})\big)$.

Proof: Since the gauge function $\phi(t)$ is increasing, we obtain

$$E[M_k] = E\Bigg[\sum_\sigma 1\!\!1_{\{l_\sigma^\alpha X_\sigma<\lambda\phi(2l_{\sigma|_{|\sigma|-1}})\}\cap\{N^{-k-1}<2l_{\sigma|_{|\sigma|-1}}\leq N^{-k}\}}\Bigg] \leq$$

$$\leq E\Big[\sum_\sigma 1\!\!1_{\{T_\sigma^\alpha X_\sigma<(2N)^\alpha\lambda g(N^{-k})\}} 1\!\!1_{\{N^{-k-1}<2l_{\sigma|_{|\sigma|-1}}\leq N^{-k}\}}\Big] =$$

$$= \sum_\tau E\Big[1\!\!1_{\{N^{-k-1}<2l_\tau\leq N^{-k}\}} \sum_{i=1}^n 1\!\!1_{\{T_{\tau*i}^\alpha X_{\tau*i}<(2N)^\alpha\lambda g(N^{-k})\}}\Big].$$

The value of the last sum does not exceed n, and it may differ from 0 only on the set $\{w|X_\tau(w) < n(2N)^\alpha \lambda g(N^{-k})\}$. We can continue as follows:

$$E[M_k] \leq n \sum_\tau E\Big[1\!\!1_{\{N^{-k-1}<2l_\tau\leq N^{-k}\}} 1\!\!1_{\{X_\tau<n(2N)^\alpha\lambda g(N^{-k})\}}\Big] \leq$$

$$\leq nP\big(X < n(2N)^\alpha\lambda g(N^{-k})\big)E[\text{card}\{\tau|N^{-k-1} < 2l_\tau \leq N^{-k}\}] \leq$$

$$\leq nMN^{k\alpha}P\big(X < n(2N)^\alpha\lambda g(N^{-k})\big). \blacksquare$$

The next lemma tells us how often the size of a cell can drop significantly from level to level.

Lemma 4. For $k_0, k \in \mathbf{N}$ let $M_k^{(k_0)} = \text{card}\{\sigma|N^{-k-1} < 2l_{\sigma|_{|\sigma|-1}} \leq N^{-k}, 0 < l_\sigma < N^{-k_0}\}$. Then for any $\zeta > 0$, $E\big[M_k^{(k_0)}\big] \leq E\big[1/\min_{1\leq i\leq n} T_i^\zeta|T_i > 0\big]2^\zeta nMN^{k\alpha}N^{\zeta(k+1-k_0)}$.

Proof:

$$E\Big[M_k^{(k_0)}\Big] \leq E\Big[\sum_\sigma 1\!\!1_{\{0<T_\sigma<2N^{k+1-k_0}\}} 1\!\!1_{\{N^{-k-1}<2l_{\sigma|_{|\sigma|-1}}\leq N^{-k}\}}\Big] =$$

$$= \sum_\sigma P\big(0 < T_\sigma < 2N^{k+1-k_0}\big)E\Big[1\!\!1_{\{N^{-k-1}<2l_{\sigma|_{|\sigma|-1}}\leq N^{-k}\}}\Big] =$$

$$= \sum_\tau E\Big[1\!\!1_{\{N^{-k-1}<2l_\tau\leq N^{-k}\}}\Big] \sum_{i=1}^n P\big(0 < T_{\tau*i} < 2N^{k+1-k_0}\big) \leq$$

$$\leq nP\Big(\min_{1\leq i\leq n} T_i < 2N^{k+1-k_0}|T_i > 0\Big)E[\text{card}\{\tau|N^{-k-1} < 2l_\tau \leq N^{-k}\}],$$

using lemma 2 and Chebyshev's inequality,

$$\leq E[1/\min_{1\leq i\leq n} T_i^\zeta|T_i > 0]2^\zeta nMN^{k\alpha}N^{\zeta(k+1-k_0)}. \blacksquare$$

