SCALING PROPERTIES OF HAUSDORFF AND PACKING MEASURES

MARIANNA CSÖRNYEI[1] AND R. DANIEL MAULDIN[2]

Abstract

Let $m \in \mathbb{N}$. Let θ be a continuous increasing function defined on $\mathbb{R}^+$, for which $\theta(0) = 0$ and $\theta(t)/t^m$ is a decreasing function of t. Let $\|\cdot\|$ be a norm on $\mathbb{R}^m$, and let ϱ, $\mathcal{H}^\theta = \mathcal{H}^\theta_\varrho$, $\mathcal{P}^\theta = \mathcal{P}^\theta_\varrho$ denote the corresponding metric, and Hausdorff and packing measures, respectively. We characterize those functions θ such that the corresponding Hausdorff or packing measure scales with exponent α by showing it must be of the form $\theta(t) = t^\alpha L(t)$, where L is slowly varying. We also show that for continuous increasing functions θ and η defined on $\mathbb{R}^+$, for which $\theta(0) = \eta(0) = 0$, $\mathcal{H}^\theta = \mathcal{P}^\eta$ is either trivially true or false: we show that if $\mathcal{H}^\theta = \mathcal{P}^\eta$, then $\mathcal{H}^\theta = \mathcal{P}^\eta = c \cdot \lambda$ for a constant c, where λ is the Lebesgue measure on $\mathbb{R}^m$.

For some time now, Hausdorff measures of the form $\mathcal{H}^\theta$ where the gauge function θ has the form $\theta(t) = t^\alpha L(t)$ with L slowly varying have occurred with a growing frequency in stochastic processes and dynamics. One main reason for the appearance of such measures is that these measures obey a scaling law: $\mathcal{H}^\theta(cA) = c^\alpha \mathcal{H}^\theta(A)$ is satisfied for every $c > 0$ and $A \subset \mathbb{R}^n$. Usually the function L has the form of some combination of iterated logarithms raised to some power, e.g., the result of Taylor and Wendel [12] that the zero sets in Brownian bridge has positive finite measure with respect to the Hausdorff measure determined by $\theta(t) = t^{1/2}(log(log(1/t))^{1/2}$ or, more generally, the exact Hausdorff dimension of random fractals given in [2]. Packing measures $\mathcal{P}^\theta$ with θ of the same form have been making their appearance, e.g. [4], [11],[7]. Packing measures were introduced by Sullivan[10], Tricot[14] and

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Taylor and Tricot[13]. Sullivan showed that in the study of geometric limit sets sometimes it is the Hausdorff measure which is important and sometimes the packing measure, a theme which continues to appear in dynamics, e.g. [7]. Although the importance of packing measure has been becoming more and more apparent, they remain somewhat more difficult to deal with. The central issue revolves around the two stage definition of a packing measure. Let $A \subset \mathbb{R}^n$ and $\delta > 0$. We say that $\{(x_i, r_i)\}_{i=1}^n$ is a δ-packing of A if $x_i \in A$, $\delta \geq 2r_i > 0$ and $r_i + r_j < d(x_i, x_j)$ for i, $j = 1, \ldots, n$, $i \neq j$. Then the closed balls $B(x_i, r_i)$ are disjoint. We first define the prepacking measures P_δ^θ and P^θ by

$$P_\delta^\theta(A) = \sup\left\{ \sum_{i=1}^{n} \theta(2r_i) : \{(x_i, r_i)\}_{i=1}^n \text{ is a } \delta\text{-packing of } A \right\}$$

and

$$P^\theta(A) = \lim_{\delta \to 0} P_\delta^\theta(A).$$

So far this definition bears some similarity to the definition of Hausdorff measure. However, we are measuring the maximality of a packing, not the efficiency of a covering. Also, since P^θ is not countably subadditive one needs a standard modification to get an outer measure out of it. Thus, we define the packing θ-measure for $A \subset X$ by

$$\mathcal{P}^\theta(A) = \inf\left\{ \sum_{i=1}^{\infty} P^\theta(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

It is this second step which makes packing measure more problematic to deal with (This complexity is made explicit in [6]. It is an important fact that these packing measures obey the same scaling law. One major purpose of this paper is to show that under mild restictions the converse holds.

R.D. Mauldin and S.C. Williams investigated in [8] those Hausdorff measures $\mathcal{H}^\theta$, which obey a scaling law: they proved that for every continuous increasing concave function θ and for every $0 \leq \alpha \leq 1$,

$$\mathcal{H}^\theta(cA) = c^\alpha \mathcal{H}^\theta(A)$$

is satisfied for every $c > 0$ and $A \subset \mathbb{R}$, if and only if

$$\lim_{t \to 0} \frac{\theta(ct)}{\theta(t)} = c^\alpha$$

holds for every $c > 0$. In other words, for these functions θ the generalized Hausdorff measure $\mathcal{H}^\theta$ scales as the Hausdorff measure $\mathcal{H}^\alpha$ does if and only if $\theta(t) = t^\alpha L(t)$, where L is a slowly varying function in the sense of Karamata [15] (that is, $L(cx)/L(x) \to 1$ for every $c > 0$). They asked whether one can characterise the higher dimensional functions such that the corresponding Hausdorff measure scales. We prove a general theorem which as a corollary characterizes these functions not only with respect to the scaling of the Hausdorff measure but also with respect to the scaling of the packing measures. (For further properties of packing measures see in [5] and [1]).

It was also asked in [8] whether the packing measure $\mathcal{P}^\alpha$, or more generally, $\mathcal{P}^\theta$ in $\mathbb{R}^m$ can be a Hausdorff measure. H. Haase proved that $\mathcal{P}^\alpha \neq \mathcal{H}^\theta$, provided that there is a number γ such that if $\dim_H(E) < \gamma$, then $\mathcal{H}^\theta(E) = 0$, and if $\dim_H(E) > \gamma$, then $\mathcal{H}^\theta(E) = \infty$ (see in [3]). In [9], X. Saint Raymond and C. Tricot proved that for any $0 < s < m$ and $A \subset \mathbb{R}^m$ with $0 < \mathcal{P}^s(A) < \infty$, $\mathcal{P}^s(A) = \mathcal{H}^s(A)$ is satisfied if and only if s is an integer, and $\mathcal{P}^s$-a.e. point of A can be covered with countable many Lipschitz images of $\mathbb{R}^s$. In [6] P. Mattila and R.D. Mauldin proved an analogous result for doubling gauge functions: if a function θ satisfies the doubling condition, that is, $\theta(2r) \leq c \cdot \theta(r)$ for a constant c, and $\mathcal{P}^\theta(A) < \infty$ for a set $A \subset \mathbb{R}^m$, then $\mathcal{H}^\theta(A) = \mathcal{P}^\theta(A)$ holds if and only if

$$\lim_{r \to 0} \frac{\mathcal{H}^\theta(A \cap B(x,r))}{\theta(2r)} = 1 \quad \text{for } \mathcal{P}^\theta\text{-a.e. } x \in A.$$

In this paper we will first give a general answer to the question by proving that for continuous increasing functions θ and η defined on $\mathbb{R}^+$, for which $\theta(0) = \eta(0) = 0$, $\mathcal{H}^\theta = \mathcal{P}^\eta$ is either trivially true or false: we show that if $\mathcal{H}^\theta = \mathcal{P}^\eta$, then $\mathcal{H}^\theta = \mathcal{P}^\eta = c \cdot \lambda$ for a constant c, where λ is the Lebesgue measure on $\mathbb{R}^m$. We will then expand on the techniques of proof of this first result to characterize the Hausdorff and packing measures which obey a scaling law.

Notations. First we introduce some notations. We fix a metric ϱ induced by a norm, and when we take a diameter or the distance of two points with respect to this metric or take the Hausdorff or packing measure $\mathcal{H}^\theta_\varrho, \mathcal{P}^\theta_\varrho$, there will be no subscript. Otherwise, we will always indicate if a norm different from ϱ is being used.

