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HOMEOMORPHISMS OF THE PLANE

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This paper is concerned with homeomorphisms of Euclidean spaces onto themselves, with bounded orbits. The following results are obtained. (1) A homeomorphism of E^2 onto itself has both bounded orbits and an equicontinuous family of iterates iff it is a conjugate of either a rotation or a reflection; (2) An example of Bing is modified to produce a fixed point free, orientation preserving homeomorphism of E^3 onto itself, such that orbits of bounded sets are bounded; and (3) There is no homeomorphism of E^2 onto itself such that the orbit of every point is dense.

1. **Introduction.** One motivation for this paper is the well-known bounded orbit problem, "Does a homeomorphism T of E^2 onto itself, with bounded orbits, necessarily have a fixed point?" This is discussed in detail in § 2. In our investigations we were led to a study of homeomorphisms which have bounded orbits and an equicontinuous family of iterates, and we obtained a characterization of such homeomorphisms in Theorem 4. This theorem was proved earlier by Kerékjártó [13], using different methods. Our proof of this uses ε-sequential growths and is similar to the proof of the main theorem of [8].

In § 4, we study homeomorphisms with dense orbits.

2. **The bounded orbit problem.** As far as we know, this problem remains unsolved: Is there a homeomorphism T of the plane onto itself such that the orbit of each point is bounded, and which does not have a fixed point? The answer is "no" if T is orientation-preserving, and this is proved in [1, Proposition 1.2].

We wish to make the following observations:

(1) It follows from the methods of this paper that if there is a fixed point free homeomorphism T of the plane such that the orbits of bounded sets are bounded, then there is a compact continuum M in E^2, which does not separate the plane and which is invariant under T.

(2) If the orbits of points under T are bounded and closed, then T is periodic. This follows from [15].

(3) If T is orientation-reversing with bounded orbits, then T^2 is orientation-preserving with bounded orbits and thus T^2 has a fixed point. However, this does not necessarily imply that T has a fixed point. In [12], Johnson has given an example of a homeo-

morphism T of E^2 onto itself such that T is fixed point free, while T^n has a fixed point for all $n > 1$.

(4) There are homeomorphisms of the plane such that the orbits of points are bounded, but the orbits of bounded sets are not necessarily bounded. We modify the example in § 6 of [14] to show this.

Let B be the unit disk in E^2, let D be a disk in S^2 tangent to the north pole, and let $g\colon D \twoheadrightarrow B$ be a homeomorphism such that $g^{-1}(A)$ doesn't contain the north pole, where $A = \{(r, \theta) \mid \theta = 0$ and $0 \leqq r \leqq 1\}$. Let $f\colon B \twoheadrightarrow B$ be the homeomorphism defined by $f(r, \theta) = (r, \theta + 1 - r)$, where (r, θ) is in polar coordinates, and let $\varphi\colon E^2 \to S^2$ be the stereographic projection. Then $h\colon S^2 \twoheadrightarrow S^2$ defined by

$$h(x) = \begin{cases} g^{-1}fg(x) \ , & \text{if } x \in D \\ x \ , & \text{if } x \in S^2 - D \end{cases}$$

is a homeomorphism of S^2 which keeps $\mathrm{Bd}\, D$ fixed. Now the interior of A, A^0, is an open arc in B and $\varphi^{-1}g^{-1}(A^0)$ is bounded in E^2, but the orbit of this set under $\varphi^{-1}h\varphi$ is unbounded in E^2.

(5) The bounded orbit problem is a problem strictly for the plane, since the example given by Bing on page 61 of [7] may be modified to give a homeomorphism h of E^3 onto itself such that the orbits of bounded sets are bounded and yet h has no fixed point. We explain this modification in the theorem below. At this point, we wish to thank Howard Cook for pointing out to us that Bing's example could be modified.

THEOREM 2.1. *There exists a fixed point free, orientation preserving homeomorphism h of E^3 onto itself, such that the orbits of bounded sets are bounded.*

Proof. We first give a description of a subset of E^3 which does not have the fixed point property.