Theorem 6 (upper bound). Suppose that there exists a $\zeta > 0$, such that $E\Big[1/\min_{1\leq i\leq n} T_i^\zeta|T_i > 0\Big] < \infty$. Then

1. If $P(X < a) \leq Ca^\beta$ as $a \to 0$ and $\phi(t) = t^\alpha g(t)$ is an arbitrary gauge function, then $\int_{0^+} \frac{g^{\beta+1}(s)}{s}ds < +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = 0|K(w) \neq \emptyset) = 1$.

2. If $0 < r = \liminf_{a\to 0} -a^{-1/\beta}\log P(X < a) < \infty$, then for $\phi(t) = t^\alpha|\log|\log t||^\beta = t^\alpha g(t)$, $P(\mathcal{P}^\phi(K(w)) < \infty | K(w) \neq \emptyset) = 1$.

Proof: Fix an arbitrary $\lambda > 0$. For $\delta > 0$, choose $k_0 \in \mathbf{N}$ such that $N^{-k_0} \geq \delta > N^{-k_0-1}$. Consider a (random) δ-packing of $K(w)$ consisting of balls $B_i(x_i, r_i)$. Build an antichain $\Gamma = \{\sigma | J_\sigma(w) \ni x_i$ for some i and $l_\sigma(w) < r_i, l_{\sigma|_{|\sigma|-1}}(w) \geq r_i\}$. Then $\sum \phi(|B_i|) \leq \sum_{\sigma\in\Gamma} \phi(2l_{\sigma|_{|\sigma|-1}})$ and certainly we have

$$P^\phi_{0,\delta}(K(w)) \leq \sup\Big\{\sum_{\sigma\in\Gamma}\phi(2l_{\sigma|_{|\sigma|-1}})|\Gamma \text{ is an antichain,} \forall\sigma\in\Gamma\ 0 < l_\sigma < N^{-k_0}\Big\}.$$

For such an antichain Γ, let $\Gamma_1 = \{\sigma \in \Gamma | l_\sigma^\alpha X_\sigma \geq \lambda\phi(2l_{\sigma|_{|\sigma|-1}})\}$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. Then $\sum_{\sigma\in\Gamma_1}\phi(2l_{\sigma|_{|\sigma|-1}}) \leq \lambda^{-1}\sum_{\sigma\in\Gamma_1} l_\sigma^\alpha X_\sigma \leq \lambda^{-1}X$ and using the terminology from lemma 3, $\sum_{\sigma\in\Gamma_2}\phi(2l_{\sigma|_{|\sigma|-1}}) \leq \sum_{k=1}^{[\log k_0]} M_k^{(k_0)}(w)\phi(N^{-k}) + \sum_{k=[\log k_0]}^{\infty} M_k(w)\phi(N^{-k})$. Thus $\mathcal{P}^\phi_{0,\delta}(K(w)) \leq \lambda^{-1}X(w) + \sum_{k=1}^{[\log k_0]} M_k^{(k_0)}(w)\phi(N^{-k}) + \sum_{k\geq[\log k_0]} M_k(w)\phi(N^{-k})$.

Since $P^\phi_{0,\delta}(K(w))$ decreases as $\delta \searrow 0$, we obtain by lemmas 3 and 4 that

$$E[P^\phi_0(K(w))] = E[\liminf_{\delta\to 0} P^\phi_{0,\delta}(K(w))] \leq \liminf_{\delta\to 0} E[P^\phi_{0,\delta}(K(w))] \leq$$

$$\leq \liminf_{k_0\to\infty} E\Big[\lambda^{-1}X + \sum_{k=1}^{[\log k_0]} M_k^{k_0}(w)\phi(N^{-k}) + \sum_{k\geq[\log k_0]} M_k(w)\phi(N^{-k})\Big] \leq$$

$$\leq \liminf_{k_0\to\infty}\Big[\lambda^{-1} + nME\Big[1/\min_{1\leq i\leq n} T_i^\zeta | T_i > 0\Big]\sum_{k=1}^{[\log k_0]} N^{\zeta(k+1-k_0)}g(N^{-k}) +$$

$$+ nM\sum_{k\geq[\log k_0]} g(N^{-k})P\big(X < n(2N)^\alpha\lambda g(N^{-k})\big)\Big].$$