We choose a base $u_1, u_2, \ldots, u_m$ of $\mathbb{R}^m$, in the following way. Let u_1 be a unit vector (according to our metric ϱ), whose Euclidean length is maximal.

Then we choose a vector u_2 orthogonal to u_1, whose length is 1 and whose Euclidean length is maximal among all unit vectors orthogonal to u_1. If for some $m' \leq m$ the vectors $u_1, u_2, \ldots, u_{m'-1}$ have been defined, then let $u_{m'}$ be a unit vector orthogonal to $u_1, u_2, \ldots, u_{m'-1}$ whose Euclidean length is maximal among all such vectors.

If the edges of a brick have directions $u_1, u_2, \ldots, u_m$, then we say that this brick is *in standard position.* A brick C is called a *regular brick of size* s, if C is in standard position, and the distance between its opposite faces is s. Observe that for general metric ϱ this does not imply that the edges of C have length s. A brick D in standard position whose edges are of length s we call *a cube of size* s.

It is immediate to check that every convex set of diameter d can be covered by a regular brick of size d. It is also easy to see that the distance of any two vertices of a cube of size d is at least d. Indeed, let x_1 and x_2 be two distinct vertices and let j be the first coordinate which differs. Then x_1 and x_2 belong to an affine subspace orthogonal to $u_1, u_2, \ldots, u_{j-1}$, and the Euclidean distance of x_1 and x_2 is at least the Euclidean length of the edge of the cube parallel to u_j. But this edge is one of the longest vectors (according to the Euclidean metric) whose length is d according to ϱ, thus $\varrho(x, y) \geq d$.

A set of diameter s whose volume is maximal among all such sets we call a *weak ball*, and *ball* means the usual ball in the metric space $(\mathbb{R}^m, \varrho)$.

We will prove the following theorem:

Theorem 1 *Let ϱ be a metric induced by a norm on $\mathbb{R}^m$. Let θ and η be arbitrary continuous increasing functions defined on $\mathbb{R}^+$, for which $\theta(0) = \eta(0) = 0$. Let $\mathcal{H}^\theta$ and $\mathcal{P}^\eta$ denote the corresponding Hausdorff and packing measure on $(\mathbb{R}^m, \varrho)$, respectively. Then either*

$$\mathcal{H}^\theta = \mathcal{P}^\eta = c\lambda,$$

where $c > 0$ and λ is the Lebesgue measure on $\mathbb{R}^m$, or there exists a set $A \subset \mathbb{R}^m$, for which

$$\mathcal{H}^\theta(A) \neq \mathcal{P}^\eta(A).$$

PROOF. It is immediate to see that if $\limsup_{t\to 0} \eta(t)/t < \infty$, then $\mathcal{P}^\eta(Q) < \infty$ for every cube Q of $\mathbb{R}^m$, thus $\mathcal{P}^\eta$ is a σ-finite measure on $\mathbb{R}^m$. But $\mathcal{P}^\eta$ is translation invariant, thus we have $\mathcal{P}^\eta = c\lambda$ for a constant c. Hence we can assume that $\limsup_{t\to 0} \eta(t)/t = \infty$. We construct a set A for which $\mathcal{H}^\theta(A) = 0$ and $\mathcal{P}^\eta(A) = \infty$.

For every $n \in \mathbb{N}$ we define a set of pairwise disjoint regular bricks

$$\mathcal{C}^n = \{C_{11}, C_{21}, \ldots, C_{k_n 1}\}$$

of size e_n for an $e_n > 0$ and $k_n \in \mathbb{N}$. First we choose $m_1 = 1$ and choose $\mathcal{C}^1 = \{C_{11}\}$ arbitrarily. If for an $n \in \mathbb{N}$ the bricks of $\mathcal{C}^n$ have been defined, then we choose $r_{n+1} > 0$ and $\ell_{n+1} \in \mathbb{N}$ such that $r_{n+1}\ell_{n+1} < e_n$ and $\eta(r_{n+1}) \cdot \ell_{n+1}^m > 2^n$. We put $k_{n+1} = k_n \ell_{n+1}^m$, and let $e_{n+1} < r_{n+1}$ be so small that $\theta(we_{n+1}) \cdot k_{n+1} < 1/2^n$, where w denotes the diameter of the regular brick of size 1.

Now we choose a regular subbrick of size $r_{n+1}\ell_{n+1}$ inside each brick of $\mathcal{C}^n$, and consider its ℓ_{n+1}^m regular subbricks of size r_{n+1}. Let $\mathcal{C}^{n+1}$ be the set of the $k_n \ell_{n+1}^m = k_{n+1}$ middle bricks, whose midpoints are the same and whose size is e_{n+1}.

We put

$$A = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{k_n} C_{in}.$$

Then $\mathcal{C}^n$ is a we_n-covering of A, and $\theta(we_n) \cdot k_n < 1/2^{n-1}$ for every $n > 1$, thus $\mathcal{H}^\theta(A) = 0$. We prove that $\mathcal{P}^\eta(A) = \infty$.

Let μ be the 'natural' measure defined on A, that is, let μ be the unique probability measure for which $\mu(C_{in}) = 1/k_n$ for every i, n. If $K \subset A$ is compact and $\mu(K) > 0$, then $\mathcal{P}_0^\eta(K) = \infty$. Indeed, K intersects at least $\mu(K) \cdot k_{n+1}$ of the k_{n+1} bricks of $\mathcal{C}^{n+1}$, we can choose $1/2^m \cdot \mu(K) \cdot k_{n+1}$ of them whose distance is at least r_{n+1}. This gives a packing of K by $1/2^m \cdot \mu(K) \cdot k_{n+1}$ balls of diameter r_{n+1}. From $\eta(r_{n+1}) \cdot k_{n+1} \geq \eta(r_{n+1}) \cdot \ell_{n+1}^m > 2^n \to \infty$ we obtain $\mathcal{P}_0^\eta(K) = \infty$ for every compact subset $K \subset A$ with $\mu(K) > 0$, and then $\mathcal{P}(A) = \infty$ follows. ■

Let us turn now to a study of the scaling properties of Hausdorff and packing measures in higher dimensions. We generalise the result in 1 dimension, as well. We assume θ is a continuous increasing function defined on $\mathbb{R}^+$, for which $\theta(0) = 0$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^m$. As before, let ϱ, $\mathcal{H}^\theta = \mathcal{H}_\varrho^\theta$, $\mathcal{P}^\theta = \mathcal{P}_\varrho^\theta$ denote the corresponding metric, and Hausdorff and packing measures, respectively. For the next theorem we also assume that $\theta(t)/t^m$ is a decreasing function of t (this is obviously satisfied for any continuous increasing concave function θ and $m = 1$). We will prove the following theorem:

Theorem 2 *For every function $f : \mathbb{R}^+ \to \mathbb{R}^+$, the following are equivalent:*

(i)

$$\mathcal{H}^\theta(cA) \le f(c) \cdot \mathcal{H}^\theta(A) \quad \forall c > 0, A \subset \mathbb{R}^m;$$

(ii)

$$\mathcal{P}^\theta(cA) \le f(c) \cdot \mathcal{P}^\theta(A) \quad \forall c > 0, A \subset \mathbb{R}^m;$$

(iii)

$$\limsup_{t \to 0} \frac{\theta(ct)}{\theta(t)} \le f(c) \quad \forall c > 0.$$

Analogously, for every function $g : \mathbb{R}^+ \to \mathbb{R}^+$, the following are equivalent:

(iv)

$$\mathcal{H}^\theta(cA) \ge g(c) \cdot \mathcal{H}^\theta(A) \quad \forall c > 0, A \subset \mathbb{R}^m;$$

(v)

$$\mathcal{P}^\theta(cA) \ge g(c) \cdot \mathcal{P}^\theta(A) \quad \forall c > 0, A \subset \mathbb{R}^m;$$

(vi)

$$\liminf_{t \to 0} \frac{\theta(ct)}{\theta(t)} \ge g(c) \quad \forall c > 0.$$

Before giving the proof of Theorem 2, let us note the following corollary.