Let S be the surface consisting of the circle of radius 1 and center $(0, 0, 1)$ in E^3, together with the surface given by the parametric equation:

$$R(\tau, \theta) = \Big(\frac{\tau}{1+\tau}\cos\frac{\pi}{2}\tau + \frac{1}{1+2\tau}\cos\theta, \\ \frac{\tau}{1+\tau}\sin\frac{\pi}{2}\tau + \frac{1}{1+2\tau}\sin\theta, \frac{\tau}{1+\tau}\Big)$$

for $0 \leqq \tau$ and θ in E^1. Notice that for each τ, $\tau \geqq 0$, the intersection of the surface with the plane $z = \tau/(1+\tau)$ is a circle with center $(\tau/(1+\tau)\cos(\pi/2)\tau,\ \tau/(1+\tau)\sin(\pi/2)\tau,\ \tau/(1+\tau))$ and radius

$1/(1+2\tau)$. This is homeomorphic to half of Bing's example given on page 61 of [7]; that is, one cone with its narrow end spiraling toward a limit circle, together with this limit circle. The set of centers can be considered the "guiding spiral" of the example.

Consider the map $\bar{h}$ of S onto itself defined by: $\bar{h}(R(\tau, \theta)) = R(\varphi(\tau), \theta + \pi/2)$ for $0 \leqq \tau$, where $\varphi(\tau) = \tau + \tau/(1+\tau)$ and such that $\bar{h}$ is a rotation of 90° on the limit circle. We choose $\varphi(\tau)$ in this manner to insure that $\bar{h}$ is continuous on the limit circle. It can be seen that $\bar{h}$ has no fixed point and $\bar{h}$ is a homeomorphism of S onto itself.

Now let M be the surface S, together with the unit disk in the xy-plane and the bounded complementary domain of this surface. We first describe a homeomorphism $\hat{h}$ of M into itself which is an extension of $\bar{h}$ such that $\hat{h}$ does not have a fixed point. Fix $\tau \geqq 0$. Let us define $\hat{h}$ on the disk at height $\tau/(1+\tau)$ which is the intersection of M and the plane $z = \tau/(1+\tau)$. The circle having parametric equation

$$P(\theta) = \Big(\frac{\tau}{1+\tau}\cos\frac{\pi}{2}\tau + a\Big(\frac{1}{1+2\tau}\Big)\cos\theta,$$
$$\frac{\tau}{1+\tau}\sin\frac{\pi}{2}\tau + a\Big(\frac{1}{1+2\tau}\Big)\sin\theta, \frac{\tau}{1+\tau}\Big), \quad 0 \leqq a < 1$$

goes onto the circle with center on the guiding spiral at height $\tau'/(1+\tau')$ where $\tau' = \varphi(\tau) + (1-a)/(1+\tau)$, and radius $r = a/(1+2\tau')$. It is also rotation 90°. Thus the image of the disk D at height $\tau/(1+\tau)$ is a twisted cone having as base the circle on S at height $\varphi(\tau)/(1+\varphi(\tau))$ and vertex on the guiding spiral at height $z = \tau'/(1+\tau')$, where $\tau' = \varphi(\tau) + 1/(1+\tau)$. It can be seen that $\hat{h}$ is a homeomorphism of M into itself and $\hat{h}$ has no fixed points.

We next extend $\hat{h}$ to a homeomorphism h of E^3 onto itself such that h has no fixed points and the orbits of bounded sets are bounded under the action of h. We define h on the slab $n \leqq z < n+1$, for all integers n, to be a copy of the action of h on $0 \leqq z < 1$. Thus it is sufficient to define h on $0 \leqq z < 1$.

We describe h as follows. If (x, y, z) is a point, $x^2 + y^2 \geqq 1$, and $z = \tau/(1+\tau)$, $\tau \geqq 0$, then $h(x, y, z) = (-y, x, (\varphi(\tau))/(1+\varphi(\tau)))$. To complete the description of h inside the cylinder $x^2 + y^2 \leqq 1$ and $0 \leqq z < 1$, we first construct a twisted cone S', having base the circle $x^2 + y^2 = 1$ and $z = 1$. This cone will twist down and have the circle $x^2 + y^2 = 1$ and $z = 0$ as its limit. Thus it looks similar to the twisted surface S already constructed, except that it is inverted. $S \cup S'$ is very much like the illustration on page 61 of [7]. However, for the construction here, at level z, $0 < z < 1$, instead of having two tangent circles, the circles shall not meet. Further, neither of these circles touches the boundary of the cylinder $x^2 + y^2 = 1$.