In case 1 we observe that if $\int_{0+} \frac{g^{\beta+1}(s)}{s}ds < +\infty$, then $\sum_{k=1}^{\infty} g^{\beta+1}(N^{-k}) < +\infty$ and the set $\{g(N^{-k})\}_{k=1}^\infty$ is bounded. Hence, for all $\lambda > 0$ $E[P^\phi_0(K(w))] \leq \lambda^{-1}$. Thus $P^\phi_0(K(w)) = 0$ a. s., and therefore $P(\mathcal{P}^\phi(K(w)) = 0 | K(w) \neq \emptyset) = 1$.

In case 2 let $0 < t < r$. Then by the definition of r, there is some $C_t > 0$ such that for all $a > 0$, $P(X < a) \leq C_t e^{-ta^{1/\beta}}$ so that $\beta \leq 0$, the set $\{g(N^{-k})\}_{k=1}^\infty$ is also bounded and therefore the limit of the sum over first $[\log k_0]$ terms is 0. The tail sum $\leq \sum_{k\geq[\log k_0]} P(X <$

$\lambda(2N)^\alpha(\log k)^\beta) \le C_t \sum_{k \ge [\log k_0]} (\log k)^\beta k^{-t(\lambda(2N)^\alpha)^{1/\beta}}$. If $t(\lambda(2N)^\alpha)^{1/\beta} > 1$, this is the tail of a convergent series. Hence, $P(\mathcal{P}^\phi(K(w)) < \infty | K(w) \ne \emptyset) = 1$. ■

Based on the articles of Xiao(23), Liu(13) and examples that follow we conjecture that there is a corresponding lower bound result:

Conjecture (lower bound). In the setting of theorem 6, we have

1. If $P(X < a) \ge Ca^\beta$ as $a \to 0$ and $\phi(t) = t^\alpha g(t)$ is an arbitrary gauge function, then $\int_{0+} \frac{g^{\beta+1}(s)}{s} ds = +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = +\infty | K(w) \ne \emptyset) = 1$.
2. If $0 < r = \liminf_{a \to 0} -a^{-1/\beta} \log P(X < a) < \infty$, then for $\phi(t) = t^\alpha |\log|\log t||^\beta = t^\alpha g(t)$, $P(\mathcal{P}^\phi(K(w)) > 0 | K(w) \ne \emptyset) = 1$.

4. Examples and Applications.

Example 1. Mandelbrot percolation or canonical curdling.

Mandelbrot introduced the following process which he termed canonical curdling. Fix an integer $n > 1$ and a number p with $0 < p < 1$. Partition the unit square into n^2 congruent subsquares. Let each subsquare survive independently with probability p. For each subsquare which survives repeat the process. This is a n^2-ary random recursive construction. The limit set is nonempty with positive probability provided $p > 1/n^2$. The Hausdorff dimension in this case is $\alpha = 2 + (\log p / \log n)$. The exact Hausdorff gauge function is $t^\alpha (|\log|\log t||)^{1-(\alpha/2)}$ as determined by Graf, Mauldin and Williams(8) in Example 6.2. By theorem 1, the limit set of the Mandelbrot percolation process provided it is nonempty also has packing dimension $\alpha = 2 + \log p / \log n$ a.s. This also follows from the results of Gatzouras and Lalley(6). By theorem 5 its α-packing measure is infinite. It is known (see Bingham(1)) that $P(X < a) \asymp a^\beta$ as $a \to 0$ where β satisfies $p_1 m^\beta = 1$, $p_1 = P(\exists! i{:}\, T_i \ne 0) = n^2 p(1-p)^{n^2-1}$ and $m = n^2 p$ is the expected number of offspring. In this case, $\beta = -1 - \frac{\log(1-p)^{n^2-1}}{\log n^2 p}$. Hence from theorem 6, part 1 we deduce that for the gauge function $\phi(t) = t^\alpha g(t)$ such that $\int_{0+} \frac{g(s)^{\beta+1} ds}{s} < +\infty$, $\mathcal{P}^\phi(K(w)) = 0$. We conjecture that as in the article of Taylor(21) there is no exact packing measure function.