Theorem 3 *Let m be a positive integer. Let θ be a continuous increasing function defined on $\mathbb{R}^+$, for which $\theta(0) = 0$ and such that $\theta(t)/t^m$ is a decreasing function of t. Let $\mathcal{H}^\theta$ and $\mathcal{P}^\theta$ denote the corresponding Hausdorff and packing measure on $\mathbb{R}^m$, respectively. The following statements are equivalent:*

(i)

$$\mathcal{H}^\theta(cA) = c^\alpha \cdot \mathcal{H}^\theta(A) \quad \forall c > 0, A \subset \mathbb{R}^m;$$

(ii)

$$\mathcal{P}^\theta(cA) = c^\alpha \cdot \mathcal{P}^\theta(A) \quad \forall c > 0, A \subset \mathbb{R}^m;$$

(iii)

$$\lim_{t \to 0} \frac{\theta(ct)}{\theta(t)} = c^\alpha \quad \forall c > 0.$$

Again, we recall that (iii) means θ is of the form $\theta(t) = t^\alpha L(t)$, where L is slowly varying. These are precisely the types of gauge functions which are so common in dynamics and stochastic processes.

Remark 4 *It immmediately follows that if $\mathcal{H}^\theta_\varrho$ or $\mathcal{P}^\theta_\varrho$ scales with respect to the metric induced by some norm then it scales with respect to any metric induced by a norm.*

Now we turn to the proof of Theorem 2.

Proof of (iii) $\to$ (i) and (vi) $\to$ (v). If

$$\limsup_{t\to 0} \frac{\theta(ct)}{\theta(t)} \le f(c),$$

then for ε small enough and $t < \varepsilon$ we have

$$\theta(ct) \le (f(c) + M(\varepsilon)) \cdot \theta(t),$$

where $M(\varepsilon) \to 0$ as $\varepsilon \to 0$. If $\mathcal{G}$ is an ε/c-cover of A, then $c \cdot \mathcal{G}$ is an ε cover of cA, and

$$\sum_{G\in\mathcal{G}} \theta(c \cdot \operatorname{diam} G) \le (f(c) + M(\varepsilon)) \cdot \sum_{G\in\mathcal{G}} \theta(\operatorname{diam} G).$$

Therefore,

$$\mathcal{H}^\theta_{\varepsilon/c}(cA) \le (f(c) + M(\varepsilon)) \cdot \mathcal{H}^\theta_\varepsilon(A),$$

and thus, letting ε decrease to 0, $\mathcal{H}^\theta(cA) \le f(c) \cdot \mathcal{H}^\theta(A)$.

Analogously, for every ε/c-packing $\mathcal{G}$ of A, $c \cdot \mathcal{G}$ is an ε packing of cA, and assuming (vi) we have

$$\sum_{G\in\mathcal{G}} \theta(c \cdot \operatorname{diam} G) \ge (g(c) - M(\varepsilon)) \cdot \sum_{G\in\mathcal{G}} \theta(\operatorname{diam} G).$$

Hence,

$$\mathcal{P}^\theta_{\varepsilon/c}(cA) \ge (g(c) - M(\varepsilon)) \cdot \mathcal{P}^\theta_\varepsilon(A)$$

for every $A \subset \mathbb{R}^m$, and thus $\mathcal{P}^\theta_0(cA) \ge g(c) \cdot \mathcal{P}^\theta_0(A)$. From this inequality for the packing pre-measure, we obtain (v) for the θ-packing measure. ■

It is also clear that for every function f and g satisfying

$$f(x)g(1/x) = 1, \tag{1}$$

assumptions (i) and (iv) are equivalent. Indeed, (i) is equivalent to

$$\mathcal{H}^\theta(A) \geq \frac{1}{f(c)}\mathcal{H}^\theta(cA) \quad \forall c > 0, A \subset \mathbb{R}^m,$$

and by replacing A by $(1/c) \cdot A$ and c by $(1/c)$ we obtain (iv). Similarly, (ii) is equivalent to (v) and (iii) is equivalent to (vi) for every f and g for which (1) is satisfied. Since for every f there exists a (unique) function g such that (1) holds, and similarly, for every function g there exists an f with this property, it is enough to prove that (iv) implies (vi) and (ii) implies (iii).

Proof of (iv) $\to$ (vi). Fix $c > 0$. First we choose a sequence $z_1 > z_2 > \cdots \to 0$, for which

$$\lim_{n\to\infty} \frac{\theta(cz_n)}{\theta(z_n)} = \liminf_{t\to 0} \frac{\theta(ct)}{\theta(t)}. \tag{2}$$

It is enough to prove that there is a set K, probability measure μ, positive and finite number M, and for every $\varepsilon > 0$ there exists an ε-covering $\mathcal{G}_\varepsilon$ of K, such that

(a) $\mu(K) = 1$;

(b) $\theta(\operatorname{diam} A) \geq M \cdot \mu(A) \quad \forall A \subset \mathbb{R}^m$;

(c) $\sum_{B\in\mathcal{G}_\varepsilon} \theta(\operatorname{diam} B) \to M$ as $\varepsilon \to 0$;

(d) $\operatorname{diam} B \in \{z_1, z_2 \dots\} \quad \forall B \in \mathcal{G}_\varepsilon$.

Indeed, by (b) we have $\mathcal{H}^\theta(A) \geq M \cdot \mu(A)$ for every $A \subset \mathbb{R}^m$, thus $\mathcal{H}^\theta(K) \geq M$. On the other hand, (c) implies $\mathcal{H}^\theta(K) \leq M$, thus $\mathcal{H}^\theta(K) = M$. Then, applying (iv) we have

$$g(c) \cdot M = g(c) \cdot \mathcal{H}^\theta(K) \leq \mathcal{H}^\theta(cK),$$

and since $c \cdot \mathcal{G}_\varepsilon$ is a $c\varepsilon$-covering of cK, by (c)

$$g(c) \leq \liminf_{\varepsilon\to 0} \frac{\sum_{B\in\mathcal{G}_\varepsilon} \theta(c \cdot \operatorname{diam} B)}{M} = \liminf_{\varepsilon\to 0} \frac{\sum_{B\in\mathcal{G}_\varepsilon} \theta(c \cdot \operatorname{diam} B)}{\sum_{B\in\mathcal{G}_\varepsilon} \theta(\operatorname{diam} B)}.$$

Applying (d) and (2) we obtain

$$g(c) \leq \liminf_{t\to 0} \frac{\theta(ct)}{\theta(t)}$$

and (vi) is proved.

CONSTRUCTION of K, μ, M and $\mathcal{G}_\varepsilon$.

For every $n \in \mathbb{N}$ we will choose a finite set

$$\mathcal{C}^n = \{C_{1n}, C_{2n}, \dots, C_{M_n n}\}$$

of pairwise disjoint regular bricks of size $x_n \in \{z_1, z_2, \dots\}$, such that $\cup\mathcal{C}^{n+1} \subset \cup\mathcal{C}^n$; moreover, there will be M_{n+1}/M_n bricks $C_{i\,n+1}$ inside each brick C_{jn}. We will also have $C_{i\,n+1} \subset B^*_{jn} \subset C_{jn}$, where B^*_{jn} is a weak ball inside C_{jn} of diameter x_n. Additionally, we will choose x_n, M_n and M such that $x_n \to 0$ and $M_n \cdot \theta(x_n) \to M$. The $x_n's$ will be a subsequence of the $z_i's$.

