On series of translates of positive functions

by

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ABSTRACT. For Λ, a discrete infinite set of nonnegative real numbers, and a nonnegative measurable function $f : \mathbb{R} \to \mathbb{R}^+$, consider $C = C(f, \Lambda) = \{x : \sum_{\lambda \in \Lambda} f(x + \lambda) < +\infty\}$. The sets Λ naturally break into two types. Type 1 consists of Λ such that either $C = \mathbb{R}$ almost everywhere or else $C = \emptyset$ a.e., for every f. Type 2 consists of all the other Λ. We introduce a notion of asymptotic density for Λ and the complementary notion of asymptotic lacunarity. We demonstrate that Λ is of type 2 if it is asymptotically lacunary or else is asymptotically dense and exhibits asymptotically large $\mathbb{Q}$-independent sets. We also give some examples of sets of both types.

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INTRODUCTION

Recently, it was shown that there is a nonnegative measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that both of the sets $\{x : \sum_{n=1}^{\infty} f(nx) < +\infty\}$ and $\{x : \sum_{n=1}^{\infty} f(nx) = +\infty\}$ have positive measure [1]. This answered some questions which have seem to have arisen from two different sources. The first goes back to a problem of K. L. Chung treated by Hyde and Fine [5]. This direction culminated in Weizsäcker's question [10] concerning whether it is possible for both of the sets just described to have positive measure. The second direction came from a question of W. Schmidt. He asked whether there is a subset S of the positive reals with infinite measure such that x/y is never an integer whenever $x, y \in S, x \neq y$. Haight in [3] was led to this question by studying sets $E \subset \mathbb{R}^+$ which are of infinite measure and for each $x > 0$, $nx \in E$ for only a finite number of integers n. The existence of such sets, E was shown first by Lekkerkerker [8]. If $f = \mathbf{1}_E$ (the indicator function of E) then $\sum_{n=1}^{\infty} f(nx) < \infty$ but $\int_{\mathbb{R}+} f = \infty$.

Schmidt's question was answered independently by Haight [3] and by Szemerédi [9]. Haight in turn asked a more refined question in [4] which is answered in [1] by showing that the function f above can be taken to be the characteristic function of a set S. Erdős has discussed this number theoretic direction in [2]. Naturally, the question arises whether these results remain true when the positive integers are replaced by another infinite set. This paper focuses on this problem. For technical reasons we consider the equivalent additive problem.

PRELIMINARIES

The general purpose of this paper is to study the sets of convergence and divergence of some series of translates of nonnegative functions. Given Λ an infinite discrete set of nonnegative numbers, and $f : \mathbb{R} \to \mathbb{R}$, a nonnegative function, we consider the sum

$$s(x) = \sum_{\lambda \in \Lambda} f(x + \lambda),$$

and the complementary subsets of $\mathbb{R}$:

$$C = C(f, \Lambda) = \{x : s(x) < \infty\}, \qquad D = D(f, \Lambda) = \{x : s(x) = \infty\}.$$

In many cases the sets C and D are examined up to a set of zero Lebesgue measure (a null set). We shall use the notation $A = B$ a.e. instead of $\mathbf{1}_A = \mathbf{1}_B$ a.e., A and B being subsets of $\mathbb{R}$, $\mathbf{1}_A$ and $\mathbf{1}_B$ their indicator functions. We shall always assume that f is Lebesgue measurable, hence also C and D are Lebesgue measurable.

Let us first make some observations concerning the dependence of C and D on modifications of f for a given Λ.

Clearly, $C(f, \Lambda) = C(f \wedge \mathbf{1}, \Lambda)$. So, from now on we suppose that f is bounded.

Writing

$$u_\ell(x) = \sum_{\lambda\in\Lambda, \ell\le\lambda<\ell+1} f(x+\lambda),$$

it follows that there is an increasing sequence of positive integers $n_1, n_2, \ldots$ such that if we define $A_p = 0,\ p < n_1$, and $A_p = t,\ n_t \le p < n_{t+1}$, then

$$\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}\{x : \sum_{1\le\ell\le i} u_\ell(x) > A_i\} = \lim_{n\to\infty}\{x : \sum_{1\le\ell\le n} u_\ell(x) > A_n\} = D \ \text{ a.e.}$$

Now, let $\alpha(x)$ to be a positive decreasing function such that for a positive integer p, $\alpha(p) = (t+1)^{-1/2},\ n_t \le p < n_{t+1}$. Then $\lim_{x\to\infty}\alpha(x) = 0$ and for each $k\in\mathbb{N}, \lim_{n\to\infty}\alpha(n+k)A_n = +\infty$. This implies

$$D(\alpha f,\Lambda) = D(f,\Lambda) \ \text{ a.e.}$$

Thus, from now on we can additionally suppose that $f(x)$ tends to 0 at infinity provided we are interested in C and D up to a nullset.

Next, let us see the effect of modifying f on a set E which is small in the sense that

$$\text{meas }(E\cap(x,\infty)) \le \epsilon(x),$$

where $\epsilon(x)$ is a positive decreasing function and tends to 0 at infinity. If we restrict ourselves to a bounded interval, $x\in(-b,b)$, $u_\ell(x)$ is modified on a set whose measure does not exceed $\epsilon(\ell-b)\cdot\#(\Lambda\cap[\ell,\ell+1))$. Since Λ is given, $\epsilon(x)$ can be chosen in such a way that

$$\forall b>0, \qquad \sum_{\ell\in\mathbb{N}}\epsilon(\ell-b)\cdot\#(\Lambda\cap[\ell,\ell+1)) < \infty.$$

By the Borel-Cantelli lemma modifying f on E results in the same sets C and D up to sets of measure zero. In particular, we can change f into a continuous function tending to 0 at infinity without modifying C and D except on a null set. Let us state the conclusion of this discussion.

Proposition 1. *Given $f\ge 0$ and Lebesgue measurable, there exists $g\ge 0$, continuous and tending to 0 at infinity ($g\in C_0^+(\mathbb{R})$) such that*

$$C(f,\Lambda) = C(g,\Lambda) \ a.e.,\ D(f,\Lambda) = D(g,\Lambda) \ a.e.$$

Now let us see how C and D depend on Λ.

We always suppose that Λ is infinite. Given Λ we have two extreme cases

$$C(\mathbf{0},\Lambda) = \mathbb{R}, \quad D(\mathbf{0},\Lambda) = \emptyset,$$

$$C(\mathbf{1}, \Lambda) = \emptyset, \quad D(\mathbf{1}, \Lambda) = \mathbb{R}.$$

If these are the only cases that can occur up to a nullset, we say that Λ has type 1, and otherwise has type 2.