Now extend $\hat{h}$ to S' by letting $\hat{h}$ take the circle on S' at height $\tau/(1 + \tau)$ to the circle on S' at height $(\varphi(\tau))/(1 + \varphi(\tau))$. Next extend $\hat{h}$ to M', the (solid) interior of S', by pushing the interior of M' up, taking horizontal disks to twisted cones above them, as before. A portion of the interior of M' will move into the slab between $z = 1$ and $z = 2$; in fact onto the interior of the twisted cone which is the image of the unit disk at height $z = 1$.

Now for each $z = \tau/(1 + \tau)$, we have defined h on two disjoint disks at that height, both lying in the interior of the unit disk at that height, as well as on the points on or outside the unit circle at the height. It is readily seen that, for each z, h can be extended to the remainder of the plane at height z, in such a way that h is a homeomorphism of E^3 onto itself.

Clearly, h is fixed point free, and the orbits of bounded sets are bounded.

3. The main theorem. For the remainder of this section we assume that T is a homeomorphism of the plane E^2 onto itself, with an equicontinuous family $\{T^n\}_{n=-\infty}^{\infty}$ of iterates, and such that for some point x_0, $0(x_0)$ is bounded. We observe that the proofs of Theorems 1 and 2 work for E^n as well as E^2. We will use the notation $O(H)$ to mean the orbit of the set H.

THEOREM 1. *Orbits of bounded sets are bounded.*

Proof. We first show that orbits of points are bounded. Let $B = \{x \mid O(x) \text{ is bounded}\}$. It follows from pointwise equicontinuity of the family $\{T_n\}$ that B is both open and closed. Thus $B = E^2$.

Now suppose K is bounded. We show that the orbit of the closure of K is bounded.

It this isn't so, then there is a sequence $\{p_n\}_{n=1}^{\infty}$ from $\bar{K}$ converging to a point p of $\bar{K}$ such that for each n, the orbit of p_n is not a subset of the ball of radius n and center the origin.

Let δ be a positive number such that for each n, the image of the δ-neighborhood of p under T^n has diameter less than 1.

Since the orbit of p is bounded, there is a positive integer k such that $O(p) \subseteqq S_k$. It follows that if p_n is within δ of p, then $O(p_n) \subseteqq S_{k+1}$. This is a contradiction.

THEOREM 2. *There exists a continuum K such that $T(K) = K$.*

Proof. For each n, let $F_n = \{p \mid O(p) \subseteqq S_n\}$, where S_n is the ball centered at the origin and of radius n. It follows from the Baire category theorem that for some n, F_n contains an open set. Let $U = \operatorname{Int} F_n$. Then $U \subseteqq S_n$ and $T(U) = U$. Thus, since orbits of bounded sets are bounded, $K = \overline{O(S_n)}$ is an invariant, compact continuum.

THEOREM 3. *Given an invariant continuum K, there exists a disk D such that $D \supseteqq K$ and $T(D) = D$.*

Proof. By Theorem 2, there exists an invariant continuum K. We proceed as in the proof of Theorem 3.1 of [8]. Let $\varepsilon > 0$ and let $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers such that $\sum \varepsilon_i < \varepsilon$. It follows from the equicontinuity of $\{T^n\}$ and the compactness of K, that $\exists\, \delta_1 > 0 \ni$ if diam $H < \delta_1$ and $H \cap K \neq \emptyset$ then diam $T^n(H) < \varepsilon_1$ for all n. Let $\mathscr{U}_1$: $U_{1,1}, U_{1,2}, \cdots, U_{1,n_1}$ be a finite δ_1-cover of K, and let $D_1 = \bigcup_n T^n(\bigcup_{i=1}^{n_1} U_{1,i})$. Then D_1 is invariant by definition, and bounded since it lies in an ε_1-neighborhood of K. It is easy to see that D_1 is an ε_1-growth of K.