Let μ be the (unique) probability measure on $\mathbb{R}^m$ for which

$$\mu(C_{in}) = \frac{1}{M_n},$$

and put

$$K = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{M_n} C_{in} = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{M_n} B^*_{in}.$$

Then (a) is trivially satisfied. For every ε, if n is large enough then $\mathcal{B}^n = \{B^*_{1n}, B^*_{2n}, \dots, B^*_{M_n n}\}$ is an ε-cover of K, so we can choose $\mathcal{G}_\varepsilon = \mathcal{B}^n$ and then $\sum_{B\in\mathcal{G}_\varepsilon} \theta(\operatorname{diam} B) = M_n \cdot \theta(x_n) \to M$, that is, (c) holds. It is also immediate to see that (d) is satisfied. Therefore, it is enough to prove that $\mathcal{C}^n$ can be chosen such that (b) holds.

We know that $\theta(t)/t^m$ increases as $t \to 0$. We can assume that

$$\theta(t)/t^m \to \infty,$$

otherwise we have

$$\lim_{t\to 0} \frac{\theta(ct)}{\theta(t)} = c^m,$$

so (iii) and thus also (i) is satisfied for $f(c) = c^m$. Applying (i) and (iv)

$$c^m \geq g(c)$$

follows, which proves (vi).

We fix a sequence $\varrho_1 > \varrho_2 > \cdots > 0$, for which $\sum_{n=1}^{\infty} \varrho_n < 1/2$. Let $x_1 = z_1$, $M_1 = 1$ and choose $\mathcal{C}^1 = \{C_{11}\}$ arbitrarily with z_1 the size of C_{11}. We will define x_n and $\mathcal{C}^n$ by induction. We will require

$$1 - 2\varrho_n < \frac{M_{n+1} \cdot \theta(x_{n+1})}{M_n \cdot \theta(x_n)} < 1 - \varrho_n. \tag{3}$$

Then $M_n \cdot \theta(x_n) \quad (n = 1, 2 \dots)$ is a decreasing sequence, and

$$M_n \cdot \theta(x_n) > M_1 \cdot \theta(x_1) \cdot \prod_{i=1}^{n-1} (1 - 2\varrho_i) > M_1 \cdot \theta(x_1) \cdot \prod_{i=1}^{\infty} (1 - 2\varrho_i).$$

Therefore, there exists a $0 < M < \infty$ for which $M_n \cdot \theta(x_n) \to M$.

By assumption (3) we will also obtain

$$M = \lim_{n\to\infty} M_n \theta(x_n) < (1 - \varrho_n) M_n \theta(x_n) \tag{4}$$

for every $n \geq 1$. The only thing we need to check is that x_n and $\mathcal{C}^n$ can be chosen such that (b) and (3) are satisfied.

We assume that x_n and $\mathcal{C}^n$ have been defined, and we define x_{n+1} and construct $\mathcal{C}^{n+1}$. We will use the constant

$$\gamma = \frac{\mathrm{Vol}(B^*)}{\mathrm{Vol}(C)},$$

where C is a regular brick of $\mathbb{R}^n$ and B^* is a weak ball inside the brick whose diameter is the size of C. Then B^* contains approximately γN^m bricks of the regular partition of $N \times N \times \cdots \times N = N^m$ small bricks on C. More precisely: for every $\varepsilon > 0$ there exists an $N(\varepsilon)$, such that if $N > N(\varepsilon)$ then B^* contains at least $(1-\varepsilon)\gamma N^m$ bricks and B^* intersects at most $(1+\varepsilon)\gamma N^m$ bricks. Moreover, every convex set in C of the same diameter as B^* will intersect at most $(1+\varepsilon)\gamma N^m$ bricks.

We know that if x_{n+1} is small enough then $\theta(x_n)/\theta(x_{n+1})$ is large enough. First we choose an ε_n small enough (we will specify it later). Then we choose an $x_{n+1} = z_i$ so small, that there exists an integer $N = N_n > N(\varepsilon_n)$ for which

$$(1 - \frac{5}{3}\varrho_n)\frac{\theta(x_n)}{\theta(x_{n+1})} < \gamma N^m < (1 - \frac{4}{3}\varrho_n)\frac{\theta(x_n)}{\theta(x_{n+1})}.$$

If ε_n is small enough then

$$\frac{1-2\varrho_n}{1-\varepsilon_n} < 1 - \frac{5}{3}\varrho_n < 1 - \frac{4}{3}\varrho_n < \frac{1-\varrho_n}{1+\varepsilon_n},$$

that is,

$$1-2\varrho_n < (1-\varepsilon_n)\gamma N^m \cdot \frac{\theta(x_{n+1})}{\theta(x_n)} < (1+\varepsilon_n)\gamma N^m \cdot \frac{\theta(x_{n+1})}{\theta(x_n)} < 1-\varrho_n.$$

We put an $N \times N \times \cdots \times N$ grid onto each of the bricks C_{in}, choose those small regular bricks determined by the grid which are inside B^*_{in}. Let $\mathcal{C}^{n+1}$ be the set of the smaller middle regular bricks with the same midpoints as the chosen bricks and of size x_{n+1}. For this system (3) is satisfied, we will need to verify (b) only.

We can assume that

$$x_{n+1} < \varepsilon_n \cdot \frac{x_n}{N}. \tag{5}$$

Indeed,

$$\gamma N^m \cdot \frac{\theta(x_{n+1})}{\theta(x_n)} < 1 - \frac{4}{3}\varrho_n < 1,$$

thus

$$\gamma N^m \cdot \frac{\theta(x_{n+1})/x^m_{n+1}}{\theta(x_n)/x^m_n} < \frac{x^m_n}{x^m_{n+1}}.$$

That is,

$$x_{n+1} < \Big(\frac{1}{\gamma} \cdot \frac{\theta(x_n)/x^m_n}{\theta(x_{n+1})/x^m_{n+1}}\Big)^{1/m} \cdot \frac{x_n}{N},$$

and

$$\frac{\theta(x_n)/x^m_n}{\theta(x_{n+1})/x^m_{n+1}}$$

is small enough if x_{n+1} is small enough.

Let A be an arbitrary subset of $\mathbb{R}^m$, we verify (b) for this set A. We can assume that A is convex and A meets K in at least two points. Let $n = n_0$ be the first index for which A intersects only one of the bricks of $\mathcal{C}^n$ but at least 2 of the bricks of $\mathcal{C}^{n+1}$. If $\operatorname{diam} A \geq x_n$, then $\theta(\operatorname{diam} A) \geq \theta(x_n)$, and by (4) we have $\theta(x_n) > M/M_n = M\mu(C_{in})$. Since A intersects at most one of the bricks C_{in} and $\operatorname{supp}\mu \subset \bigcup_{i=1}^{M_n} C_{in}$, it is clear that (b) holds. So we can assume that $\operatorname{diam} A < x_n$.

We put

$$\operatorname{diam} A = r \cdot \frac{x_n}{N},$$

where $N = N_n$. Then $0 < r < N$. We will use the notation $k = [r] + 2$, where $[r]$ denotes the integer part of r. Let C_{in} be the only brick of $\mathcal{C}^n$ which intersects A. We extend the $N \times N \times \cdots \times N$ grid from C_{in} into the whole space, and choose a $k \times k \times \cdots \times k$ piece which covers A.