Example 2. The zero set of the Brownian bridge.

Graf, Mauldin and Williams(8) have shown that this set can be represented as a random recursive construction and the distribution density of the vector (T_1, T_2) has been found. The Hausdorff dimension of this set is known to be 1/2. Therefore its packing and box-counting dimensions are 1/2, and packing measure in dimension 1/2 is infinite.

Using the distribution density it is easy to show that $P(T_{1,2} < a) = O(\sqrt{a}), a \to 0$. By a result of Liu[14], we obtain that $P(X < a) = O(a), a \to 0$. Graf, Mauldin and Williams[8] in example 6.1 show that the condition of theorem 6 is satisfied, therefore $\int_{0+} \frac{g^2(s)}{s} ds < +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = 0 | K(w) \neq \emptyset) = 1$, and our hypothesis would say that $\int_{0+} \frac{g^2(s)}{s} ds = +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = +\infty | K(w) \neq \emptyset) = 1$. This is actually proven by Feng and Sha[3] from the view point of subordinators.

Example 3. A random Cantor set.

Choose two numbers independently with respect to the uniform distribution on $J_\emptyset = [0, 1]$. J_1 is the left most interval and J_2 is the right most interval in the partition of $J_\emptyset$ thus obtained. Its Hausdorff dimension α has been found to be $(\sqrt{17} - 3)/2$, and the exact Hausdorff dimension function is $t^\alpha |\log|\log t||^{1-\alpha}$ (see Graf, Mauldin and Williams[8], Mauldin and Williams[18]). By theorem 1, it has the same packing and box-counting dimensions and by theorem 5, its packing measure in dimension $(\sqrt{17} - 3)/2$ is infinite.

One can calculate $P(T_1 < a) = P(T_2 < a) = 2a - a^2$. Hence, $P(T_{1,2} < a) = O(a), a \to 0$ and again according to Liu[14] $P(X < a) = O(a^2), a \to 0$. Therefore by theorem 6, $\int_{0+} \frac{g^3(s)}{s} ds < +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = 0) = 1$ and our hypothesis would say that $\int_{0+} \frac{g^3(s)}{s} ds = +\infty$ implies $P(\mathcal{P}^\phi(K(w)) = +\infty) = 1$.

Example 4. Modified Mandelbrot percolation or modified curdling.

This process was proposed by Dekking and Grimmett. It was discussed in detail by Graf, Mauldin and Williams[8] in example 6.12 and they found the exact Hausdorff gauge function for this construction. Fix a positive integer $n > 1$ and a probability measure μ on the power set of $\{1, ..., n^2\}$. Let $J_1, ..., J_{n^2}$ be a labelling of the partition of $[0, 1] \times [0, 1]$ into congruent subsquares. If the square J_σ has been constructed, then choose $A \subset \{1, ..., n^2\}$ according to μ and let $J_{\sigma * i}, i \in A$ be the subsquares of J_σ obtained by scaling J_i into J_σ via the natural map. If m is the average number of offspring, then $\alpha = \log mp / \log n$. If $\underline{m}$, the essential infimum of the number of offspring, is at least 2, then according to Liu[13], (2.3a) the second case in theorem 6 holds, and for $\beta = 1 - \log m / \log \underline{m}$, the gauge function $\phi(t) = t^\alpha |\log|\log t||^\beta$, we have $\mathcal{P}^\phi(K(w)) < \infty$ a.s. We conjecture that it is positive a.s.