Definition 1. Λ has type 1 if, for every f, either $C(f, \Lambda) = \mathbb{R}$ a.e. or $C(f, \Lambda) = \emptyset$ a.e. (or equivalently $D(f, \Lambda) = \emptyset$ a.e. or $D(f, \Lambda) = \mathbb{R}$ a.e.). Otherwise, Λ has type 2.

We shall discuss two examples in particular.

Example 1. $\Lambda = \cup_{k \in \mathbb{N}} \Lambda_k$, $\Lambda_k = 2^{-k}\mathbb{N} \cap [k, k+1)$. In Theorem 1 we shall prove that Λ has type 1.

Example 2. Let $\Lambda = \{\log n, n = 1, 2, ...\}$. It was proved in [1] that Λ has type 2. This result will also be proven here from a different perspective which, in addition, allows us to see that this is the case for very many sets Λ.

It will be helpful to introduce a few other definitions concerning the density of Λ at $+\infty$.

Definition 2. Λ is asymptotically dense provided when it is ordered in the increasing order, the distance between two consecutive points tends to zero, or equivalently:

$$\forall a > 0, \quad \lim_{x \to \infty} \#(\Lambda \cap [x, x+a]) = \infty.$$

Otherwise, Λ is asymptotically lacunary, or equivalently, Λ has lacunae of the form $(x_j, x_j + l)$, $l > 0$, $x_j \to +\infty$.

In both of the preceding examples Λ is asymptotically dense. We note that our notion of asymptotically lacunary is not the same as the usual one of a "lacunary sequence."

We shall prove that all asymptotically lacunary sets are of type 2 (Theorem 4). In order to discuss the behaviour of the asymptotically dense sets more refined notions are necessary.

Definition 3. Let us say that $t > 0$ is a translator of Λ if $(\Lambda + t) \setminus \Lambda$ is finite, that is, $\lambda + t \in \Lambda$ for all but a finite number of $\lambda \in \Lambda$. The set of translators of Λ is a countable additive semigroup that we denote by $T(\Lambda)$. When $T(\Lambda)$ is dense in $\mathbb{R}^+$, that is, when Λ has arbitrarily small translators, we say that condition$(*)$ is satisfied.

Examples. In Example 1, $T(\Lambda)$ consists of all dyadic positive rationals; condition$(*)$ is satisfied. In Example 2, $T(\Lambda) = \Lambda = \{\log n\}$. Of course, $T(\mathbb{N}) = \mathbb{N}$. In Theorem 3 we construct type 2 sets Λ which are subsets of dyadic rationals and $T(\Lambda)$ consists of all dyadic positive rationals. Assume that the numbers $\{q_n\}_{n=1}^{\infty}$ are independent over the rationals and they converge to ∞. If we let $\Lambda = \cup_{n \in \mathbb{N}} \{q_n + 2 \cdot \mathbb{N}\} \cup \cup_{n \in \mathbb{N}} \{q_n + 3 + 2 \cdot \mathbb{N}\}$ then $T(\Lambda) = \mathbb{N} \setminus \{1\}$. This $T(\Lambda)$ is asymptotically lacunary but not generated by one element though it is a subset of $\mathbb{N}$. It is not difficult to see that if $T(\Lambda)$ is asymptotically lacunary then it is contained in an arithmetic progression.

Whenever Λ itself is an additive semigroup $T(\Lambda) = \Lambda$. If Λ consists of independent points over the rationals (e.g., $\Lambda = \{\log p : p \text{ prime}\}$), then $T(\Lambda) = \emptyset$.

Our interest in $T(\Lambda)$ is that $C = C(f, \Lambda)$ is invariant under the translations belonging to $T(\Lambda)$ and $D = D(f, \Lambda)$ under the opposite translations:

$$x \in C,\ t \in T(\Lambda) \Rightarrow x + t \in C, \qquad x \in D,\ t \in T(\Lambda) \Rightarrow x - t \in D.$$

This has an interesting consequence provided $T(\Lambda)$ is non empty: knowing C or D on an interval gives information about C to the right of this interval, or on D to the left.

Proposition 2. *Given an interval I suppose that C (resp. D) enjoys one of the following properties:*

a) $C \cap I$ (resp. $D \cap I$) is dense on I,

b) $C \cap I$ (resp. $D \cap I$) has positive Lebesgue measure,

c) $C \cap I$ (resp. $D \cap I$) has full measure on I,

d) C (resp. D) contains I.

Then the same property holds when I is replaced by $I + T(\Lambda)$ (resp. $I - T(\Lambda)$). In particular, if $T(\Lambda)$ is not contained in an arithmetic progression the property holds on some right (resp. left) half line.

This proposition can be applied when $\Lambda = \{\log n\}$ (Example 2).

Proposition 3. *Suppose that condition$(*)$ is satisfied (Λ has arbitrarily small translators). Then the topological closure of C (resp. D) is either $\emptyset$, or $\mathbb{R}$, or else a closed right half line (resp. left half line). The same holds for the support of $\mathbf{1}_C$ (resp. $\mathbf{1}_D$) meaning the smallest closed set carrying C (resp. D) except for a null set. The interior of C (resp. D) is either $\emptyset$ or $\mathbb{R}$ or else an open right (resp. left) half line.*

This applies in particular to Example 1.

THE MAIN RESULTS

Let us state and then prove the main results.

Theorem 1. *Let (n_k) be an increasing sequence of positive integers and let $\Lambda = \cup_{k \in \mathbb{N}} \Lambda_k$ where $\Lambda_k = 2^{-k}\mathbb{N} \cap [n_k, n_{k+1})$. Then Λ is of type 1.*

Theorem 2. *The set $\Lambda = \{\log n : n = 1, 2, ...\}$ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, C has full measure on the half-line $(0, \infty)$ and D contains the half line $(-\infty, 0)$. If for each c, $\int_c^{+\infty} e^y g(y)dy < +\infty$, then $C(g, \Lambda) = \mathbb{R}$ a.e. If $g \in C_0^+(\mathbb{R})$ and $C(g, \Lambda)$ is not of the first category (meager), then $C = \mathbb{R}$ a.e. Finally, there is some $g \in C_0^+(\mathbb{R})$ such that $C(g, \Lambda) = \mathbb{R}$ a.e. and $\int_0^{+\infty} e^y g(y)dy = +\infty$.*

Theorem 3. *Let (n_k) be a given increasing sequence of positive integers. There is an increasing sequence of integers $(m(k))$ such that the set $\Lambda = \cup_{k \in \mathbb{N}} \Lambda_k$ with $\Lambda_k = 2^{-m(k)}\mathbb{N} \cap [n_k, n_{k+1})$ is of type 2.*

Theorem 4. *If Λ asymptotically lacunary, then Λ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, there exist intervals I and J, I to the left of J, such that $C(f,\Lambda)$ contains I and $D(f,\Lambda)$ contains J.*

Theorem 5. *Suppose that there exist three intervals I, J, K such that $J = K+I-I$ (algebraic sum), I is to the left of J, and dist$(I,J) \geq |I|$, and two sequences (y_j) and (N_j) tending to infinity ($y_j \in \mathbb{R}^+$, $N_j \in \mathbb{N}$) such that, for each j, $y_j - I$ contains a set of N_j points of Λ independent from $\Lambda \cap (y_j - J)$ in the sense that the additive groups generated by these sets have only 0 in common. Then Λ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, $D(f,\Lambda)$ contains I and $C(f,\Lambda)$ has full measure on K.*

Remarks. If Λ is asymptotically dense and consists of elements independent over $\mathbb{Q}$ then using Theorem 5 it is easy to show that Λ is of type 2. The next example illustrates that in the statement of Theorem 5 a weaker independence assumption is not sufficient.