Now we fix a number ε_n^* small enough (we will specify it later). If $k > N(\varepsilon_n^*)$, then A intersects at most $(1 + \varepsilon_n^*)\gamma k^m$ bricks of the grid. We also know that B_{in} contains at least $(1 - \varepsilon_n)\gamma N^m$ bricks, thus in this case

$$\mu(A) \leq \frac{(1 + \varepsilon_n^*)\gamma k^m}{(1 - \varepsilon_n)\gamma N^m} \cdot \mu(B_{in}).$$

Since $\theta(t)/t^m$ decreases,

$$\theta(\operatorname{diam} A) = \theta\big(\frac{r x_n}{N}\big) \geq \frac{r^m}{N^m} \cdot \theta(x_n),$$

and by (4) we have

$$\theta(x_n) \geq \frac{M \cdot \mu(C_{in})}{1 - \varrho_n} = \frac{M \cdot \mu(B_{in})}{1 - \varrho_n}.$$

Hence

$$M \cdot \mu(A) \leq \frac{(1 + \varepsilon_n^*)\gamma k^m}{(1 - \varepsilon_n)\gamma N^m} \cdot (1 - \varrho_n) \cdot \frac{N^m}{r^m} \cdot \theta(\operatorname{diam} A) \leq$$

$$\leq \frac{(1 + \varepsilon_n^*)}{(1 - \varepsilon_n)} \cdot (1 - \varrho_n) \cdot \big(\frac{k}{k - 2}\big)^m \cdot \theta(\operatorname{diam} A).$$

If ε_n^* is small enough then $k \geq N(\varepsilon_n^*)$ is large enough, thus if we choose both ε_n and ε_n^* small enough then $M \cdot \mu(A) \leq \theta(\operatorname{diam} A)$ is satisfied.

Now we consider the case $3 \leq k < N(\varepsilon_n^*)$. Since A intersects at most k^m bricks, we have

$$\mu(A) \leq \frac{k^m}{(1 - \varepsilon_n)\gamma N^m} \cdot \mu(B_{in}) \leq \frac{k^m}{(1 - \varepsilon_n)\gamma N^m} \cdot \frac{\theta(x_n)}{M}.$$

We know that $\theta(t)/t^m \to \infty$ as $t \to 0$, thus for every constant C there exists an α_0, such that for $\alpha < \alpha_0 = \alpha_0(C)$ we have

$$\theta(\alpha x_n) \geq C\alpha^m \cdot \theta(x_n).$$

Hence if

$$\frac{r}{N} \le \frac{k-1}{N} < \frac{N(\varepsilon_n^*)}{N(\varepsilon)} \le \alpha_0(C),$$

then

$$M \cdot \mu(A) \le \frac{k^m}{(1-\varepsilon_n)\gamma N^m} \cdot \frac{\theta(\operatorname{diam} A)}{C} \cdot \frac{N^m}{r^m} \le$$

$$\le \frac{1}{(1-\varepsilon_n)\gamma} \cdot \frac{1}{C} \cdot \big(\frac{k}{k-2}\big)^m \cdot \theta(\operatorname{diam} A) \le$$

$$\le \frac{1}{(1-\varepsilon_n)\gamma} \cdot \frac{3^m}{C} \cdot \theta(\operatorname{diam} A).$$

First we choose C so large that

$$\frac{3^m}{\gamma C} < 1.$$

Then we can choose both ε_n^* and ε_n arbitrarily small, such that, in addition,

$$\frac{N(\varepsilon_n^*)}{N(\varepsilon_n)} \le \alpha_0(C)$$

is satisfied. Then (b) follows for every $k \ge 3$.

Finally, for $k = 2$ we have

$$M \cdot \mu(A) \le \frac{1}{(1-\varepsilon_n)\gamma} \cdot \frac{1}{C} \cdot \big(\frac{2}{r}\big)^m \cdot \theta(\operatorname{diam} A).$$

Since A intersects at least 2 bricks of $\mathcal{C}^{n+1}$,

$$r \cdot \frac{x_n}{N} = \operatorname{diam} A \ge \frac{x_n}{N} - x_{n+1},$$

thus by (5) we have $r \ge 1 - \varepsilon_n$. Hence,

$$M \cdot \mu(A) \le \frac{1}{(1-\varepsilon_n)\gamma} \cdot \frac{1}{C} \cdot \big(\frac{2}{1-\varepsilon_n}\big)^m \cdot \theta(\operatorname{diam} A) \le$$

$$\le \frac{3^m}{\gamma C} \cdot \frac{1}{(1-\varepsilon_n)^{m+1}} \cdot \theta(\operatorname{diam} A) < \theta(\operatorname{diam} A),$$

if ε_n is small enough. ■

Proof of and (ii) $\to$ (iii). As before, we can assume that

$$\theta(t)/t^m \to \infty.$$

Fix $c > 0$. We choose a sequence $v_1 > v_2 > \cdots \to 0$ for which

$$\lim_{n\to\infty} \frac{\theta(cv_n)}{\theta(v_n)} = \limsup_{t\to 0} \frac{\theta(ct)}{\theta(t)}. \tag{6}$$

Let w be the diameter of the unit cube, that is, the diameter (according to ϱ) of the cube D whose edges have direction $u_1, u_2, \ldots, u_m$ and (according to ϱ) length 1. We fix a number q_0 so small that $\theta(2wq_0) < 1/2^m$. It is enough to prove that there is a compact set $L \subset \mathbb{R}^m$ and a probability measure ν, such that

(e) $\nu(L) = 1$;

(f) $\theta(\operatorname{diam} B) \le \nu(B)$ for every ball $B = B(x, r)$ for which $x \in L$ and $r \le wq_0$;

and for every $\varepsilon, \delta > 0$ and for every subset $A \subset L$, there exists an ε-packing $\mathcal{F} = \mathcal{F}^A_{\varepsilon,\delta}$ of A for which

(g) $\sum_{B\in\mathcal{F}} \theta(\operatorname{diam} B) \ge \nu(A) - \delta$;

(h) $\operatorname{diam} B \in \{v_1, v_2 \ldots\} \quad \forall B \in \mathcal{F}$.

Indeed, from (g) it follows that $\mathcal{P}^\theta_0(A) \ge \nu(A)$ for every $A \subset L$, thus

$$\mathcal{P}^\theta_0(A) \ge \mathcal{P}^\theta(A) \ge \nu(A) \quad \forall\, A \subset L.$$

On the other hand, by (f) we have $\mathcal{P}^\theta_0(A) \le \nu(A)$ for every compact set A, thus for every compact set $A \subset L$

$$\nu(A) = \mathcal{P}^\theta(A) = \mathcal{P}^\theta_0(A)$$

is satisfied. Hence, we can see by regularity that ν is the restriction of the packing measure to L. It also follows that

$$\lim_{\varepsilon,\delta\to 0} \sum_{B\in\mathcal{F}^A_{\varepsilon,\delta}} \theta(\operatorname{diam} B) = \nu(A) = \mathcal{P}^\theta(A)$$

whenever $A \subset L$ is compact.

By (ii) we have

$$f(c) \geq \frac{\mathcal{P}^\theta(cL)}{\mathcal{P}^\theta(L)},$$

and L is compact, thus

$$\mathcal{P}^\theta(cL) = \inf\{\sum \mathcal{P}_0^\theta(cL_i) : \ L \subset \bigcup_i L_i, L_i \subset L, L_i \text{ is compact}\}.$$

But

$$\mathcal{P}^\theta(L) \leq \sum_i \mathcal{P}^\theta(L_i) = \sum_i \lim_{\varepsilon,\delta\to 0} \sum_{B\in\mathcal{F}_{\varepsilon,\delta}^{L_i}} \theta(\operatorname{diam} B),$$

and $c \cdot \mathcal{F}_{\varepsilon,\delta}^{L_i}$ is a $c\varepsilon$-packing of cL_i, thus

$$\mathcal{P}_0^\theta(cL_i) \geq \lim_{\varepsilon,\delta\to 0} \sum_{B\in\mathcal{F}_{\varepsilon,\delta}^{L_i}} \theta(c \cdot \operatorname{diam} B).$$

Therefore,

$$f(c) \geq \inf \lim_{\varepsilon,\delta\to 0} \frac{\sum_i \sum_{B\in\mathcal{F}_{\varepsilon,\delta}^{L_i}} \theta(c \cdot \operatorname{diam} B)}{\sum_i \sum_{B\in\mathcal{F}_{\varepsilon,\delta}^{L_i}} \theta(\operatorname{diam} B)}.$$

Applying (h) and (6) we obtain (iii).