5. Connection between random constructions and Galton-Watson trees.

As mentioned before, there is a connection between Galton-Watson tree processes and random recursive constructions. Let N_σ, $\sigma \in \{1,\ldots,n\}^*$ be a sequence of i.i.d. random variables with non-negative integer values. The Galton-Watson tree T corresponding to this sequence is a subset of $\{1,\ldots,n\}^*$ such that $\emptyset \in T$ and $\sigma \in T \iff \sigma * i \in T$ for all $1 \le i \le N_\sigma$. The boundary, ∂T, of the random tree is the set of all infinite paths through the tree. The tree metric on ∂T is defined by setting for $\sigma, \tau \in \partial T$, $d_T(\sigma,\tau) = c^{|\sigma\wedge\tau|}$ when $\sigma \neq \tau$ and $d_T(\sigma,\tau) = 0$ if $\sigma = \tau$, where $c \in (0,1)$ and $\sigma \wedge \tau$ denotes the largest common subsequence of σ and τ. Liu[12,13] has studied the dimension properties of ∂T with respect to the tree metric.

Suppose the random recursive construction satisfies $P(T_i = c | T_i \neq 0) = 1$. To simplify the matter relabel the cells on each level so that non-empty ones go first. Then a random map $\kappa_w : \partial T(w) \to K(w)$ can be considered, defined by $\sigma \mapsto \bigcap_k J_{\sigma|_k}$. If for some $\rho > 0$ for all σ and $i \neq j$ $P(\mathrm{dist}(J_{\sigma*i}, J_{\sigma*j}) \ge \rho \mathrm{diam}(J_\sigma)) = 1$, then κ_w is 1–1. The question arises as to the relationship between the tree metric on the limit set and the usual Euclidean metric from $\mathbf{R}^d$.

Proposition. If for all σ $P(\exists x_\sigma \in J_\sigma \cap K(w) \colon \mathrm{dist}(x_\sigma, \partial J_\sigma) \ge \rho \mathrm{diam}(J_\sigma)) = 1$, then these two metrics are bi-Lipschitz equivalent.

Proof: For all $x, y \in K(w)$ we obviously have $d(x,y) \le d_T(x,y)$. On the other hand, if there is a point inside each J_σ as in the condition of the proposition, then $d(x,y) \ge c\rho d_T(x,y)$. ■

We note that for the proofs of theorems about the packing and Hausdorff measures it suffices to have the second condition satisfied and $P(X = 1) < 1$, then they are valid (or invalid) for trees and random recursive constructions of this kind simultaneously (see Graf, Mauldin and Williams[8] , Liu[12,13]).

Liu[13] on pages 25–26 attempts to show that under certain conditions there exists the exact packing dimension for the branching process on a Galton-Watson tree when the number of offspring is at least 2. However the proof that the packing measure with respect to the gauge function is positive contains a mistake. We consider a gauge function $\phi(t) = t^\alpha g(t)$. By theorem 5 it is natural to assume that $\lim_{t\to 0^+} g(t) = 0$, otherwise the corresponding packing measure will be infinite.

For a natural number $k > 3$ and $K > 0$ one constructs an antichain $\Gamma(w) = \Gamma_k(w) = \{\sigma||\sigma| = k \text{ and for all } [\log k] \leq j \leq k-1\ l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > K\phi(l_{\sigma|_j})\} \cup \{\sigma|[\log k] \leq |\sigma| \leq k-1, l^\alpha_\sigma X_{\sigma^*} \leq K\phi(l_\sigma) \text{ and for all } [\log k] \leq j \leq |\sigma|-1\ l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > K\phi(l_{\sigma|_j})\}$ where for $\sigma = (\sigma_1, \ldots, \sigma_{l-1}, \sigma_l)$, σ^* is the cyclic permutation of σ, given by

$$\sigma^* = \begin{cases} (\sigma_1, \ldots, \sigma_{l-1}, \sigma_l + 1), & \text{if } \sigma_l < n \\ (\sigma_1, \ldots, \sigma_{l-1}, 1), & \text{if } \sigma_l = n \end{cases}.$$