Example 3. Assume Λ_0 denotes the set of Example 1. It is of type 1. For a real number τ set $\Lambda_\tau = \Lambda - \tau$. It is obvious that $\sum_{\lambda\in\Lambda_0} f(x+\lambda)$ converges almost everywhere if and only if $\sum_{\lambda\in\Lambda_\tau} f(x+\lambda)$ does so. Set $\Lambda = \Lambda_0 \cup (\Lambda_{\sqrt{2}} \cap \cup_{j=1}^{\infty}[8j, 8j+1))$. Then choosing $I = [-1,0]$, $K = [2,3]$, $J = [1,4]$, $y_j = 8j$ it is clear that $y_j - I$ contains N_j points of Λ which are independent from $\Lambda \cap (y_j - J)$, with $N_j \to \infty$. This assumption is a little weaker than the assumption of Theorem 5. Next we show that Λ is of type 1. Hence this example illustrates that in Theorem 5 we need independence of the generated groups, independence of individual elements is not sufficient. Indeed, take a nonnegative measurable function f. If there is a set of positive measure where $\sum_{\lambda\in\Lambda} f(x+\lambda)$ converges then on this set $\sum_{\lambda\in\Lambda_0} f(x+\lambda)$ also converges. Since Λ_0 is of type 1, $\sum_{\lambda\in\Lambda_0} f(x+\lambda)$ converges almost everywhere. Then $\sum_{\lambda\in\Lambda_\tau} f(x+\lambda)$, and hence $\sum_{\lambda\in\Lambda} f(x+\lambda)$ both converge almost everywhere. Therefore Λ is of type 1.

¿From the sufficient condition of Theorem 5 one might think that for sets satisfying condition($*$) some sort of independence determines whether the set is type 2. However, Theorem 3 shows that this is not the case, there are sets Λ any two elements of which are dependent, Λ satisfies condition($*$) and yet Λ has type 2.

The sets of type 2 form a dense open set in the topology of "small perturbations" of discrete sets or box topology. Specifically, let us introduce a topology $\mathcal{T}$ on the space $\mathcal{D}$ of infinite discrete subsets of $\mathbb{R}^+$ as follows. Let $\Lambda \in \mathcal{D}$ and let $r_n \in \mathbb{R}^+$ for $n = 1, 2, \ldots$. We say that Λ' belongs to the $\mathcal{N}((\ell_1, r_1), (\ell_2, r_2), \ldots)$ neighborhood of Λ provided we can order the elements of Λ into a sequence $\{\ell_1, \ell_2, \ldots\}$ and we can order the elements of Λ' into a sequence (ℓ'_n) such that $\ell'_n \in (\ell_n - r_n, \ell_n + r_n)$. The topology $\mathcal{T}$ will be generated by these neighborhoods. We recall

Proposition 4. *The space $(\mathcal{D}, \mathcal{T})$ is a Baire space.*

Proof. Assume that $\mathcal{G} \subset \mathcal{D}$ is a non-empty open set and, proceeding towards a contradiction, $\mathcal{G} = \cup_{n=1}^{\infty} H_n$, where the H_n's are nowhere dense sets.

Arguing as in the proof of Baire's category theorem one can choose a nested sequence of neighborhoods $\mathcal{N}_n = \mathcal{N}((\ell_{1,n}, r_{1,n}), (\ell_{2,n}, r_{2,n}), ...) \subset \mathcal{G}$ such that $\mathcal{N}_n \cap H_n = \emptyset$, and we can also assume that for all n,

$$\ell_{1,n} < \ell_{2,n} < ..., \quad [\ell_{i,n} - r_{i,n}, \ell_{i,n} + r_{i,n}] \cap [\ell_{j,n} - r_{j,n}, \ell_{j,n} + r_{j,n}] = \emptyset \text{ if } i \neq j;$$

furthermore, for all i

$$[\ell_{i,n+1} - r_{i,n+1}, \ell_{i,n+1} + r_{i,n+1}] \subset (\ell_{i,n} - r_{i,n}, \ell_{i,n} + r_{i,n}) \text{ and } r_{i,n} \to 0 \text{ as } n \to \infty.$$

Then $\lambda_i = \lim_{n\to\infty} \ell_{i,n}$ exists and $\Lambda = \{\lambda_1, \lambda_2, ...\} \in \mathcal{G} \setminus \cup_{n=1}^{\infty} H_n$, a contradiction proving Proposition 4. ∎

Theorem 6. *The sets of type* 2 *form a dense open subset in* $\mathcal{T}$, *while sets of type* 1 *form a closed nowhere dense set. Therefore type* 2 *is typical in the Baire category sense.*

Proof of Theorem 6. If Λ is asymptotically lacunary, then it is of type 2 by Theorem 4. If Λ is asymptotically dense and (r_n) is given, then one can choose $\ell'_n \in (\ell_n - r_n, \ell_n + r_n)$ such that $\Lambda' = \{\ell'_1, \ell'_2 ...\}$ is asymptotically dense and consists of elements which are independent over $\mathbb{Q}$, the rationals. Then by the remark after Theorem 5 Λ' is of type 2. Therefore, the type 2 sets are dense in $\mathcal{T}$.

Assume now that $\Lambda = \{\ell_1, \ell_2, ...\}$ is of type 2. By Proposition 1 we can choose $f \in C_0^+(\mathbb{R})$ such that $C(f, \Lambda) \neq \mathbb{R}$ and $C(f, \Lambda) \neq \emptyset$ a.e. There are compact sets $F_C \subset C(f, \Lambda)$ and $F_D \subset D(f, \Lambda)$ both of positive measure. By using the continuity of f for a given n, one can choose $r_n > 0$ such that $|f(x + \ell') - f(x + \ell_n)| < 1/n^2$ holds for all $x \in F_C \cup F_D$ and for all $\ell' \in (\ell_n - r_n, \ell_n + r_n)$. This easily implies that if Λ' belongs to the (r_n) neighborhood of Λ then $F_C \subset C(f, \Lambda')$ and $F_D \subset D(f, \Lambda')$. ∎

We shall prove Theorems 4, 5, 2, 3, and 1 successively.