Construction of L and ν.

For every $n \in \mathbb{N}$ we will construct a finite set $\mathcal{D}^n = \{D_{1n}, D_{2n}, \ldots, D_{N_n n}\}$ of pairwise disjoint cubes for which $\cup\mathcal{D}^{n+1} \subset \cup\mathcal{D}^n$. For every $1 \leq i \leq N_n$ the cube D_{in} will contain $2^m k_{in}^m$ cubes of $\mathcal{D}^{n+1}$ for an even integer $k_{in} > w$ (and of course we will have $N_{n+1} = 2^m \sum_{i=1}^{N_n} k_{in}^m$). We will also define positive numbers $p_{1n}, p_{2n}, \ldots, p_{N_n n}$, for which

$$\sum_{i=1}^{N_n} p_{in} = 1.$$

Moreover, for every fixed i, n and $k = k_{in}$ if $D_{i_1 n+1}, D_{i_2 n+1}, \ldots, D_{i_k n+1}$ are those cubes of $\mathcal{D}^{n+1}$ which belong to D_{in}, then

$$p_{in} = \sum_{j=1}^{k} p_{i_j n+1}.$$

Then there exist a (unique) probability measure ν, for which $\nu(D_{in}) = p_{in}$, and for this ν and $L = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N_n} D_{in}$ assumption (e) is satisfied. We note that ν is a "redistribution of mass" measure; however, unlike μ it is a not uniform redistribution. For an arbitrary sequence $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_m) \in \{0,1\}^m$, we put

$$\mathbb{Z}^\epsilon = \{(z_1, z_2, \dots, z_m) : \ z_i \equiv \epsilon_i \pmod 2\},$$

where $z_1, z_2, \dots, z_m$ are coordinates according to $u_1, u_2, \dots, u_m$. Also, define

$$c_k = \begin{cases} |\bar{B}(0,k) \cap (\mathbb{Z}^{\geq 0})^m| & \text{if } k < w \\ |\mathbb{Z}^m \cap [0, k/w]^m| & \text{if } k \geq w \end{cases}$$

and

$$d_k = \begin{cases} \min_\epsilon |(\bar{B}(0,k) \cap (\mathbb{Z}^{\geq 0})^m) \setminus \mathbb{Z}^\epsilon| & \text{if } k < w \\ \min_\epsilon |(\mathbb{Z}^m \cap [0, k/w]^m) \setminus \mathbb{Z}^\epsilon| & \text{if } k \geq w, \end{cases}$$

where $\bar{B}(0,k)$ denotes the closed ball around the origin of $\mathbb{R}^m$ of radius k. Then $c_0 = 1, d_0 = 0$, $d_k \geq 1$ for $k \geq 1$, and $|\bar{B}(0,k) \cap \mathbb{Z}^{\geq 0}| \geq c_k$ for every k. It is also easy to see that $c_k/(k+1)^m \to 1/w^m$ and $d_k/(k+1)^m \to (1-1/2^m)/w^m$ as $k \to \infty$. Thus we can choose a constant σ (which depends only on ϱ), such that

$$c_k + \sigma \cdot d_k \geq (k+1)^m \tag{7}$$

for every $k \geq 1$.

We choose $N_1 = 1$, and choose a cube D_{11} of size q_0 arbitrarily. Assume that a cube $D = D_{in}$ and the number $p = p_{in}$ have been defined. Let $s = s_{in}$ be the size of D_{in}.

We will choose $k = k_{in}$, positive numbers $q = q_{in}$, $r = r_{in}$, and put $t = t_{in} = r_{in} - (w+1)q_{in}$, satisfying:

(A) $2rk < s$;

(B) $q \leq wq < t < r < 1/2^n$;

(C) $2t \in \{v_1, v_2, \dots\}$.

Then we choose a subcube $D' = D'_{in} \subset D$ of size $2rk$, and consider its $(2k) \times (2k) \times \dots \times (2k)$ subcubes of size r. We denote the small middle cubes of these of size q by $E(i_1, i_2, \ldots, i_m)$ $(1 \le i_1, i_2, \ldots, i_m \le 2k)$. First, let us give a recipe for the redistribution of mass to the next level. We define

$$\mathcal{I}_1 = \mathcal{I}_1^{in} = \{(i_1, i_2, \ldots, i_m) \in \mathbb{Z}^m : 1 \le i_1, i_2, \ldots, i_m \le 2k\}$$

and

$$\mathcal{I}_2 = \mathcal{I}_2^{in} = \mathcal{I}_1 \cap (2\mathbb{Z})^m,$$

then we define

$$\mathcal{E}_1^{in} = \{E(i_1, i_2, \ldots, i_m) : (i_1, i_2, \ldots, i_m) \in \mathcal{I}_1 \setminus \mathcal{I}_2\},$$

$$\mathcal{E}_2^{in} = \{E(i_1, i_2, \ldots, i_m) : (i_1, i_2, \ldots, i_m) \in \mathcal{I}_2\}.$$

Now we put

$$\lambda = \lambda_{in} = \frac{p}{2^m k^m + \sigma \cdot (2^m k^m - k^m)},$$

for $p = p_{in}$ and $k = k_{in}$, and define

$$r(i_1, i_2, \ldots, i_m) = \begin{cases} \lambda & \text{if } (i_1, i_2, \ldots, i_m) \in \mathcal{I}_2; \\ (1+\sigma)\lambda & \text{if } (i_1, i_2, \ldots, i_m) \in \mathcal{I}_1 \setminus \mathcal{I}_2. \end{cases}$$

It is easy to check that

$$\sum_{(i_1, i_2, \ldots, i_m) \in \mathcal{I}_1} r(i_1, i_2, \ldots, i_m) = p.$$

Hence we can choose

$$\mathcal{D}^{n+1} = \bigcup_{i=1}^{M_n} (\mathcal{E}_1^{in} \cup \mathcal{E}_2^{in})$$

and $p_{j\,n+1} = r(i_1, i_2, \ldots, i_m)$ for the cubes $D_{j\,n+1} = E(i_1, i_2, \ldots, i_m)$. That is, $p_{j\,n+1} = \lambda_{in}$ if $D_{j\,n+1} \in \mathcal{E}_2^{in}$ and $p_{j\,n+1} = (1+\sigma)\lambda_{in}$ if $D_{j\,n+1} \in \mathcal{E}_1^{in}$.

We need to show that for suitable parameters k_{in}, q_{in}, r_{in} (f), (g) and (h) are satisfied.

First we fix a sequence $\eta_1 > \eta_2 > \cdots \to 0$. We will choose $k = k_{in}$, $q = q_{in}$, and $r = r_{in}$ such that

(D) $(1 - \eta_n)\lambda \le \theta(2t) \le \theta(2r + 2wq) \le \lambda$;

(E) $t < \min_j q_{j\,n-1} < \min_j t_{j\,n-1}$;

(F) $\theta(2ws) \le p/2^m$.

(Observe that $s_{in} = q_{j\,n-1}$ for $D_{in} \subset D_{j\,n-1}$).