It is claimed that $E\left[\sum_{\sigma \in \Gamma(w)} l^\alpha_\sigma X_{\sigma^*}\right] = E\left[\sum_{\sigma \in \Gamma(w)} l^\alpha_\sigma\right] = 1$. But because the choice of $\Gamma(w)$ depends on X_{σ^*}, we have the following

Theorem 7. For large k and $\Gamma(w) = \Gamma_k(w)$, $E\left[\sum_{\sigma \in \Gamma(w)} l^\alpha_\sigma X_{\sigma^*}\right] < E\left[\sum_{\sigma \in \Gamma(w)} l^\alpha_\sigma\right] = 1$.

Moreover, in the proof of proposition 4.1 in the paper of Liu(13) $\varliminf_{k\to\infty} E\left[\sum_{\sigma \in \Gamma_k(w)} l^\alpha_\sigma X_{\sigma^*}\right] = 0$.

Proof: This can be seen as follows:

$$E\left[\sum_{\sigma \in \Gamma(w)} l^\alpha_\sigma X_{\sigma^*}\right] = \sum_{|\sigma|=k} E\left[l^\alpha_\sigma X_{\sigma^*} \prod_{j=[\log k]}^{k-1} 1\!\!1_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\right] +$$

$$\sum_{l=[\log k]}^{k-1} \sum_{|\sigma|=l} E\left[l^\alpha_\sigma X_{\sigma^*} 1\!\!1_{\{l^\alpha_\sigma X_{\sigma^*} \leq \phi(l_\sigma)\}} \prod_{j=[\log k]}^{l-1} 1\!\!1_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\right].$$

So, for $[\log k] \leq l \leq k-1$ we let $r_l = \sum_{|\sigma|=l} E\left[X_{\sigma^*} 1\!\!1_{\{l^\alpha_\sigma X_{\sigma^*} \leq \phi(l_\sigma)\}} \prod_{j=[\log k]}^{l-1} 1\!\!1_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\right]$,

$q_l = \sum_{|\sigma|=l} E\left[1\!\!1_{\{l^\alpha_\sigma X_{\sigma^*} \leq \phi(l_\sigma)\}} \prod_{j=[\log k]}^{l-1} 1\!\!1_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\right]$,

$r_k = \sum_{|\sigma|=k} E\left[X_{\sigma^*} \prod_{j=[\log k]}^{k-1} 1\!\!1_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\right]$, $q_k = \sum_{|\sigma|=k} E\left[\prod_{j=[\log k]}^{k-1} 1\!\!1_{\{l^\alpha_{\sigma|_j} X_{\sigma|^*_j} > \phi(l_{\sigma|_j})\}}\right]$.

Then the left-hand side of the inequality becomes $\sum_{l=[\log k]}^{k} r_l$, and the right-hand side is $\sum_{l=[\log k]}^{k} q_l$. By independence $q_k = r_k$. On the other hand for $[\log k] \leq l \leq k-1$ we have $r_l/q_l \leq \sup_{|\tau|=l} g(l_\tau) \to 0$ as $k \to \infty$. This gives $\lim_{k\to\infty} \sum_{l=[\log k]}^{k-1} r_l = 0$. Lines (4.3a)–(4.4), proposition 4.1 in Liu's paper yield $\liminf_{k\to\infty} r_k = 0$. The result follows. ■

Therefore it remains unknown, if there is a gauge function in the exponential case (case 2 of theorem 6) that gives a.s. positive packing measure.

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