Proof of Theorem 4. Since Λ is asymptotically lacunary, there is a sequence x_j tending to infinity such that

$$(x_j - \Lambda) \cap (\alpha, \beta) = \emptyset \quad (j = 1, 2, ...) \tag{1}$$

for some open interval (α, β). The same remains true if we move each x_j to the left, if necessary, until

$$\beta \in x_j - \Lambda \quad (j = 1, 2, ...), \tag{2}$$

and from this point on, we suppose this is so. With these adjustments note that we still have $x_j \to +\infty$. Let $0 < \delta < (\beta - \alpha)/2$,

$$I = (\alpha + \delta, \beta - \delta), \ J = (\beta - \frac{\delta}{2}, \beta + \frac{\delta}{2}).$$

Let Δ be the triangular function based on $[-\delta, \delta]$, that is $\Delta(x) = (1 - |x/\delta|) \vee \mathbf{0}$, and

$$f(x) = \sum_j m_j \Delta(x - x_j),$$

with $m_j \downarrow 0$ and $\sum m_j = \infty$. Then $f \in C_0^+(\mathbb{R})$. If $x \in J$, then $s(x) = \sum_{\lambda \in \Lambda} f(x + \lambda) \geq \sum_j f(x + x_j - \beta) \geq \sum_j \sum_l m_l \Delta(x + x_j - \beta - x_l) \geq \sum_j m_j \Delta(x - \beta) \geq (1/2) \sum_j m_j = +\infty$. If $x \in I$, then for all j and $\lambda \in \Lambda, x + \lambda - x_j \notin (-\delta, \delta)$ and $s(x) = 0$ on I. ■

Proof of Theorem 5. Using translation and dilatation if necessary, we can suppose $I = (-1, 0)$; therefore $J = (a-1, b+1)$ and $K = (a, b)$ with $a \geq 2$. We intend to construct $f \in C_0^+(\mathbb{R})$ such that $s(x) = \infty$ on I and $s(x) < \infty$ a.e. on K. Firstly, we shall construct a continuous f; a trivial change gives $f \in C_0^+(\mathbb{R})$.

The principle is to select a subsequence of j, still denoted by j, such that N_j is a very rapidly increasing sequence, then choose the functions f_j carried by the intervals $[y_j - 1, y_j + 1]$, which we may assume disjoint, such that $s_j(x) = \sum_{\lambda \in \Lambda} f_j(x + \lambda) \geq 0.99$ on I and $s_j(x) = 0$ on K except on a set of measure μ_j with $\sum \mu_j < \infty$. To do this we need the following Lemma.

Lemma 1. *Suppose that E, a finite subset of $\mathbb{R}$, has the following property:*
(♣) *all non-trivial linear combinations $\sum_{\lambda \in E} n_\lambda \lambda$ with $n_\lambda \in \mathbb{Z}$, $|n_\lambda| \leq q \in \mathbb{N}$ are different.*
Then, given any linear combination of the $e^{i\lambda t}$, $(\lambda \in E)$,

$$P(t) = \sum_{\lambda \in E} r_\lambda e^{i\lambda(t - \phi_\lambda)} \quad (r_\lambda \geq 0,\ \phi_\lambda \in \mathbb{R})$$

we have

$$\sup_{t \in \mathbb{R}} Re\, P(t) = \limsup_{t \mapsto +\infty} Re\, P(t) \geq \left(1 - \frac{1}{q}\right) \sum_{\lambda \in E} r_\lambda.$$

Proof of Lemma 1. This fact is well known in the theory of Sidon sets [7]. For the convenience of the reader we provide a proof. Let

$$K(t) = \sum_{-q < n < q} \left(1 - \frac{|n|}{q}\right) e^{int} \quad \text{the Fejér kernel}$$

and

$$R(t) = \prod_{\lambda \in E} K(\lambda(t - \phi_\lambda)) \text{ (kind of Riesz product)},$$

then $R(t) \geq 0$ and

$$\mathcal{M}R = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R(t) dt = 1.$$

Recalling, that $\mathcal{M}e^{i\mu t} = 1$ if $\mu = 0$, and 0 if $\mu \neq 0$, due to the assumption on E,

$$\mathcal{M}PR = \left(1 - \frac{1}{q}\right) \sum_{\lambda \in E} r_\lambda.$$

The result follows from $\mathcal{M}PR \leq \sup_{t\in\mathbb{R}} \mathrm{Re}(P(t)) \cdot \mathcal{M}R$. ■

Remark. Let F be an arbitrary finite set of real numbers. One can use Lemma 1 to show $\limsup_{t\to\infty} \sum_{\lambda\in F} \cos(\lambda t) = \#F$. Of course, this fact is well known. The function $p(t) = \sum_{\lambda\in F} \cos \lambda t$ is almost periodic and $|p(t)| \leq p(0) = \#F$. On the other hand, by almost periodicity [7], there exists $t_n \to \infty$ such that $p(t_n) \to p(0)$. Hence, $\limsup_{t\to\infty} \sum_{\lambda\in F} \cos(\lambda t) = \#F$.

Proof of Theorem 5 (continuation). Now, for any fixed q, given any set $\mathcal{E}$ of N nonzero real numbers we inductively select points $\lambda \in \mathcal{E}$ in such a way that condition ♣ is satisfied. At step n we have already selected $n-1$ numbers λ and these numbers forbid less than $(4q+1)^n$ numbers for the next choice. Thus, we can proceed to step $\nu > 1$, if $\sum_{n=2}^{\nu}(4q+1)^n \leq N$. We define

$$\nu(q,N) = \max\{\nu \in \mathbb{N} : \sum_{n=2}^{\nu}(4q+1)^n \leq N\};$$

hence from $\mathcal{E}$ we can always select a subset E of cardinality at least $\nu(q,N)$ for which Lemma 1 applies. Also observe that $\lim_{N\to\infty} \nu(q,N) = \infty$.

Fix $0 < \epsilon < 10^{-4}$ with ϵ irrational. Set $q_j = j^6([1/\epsilon]+1)$. By taking a suitable subsequence, still denoted by N_j, we can assume that $\nu(q_j, N_j) \geq (\epsilon q_j)^{1/5}$. Then there exists $\mathcal{E}_j \subset \Lambda \cap (y_j - I)$ of cardinality N_j, consisting of points which are not rational linear combinations of points in $\Lambda \cap (y_j - J)$. In $\mathcal{E}_j$ choose a set E_j of cardinality $\nu_j = [(\epsilon q_j)^{1/5}]$ satisfying condition ♣. Observe that $\sum \frac{1}{\nu_j} < \infty$. Let us write $q = q_j$, $E = E_j$, $\nu = \nu_j$, and $F = \Lambda \cap (y_j - J)$.