Let A be an arbitrary subset of L, and let ε, δ be positive numbers. First we show that there exists an $\mathcal{F}$ satisfying (g) and (h). We choose n_0 so large that $1/2^{n_0-1} < \varepsilon$ and $(1-\eta_{n_0})\nu(A) \ge \nu(A) - \delta$. We define A^n by induction for every $n \ge n_0$. Let

$$A^{n_0} = \{D_{j\,n_0+1} : A \cap D_{j\,n_0+1} \ne \emptyset, D_{j\,n_0+1} \in \mathcal{E}_2^{in_0}, \text{for some } i, 1 \le i \le N_{n_0}\}.$$

If A^{n-1} has been defined for an $n > n_0$, then let

$$A^n = \{D_{j\,n+1} : (A \setminus \bigcup_{k=n_0}^{n-1} \cup A^k) \cap D_{j\,n+1} \ne \emptyset, D_{j\,n+1} \in \mathcal{E}_2^{in}, \text{for some } i, 1 \le i \le N_n\}.$$

Now, for every $n \ge n_0$ and for every $D_{j\,n+1} \in A^n$ we choose a point $x_{jn+1} \in A \cap D_{j\,n+1}$, and let

$$\mathcal{F} = \{B(x_{j\,n+1}, t_{in}) : D_{j\,n+1} \in A^n, D_{j\,n+1} \in \mathcal{E}_2^{in}, 1 \le i \le N_n, n \ge n_0\}.$$

By definition, centers of all the balls of $\mathcal{F}$ belong to A. It is also clear from (B) that the balls have diameter at most $1/2^{n_0-1} < \varepsilon$. Assumption (h) follows from (C).

The diameter of $D_{j\,n+1}$ being wq_{in}, it follows from (B) that $B = B(x_{j\,n+1}, t_{in})$ covers the whole cube $D_{j\,n+1}$. On the other hand, the minimal distance of the midpoints of the cubes of $\mathcal{E}_1^{in} \cup \mathcal{E}_2^{in}$ is r_{in}, thus the minimal distance of these cubes is at least $r_{in} - wq_{in} = t_{in} + q_{in} > t_{in}$. Therefore this B is disjoint from all the cubes in $\mathcal{E}_1^{in} \cup \mathcal{E}_2^{in}$ but $D_{j\,n+1}$. From (E) it is also immediate to see that $D_{j\,n+1}$ is the only cube of $\mathcal{D}^{n+1}$ which intersects $B = B(x_{j\,n+1}, t_{in})$.

Thus $\nu(B) = \nu(D_{j\,n+1}) = \lambda_{in}$, and then by (D) we have

$$\theta(\operatorname{diam} B) \ge (1-\eta_n)\lambda_{in} \ge (1-\eta_{n_0})\nu(B).$$

Moreover, the minimal distance between the elements of $\mathcal{E}_2^{in}$ is at least $2r_{in} - wq_{in} > 2t_{in}$, from this and from (E) we can see that the balls of $\mathcal{F}$ are pairwise disjoint.

Hence $\mathcal{F}$ is an ε-packing of A, for which

$$\sum_{B \in \mathcal{F}} \theta(\operatorname{diam} B) \ge \sum_{B \in \mathcal{F}} (1-\eta_{n_0})\nu(B) = (1-\eta_{n_0})\nu(\cup \mathcal{F}).$$

We also know that

$$\cup\mathcal{F} \supset \bigcup_{n=n_0}^{\infty} \cup A^n,$$

thus

$$A \setminus \cup\mathcal{F} \subset \bigcap_{n=n_0}^{\infty} A \setminus \cup A^n \subset \bigcap_{n=n_0}^{\infty} \bigcup_{i=1}^{N_n} \cup\mathcal{E}_1^{in}.$$

But

$$\nu(\cup\mathcal{E}_1^{in}) = (1+\sigma)\lambda_{in}(2^m k^m - k^m) = \frac{p(1+\sigma)(2^m k^m - k^m)}{2^m k^m + \sigma\cdot(2^m k^m - k^m)} =$$

$$= \nu(D_{in})\cdot\frac{(1+\sigma)2^m - 1 - \sigma}{(1+\sigma)2^m - \sigma},$$

thus

$$\nu\big(\bigcap_{n=n_0}^{n_0+k-1} \bigcup_{i=1}^{N_n} \cup\mathcal{E}_1^{in}\big) \le \nu(\cup\mathcal{D}^{n_0})\cdot\Big(\frac{(1+\sigma)2^m - 1 - \sigma}{(1+\sigma)2^m - \sigma}\Big)^k \to 0 \quad \text{as } k\to\infty.$$

Therefore

$$\sum_{B\in\mathcal{F}} \theta(\operatorname{diam} B) \ge (1-\eta_{n_0})\nu(A) \ge \nu(A) - \delta,$$

and (g) is proved.

To check that (f) holds, let $B = B(x_0, r_0)$ be a ball for which $x_0 \in L$ and $r_0 \le wq_0$. Let $n+1$ be the first index for which B intersects at least 2 of the cubes of $\mathcal{D}^{n+1}$. Since $x_0 \in L$, we can choose indexes i, j for which $x_0 \in D_{in}$ and $x_0 \in D_{j\,n+1}$. The distance between the cubes of $\mathcal{D}^{n+1}$ inside D_{in} is greater than t, thus $r_0 > t$. We put $\ell = [r_0/(r+wq)]$, where $[\,\cdot\,]$ denotes the integer part. If $\ell = 0$ then $r_0 < r + wq$, thus by (D)

$$\theta(\operatorname{diam} B) \le \theta(2r + 2wq) \le \lambda.$$

On the other hand, by (B) $\operatorname{diam} D_{j\,n+1} = wq < t < r_0$, thus B covers $D_{j\,n+1}$ and

$$\nu(B) \ge \nu(D_{j\,n+1}) \ge \lambda.$$

So if $\ell = 0$ then (f) is satisfied.

If $\ell \geq 1$ then $\theta(\operatorname{diam} B) = \theta(2r_0) \leq \theta((2r+2wq)(\ell+1))$, and since $\theta(t)/t^m$ is monotone decreasing we have

$$\theta(\operatorname{diam} B) \leq \theta\big((2r+2wq)(\ell+1)\big) \leq (\ell+1)^m \theta(2r+2wq) \leq (\ell+1)^m \lambda.$$

So it is enough to prove that

$$\nu(B) \geq (\ell+1)^m \lambda. \tag{8}$$

Let the midpoint of $D_{j\,n+1}$ be $P = (p_1, p_2, \ldots, p_m)$. The midpoints of the cubes of $\mathcal{D}^{n+1}$ inside D_{in} form a regular lattice with distance r, and the diameter of the cubes is wq. Since $r_0 \geq \ell(r+wq) \geq \ell r + wq$, B covers all the cubes of $\mathcal{D}^{n+1}$ whose midpoint is inside the closed ball $B^* = \bar{B}(P, \ell r)$. It is easy to see that there exists an $\epsilon \in \{-1,+1\}^m$ for which all the points of

$$\mathcal{K} = \{(p_1 + \epsilon_1 i_1 r, p_2 + \epsilon_2 i_2 r, \ldots, p_m + \epsilon_m i_m r) : 1 \leq i_1, i_2, \ldots, i_m \leq k\}$$

are midpoints of cubes of $\mathcal{D}^{n+1}$ inside D_{in}. Thus, for $\ell \leq kw$, the ball B^* contains at least c_ℓ midpoints, and at least d_ℓ of them belong to cubes in $\mathcal{E}_1^{in}$. Therefore, in this case

$$\nu(B) \geq c_\ell \lambda + \sigma \cdot d_\ell \lambda,$$

thus by (7)

$$\nu(B) \geq (\ell+1)^m \lambda,$$

and (8) is proved.