Now $\bar{D}_1$ is an invariant continuum, so for ε_2 there is $\delta_2 > 0$ such that diam $T^n(\delta_2\text{-set}) < \varepsilon_2$. We choose a finite cover of $\bar{D}_1$ by open sets of diam $< \delta_2$; $\mathscr{U}_2$: $U_{2,1}, U_{2,2}, \cdots, U_{2,n_2}$. Let $D_2 = \bigcup_n T^n(\bigcup_{i=1}^{n_2} U_{2,i})$. Then D_2 is bounded and invariant, and $\bar{D}_2$ is an invariant continuum.

Continue the process inductively, and let $E' = \bigcup_{i=1}^{\infty} D_i$. Now $\bar{E}'$ is a locally connected continuum by Proposition 2.4 of [8], and is invariant. Further, as in [8], $\bar{E}'$ has no cut points.

Thus it follows from Theorem 9 of [16], that the boundary of each of its complementary domains is a simple closed curve. Let D be the disk which is the closure of the complement of the unbounded component of $C(E')$. Then D is an invariant disk containing K.

THEOREM 4. *T is a conjugate of either a rotation or reflection.*

Proof. We first show that E^2 is the union of an increasing sequence of disks $\{B_i\}_{i=1}^{\infty}$ such that

(1) $B_1 \subseteqq B_2^0 \subseteqq B_2 \subseteqq B_3^0 \subseteqq B_3 \subseteqq \cdots \subseteqq B_n^0 \subseteqq B_n \subseteqq \cdots$ and (2) $T(B_n) = B_n$ for all n.

By Theorems 2 and 3, there exists an invariant disk $B_1 \ni T(B_1) = B_1$. Let C_2 be the circle of radius n_2 about the origin, where $n_2 \geqq 2$, and such that C_2 contains B_1 in its interior. By Theorem 1, C_2 plus its interior has bounded orbit. By Theorem 3, there is a disk B_2 containing C_2 such that $T(B_2) = B_2$.

We continue inductively, requiring at the i^{th} stage, that C_i be a circle of radius n_i about the origin, $n_i \geqq i$, and B_{i-1} be a subset of the interior of C_i. Thus we have proved the claim of the first paragraph.

From this point on, the proof is exactly as in [8], if one replaces "almost periodic homeomorphism" by "homeomorphism with a family of equicontinuous iterates".

4. Dense orbits. Besicovitch in [4] and [5] gave an example of a homeomorphism of the plane such that the positive semi-orbit of some point is dense in E^2. It is known that there is no homeomorphism of E^n such that the positive semiorbit of each point is dense [11]. Here we give a short argument that there is no homeomorphism of the plane such that the orbit of every point is dense. Certainly this fact is known but we have been unable to find it in the literature. The question as to the existence of such a homeomorphism in E^3 or S^3 seems to be unanswered.

THEOREM 5. *There is no homeomorphism of E^2 such that the orbit of each point is dense.*

Proof. Let us assume that h is a homeomorphism of E^2 such that the orbit of each point is dense.

Then h^2 is an orientation preserving homeomorphism of E^2 and h^2 cannot have a fixed point.

Let D be a bounded disk in the plane such that $h^2(D) \cap D = \varnothing$, but $\bar{D} \cap \overline{h^2(D)} \neq \varnothing$. Let p be a point of the boundary of D such that $h^2(p)$ is a boundary point of D and let γ be an arc from p to $h^2(p)$ such that $\gamma - \{p, h^2(p)\} \subset D$. Then $F = \bigcup_{n=-\infty}^{\infty} h^{2n}(\gamma)$ is a flow line of h^2 [1].

Since $D \cap h^2(D) = \varnothing$, $h^{2n}(D) \cap D = \varnothing$ for all nonzero integers n. This follows from Proposition 1.1 of [1]. Thus, F is nowhere dense.

Let $N = F \cup h(F)$. Then N is a nowhere dense subset of the plane and $h(N) \subset N$. Thus, the orbit of every point of N is nowhere dense. This is a contradiction.

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