Since the $\mathbb{Q}$-linear spaces generated by E and F have only 0 in common, whatever may be the complex numbers $a(\lambda)$, $(\lambda \in E)$ and $b(\lambda)$, $(\lambda \in F)$, we can choose t such that

$$\mathrm{Re} \sum_{\lambda\in E} a(\lambda) e^{2\pi i \lambda t} \text{ and } \mathrm{Re} \sum_{\lambda\in F} b(\lambda) e^{2\pi i \lambda t}$$

are arbitrarily close to

$$\limsup_{t\to\infty} \mathrm{Re} \sum_{\lambda\in E} a(\lambda) e^{2\pi i \lambda t} \text{ and } \limsup_{t\to\infty} \mathrm{Re} \sum_{\lambda\in F} b(\lambda) e^{2\pi i \lambda t}.$$

Let $\ell(\cdot)$ be a one-to-one mapping of E onto $\{1, 2, ..., \nu\}$. We choose $a(\lambda) = \exp(2\pi i \frac{\ell(\lambda)}{\nu})$, $(\lambda \in E)$ and $b(\lambda) = 1$, $(\lambda \in F)$. By using Lemma 1, the fact that

$(1-\frac{1}{q})\nu > \nu - \frac{\epsilon}{\nu^4}$, and the remark after Lemma 1, we choose t so that

$$\sum_{\lambda\in E} \cos\left(2\pi\left(\lambda t + \frac{\ell(\lambda)}{\nu}\right)\right) > \nu - \frac{\epsilon}{\nu^4}, \text{ and}$$

$$\sum_{\lambda\in F} \cos 2\pi\lambda t > \#F - \frac{\epsilon}{\nu^4}.$$

These inequalities imply

$$\lambda t \simeq -\frac{\ell(\lambda)}{\nu} \bmod 1 \text{ when } \lambda\in E,$$

(more precisely we know $\lambda t + \frac{\ell(\lambda)}{\nu} \in \mathbb{Z} + (\frac{-\sqrt{\epsilon}}{\nu^2}, \frac{\sqrt{\epsilon}}{\nu^2})$), and

$$\lambda t \simeq 0 \bmod 1 \text{ when } \lambda \in F$$

(more precisely we know $\lambda t \in \mathbb{Z} + (\frac{-\sqrt{\epsilon}}{\nu^2}, \frac{\sqrt{\epsilon}}{\nu^2})$).

Now we define

$$f_j(y) = \sum_{k\in\mathbb{Z}} \Delta(ty-k)\mathbf{1}_{[-1,1]}(y-y_j),$$

where Δ is the triangular function based on $[-\delta,\delta]$ with $\delta = 1/\nu$ and $\Delta(0)=1$. When $x\in I=(-1,0)$,

$$s_j(x) = \sum_{\lambda\in\Lambda} f_j(x+\lambda) = \sum_{k\in\mathbb{Z},\lambda\in\Lambda} \Delta(tx+t\lambda-k)\mathbf{1}_{[-1,1]}(x+\lambda-y_j)$$

$$\geq \sum_{k\in\mathbb{Z},\lambda\in E} \Delta(tx+t\lambda-k) = \sum_{\lambda\in E}\sum_{k\in\mathbb{Z}} \Delta(tx+m_\lambda-\frac{\ell(\lambda)}{\nu}+w_\lambda-k),$$

where $m_\lambda\in\mathbb{Z}$ and $|w_\lambda| < \frac{\sqrt{\epsilon}}{\nu^2}$. Notice that for each $\lambda\in E$, there is at most one value of $k=k_\lambda\in\mathbb{Z}$ such that $|tx-\frac{\ell(\lambda)}{\nu}-k_\lambda| < \delta$. Thus,

$$s_j(x) \geq \sum_{\lambda\in E} \Delta(tx - \frac{\ell(\lambda)}{\nu} + w_\lambda - k_\lambda).$$

Since Δ has Lipshitz constant ν, we have

$$s_j(x) \geq \sum_{\lambda\in E}(\Delta(tx-\frac{\ell(\lambda)}{\nu}-k_\lambda) - \nu|w_\lambda|) \geq \sum_{\ell=1}^{\nu} \Delta(tx-\frac{\ell}{\nu}-k_\lambda) - \sqrt{\epsilon} = 1-\sqrt{\epsilon} \geq 0.99.$$

On the other hand, when $x \in K = (a, b)$, we have

$$s_j(x) = \sum_{k\in\mathbb{Z},\lambda\in F} \Delta(tx + t\lambda - k)\mathbf{1}_{[-1,1]}(x + \lambda - y_j)$$

because $|x + \lambda - y_j| > 1$ when $x \in K$ and $\lambda \notin y_j - J$. Thus, if $x \in K$,

$$s_j(x) \le \sum_{\lambda\in F}\sum_{k\in\mathbb{Z}} \Delta(tx + t\lambda - k).$$

Now, for each $\lambda \in F$, $\sum_{k\in\mathbb{Z}} \Delta(tx + t\lambda - k)$ is carried on a subset of $\cup_{k\in\mathbb{Z}}(k/t - 1/t\nu - \sqrt{\epsilon}/\nu^2, k/t + 1/t\nu + \sqrt{\epsilon}/\nu^2)$. These intervals are uniformly spaced and have length $2/t\nu + 2\sqrt{\epsilon}/t\nu^2$. Taking into account the maximum number of these intervals which can hit K, we have $s_j(x) = 0$, for $x \in K$ except on a set of measure $2(|K| + (1/t))(1/\nu_j + \sqrt{\epsilon}/\nu_j^2) = \mu_j$ and $\sum \mu_j < \infty$ as wanted.

These considerations provide a continuous and bounded f. In order to have f tending to 0 at infinity, it suffices to multiply f_j by a scalar $\alpha_j \downarrow 0$ such that $\sum \alpha_j = \infty$, and to define $f = \sum \alpha_j f_j$. ■

Proof of Theorem 2. For $i = 0, 1, ...,$ set $a_i = \left(\frac{3}{4}\right)^i$. Define the sequence c_j by setting $c_1 = 1$ and for $j \ge 1, c_{j+1} = (3/2)c_j + 1$. For each $i \ge 0$, set $u_{0,i} = 0$ and $u_{j,i} = c_j a_i, j \ge 1$. For each fixed i, the intervals $I_{j,i} = [-u_{j+1,i}, -u_{j,i}], K_{j,i} = [2a_i, 4]$ and $J_{j,i} = [u_{j,i} - u_{j+1,i} + 2a_i, u_{j+1,i} - u_{j,i} + 4]$ satisfy the first part of the hypothesis of Theorem 5. For each i, let $y_{j,i}$ tend to $+\infty$ and let $N_{j,i} = \#G_{j,i}$, where $G_{j,i} = \{\log p : p \text{ is a prime and} \log p \in y_{j,i} - I_{j,i}\}$. It now follows from the Prime Number Theorem that all the conditions of Theorem 5 are satisfied. Thus, there is some function $f_{j,i} \in C_0^+(\mathbb{R})$ such that $I_{j,i} \subset D(f_{j,i}, \Lambda)$ and $C(f_{j,i})$ has full measure on $[2a_i, 4]$. If necessary, by truncation, we may assume $f_{j,i}(x) \le 1$, for all x. Now choose $0 < \beta_{j,i} < 2^{-j}$ such that

$$\text{meas } \{x \in [2a_i, 4] : \sum_{n=1}^{\infty} \beta_{j,i} f_{j,i}(x + \log n) \ge \frac{1}{2^j}\} < \frac{1}{2^j}.$$