If $\ell > kw$ then B^* covers the whole set $\mathcal{K}$. Since k is even $\mathcal{K}$ contains the midpoint of $(k/2)^m$ elements of $\mathcal{E}_2^{in}$ and $k^m - (k/2)^m$ elements of $\mathcal{E}_1^{in}$; therefore the measure of the union of the cubes with midpoint in $\mathcal{K}$ is $p/2^m$. So $\nu(B) \geq p/2^m$, thus for $r_0 \leq ws$, (f) follows from (F). Finally, if $r_0 > ws$ then B covers the whole cube D_{in}, thus $\nu(B) = p$. In this case $n \geq 2$, we can choose an index h for which $D_{in} \subset D_{h\,n-1}$. Since B intersects only one of the cubes of $\mathcal{D}^n$ and for every cube of $\mathcal{D}^n$ we can find an other one for which the distance of the midpoints is r, it is clear that $r_0 < r_{h\,n-1} + wq_{h\,n-1}$, thus applying (D) we have

$$\theta(\operatorname{diam} B) \leq \lambda_{h\,n-1}.$$

Then (f) follows from

$$\nu(B) = p \geq \lambda_{h\,n-1}.$$

So it is enough to prove that q_{in}, r_{in} and k_{in} can be chosen such that (A)-(F) is satisfied for every i, n.

DEFINITION of q_{in}, r_{in} and k_{in}.

We know that (F) is satisfied for $s_{11} = q_0$ (recall that $p_{11} = 1$). For $n \geq 2$ we have $s_{in} = q_{j\,n-1}$ and either $p_{in} = \lambda_{j\,n-1}$ or $p_{in} = (1+\sigma)\lambda_{j\,n-1}$, thus it is enough to have

(F*) $\theta(2wq) \leq \lambda/2^m$

for every i, n, $q = q_{in}$ and $\lambda = \lambda_{in}$. Assume that $q_{in'}, r_{in'}$ and $k_{in'}$ have been defined for every $n' < n$ and every i, and we define q_{in}, r_{in} and k_{in}. We put

$$\varepsilon = 2 \cdot \min(1/2^n, \min_j q_{j\,n-1});$$

$$c = p/(2^m + \sigma(2^m - 1)).$$

Then we choose a number $2t < \varepsilon$ from the set $\{v_1, v_2, \dots\}$ so small, for which there exists an even integer $k > w$ such that

$$(1-\eta_n)\frac{c}{\theta(2t)} < k^m < \frac{c}{\theta(2t)}$$

and $(2t)^m/\theta(2t) < s^m/c$. Then we have

(A') $(2tk)^m < (2t)^m \frac{c}{\theta(2t)} < c \cdot s^m/c = s^m$;

(B') $t < 1/2^n$;

(C') $2t \in \{v_1, v_2, \dots\}$;

(D') $(1-\eta_n)c/k^m < \theta(2t) < c/k^m$ and $c/k^m = \lambda$;

(E') $t < \min_j q_{j\,n-1}$.

Now we can choose r and q such that $t = r - (w+1)q$, and q is so small that

(A") $2tk + 2(w+1)qk < s$;

(B") $wq < t$;

(D") $\theta(2t + 4wq + 2q) < \lambda$;

and (F*) hold. Then (A)-(F) is satisfied, and Theorem 2 is proved. ∎

Remark 5 *Remark 4 has a surprising corollary. First of all notice that if ϱ and ϱ' are two different metrics on $\mathbb{R}^m$ induced by norms on $\mathbb{R}^m$, then for any gauge function θ, $\mathcal{H}^\theta_\varrho$ and $\mathcal{H}^\theta_{\varrho'}$ are equivalent measures (in the sense that there exists a constant L such that $L^{-1}\mathcal{H}^\theta_\varrho \le \mathcal{H}^\theta_{\varrho'} \le L\mathcal{H}^\theta_\varrho$). In particular, if $\mathcal{H}^\theta_\varrho$ satisfies the assumption (i) of Theorem 3, that is,*

(i) $\mathcal{H}^\theta_\varrho(cA) = c^\alpha \cdot \mathcal{H}^\theta_\varrho(A) \quad \forall c > 0, A \subset \mathbb{R}^m$,

then

(i') $\exists L \ge 1 \quad L^{-1}c^\alpha \cdot \mathcal{H}^\theta_{\varrho'}(A) \le \mathcal{H}^\theta_{\varrho'}(cA) \le Lc^\alpha \cdot \mathcal{H}^\theta_{\varrho'}(A) \quad \forall c > 0, A \subset \mathbb{R}^m$.

From Theorem 2 we know that (i') is equivalent to

(iii') $\exists L \ge 1 \quad L^{-1}c^\alpha \le \liminf_{t\to 0} \frac{\theta(ct)}{\theta(t)} \le \limsup_{t\to 0} \frac{\theta(ct)}{\theta(t)} \le Lc^\alpha \quad \forall c > 0$,

whenever $\theta(t)/t^m$ is a decreasing function of t. On the other hand, if $\theta(t)/t^m$ is decreasing then (i) is equivalent to

(iii) $\lim_{t\to 0} \frac{\theta(ct)}{\theta(t)} = c^\alpha \quad \forall c > 0$,

that is, we can choose $L = 1$ in (iii'). Then (i') is also satisfied for $L = 1$. So (i) is equivalent to

(i") $\mathcal{H}^\theta_{\varrho'}(cA) = c^\alpha \cdot \mathcal{H}^\theta_{\varrho'}(A) \quad \forall c > 0, A \subset \mathbb{R}^m$.

However, (iii') does not imply (iii) (and thus (i') does not imply (i")) for general gauge functions θ for which $\theta(t)/t^m$ is decreasing. Indeed, for example for $m = 2$, $\alpha = 1$ and $\theta(t) = t \cdot \tilde{\theta}(t)$,

$$\tilde{\theta}(t) = \begin{cases} 3^n t & t \in [\frac{2}{3^{n+1}}, \frac{3}{3^{n+1}}],\ n \in \mathbb{N} \\ \frac{4}{3} - 3^n t & t \in [\frac{1}{3^{n+1}}, \frac{2}{3^{n+1}}],\ n \in \mathbb{N} \end{cases}$$

one can check that θ is a well-defined monotone increasing function of t, $\theta(t) \to 0$ as $t \to 0$, and $\theta(t)/t^2$ is decreasing. But

$$\frac{\theta(ct)}{\theta(t)} = c \cdot \frac{\tilde{\theta}(ct)}{\tilde{\theta}(t)},$$

where $\tilde{\theta}$ is a bounded function for which

$$\inf_{c>0} \liminf_{t\to 0} \frac{\tilde{\theta}(ct)}{\tilde{\theta}(t)} = 2/3 \text{ and } \sup_{c>0} \limsup_{t\to 0} \frac{\tilde{\theta}(ct)}{\tilde{\theta}(t)} = 3/2,$$

that is, $\theta(t)$ satisfies (iii') for $L = 3/2$ and does not satisfy (iii).

But if $\mathcal{H}_\varrho^\theta$ satisfies the scaling property for at least one metric ϱ induced by the norm on $\mathbb{R}^m$, then it satisfies the scaling property for all such metrics, assuming that $\theta(t)/t^m$ is decreasing. We do not know whether this assumption is necessarily.

Problem 6 *Let m be a positive integer. Let θ be a continuous increasing function defined on $\mathbb{R}^+$. We conjecture that if the Hausdorff measure $\mathcal{H}^\theta$ satisfies*

$$\mathcal{H}_\varrho^\theta(cA) = c^\alpha \cdot \mathcal{H}_\varrho^\theta(A) \quad \forall c > 0, A \subset \mathbb{R}^m$$

with respect to some metric ϱ induced by a norm on $\mathbb{R}^m$, then the Hausdorff measure satisfies this scaling property with respect to all such metrics. We make the same conjecture with respect to the corresponding packing measure. It also may be true that if the Hausdorff measure satisfies this scaling law, then $\theta(t) = t^\alpha L(t)$, where L is slowly varying.

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Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK

E-mail address: **mari@math.ucl.ac.uk**

Department of Mathematics, University of North Texas, Box 311430, Denton, TX 76203, USA

E-mail address: **mauldin@unt.edu**