Let $g_i = \sum_{j=0}^{\infty} \beta_{j,i} f_{j,i}$. Then $D(g_i, \Lambda) \supset \cup D(f_{j,i}, \Lambda) \supset \cup I_{j,i} = (-\infty, 0)$. Also, by the Borel-Cantelli Lemma, $C(g_i, \Lambda)$ has full measure on $[2a_i, 4]$. Again, we can truncate each g_i so that for all $x, g_i(x) \le 1$. Now, choose $0 < \beta_i < 2^{-i}$ such that

$$\text{meas } \{x \in [2a_i, 4] : \sum_{n=1}^{\infty} \beta_i g_i(x + \log n) \ge \frac{1}{2^i}\} < \frac{1}{2^i}.$$

Set $b_0 = 0$, $b_i = \sum_{j=0}^{i-1} a_j = 4 - 4a_i$ and

$$f(x) = \sum_{i=0}^{\infty} \beta_i g_i(x - a_i - b_i) = \sum_{i=0}^{\infty} \beta_i g_i(x - 4 + 3a_i).$$

Observe that $f \in C_0^+(\mathbb{R})$. Furthermore, $s_{g_i}(x) = \sum_{n=1}^{\infty} \beta_i g_i(x - 4 + 3a_i + \log n)$ diverges on $(-\infty, 4 - 3a_i)$, converges a.e. on $(4 - a_i, 8 - 3a_i)$ and

$$\text{meas }\{x \in [4,5] : \ s_{g_i}(x) \geq \frac{1}{2^i}\} \leq \text{meas }\{x \in [4 - a_i : 8 - 3a_i],\ s_{g_i}(x) \geq \frac{1}{2^i}\} < \frac{1}{2^i}.$$

By the Borel-Cantelli lemma, $\sum_{i=1}^{\infty} s_{g_i}(x)$ converges a.e. on $[4,5]$. Also, if $\sum_{n=1}^{\infty} f(x+\log n)$ converges, $\sum_{n=1}^{\infty} f(x+\log 2+\log n)$ converges. Hence, $\sum_{n=1}^{\infty} f(x+\log n)$ converges a.e. on $(4,\infty)$ and diverges on $(-\infty,4)$. Translating f by 4, one can obtain the desired function. This completes the proof of the first two statements of the theorem.

Fix $-\infty < c < d < \infty$. Then

$$\int_c^d s(x)dx = \int_c^d \sum_{n=1}^{\infty} g(x + \log n)dx = \sum_{n=1}^{\infty} \int_{c+\log n}^{d+\log n} g(y)dy = \int_c^{\infty} \psi(y)g(y)dy,$$

where $\psi(y) = \#\{n : \ c + \log n < y < d + \log n\ \} = \#\{n : e^y/e^d < n < e^y/e^c\ \}$. So, there exists $K_0 \in \mathbb{R}$ and a constant $\tau > 1$ such that $\tau^{-1}e^y \leq \psi(y) \leq \tau e^y$, if $y > K_0$. From this it follows that if $\int_c^{+\infty} e^y g(y)dy < \infty$, for each c, then $C(g,\Lambda) = \mathbb{R}$ a.e.

To prove the fourth statement, assume $g \in C_0^+(\mathbb{R})$ and $C = C(g,\Lambda)$ is not of the first category. Since the function s is lower semi-continuous, the sets $A_M = \{x : s(x) \leq M\}$ are closed and $C = \cup_{M\in\mathbb{N}} A_M$, there is some M and an interval $(c_0, d_0) \subset C$. Then $\int_{c_0}^{d_0} s(x)dx < M(d_0 - c_0)$. Therefore, as we have just seen, $\int_{c_0}^{\infty} e^y g(y)dy < +\infty$. But, then this last integral is finite for all c and therefore $C = \mathbb{R}$ a.e.

To see the final part of Theorem 2, we note that Haight [4] gave an example of a function f such that $C(f,\Lambda) = \mathbb{R}$ a.e., but $\int_0^{\infty} e^y f(y)dy = +\infty$. The function f is the characteristic function of a sequence of disjoint intervals, (c_n, d_n) converging to $+\infty$. It is straightforward to see that this function may be modified as indicated in the proof of Proposition 1 to yield a function $g \in C_0^+(\mathbb{R})$ with the same properties and $\int_0^{\infty} e^y g(y)dy = +\infty$. ■

Remarks. Haight's example actually concerns nonnegative functions defined on the positive reals and sums of the form $\sum f(nx)$. However, as we mentioned at the beginning a simple transformation converts this to our setup. The third part of Theorem 2 has been known for quite some time and the fourth part is noted by Fine and Hyde [5].

Proof of Theorem 3. For convenience we assume that the n_k's are multiples of 8 (the general case follows by a scaling of 1/8). Set $I = (-1,0)$, $J = (1,4)$.

Set $m(0) = 0$ and $\Lambda_0 = \emptyset$. Assume that at step $j \geq 1$, we have already defined the numbers $m(k)$, and hence the sets Λ_k for $k = 1,...,j-1$.

We will use a modification of the argument of Theorem 5. Set $y_j = n_j$. Using the definitions of ϵ, q_j and $\nu(q,N)$ given in the proof of Theorem 5, choose N_j such that $\nu(q_j, N_j) > \nu_j = [(\epsilon q_j)^{1/5}]$. We also choose a set $\Lambda_j' \subset (n_j, n_j+1) = y_j - I$ consisting of N_j numbers none of which is a dyadic rational. Using the definition of ν_j, choose an

$E = E'_j = \{\ell_1, ..., \ell_{\nu_j}\} \subset \Lambda'_j$ satisfying condition ♣ and set $F = \cup_{k=1}^{j-1} \Lambda_k \cap (n_j - 4, n_j - 1)$, (when $j = 1$ one can use $F = \emptyset$). Next, by choosing a suitable t define $f_j(y)$ as in the proof of Theorem 5. By continuity we can choose $m(j) > m(j-1)$ and $\{\lambda_1, ..., \lambda_{\nu_j}\} = E_j \subset 2^{-m(j)}\mathbb{N} \cap (n_j, n_j + 1)$ such that

$$\left| \sum_{k\in\mathbb{Z},\lambda'\in E'_j} \Delta(tx + t\lambda' - k) - \sum_{k\in\mathbb{Z},\lambda\in E_j} \Delta(tx + t\lambda - k) \right| < 0.01. \tag{3}$$

Set $\Lambda_j = 2^{-m(j)}\mathbb{N} \cap [n_j, n_{j+1})$, and $\Lambda_j^{\leq} = \cup_{k\leq j}\Lambda_k$. Then (3) and an argument similar to the one used in the proof of Theorem 5 shows that for $x \in I = (-1, 0)$

$$s_j(x) = \sum_{k\in\mathbb{Z},\lambda\in\Lambda_j^{\leq}} \Delta(tx + t\lambda - k)\mathbf{1}_{[-1,1]}(x + \lambda - y_j) \geq 0.98$$

holds.

For $x \in K = (2, 3)$ we will still have the property that $s_j(x) = 0$ except on a set of small measure μ_j, satisfying $\sum \mu_j < \infty$.

Using the above steps we can define $m(j)$ and Λ_j for $j = 1, 2, ...$

Finally, observe that for $x \in I \cup K$ and for $j' \geq j$ we have

$$s_j(x) = \sum_{k\in\mathbb{Z},\lambda\in\Lambda_{j'}^{\leq}} \Delta(tx+t\lambda-k)\mathbf{1}_{[-1,1]}(x+\lambda-y_j) = \sum_{k\in\mathbb{Z},\lambda\in\Lambda} \Delta(tx+t\lambda-k)\mathbf{1}_{[-1,1]}(x+\lambda-y_j).$$

This will imply that one can obtain a suitable f as in Theorem 5. ∎

To prove Theorem 1 we need a lemma.

In the sequel functions defined on the circle, $\mathbb{T}$ and functions on $\mathbb{R}$, periodic by 1 will be identified.

Lemma 2. *Let $\varphi_n : \mathbb{T} \to \mathbb{R}$ be positive measurable functions. If $\sum_{n=1}^{\infty} \varphi_n(2^n t) < \infty$ a.e., it then follows that $\sum_{n=1}^{\infty} \varphi_n(2^n t + \frac{1}{2}) < \infty$.*

Proof of Lemma 2. First, it is easy to see that due to the periodicity property there is a $0-1$ law, that is, if $\sum \varphi_n(2^n t+\frac{1}{2}) < \infty$ does not hold a.e. then $\sum \varphi_n(2^n t+\frac{1}{2}) = \infty$ a.e. and this implies that $\sum \int_A \varphi_n(2^n t + \frac{1}{2})dt = \infty$ holds when $|A| > 0$.

Choose $H \subset \mathbb{T}$ closed such that $|H| > 0.99$ and $\sum \varphi_n(2^n t) \to \varphi(t)$ uniformly on H and $|\varphi| \leq M$, where $M \in \mathbb{R}$.

Set $H_0 = H$, $H_n = \cup_{j=0}^{2^n-1}(H + \frac{j}{2^n})$, and for $n = 0, 1, ...$ define $G_n = H_{n+1} \setminus H_n$. We observe that $H_n \subset H_{n+1}$, therefore $\sum_{n=0}^{\infty} |G_n| \leq |\mathbb{T}| - |H| = 0.01$.

Set $H' = H \setminus \cup_{n=0}^{\infty}(G_n - \frac{1}{2^{n+1}})$. Then $|H'| \geq |H| - \sum |G_n| > 0.9$. Observe that $t \in H'$ implies $t \in H$ and $t + \frac{1}{2^{n+1}} \notin G_n$. Since $t \in H$ implies $t \in H_{n+1}$, $t + \frac{1}{2^{n+1}} \in H_{n+1}$, and $H_{n+1} \setminus G_n = H_n$, we see that $t \in H'$ implies $t + \frac{1}{2^{n+1}} \in H_n$ holds for each n. Thus

$$\int_{H'} \varphi_n(2^n t + \frac{1}{2})dt = \int_{H'} \varphi_n(2^n(t + \frac{1}{2^{n+1}}))dt \leq \int_{H_n} \varphi_n(2^n t)dt.$$

Therefore if we can show that $\sum \int_{H_n} \varphi_n(2^n t)dt < \infty$ then we showed that $\sum \int_{H'} \varphi_n(2^n t + \frac{1}{2})dt < \infty$ and this implies that $\sum \varphi_n(2^n t + \frac{1}{2}) < \infty$ holds a.e.

Recalling that $\int_{H_n} = \int_H + \int_{G_0} + ... + \int_{G_{n-1}}$, we have

$$\sum_{n=1}^{\infty} \int_{H_n} \varphi_n(2^n t)dt = \int_H \sum_{n=1}^{\infty} \varphi_n(2^n t)dt + \sum_{m=0}^{\infty} \int_{G_m} \sum_{n=m+1}^{\infty} \varphi_n(2^n t)dt \leq$$

$$M(|H| + \sum_m |G_m|) \leq M.$$

This completes the proof. ■

Proof of Theorem 1. Given m choose $k(m)$ such that $[n_{k(m)}, n_{k(m)+1}) \supset [m, m+1)$. Set $\Lambda'_m = \Lambda \cap [m, m+1) = 2^{-k(m)}\mathbb{N} \cap [m, m+1)$.

Let us introduce $f_m(x) = f(x+m)$, $g_m = f_m \mathbf{1}_{[1,3]}$,

$$s_m(x) = \sum_{\lambda \in \Lambda'_m} f(x + \lambda) = \sum_{0 \leq j < 2^{k(m)}} f_m(x + \frac{j}{2^{k(m)}}),$$

$$u_m(x) = \sum_{0 \leq j < 2^{k(m)}} g_m(x + \frac{j}{2^{k(m)}}), \qquad v_m(x) = \sum_{0 \leq j < 2^{k(m+1)}} g_m(x + \frac{j}{2^{k(m+1)}}).$$

Then $s = \sum s_m = \sum_{\lambda \in \Lambda} f(x + \lambda)$, u_m and v_m are carried by $[0, 3]$.

Set $\Gamma = \{m \in \mathbb{N} : m \neq n_{k(m)+1} - 1\}$.

Observe that if $m \in \Gamma$ then $k(m+1) = k(m)$ and hence $v_m(x) = u_m(x)$; if $m \notin \Gamma$ then $k(m+1) = k(m)+1$ and $v_m(x) = u_m(x) + u_m(x + \frac{1}{2^{k(m+1)}})$. For all x, $s_m(x+1) \leq s_{m+1}(x)$. So, for all x, $u_m(x) \leq s_m(x)$, $u_m(x+1) \leq s_{m+1}(x)$, and $u_m(x+2) \leq s_{m+2}(x)$. Moreover, for $x \in [0,1]$, $v_m(x+1) = s_{m+1}(x)$. We define

$$u^*_m(x) = \sum_{n \in \mathbb{Z}} u_m(x+n), \quad v^*_m(x) = \sum_{n \in \mathbb{Z}} v_m(x+n), \quad (x \in \mathbb{R}).$$

Then $u^*_m(x)$ is periodic by $2^{-k(m)}$, indeed,

$$u^*_m(x + \frac{1}{2^{k(m)}}) = \sum_{n \in \mathbb{Z}} \sum_{0 \leq j < 2^{k(m)}} g_m(x + \frac{j+1}{2^{k(m)}} + n) =$$

$$\sum_{n\in\mathbb{Z}}\sum_{0<j'\le 2^{k(m)}} g_m(x+\frac{j'}{2^{k(m)}}+n)=u_m^*(x).$$

Denote by C^* the set of those points where $\sum u_m^*$ converges. If $\sum s_m(x)<\infty$ on a set of positive measure in $[0,1]$ the same holds for $\sum u_m^*(x)\le 3\sum s_m(x)$; hence $C^*\cap[0,1]$ is of positive measure. Since $k(m)$ is monotone increasing and tends to ∞ one can easily see that C^* is periodic by 2^{-k} for all k. Therefore $\sum u_m^*(x)<\infty$ a.e.

Observe that $\sum_{m\in\Gamma} u_m^*(x)=\sum_{m\in\Gamma} v_m^*(x)$. Set $\psi_k(x)=u_{n_{k+1}-1}^*(x)$. Then ψ_k is periodic by 2^{-k}. Clearly, $\sum\psi_k(x)<\infty$ a.e. and Lemma 2 applied for the functions $\varphi_k(x)=\psi_k(2^{-k}x)$ yields $\sum\varphi_k(2^kx+\frac{1}{2})=\sum\psi_k(x+\frac{1}{2^{k+1}})<\infty$ a.e. This implies $\sum_{m\in\mathbb{N}\setminus\Gamma} v_m^*(x)<\infty$ a.e.. Hence $\sum_m s_{m+1}(x)\le\sum_m v_m^*(x)<\infty$ a.e. on $[0,1]$. In other words, either $\sum s_m(x)=\infty$ a.e. on $[0,1]$, or $\sum s_m(x)<\infty$ a.e. on $[0,1]$.

If f is arbitrary and for an $x_0\in\mathbb{R}$, $C(f,\Lambda)\cap[x_0,x_0+1)$ is of positive measure. Then for $f_{x_0}(x)=f(x+x_0)$ we have $C(f_{x_0},\Lambda)\cap[0,1)$ is of positive measure and hence $C(f_{x_0},\Lambda)$ contains almost every point of $[0,1)$, that is, $C(f,\Lambda)$ contains almost every point of $[x_0,x_0+1)$. Using this argument for $x_0\pm(k/2)$ for $k=1,2,...$, by induction on $|k|$, we obtain that $C(f,\Lambda)$ contains almost every point of $\mathbb{R}$.

Thus either $s(x)<\infty$ or $s(x)=\infty$ a.e. on $\mathbb{R}$. ■

COMMENTS AND QUESTIONS

In Theorems 2,3,4 and 5 it is shown that there is some $f\in C_0^+(\mathbb{R})$ such that both $C(f,\Lambda)$ and $D(f,\Lambda)$ have positive Lebesgue measure. It is easy to construct a measurable function f with these same properties which has values in $\{0,1\}$. This leads to

QUESTION 1. Is it true that Λ is of type 2 if and only if there is a $\{0,1\}$ valued measurable function f such that both $C(f,\Lambda)$ and $D(f,\Lambda)$ have positive Lebesgue measure?

When Λ is asymptotically lacunary, it is possible to construct $f\in C_0^+(\mathbb{R})$ such that both $C(f,\Lambda)$ and $D(f,\Lambda)$ have interior points, say x_C and x_D. In fact, Theorem 4 states that we may require $x_C<x_D$. One can obtain $x_D<x_C$ as well.

QUESTION 2. Given open sets G_1 and G_2 when is it possible to find Λ and f such that $C(f,\Lambda)$ contains G_1 and $D(f,\Lambda)$ contains G_2?

We note a partial answer to Question 2. Suppose the counting function of Λ, $n(x)=\#\{\Lambda\cap[0,x]\}$ satisfies a condition of the type

$$\text{(A)}\qquad \forall\ell<0\ \forall a\in\mathbb{R}\quad \limsup_{x\to\infty}\frac{n(x+\ell+a)-n(x+a)}{n(x+\ell)-n(x)}<+\infty$$

(as is the case for $\Lambda=\{\log n\}$). Then either C has full measure on $\mathbb{R}$ or C does not contain any interval. To see this note that if C contains an interval, Baire's theorem shows that $s(x)$ is bounded on some subinterval $(x_0,x_0+\ell)$. Therefore,

$$\int f(y)(n(y-x_0)-n(y-x_0-\ell))dy = \int_{x_0}^{x_0+\ell} s(x)dx < \infty.$$

According to (A), the same holds when x_0 is replaced by $x_0 - a$. Thus, $s(x) < \infty$ a.e. The meaning of (A) is that Λ is asymptotically dense in a regular way. If we relax condition (A) by replacing $(\forall a \in \mathbb{R})$ by $(\forall a \in \mathbb{R}^+)$ or by $(\forall a \in \mathbb{R}^-)$, we obtain a weaker conclusion, namely C has full measure on either a left or right half line.

Finally, let us make comments on the assumption that Λ is a discrete set of non-negative numbers. Types 1 and 2 can be defined without this assumption. For example, if Λ consists of all dyadic rationals, then $C(f,\Lambda) = \emptyset$ or $C(f,\Lambda) = \mathbb{R}$ a.e. according to $\int |f(x)|dx = 0$ or > 0. This may be proved as a very simple application of the zero-one law, while the proof of Theorem 1 uses the same tool in a much more elaborated way. More generally, suppose λ_∞ is an accumulation point of Λ. Then $D(f,\Lambda) \supset \{x : f(x) > 0\} - \lambda_\infty$ a.e. To see this, suppose F is compact, $\text{meas}(F) > 0, c > 0$ and $f > c$ on F. Let λ_n be distinct elements of Λ converging to λ_∞. Set $H = \limsup_{n\to\infty}(F - \lambda_n)$. Then $H \subset F - \lambda_\infty$, $\text{meas}(H) = \text{meas}(F)$ and $H \subset D(f,\Lambda)$. In particular, if $f > 0$ a.e. and there is a sequence, λ_N, of accumulation points of Λ such that $\lim \lambda_N = +\infty$, then $D(f,\Lambda) = \mathbb{R}$ a.e.

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